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The forcing hull and forcing geodetic numbers of graphs

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ABSTRACT

For every pair of vertices u, v in a graph, a u–v geodesic is a shortest path from u to v. For a graph G, let $I_G[u, v]$ denote the set of all vertices lying on a u–v geodesic. Let $S \subseteq V(G)$ and $I_G[S]$ denote the union of all $I_G[u, v]$ for all $u, v \in S$. A subset $S \subseteq V(G)$ is a convex set of G if $I_G[S] = S$. A convex hull $[S]_G$ of S is a minimum convex set containing S. A subset S of $V(G)$ is a hull set of G if $[S]_G = V(G)$. The hull number $h(G)$ of a graph G is the minimum cardinality of a hull set in G. A subset S of $V(G)$ is a geodetic set if $I_G[S] = V(G)$. The geodetic number $g(G)$ of a graph G is the minimum cardinality of a geodetic set in G. A subset $F \subseteq V(G)$ is called a forcing hull (or geodetic) subset of G if there exists a unique minimum hull (or geodetic) set containing F. The cardinality of a minimum forcing hull subset in G is called the forcing hull number $f_h(G)$ of G and the cardinality of a minimum forcing geodetic subset in G is called the forcing geodetic number $f_g(G)$ of G. In the paper, we construct some 2-connected graph G with $(f_h(G), f_g(G)) = (0, 0), (1, 0)$, or $(0, 1)$, and prove that, for any nonnegative integers a, b, and c with $a + b \geq 2$, there exists a 2-connected graph G with $(f_h(G), f_g(G), h(G), g(G)) = (a, b, a + b + c, a + 2b + c)$ or $(a, 2a + b, a + b + c, 2a + 2b + c)$. These results confirm a conjecture of Chartrand and Zhang proposed in [G. Chartrand, P. Zhang, The forcing hull number of a graph, J. Combin. Math. Combin. Comput. 36 (2001) 81–94].

1. Introduction

All graphs considered in the paper are finite and simple. Let S be a subset of $V(G)$. Denote by *induced subgraph* $\langle S\rangle_G$ the subgraph of G induced by S. The *neighborhood* of a vertex v in a graph G is the set of vertices of G adjacent to v, denoted by $N_G(v)$ or $N(v)$. Define $N_G(S)$ as the union of all $N_G(v)$ for all $v \in S$. A u_0-u_k *path* $(u_0, u_1, \ldots, u_k)$ is a sequence of vertices in G such that $u_0u_1, u_1u_2, \ldots, u_{k-1}u_k \in E(G)$ and $u_0, u_1, \ldots, u_k$ are distinct. A $u - v$ *geodesic* of G is a $u - v$ shortest path in G. The *distance* between two vertices u and v in a graph G, written $d_G(u, v)$, is the length of a $u - v$ geodesic of G. A graph G is called *vertex-transitive on a vertex subset* S if, for every pair $x, y \in S$, there exists an isomorphism f on G such that $f(x) = y$.

For a graph G, let $I_G[u, v]$ denote the set of all vertices lying on a u–v geodesic. For $S \subseteq V(G)$, let $I_G[S]$ denote the union of all $I_G[u, v]$ for all $u, v \in S$. A subset S of $V(G)$ is *convex* if $I_G[S] = S$. Harary and Nieminen [7] defined the *convex hull* of a set S of vertices of a graph as the smallest vertex subset T containing S satisfying the property that all vertices of any geodesic between each pair of vertices of T belong to T. Let $I_G^k[S] = I_G[I_G^{k-1}[S]]$ for $k \geq 2$. It is easy to see that $[S]_G = I_G^k[S]$ for some k. A subset S of $V(G)$ is a *hull set* of G if $[S]_G = V(G)$. Everett and Seidman [5] gave the definition of the hull number of a graph as the cardinality of a minimum vertex subset S with the convex hull of S is the vertex set. A subset S of $V(G)$ is a *geodetic set* of G if $I_G[S] = V(G)$, and the *geodetic number* $g(G)$ of a graph G is the minimum cardinality of a geodetic set S in G. The geodetic sets of a connected graph were introduced by Harary, Loukakis, and Tsouros [6], as a tool for studying metric properties of connected graphs. A vertex v is an *extreme vertex* of G if, for every two distinct vertices $x, y \in N(v)$, $xy \in E(G)$. It is obvious that every hull (or geodetic) set of G contains all its extreme vertices.

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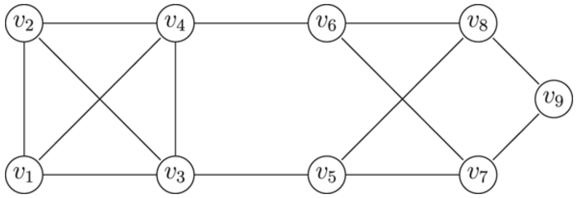

Fig. 1. The graph G_1.

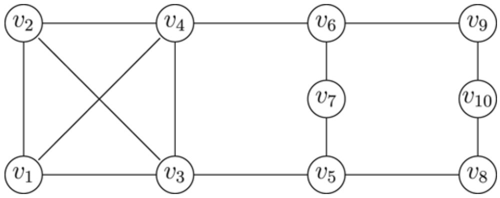

Fig. 2. The graph G_2.

A subset $F \subseteq V(G)$ is called a *forcing hull (geodetic) subset* of G, if there exists a unique minimum hull (geodetic) set containing F. The cardinality of a minimum forcing hull (geodetic) subset in G is called *the forcing hull (geodetic) number* $f_h(G)$ ($f_g(G)$) of G. The study of forcing concepts have been widely investigated in graph theory, such as forcing convexity number [4], forcing domination number [1], forcing geodetic number [2,9,10], forcing hull number [3], forcing perfect matching [8], and so on.

Chartrand and Zhang in [3] posed a conjecture that:

Conjecture 1. *For every pairs a, b of nonnegative integers, there exists a connected graph G with $f_h(G) = a$ and $f_g(G) = b$.*

In the paper, we construct some 2-connected graph G with $(f_h(G), f_g(G)) = (0, 0), (1, 0)$, or $(0, 1)$, and prove that, for any nonnegative integers a and b with $a + b \geq 2$, there exists a 2-connected graphs G with $(f_h(G), f_g(G), h(G), g(G)) = (a, b, a + b + c, a + 2b + c)$ or $(a, 2a + b, a + b + c, 2a + 2b + c)$. These results confirm the conjecture above.

2. Forcing hull and geodetic graphs

In this section, we construct 2-connected graphs with fixed forcing hull number and forcing geodetic number. First, we construct a 2-connected graph G_1 with $h(G_1) = g(G_1) = 3$, $f_h(G_1) = 1$, and $f_g(G_1) = 0$.

Define G_1 as the 2-connected graph with the vertex set $\{v_i : i = 1, 2, \ldots, 9\}$ and the edge set $\{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4, v_3v_5, v_4v_6, v_5v_7, v_5v_8, v_6v_7, v_6v_8, v_7v_9, v_8v_9\}$, see Fig. 1.

Proposition 1. *$h(G_1) = g(G_1) = 3$, $f_h(G_1) = 1$, and $f_g(G_1) = 0$.*

Proof. Observe that v_1 and v_2 are the only two extreme vertices of G_1. Therefore every hull set or geodetic set contains v_1 and v_2. By $[\{v_1, v_2\}]_{G_1} = \{v_1, v_2\}$ and $I_{G_1}[\{v_1, v_2, v_9\}] = V(G_1)$, $h(G_1) = g(G_1) = 3$. Since $\{v_1, v_2, v_9\}$ is the unique minimum geodetic set, $f_g(G_1) = 0$. And, by $\{v_1, v_2, v_8\}$ being a minimum hull set, $f_h(G_1) = 1$. ■

Let G_2 be the 2-connected graph with the vertex set $\{v_i : i = 1, 2, \ldots, 10\}$ and the edge set $\{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4, v_3v_5, v_4v_6, v_5v_7, v_5v_8, v_6v_7, v_6v_9, v_8v_{10}, v_9v_{10}\}$, see Fig. 2.

Proposition 2. *$h(G_2) = 3$, $g(G_2) = 4$, $f_h(G_2) = 0$, and $f_g(G_2) = 1$.*

Proof. Observe that v_1 and v_2 are the only two extreme vertices of G_2. Therefore every hull set or geodetic set contains v_1 and v_2. Since $[\{v_1, v_2\}]_{G_2} = \{v_1, v_2\}$, $h(G_2), g(G_2) \geq 3$. Since $[\{v_1, v_2, v_{10}\}]_{G_2} = V(G_2)$ and $\{v_1, v_2, v_{10}\}$ is the unique minimum hull set of G_2, $h(G_2) = 3$ and $f_h(G_2) = 0$.

Since $I_{G_2}[\{v_1, v_2, v_x\}] \neq V(G_2)$ for any $x \in V(G_2)$ and $I_{G_2}[\{v_1, v_2, v_7, v_{10}\}] = V(G_2)$, $g(G_2) = 4$. It follows that $I_{G_2}[\{v_1, v_2, v_6, v_8\}] = V(G_2)$. This implies that $f_g(G_2) \geq 1$. Now, since the set $\{v_1, v_2, v_7, v_{10}\}$ is the only minimum geodetic set containing v_7, it follows that $f_g(G_2) = 1$. ■

Let G_3 be a 2-connected graph with the vertex set $\{v_i : i = 1, 2, \ldots, 15\}$ and the edge set $\{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4, v_3v_5, v_3v_6, v_4v_7, v_4v_8, v_5v_9, v_6v_9, v_7v_{10}, v_8v_{10}, v_9v_{12}, v_{10}v_{13}, v_{11}v_{12}, v_{11}v_{13}, v_{12}v_{14}, v_{12}v_{15}, v_{13}v_{14}, v_{13}v_{15}\}$, see Fig. 3.

Proposition 3. *$h(G_3) = 3$, $g(G_3) = 4$, $f_h(G_3) = 1$, and $f_g(G_3) = 2$.*

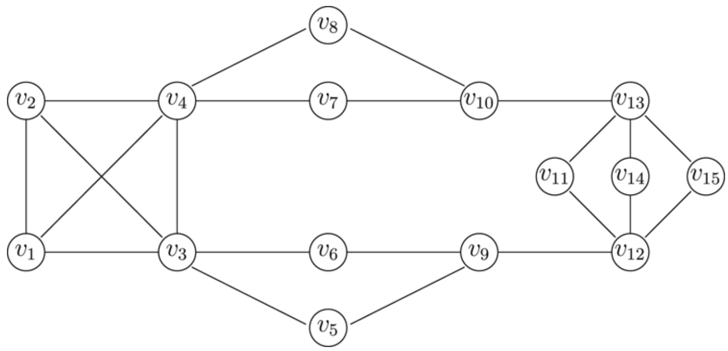

Fig. 3. The graph G_3.

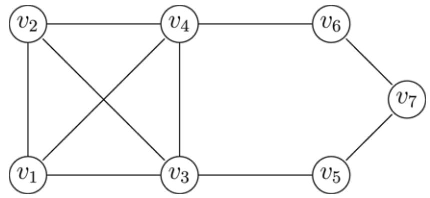

Fig. 4. The graph H_0.

Proof. Observe that v_1 and v_2 are the only two extreme vertices of G_3. Therefore every hull set or geodetic set contains v_1 and v_2. Also, since $[\{v_1, v_2\}]_{G_3} = \{v_1, v_2\}$, $h(G_3), g(G_3) \geq 3$. Since $[\{v_1, v_2, v_{15}\}]_{G_3} = [\{v_1, v_2, v_{14}\}]_{G_3} = V(G_3)$, $h(G_3) = 3$ and $f_h(G_3) = 1$.

Consider the values of $f_g(G_3)$ and $g(G_3)$. Since $I_{G_3}[\{v_1, v_2, v_a\}] \neq V(G_3)$ for any $a \in \{3, 4, \ldots, 15\}$ and $I_{G_3}[\{v_1, v_2, v_{10}, v_{12}\}] = V(G_3)$, we get $g(G_3) = 4$. Let $T = \{v_1, v_2, x, y\}$ be a minimum geodetic set of G_3. If $x = v_5$, then, by $I_{G_3}[\{v_1, v_2, v_5\}] = \{v_1, v_2, v_3, v_5\}$. This implies that $v_6 \in I_{G_3}[v_1, y]$. Thus, $y \in \{v_9, v_{11}, v_{12}, v_{14}, v_{15}\}$. We find that $I_{G_3}[\{v_1, v_2, v_5, v_t\}] \neq V(G_3)$ for $t \in \{9, 11, 12, 14, 15\}$. Then $v_5 \notin T$. By a similar argument, $v_t \notin T$ for $t = 5, 6, 7, 8$. If $x = v_{11}$, then, by $I_{G_3}[\{v_1, v_2, v_{11}\}] = V(G_3) - \{v_{14}, v_{15}\}$; that is $v_{14} \in I_{G_3}[v_1, y]$. Since only $I_{G_3}[v_1, v_{14}]$ contains v_{14}, $y = v_{14}$. But $I_{G_3}[\{v_1, v_2, v_{11}, v_{14}\}] = V(G_3) - \{v_{15}\}$. Therefore, $v_{11} \notin T$. Similarly, $v_t \notin T$ for $t = 14, 15$. And, we have that $I_{G_3}[\{v_1, v_2, v_9, v_{10}\}] = I_{G_3}[\{v_1, v_2, v_9, v_{13}\}] = I_{G_3}[\{v_1, v_2, v_{10}, v_{12}\}] = I_{G_3}[\{v_1, v_2, v_{10}, v_{13}\}] = V(G_3)$. These imply that $f_g(G_3) = 2$. ■

Let H_0 be a 2-connected graph with the vertex set $\{v_i : i = 1, 2, \ldots, 7\}$ and the edge set $\{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4, v_3v_5, v_4v_6, v_5v_7, v_6v_7\}$, see Fig. 4.

Proposition 4. *In the graph H_0, $h(H_0) = g(H_0) = 3$ and $f_h(H_0) = f_g(H_0) = 0$.*

Proof. Since $\{v_1, v_2, v_7\}$ is the unique minimum hull (geodetic) set of H_0, $h(H_0) = g(H_0) = 3$ and $f_h(H_0) = f_g(H_0) = 0$. ■

For nonnegative integers a and b, we construct a graph G with $f_h(G) = a$ and $f_g(G) = b$ as follows. Let $m \geq 2$ and t be positive integers. Assume that $G_1, G_2, \ldots, G_m$ are vertex disjoint graphs, each has at least $t + 1$ vertices and contains a t-subset S_i for which $\langle S_i \cup N_{G_i}(S_i)\rangle_{G_i}$ is complete. Let G be obtained from the disjoint union of $G_1, G_2, \ldots, G_m$ by identifying $S_1, S_2, \ldots, S_m$. Let $S = S_1 = S_2 = \cdots = S_m$. It is easy to see that $G_1, G_2, \ldots, G_m$, and G are vertex-transitive on S, and if G_1, $G_2, \ldots, G_m$ are 2-connected and $t \geq 2$, then G is also 2-connected.

Lemma 5. *Suppose that G is the graph defined above and T is a minimum hull (or geodetic) set. Then T and S are disjoint.*

Proof. Let T be a minimum hull (or geodetic) set of G and u be a vertex of S. Since $\langle S_i \cup N_{G_i}(S_i)\rangle_{G_i}$ is complete, $\langle S_i \cup N_{G_i}(S_i)\rangle_{G_i}$ is convex. Then $T \cap (V(G_i) - S_i)$ is nonempty. Let $x_i \in T \cap (V(G_i) - S_i)$ for $i = 1, 2, \ldots, m$. As G is vertex-transitive on S, and $x_1, x_2, \ldots, x_m$ are in the different components of $G - S$, we have that $u \in I_G[x_i, x_j]$ for $i \neq j$. If T is a minimum hull set and $u \in I_G[x_i, x_j]$ for some $i \neq j$, then T and S are disjoint. For T being a minimum geodetic set, if $v \in I_G[x, y]$ for some $x \in T \cap (V(G_i) - S_i)$ and $y \in S$, then $v \in I_G[x, x_j]$ for some $j \neq i$. Then T and S are disjoint. ■

Lemma 6. *Suppose that G is the graph defined in Lemma 5. Then T is a minimum hull (or geodetic) set of G if and only if $(T \cap V(G_i)) \cup S_i$ is a minimum hull (or geodetic) set of G_i for $i = 1, 2, \ldots, m$.*

Proof. Since G is vertex-transitive on S and $V(G) - S = \cup_{i=1}^{m}(V(G_i) - S_i)$, we have that, for $v_i \in V(G_i) - S_i$ and $v_j \in V(G_j) - S_j$ with $i \neq j$, $x \in I_G[v_i, v_j]$ for all $x \in S$. It follows that:
Property (*): For $v_i \in V(G_i) - S_i$ and $v_j \in V(G_j) - S_j$ with $i \neq j$, $S \subseteq I_G[v_i, v_j]$.

Suppose that T is a minimum hull set in G. Let $v_i \in V(G_i) - S_i$ and $v_j \in V(G_j) - S_j$ with $i \neq j$. Since v_i and v_j are in different components of $G-S$, every geodesic between v_i and v_j passes through some vertex of S. Then $V(G_i) \cap I_G^k[T] = I_G^k[(T \cap V(G_i)) \cup S_i]$ for all positive integer k. This implies that $[(T \cap V(G_i)) \cup S_i]_{G_i} = V(G_i)$; that is, $(T \cap V(G_i)) \cup S_i$ is a hull set of G_i. Thus, by Lemma 5, $h(G) + m|S| \geq h(G_1) + h(G_2) + \cdots + h(G_m)$. If T_i is a minimum hull set of G_i and every vertex of S_i is an extreme vertex in G_i, then $S_i \subseteq T_i$ for $i = 1, 2, \ldots, m$. By the property (*), $\cup_{i=1}^m (T_i - S_i)$ is a hull set of G. We thus get $h(G) \leq h(G_1) + h(G_2) + \cdots + h(G_m) - m|S|$. By the two inequalities above, we conclude that $h(G) = h(G_1) + h(G_2) + \cdots + h(G_m) - m|S|$. Then it follows that T is a minimum hull set of G if and only if $(T \cap G(G_i)) \cup S_i$ is a minimum hull set of G_i for $i = 1, 2, \ldots, m$.

Assume that T is a minimum geodetic set in G. Since G is vertex-transitive on S, $V(G) - S = \cup_{i=1}^m (V(G_i) - S_i)$, $u \in V(G_i) - S_i$, and $v \in V(G_j) - S_j$ with $i \neq j$, we have that $x \in I_G[u, v]$ for all $x \in S$. For each $x \in V(G_i) - S_i$, by Lemma 5, there exist $v_i \in (T \cap V(G_i)) - S_i$ and $v_j \in (T \cap V(G_j)) - S_j$ such that $x \in I_G[v_i, v_j]$. So either $i = j$ or $x \in I_G[v_i, y]$ for some $y \in S$. Then $(T \cap V(G_i)) \cup \{S_i\}$ is a geodetic set of G_i. Thus, $g(G) + m|S| \geq g(G_1) + g(G_2) + \cdots + g(G_m)$. Assume that T_i is a minimum geodetic set of G_i for $i = 1, 2, \ldots, m$. Since $\langle S_i \cup N_{G_i}(S_i) \rangle_{G_i}$ is complete in G_i for $i = 1, 2, \ldots, m$, $T_i - S_i$ is nonempty. By the property (*), $S \subseteq I_G[\cup_{i=1}^m (T_i - S_i)]$. For each $x \in V(G_i) - S_i$, there exist $u, v \in T_i$ such that $x \in I_{G_i}[u, v]$. If $u, v \notin S_i$, then $x \in I_G[\cup_{k=1}^m (T_k - S_k)]$. If $u \in S_i$ and $v \notin S_i$, then $x \in I_G[v, w]$ for some $w \in T_j - S_j$ with $i \neq j$; that is, $x \in I_G[\cup_{k=1}^m (T_k - S_k)]$. Hence $\cup_{k=1}^m (T_k - S_k)$ is a geodetic set in G. This implies that $g(G) \leq g(G_1) + g(G_2) + \cdots + g(G_m) - m|S|$. By above, $g(G) = g(G_1) + g(G_2) + \cdots + g(G_m) - m|S|$. Then it follows that T is a minimum geodetic set of G if and only if $(T \cap G(G_i)) \cup S_i$ is a minimum geodetic set of G_i for $i = 1, 2, \ldots, m$. ■

Theorem 7. *Suppose that G is the graph defined in Lemma 5. Then*

(a) $h(G) = h(G_1) + h(G_2) + \cdots + h(G_m) - m|S|$,
(b) $g(G) = g(G_1) + g(G_2) + \cdots + g(G_m) - m|S|$,
(c) $f_h(G) = f_h(G_1) + f_h(G_2) + \cdots + f_h(G_m)$, *and*
(d) $f_g(G) = f_g(G_1) + f_g(G_2) + \cdots + f_g(G_m)$.

Proof. By Lemma 6, we have that $h(G) = h(G_1) + h(G_2) + \cdots + h(G_m) - m|S|$ and $g(G) = g(G_1) + g(G_2) + \cdots + g(G_m) - m|S|$. According to Lemma 6 and definitions, F is a forcing hull (or geodetic) subset of G if and only if $F \cap V(G_i)$ is a forcing hull (or geodetic) subset of G_i for $i = 1, 2, \ldots, m$. This implies that $f_h(G) = f_h(G_1) + f_h(G_2) + \cdots + f_h(G_m)$, and $f_g(G) = f_g(G_1) + f_g(G_2) + \cdots + f_g(G_m)$. ■

Theorem 8. *Let a, b, and c be nonnegative integers with $a + b \geq 2$. Then there exists a 2-connected graph G with $f_h(G) = a$, $f_g(G) = b$, $h(G) = a + b + c$, and $g(G) = a + 2b + c$.*

Proof. Suppose that X_i is a graph isomorphic to G_1 for $i = 1, 2, \ldots, a$, X_j is a graph isomorphic to G_2 for $j = a+1, a+2, \ldots, a+b$, X_k is a graph isomorphic to H_0 for $k = a + b + 1, a + b + 2, \ldots, a + b + c$, and S_l is the set of all extreme vertices in X_i for $l = 1, 2, \ldots, a + b + c$. Let G be the graph obtained from $X_1, X_2, \ldots, X_{a+b+c}$ identifying $S_1, S_2, \ldots, S_{a+b+c}$. By Propositions 1, 2 and 4, and Theorem 7, G is a 2-connected graph with $f_h(G) = a$, $f_g(G) = b$, $h(G) = a + b + c$, and $g(G) = a + 2b + c$. ■

Corollary 9. *Suppose that a and b are nonnegative integers with a and b. Then there exists a 2-connected graph G with $f_h(G) = a$ and $f_g(G) = b$.*

Proof. Let H be a complete graph with $V(H) = \{x_1, x_2, x_3, x_4\}$ and H' be the graph obtained from H by deleting the edge x_1x_2. Then, it is obvious that $\{x_1, x_2\}$ is the unique maximum hull (geodetic) set in H'; that is, $f_h(H') = 0$ and $f_g(H') = 0$. For $(a, b) \neq (0, 0)$, by Propositions 1 and 2, and Theorem 8, there exists a 2-connected graph G with $f_h(G) = a$ and $f_g(G) = b$ for nonnegative integers a and b with $a + b \geq 1$. ■

Corollary 9 proves Conjecture 1.

Corollary 10. *Suppose that a and b are nonnegative integers with $a \leq b$ and $b \geq 2$. Then there exists a 2-connected graph G with $f_h(G) = a$ and $h(G) = b$.*

Theorem 11. *Let a, b, and c be nonnegative integers with $a + b \geq 2$. Then there exists a 2-connected graph G with $f_h(G) = a$, $f_g(G) = 2a + b$, $h(G) = a + b + c$, and $g(G) = 2a + 2b + c$.*

Proof. Suppose that X_i is a graph isomorphic to G_3 for $i = 1, 2, \ldots, a$, X_j is a graph isomorphic to G_2 for $j = a+1, a+2, \ldots, a+b$, X_k is a graph isomorphic to H_0 for $k = a + b + 1, a + b + 2, \ldots, a + b + c$, and S_l is the set of all extreme vertices in X_i for $l = 1, 2, \ldots, a + b + c$. Let G be the graph obtained from $X_1, X_2, \ldots, X_{a+b+c}$ identifying $S_1, S_2, \ldots, S_{a+b+c}$. By Propositions 2–4, and Theorem 7, we have that G is a 2 connected graph with $f_h(G) = a$, $f_g(G) = 2a+b$, $h(G) = a+b+c$, and $g(G) = 2a+2b+c$. ■

Chartrand and Zhang showed that if G is a connected graph with $g(G) = 2$, then $f_g(G) < 2$ in [2].

Corollary 12. *Suppose that a and b are nonnegative integers with $a + 1 \leq b$ and $b \geq 2$. Then there exists a 2-connected graph G with $f_g(G) = a$ and $g(G) = b$.*

Proof. It is obvious that a complete graph of order $n \geq 3$ is a 2-connected graph with geodetic number n and the forcing geodetic number 0. Since every even cycle is a 2-connected graph with geodetic number 2 and the forcing geodetic number 1. For $(a, b) \neq (1, 2)$ and $2 \leq a + 1 \leq b$, by Theorem 11, there exists a 2-connected graph G with $f_g(G) = a$ and $g(G) = b$. ■

According to the above study, if G is a connected graph G with $f_h(G) = a$, $f_g(G) = b$, $h(G) = c$ and $g(G) = d$, then $(b, d) \neq (2, 2)$ and $a > 0$ for $c = d$, $b > 0$. We offer the following open problem.

Problem 1. For which nonnegative integers ab, c, and d with $a \leq c \leq d$, $b \leq d$, $c \geq 2$, does there exist a connected graph G with $f_h(G) = a$, $f_g(G) = b$, $h(G) = c$ and $g(G) = d$?

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References

[1] G. Chartrand, H. Gavlas, R.C. Vandell, F. Harary, The forcing domination number of a graph, J. Combin. Math. Combin. Comput. 25 (1997) 161–174.
[2] G. Chartrand, P. Zhang, The forcing geodetic number of a graph, Discuss. Math. Graph Theory 19 (1999) 45–58.
[3] G. Chartrand, P. Zhang, The forcing hull number of a graph, J. Combin. Math. Combin. Comput. 36 (2001) 81–94.
[4] G. Chartrand, P. Zhang, The forcing convexity number of a graph, Czech. Math. J. 51 (2001) 847–858.
[5] M.G. Evertt, S.B. Seidman, The hull number of a graph, Discrete Math. 57 (1985) 217–223.
[6] F. Harary, E. Loukakis, C. Tsourus, The geodetic number of a graph, Math. Comput. Modeling 17 (1993) 89–95.
[7] F. Harary, J. Nieminen, Convexity in graphs, J. Differential Geom. 16 (1981) 185–190.
[8] M.E. Riddle, The minimum forcing number for the torus and hypercube a graph, Discrete Math. 245 (2002) 283–292.
[9] L.-D. Tong, The (a, b)-forcing geodetic graphs, preprint.
[10] P. Zhang, The upper forcing geodetic number of a graph, Ars Combin. 62 (2002) 3–15.