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Boundary Controllability Problems for the Wave Equation in a Parallelepiped

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Abstract—The wave equation in an N-dimensional parallelepiped with boundary control equal zero everywhere except of an edge of dimension $N-2$ is considered. The other case which is investigated is the boundary control acting on a face of dimension $N-1$ and depending on $N-1$ independent variables (including t). It is proved that, in both cases, the system is not approximately controllable for any $T > 0$.

Keywords—Boundary controllability, Wave equation.

SECTION 1

The controllability problems for the systems described by the hyperbolic type equations take up a prominent place in control theory of distributed parameter systems (see e.g., [1]). New fruitful approaches to these problems—Hilbert Uniqueness Method [2], microlocal analysis [3] and some others (see e.g., [4])—has been developed in the last years. Aside from numerous "positive" results on controllability, there are interesting "negative" results which determine capability limits of certain classes of controls in certain kinds of systems. We refer to some of those results related to the wave equation $u_{tt} = \Delta u$ in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$.

In the paper [3], geometrical conditions are given to a part of boundary $\Gamma_0 \subset \Gamma$ and to time T such that the wave equation is not exactly controllable in space $L^2(\Omega) \times H^{-1}(\Omega)$ under the action of boundary control v,

$$u = v \qquad \text{on } \Gamma_0 \times (0,T) \quad \text{and} \quad u = 0 \qquad \text{on } (\Gamma \backslash \Gamma_0) \times (0,T).$$

In [5, Chapters 4,5], it is shown that the wave equation (and more general equations of hyperbolic and parabolic types) with finite number of controls—boundary,

$$u = \sum_{j=1}^{m} g_j(x) v_j(t) \qquad \text{on } \Gamma \times (0,T), \tag{1}$$

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pointwise or other kind—is not exactly controllable in $H^r(\Omega) \times H^{r-1}(\Omega)$ for any r and any $T > 0$. In (1), functions $g_i \in L^2(\Omega)$ are fixed and $v_i \in L^2(0,T)$ are controls. Moreover, these systems are not M-controllable [5, Chapter 3] (are not spectral controllable or eigenfunction controllable in terms of [1]). Triggiani [6] proved lack of the exact controllability with control of the form (1) for the various equations in "natural" spaces.

There are also papers which contain more strong negative results, they prove lack of approximate controllability. The first of these results seems to has been obtained in [5, Chapter 5;7]. It states that the wave equation in a rectangle with a boundary control v,

$$\begin{aligned} u_{tt} &= \Delta u, \\ \frac{\partial}{\partial n} u &= g(x) v(t), \qquad \text{on } \Gamma \times (0,T), \end{aligned}$$

is not approximately controllable for any $g \in L^2(\Gamma)$ and for any $T > 0$. The reachability set $R(T)$ "strongly increases" on $T \in [0,\infty)$:

$$R(T_2) \supsetneqq R(T_1), \qquad \text{for } T_2 > T_1.$$

Then the similar result has been proved for the wave equation in a rectangle with finite number of pointwise controls [8;9, Chapter 6],

$$u_{tt} = \Delta u + \sum_{j=1}^{m} \delta(x - x_i)\, v_i(t), \qquad x_i \in \Omega.$$

Lebeau [10] showed that similar negative result is valid for arbitrary domain Ω with analytic boundary (under certain geometrical conditions).

In the present paper, we prove lack of the approximate controllability for an even more powerful kind of control. Namely, we consider the wave equation in an N-dimensional parallelepiped with boundary control equal to zero everywhere except on an edge of dimension $N-2$. The other case which is investigated is the boundary control acting on a face of dimension $N-1$ and depending on $N-1$ independent variables (including t). We show that in both cases the system is not approximately controllable for any $T > 0$.

To solve the posed problems, we apply the Fourier method and the theory of exponential families in spaces of vector-valued functions [5,9].

The model problems of such kind allow us to put forward the following hypothesis concerning controllability of hyperbolic equations of the second order.

HYPOTHESIS. *If*

(i) *a control acts on an m-dimensional part of boundary and/or on an m-dimensional part of domain $\Omega \subset \mathbb{R}^N$, and $m < N-1$,*

or if (more general formulation)

(ii) *a control function depends on less than N independent variables (including t),*

then a system described by hyperbolic equation of the second order is not approximately controllable in any finite time.

SECTION 2

Let Ω be a parallelepiped in $\mathbb{R}^N$, $N \geq 2$,

$$\begin{aligned} \Omega &:= \{x = (x_1, x_2, \ldots, x_N) \mid 0 < x_j < a_j\}, \\ Q &:= \Omega \times (0,T), \quad \Gamma := \partial\Omega, \quad \Sigma := \Gamma \times (0,T). \end{aligned}$$

Let A be an operator $-\Delta + I$ with domain $D(A) = H^2(\Omega)$. For $r \geq 0$, we set $W_r := D(A^{r/2})$ and let W_{-r} be a dual to W_r space, $W_{-r} = W_r'$.

Let Γ_j be $(N-j)$-dimensional part of Γ,

$$\Gamma_j := \{x \in \Gamma \mid x_1 = x_2 = \cdots = x_j = 0\}, \qquad 1 \leq j \leq N, \quad \Sigma_j := \Gamma_j \times (0,T).$$

We consider the initial boundary value problem

$$\begin{aligned} w_{tt} - \Delta w = 0, \qquad \text{in } Q, \\ w|_{t=0} = w_0, \quad \partial_t w|_{t=0} = w_1, \quad \left.\frac{\partial w}{\partial n}\right|_{\Sigma} = 0. \end{aligned} \tag{2}$$

THEOREM 1. *Let $(w_0, w_1) \in W^p \times W^{p-1}$, $s := p - 1/4$ for $j = 1$ and $s := p - (j-1)/2$ for $j > 1$. If $s \geq 0$, then $w|_{\Sigma_j} \in H^s(\Sigma_j)$.*

REMARK 1. Analogous results take place for all other faces and edges. Moreover, compatibility results are valid on mutual parts of Γ of smaller dimension. In this sense, we can say, for instance, that $w|_\Sigma \in H^{p-1/4}$.

We need also in result dual to Theorem 1. Let

$$u_{tt} - \Delta u \text{ in } Q, \quad u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0, \tag{3}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\Sigma_j} = f, \quad \left.\frac{\partial u}{\partial n}\right|_{\Sigma\setminus\Sigma_j} = 0. \tag{4}$$

The solution of the problem (3),(4) is understood in a weak sense and can be defined with the help of the method of transposition [11].

THEOREM 2. *Let $f \in [H^p(\Sigma_j)]'$, $p \geq 0$, $s = 3/4 - p$ for $j = 1$ and $s = 3/2 - j/2 - p$ for $j > 1$.*

Then there exists the unique solution of the problem (3), (4) *such that $(u, u_t) \in C([0,T]; W_s \times W_{s-1})$.*

REMARK 2. The exponents s in Theorems 1 and 2 are sharp.

REMARK 3. For $j = 1$, $p = 0$, we get from Theorem 2 the following result. If $f \in L^2(\Sigma)$, then the solution of (3) with boundary condition $\frac{\partial u}{\partial n}|_\Sigma = f$ satisfies the inclusion

$$(u, u_t) \in C\left([0,T]; W_{3/4} \times W_{-1/4}\right) = C\left([0,T]; H^{3/4}(\Omega) \times H^{-1/4}(\Omega)\right). \tag{5}$$

This result was obtained in [5, Chapter 5;7] with the help of the Fourier method. It can be also derived from the results of [12] using other techniques.

The proof of Theorems 1 and 2 is presented in [13]. In this paper, we are interested in control and observation problems for systems (3),(4) and (2), correspondingly.

SECTION 3

Let a function u satisfy the wave equation with zero initial conditions (3) and the following boundary conditions

$$\left.\frac{\partial u}{\partial n}\right|_{\Sigma_1} = f, \qquad \left.\frac{\partial u}{\partial n}\right|_{\Sigma\setminus\Sigma_1} = 0. \tag{6}$$

Suppose that function f has the form

$$f(x_2, \dots, x_N, t) = g(x_2)\, v(x_3, \dots, x_N, t), \tag{7}$$

where $g \in L^2(0, a_2)$ is a fixed function and v is a control function, $v \in U := L^2(\Sigma_2)$.

Denote by $R(T)$ the reachability set of the system (3),(6) in time T:

$$R(T) := \{(u(\cdot,T), u_t(\cdot,T)) \mid v \in U\}.$$

According to (5), $R(T) \subset \mathcal{W} := W_{3/4} \times W_{-1/4}$.

THEOREM 3. *For any time $T > 0$ and any $g \in L^2(0, a_2)$, the set $R(T)$ is not dense in $\mathcal{W}$ and* $\operatorname{codim} R(T) = \infty$.

PROOF. We shall prove the more exact statement. Namely,

— the set $R_0(T) := \{u(\cdot,T) \mid v \in U\}$ is not dense in $W_{3/4}$,

— the set $R_1(T) := \{u_t(\cdot,T) \mid v \in U\}$ is not dense in $W_{-1/4}$,

and corresponding codimensionals equal to infinity.

For simplicity of notations, we give the proof for the case $N = 3$. In the general case, the proof can be carried out in a quite similar way.

Operator $-\Delta$ with the domain $D(-\Delta) = H^2(\Omega)$ has eigenvalues

$$\lambda_{kmn} = \left(\frac{\pi k}{a_1}\right)^2 + \left(\frac{\pi m}{a_2}\right)^2 + \left(\frac{\pi n}{a_3}\right)^2, \qquad k, m, n = 0, 1, 2, \ldots,$$

and eigenfunctions

$$\varphi_{kmn}(x) = \alpha_{1k}\, \alpha_{2m}\, \alpha_{3n} \cos\left(\frac{\pi x_1 k}{a_1}\right) \cos\left(\frac{\pi x_2 m}{a_2}\right) \cos\left(\frac{\pi x_3 n}{a_3}\right),$$

$$\alpha_{jp} := \begin{cases} \sqrt{2/a_j}, & \text{for } p \neq 0, \\ \sqrt{1/a_j}, & \text{for } p = 0. \end{cases}$$

The functions φ_{kmn} form an orthonormal basis in $L^2(\Omega)$.

Let us present function $u(x,T)$ in the form

$$u(x,T) = \sum_{k,m,n} c_{kmn}\, \varphi_{kmn}(x). \tag{8}$$

Inclusion $u(\cdot,T) \in W_{3/4}$ means that

$$\sum_{k,m,n} \left| c_{kmn}\, \omega_{kmn}^{3/4} \right|^2 < \infty,$$

where

$$\omega_{kmn} := \begin{cases} \sqrt{\lambda_{kmn}}, & \text{for } k+m+n \neq 0, \\ 1, & \text{for } k = m = n = 0. \end{cases}$$

Inequality $\operatorname{cl} R(T) \neq W_{3/4}$ is equivalent to existence of a sequence $\{b_{kmn}\}$ such that

$$\sum_{k,m,n} c_{kmn}\, b_{kmn} = 0, \quad \sum_{k,m,n} \left| b_{kmn}\, \omega_{kmn}^{-3/4} \right|^2 < \infty. \tag{9}$$

With the help of standard calculations using the Fourier method (see e.g., [1]), we obtain the following formula for coefficients c_{kmn}:

$$c_{kmn} = \int_0^T \int_{\Gamma_1} f(x_2, x_3, T-t)\, \alpha_{1k}\, \alpha_{2m}\, \alpha_{3n}$$

$$\times \cos\left(\frac{\pi x_2 m}{a_2}\right) \cos\left(\frac{\pi x_3 n}{a_3}\right) \sin(\omega_{kmn} t)\, \omega_{kmn}^{-1}\, dt\, dx_2\, dx_3.$$

Taking into account (7) and introducing the notation

$$\gamma_m := \int_0^{a_2} g(x_2)\, \alpha_{2m} \cos\left(\frac{\pi x_2 m}{a_2}\right) dx_2,$$

we get

$$c_{kmn} = \int_0^T \int_0^{a_2} v(x_3, T-t)\, \alpha_{1k}\, \gamma_m\, \alpha_{3n} \cos\left(\frac{\pi x_3 n}{a_3}\right) \sin(\omega_{kmn} t)\, \omega_{kmn}^{-1}\, dt\, dx_3. \tag{10}$$

In the following proposition, we use eigenfrequencies

$$\omega_{km0} = \sqrt{\left(\frac{\pi k}{a_1}\right)^2 + \left(\frac{\pi m}{a_2}\right)^2}, \qquad k, m = 0, 1, 2, \ldots,$$

corresponding to $n = 0$. The analogous statement is valid for all sequences ω_{kmn} with fixed $n \in \mathbb{N}$.

PROPOSITION 1. [5, Section II.6] *For any $T > 0$, there exist a number $M(T)$ and sequences $\{a_{km}^+\}$, $\{a_{km}^-\}$ such that*

(i) $\sum_{k,m} [a_{km}^+ \exp(i\omega_{km0} t) + a_{km}^- \exp(-i\omega_{km0} t)] = 0$ in $L^2(-T, T)$,

(ii) $\sum_{k,m} (|a_{km}^+|^2 + |a_{km}^-|^2)\, \omega_{km0}^2 < \infty$,

(iii) $a_{km}^+ = a_{km}^- = 0$ for $m > M(T)$.

Using Euler's formulas and separating in (i) even and odd parts, we obtain the following statement.

PROPOSITION 2. *For any $T > 0$, there exist a number $M(T)$ and a sequence $\{a_{km}\}$, such that*

(i) $\sum_{k,m} a_{km} \sin(\omega_{km0} t) = 0$ in $L^2(0, T)$,

(ii) $\sum_{k,m} |a_{km}|^2 \omega_{km0}^2 < \infty$,

(iii) $a_{km} = 0$ for $m > M(T)$, $a_{00} = 0$.

Now we are able to prove that the set $R_0(T)$ is not dense in $W_{3/4}$.

If $\gamma_m = 0$ for some m, this statement is trivial (see (8),(10)).

If $\gamma_m \neq 0$ for all m, we define the sequence $\{b_{kmn}^{(0)}\}$ by the formula

$$b_{kmn}^{(0)} = \begin{cases} a_{km}\, \omega_{km0} (\alpha_{1k} \gamma_m \alpha_{30})^{-1}, & \text{for } n = 0, \\ 0, & \text{for } n \neq 0. \end{cases}$$

From (10) and Proposition 2, it follows that this sequence satisfies conditions (9). So we proved that cl $R_0(T) \neq W_{3/4}$.

In a quite similar way, we can construct sequences $\{b_{kmn}^{(l)}\}$ for any $l = 1, 2, \ldots$. These sequences are mutually orthogonal and all of them are orthogonal to $\{c_{kmn}\}$. Hence, codim $R_0(T) = \infty$ in $W_{3/4}$.

Analogously, it can be proved that $R_1(T)$ is not dense in $W_{-1/4}$ and codim $R_1(T) = \infty$.

Theorem 3 is proved.

SECTION 4

Let us consider now the equation (3) with boundary control acting on

$$\Gamma_2 = \{x \in \Gamma \mid x_1 = x_2 = 0\}, \quad \Sigma_2 = \Gamma_2 \times (0, T),$$
$$\left.\frac{\partial u}{\partial n}\right|_{\Sigma_2} = f, \quad \left.\frac{\partial u}{\partial n}\right|_{\Sigma \setminus \Sigma_2} = 0. \tag{11}$$

By virtue of Theorem 2, $(u, u_t) \in C([0, T]; W_{1/2} \times W_{-1/2})$.

THEOREM 4. *For any $T > 0$, reachability set $R(T)$ of system* (3), (11) *is not dense in* $W_{1/2} \times W_{-1/2}$ *and* codim $R(T) = \infty$.

The proof is quite similar to the proof of Theorem 3 (formally in this case, all factors γ_m equal 1).

The assertion dual to Theorem 4 gives us a nonobservability result for system (2).

THEOREM 5. *For any $T > 0$, there exist nonzero functions $w_0 \in W_{1/2}$ and $w_1 \in W_{-1/2}$ such that, for the solution w of the system (2), we have $w|_{\Sigma_2} = 0$ in $L^2(\Sigma_2)$.*

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