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J. Math. Anal. Appl. 302 (2005) 30–55

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

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Exponential decay for Maxwell equations with a boundary memory condition ☆

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Received 14 April 2004

Available online 2 September 2004

Submitted by B. Straughan

Abstract

We study the asymptotic behavior of the solution of the Maxwell equations with the following boundary condition of memory type:

$$\mathbf{E}_\tau(t) = \eta_0 \mathbf{H}(t) \times \mathbf{n} + \int_0^\infty \eta(s)\mathbf{H}(t-s) \times \mathbf{n}\, ds. \tag{0.1}$$

We consider a 'Graffi' type free energy and we prove that, if the kernel η satisfies the condition $\eta'' + \kappa\eta' > 0$ and the domain Ω is strongly star shaped, then the energy of the solution exponentially decays. We also prove that the exponential decay of η is a necessary condition for the exponential decay of the solution.

Keywords: Maxwell equations; Boundary memory condition

☆ Investigation supported by the University of Bologna, Funds for selected research topics.

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doi:10.1016/j.jmaa.2004.08.004

1. Introduction

The system of equations governing the evolution of the electromagnetic field (in the absence of free charges) in a regular domain $\Omega \subset \mathbb{R}^3$ is given by the Maxwell equations

$$\dot{\mathbf{D}} - \nabla \times \mathbf{H} + \mathbf{J} = -\mathbf{J}_f, \qquad \dot{\mathbf{B}} + \nabla \times \mathbf{E} = \mathbf{0}, \tag{1.1}$$

$$\operatorname{div} \mathbf{D} = 0, \qquad \operatorname{div} \mathbf{B} = 0, \tag{1.2}$$

where $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{E}$, $\mathbf{H}$ represent respectively the electric and magnetic inductions and the electric and magnetic fields; $\mathbf{J}$ denotes the electric current density and $\mathbf{J}_f$ the external source. Here we consider a linear homogeneous isotropic dielectric, whose constitutive equations are

$$\mathbf{D}(t) = \varepsilon \mathbf{E}(t), \qquad \mathbf{B}(t) = \mu \mathbf{H}(t), \qquad \mathbf{J} = \mathbf{0}$$

with ε and μ positive scalar functions. In this case the Maxwell equations (1.1)–(1.2) assume the following form:

$$\varepsilon \dot{\mathbf{E}} - \nabla \times \mathbf{H} = -\mathbf{J}_f, \quad \mu \dot{\mathbf{H}} + \nabla \times \mathbf{E} = q\mathbf{0}, \tag{1.3}$$

$$\operatorname{div} \mathbf{E} = 0, \qquad \operatorname{div} \mathbf{H} = 0. \tag{1.4}$$

We assume that Ω is a bounded simply connected domain with a connected complement and situated, locally, on one side of its C^2-boundary $\partial\Omega$. Moreover, $\partial\Omega$ is realized by a "good" conductor, that is a medium with a high but finite electric conductivity, so that the relation between the electric and magnetic intensity on the boundary is described by the condition:

$$\mathbf{E}_\tau(x,t) = \eta_0(x)\mathbf{H}(x,t) \times \mathbf{n}(x) + \int_0^\infty \eta(x,s)\mathbf{H}^t(x,s) \times \mathbf{n}(x)\,ds, \tag{1.5}$$

where $\mathbf{E}_\tau$ denotes the tangential component of the electric field on $\partial\Omega$ and $\mathbf{n}$ is the unit outward normal to the boundary. Moreover, given a function f on $\mathbb{R}$, the notation f^t stands for the past history of f up to time t, i.e., $f^t(s) = f(t-s)$, $s \in \mathbb{R}^+$.

The aim of this paper is the study of the asymptotic behavior of the solution of the system (1.3)–(1.5) with a memory kernel satisfying a "weak fading memory principle," namely $\eta \in L^1(\mathbb{R}^+) \cap H^2(\mathbb{R}^+)$, we also assume the restriction

$$\eta_0 + \int_0^\infty \cos(\omega s)\eta(s)\,ds > 0, \quad \forall \omega \neq 0. \tag{1.6}$$

As usual, when no ambiguity arises, the dependence on x will be omitted. Condition (1.6) is obtained as a consequence of the fact that the boundary is assumed to be locally dissipative [4], i.e.,

$$\oint_0^d \mathbf{E}(x,t) \times \mathbf{H}(x,t) \cdot \mathbf{n}(x)\,dt > 0$$

holds for every cycle of period d and $x \in \partial\Omega$. Note that, as a direct consequence of the application of the Riemann–Lebesgue lemma to (1.6), we find $\eta_0 \geqslant 0$.

The problem considered in this note has been studied for the first time by Fabrizio and Morro in [4]. The existence and the uniqueness of the (local in time) solution of the system (1.3)–(1.5) and a result of asymptotic stability have been proved in [4], under the assumptions (1.6). The problem of the exponential decay has been considered by Kapitanov and Perla Menzala in [7]. We also recall the paper by Propst and Prüss [13], where an evolutive problem with a boundary condition analogous to (1.5), though for a mechanical system, has been studied. Berti [1] considered the Maxwell equations for an ohmic conductor with a boundary of the type (1.5), while in [2] an analogous problem related to the theory of transonic gases is considered together with nonlinear boundary damping and boundary source term.

In this paper we study the system (1.3)–(1.5) in the framework of the semigroup theory, the function spaces are defined in terms of the 'free energy' of the solutions. We recall [4] that a boundary free energy density is a functional that, to each history $\mathbf{H}^t$, associates a non-negative function $\psi_{\partial\Omega}$ of the time t, such that

$$\dot{\psi}_{\partial\Omega}(t) \leqslant \mathbf{E}(t) \times \mathbf{H}(t) \cdot \mathbf{n} \tag{1.7}$$

almost everywhere in $\partial\Omega$. It is well known that, in presence of a boundary condition with memory, the expression of the boundary free energy is non unique. In the sequel we will use two different functionals, which we will refer to as the 'Graffi' free energy and the 'maximal' free energy, defined as follows:

- 'Graffi' free energy:

$$\psi_{\partial\Omega}(t) = -\frac{1}{2}\int_{\partial\Omega}\int_0^\infty \eta'(s)\left|\bar{\mathbf{H}}^t(s) \times \mathbf{n}\right|^2 ds\, d\sigma, \tag{1.8}$$

- 'Maximal' free energy:

$$\phi_{\partial\Omega}(t) = \frac{1}{2}\int_{\partial\Omega}\int_0^\infty\int_0^\infty \frac{\partial^2\eta(|s_1 - s_2|)}{\partial s_1 \partial s_2}\bar{\mathbf{H}}^t(s_1) \times \mathbf{n} \cdot \bar{\mathbf{H}}^t(s_2) \times \mathbf{n}\, ds_1\, ds_2\, d\sigma, \tag{1.9}$$

where

$$\bar{\mathbf{H}}^t(s) = \int_0^s \mathbf{H}^t(\tau)\, d\tau$$

is the "backward" integrated history, which is defined when $\mathbf{H}^t \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$.

While the maximal free energy is well-defined when the memory kernel satisfies the general hypothesis (1.6), the Graffi free energy needs more restrictive assumptions that is

$$\eta' < 0, \qquad \eta'' \geqslant 0. \tag{1.10}$$

In Section 2 (and in Section 3, respectively) we rewrite the initial value problem for the Maxwell system as $\dot{u} = \mathbf{A}u + f$ in a function space related to the Graffi (respectively

Maximal) type energy and we prove that **A** generates a strongly continuous semigroup of linear contractions. We then recover the same existence, uniqueness and asymptotic stability results proved by Fabrizio and Morro in [4] by a different technique (see Theorems 2.3 and 3.3).

In Section 4 we prove some energy estimates of the type *Hilbert uniqueness method* (H.U.M.) that are used in Section 5 to prove the exponential decay of the solution. We recall that the Hilbert uniqueness method was introduced by Lions [10] in the study of the wave equation. Lagnese in [9] adapted the method for the study of a boundary control problem for the Maxwell system, then Komornik [8] and Kapitonov [6] used the same kind of estimates in the study of the Maxwell equations with a boundary condition of Leontovich type. In a recent work, the H.U.M. has been used by Kapitanov and Perla Menzala in [7] to prove the exponential decay of the solution of the system (1.3)–(1.5). The main assumptions made in [7] are that the kernels appearing in (1.5) has the form $\eta(x,t) = a(x)\exp(-\sigma(x)t)$, with $a, \sigma \in C^1(\partial\Omega)$, $a(x) \geqslant 0$, $\sigma(x) > 0$ and that the domain is star shaped. In Section 4 we assume that

$$\eta_0 > 0, \quad \exists\kappa > 0\colon \quad \eta''(s) + \kappa\eta'(s) > 0, \quad \forall s \in \mathbb{R}^+ \tag{1.11}$$

and we show that the 'Graffi' type energy of the solution exponentially decays (see Theorem 5.2). In order to compare our hypothesis with the assumption made in [7], we note that (1.11) yields

$$\left|\eta'(x,s)\right| = -\eta'(x,s) < c_0(x)e^{-\kappa(x)s}, \quad \forall s \in \mathbb{R}^+,$$

for some positive function c_0, however we do not require any regularity on c_0 and κ. Moreover the assumption $\eta'' \in L^2$ c an be relaxed so that η' is allowed to have a weak singularity at the origin (see Remark 5.3). On the other hand, we suppose that the domain Ω is *strongly* star shaped (namely, there exists $x_0 \in \Omega$ such that $(x - x_0) \cdot \mathbf{n} > 0$, for any $x \in \partial\Omega$). As a final remark, we note that the function spaces considered here, which are related to the Graffi and Maximal free energies, are different from the one considered in [7].

In the last section we prove that the exponential decay of the kernel is also a necessary condition for the exponential decay of the solution. To be more specific, we prove that, if $\eta \geqslant 0$ and the $L^2(\partial\Omega)$ norm of $(\mathbf{E}, \mathbf{H})$ exponentially decays, in the following sense

$$\int_0^\infty e^{2\alpha t}\left(\int_{\partial\Omega}\left(\left|\mathbf{E}_\tau(t)\right|^2 + \left|\mathbf{H}_\tau(t)\right|^2\right)d\sigma\right)dt < \infty,$$

(for some positive constant α), then it exists a positive constant β such that

$$\int_0^\infty e^{\beta t}\eta(t)\,dt < \infty$$

(see Proposition 6.1).

2. Graffi energy

In this section we rewrite problem (1.3)–(1.4) as an abstract Cauchy problem $\dot{u} = \mathbf{A}u + f$ and prove that $\mathbf{A}$ generates a strongly continuous semigroup of linear contractions in a function space related to the Graffi-type free energy (1.8), under the assumptions $\eta \in L^1(\mathbb{R}^+) \cap H^2(\mathbb{R}^+)$ and (1.10).

We first show that (1.8) defines a boundary free energy density in the sense that the relation (1.7) is satisfied. Let us observe that Eq. (1.5) can be rewritten as

$$\mathbf{E}_\tau(t) = \eta_0 \mathbf{H}(t) \times \mathbf{n} - \int_0^\infty \eta'(s) \bar{\mathbf{H}}^t(s) \times \mathbf{n}\, ds. \tag{2.1}$$

Thanks to the relations

$$\dot{\bar{\mathbf{H}}}^t(s) = \mathbf{H}(t) - \mathbf{H}^t(s), \qquad \frac{\partial \bar{\mathbf{H}}^t(s)}{\partial s} = \mathbf{H}^t(s), \tag{2.2}$$

it plainly follows that

$$\begin{aligned} \dot{\psi}_{\partial\Omega}(t) &= \int_0^\infty \eta'(s) \bar{\mathbf{H}}^t(s) \times \mathbf{n} \cdot \big[\mathbf{H}^t(s) - \mathbf{H}(t)\big] \times \mathbf{n}\, ds \\ &= \mathbf{E}_\tau(t) \times \mathbf{H}(t) \cdot \mathbf{n} - \eta_0 \big|\mathbf{H}(t) \times \mathbf{n}\big|^2 - \frac{1}{2} \int_0^\infty \eta''(s) \big|\bar{\mathbf{H}}^t(s) \times \mathbf{n}\big|^2 ds. \end{aligned} \tag{2.3}$$

The non-negativeness of η'' assures the validity of (1.7).

If we now define

$$\Psi_\Omega(t) = \frac{1}{2} \int_\Omega \left[\frac{1}{\mu} \big|\mathbf{E}(t)\big|^2 + \frac{1}{\varepsilon} \big|\mathbf{H}(t)\big|^2 \right] dx, \tag{2.4}$$

then the function

$$\Psi(t) = \varepsilon\mu \Psi_\Omega(t) + \int_{\partial\Omega} \psi_{\partial\Omega}(t)\, d\sigma(x) \tag{2.5}$$

may be considered as the energy of the system and, if $(\mathbf{E}, \mathbf{H})$ is a solution of the Maxwell equations (1.3)–(1.4) with a vanishing source, we have $\dot{\Psi}(t) \leqslant 0$ (see Remark 2.2).

In order to set the problem (1.3)–(1.4) in the semigroup theory, we introduce the space $\mathcal{K}$ of the triplets $(\mathbf{E}(t), \mathbf{H}(t), \bar{\mathbf{H}}^t)$ with $\mathbf{E}(t), \mathbf{H}(t) \in L^2(\Omega)$ and $\bar{\mathbf{H}}^t$ such that, for almost every $t \in \mathbb{R}^+$,

$$\int_0^\infty \int_{\partial\Omega} \big[-\eta'(s) \big|\bar{\mathbf{H}}^t(s) \times \mathbf{n}\big|^2\big]\, d\sigma\, ds < \infty,$$

with the inner product

$$\begin{aligned}
&\big\langle \big(\mathbf{E}_1(t), \mathbf{H}_1(t), \bar{\mathbf{H}}_1^t\big), \big(\mathbf{E}_2(t), \mathbf{H}_2(t), \bar{\mathbf{H}}_2^t\big)\big\rangle_{\mathcal{K}} \\
&\quad = \int_{\Omega} \big[\varepsilon \mathbf{E}_1(t) \cdot \mathbf{E}_2(t) + \mu \mathbf{H}_1(t) \cdot \mathbf{H}_2(t)\big]\, dx \\
&\qquad + \int_0^{\infty} \int_{\partial\Omega} \big[-\eta'(s) \bar{\mathbf{H}}_1^t(s) \times \mathbf{n} \cdot \bar{\mathbf{H}}_2^t(s) \times \mathbf{n}\big]\, d\sigma\, ds,
\end{aligned} \tag{2.6}$$

so that

$$\frac{1}{2} \big\| \big(\mathbf{E}, \mathbf{H}, \bar{\mathbf{H}}^t\big) \big\|_{\mathcal{K}}^2 \equiv \frac{1}{2} \big\langle \big(\mathbf{E}, \mathbf{H}, \bar{\mathbf{H}}^t\big), \big(\mathbf{E}, \mathbf{H}, \bar{\mathbf{H}}^t\big)\big\rangle$$

is the Graffi-type energy (2.5). We then define the operator $\mathbf{A}$ as follows:

$$\mathbf{A}\big(\mathbf{E}, \mathbf{H}, \bar{\mathbf{H}}^t\big) = \left(\frac{1}{\varepsilon} \nabla \times \mathbf{H}, -\frac{1}{\mu} \nabla \times \mathbf{E}, \mathbf{H}(t) - \frac{\partial}{\partial s} \bar{\mathbf{H}}^t(s)\right), \tag{2.7}$$

and we define $\mathcal{D}(\mathbf{A}) \subset \mathcal{K}$ as the set of the triplets $(\mathbf{E}, \mathbf{H}, \bar{\mathbf{H}}^t)$ satisfying the boundary condition (2.1) and such that $\mathbf{A}(\mathbf{E}, \mathbf{H}, \bar{\mathbf{H}}^t)$ belongs to $\mathcal{K}$. Clearly, if $(\mathbf{E}, \mathbf{H}, \bar{\mathbf{H}}^t)) \in \mathcal{D}(\mathbf{A})$, then $\nabla \times \mathbf{E}, \nabla \times \mathbf{H} \in L^2(\Omega)$, as a consequence $\mathbf{H} \times \mathbf{n} \in L^2(\partial\Omega)$ and

$$\int_0^{\infty} \int_{\partial\Omega} \big[-\eta'(s) \big|\mathbf{H}^t(s) \times \mathbf{n}\big|^2\big]\, d\sigma\, ds < \infty,$$

for almost every $t \in \mathbb{R}^+$. Moreover, thanks to the boundary condition (1.5), also $\mathbf{E}_\tau \in L^2(\partial\Omega)$.

Theorem 2.1. *If η satisfy condition* (1.10) *and $\eta_0 \geqslant 0$, then $(\mathbf{A}, \mathcal{D}(\mathbf{A}))$ generates a strongly continuous semigroup of linear contractions.*

Proof. Thanks to Lumer–Phillips theorem, we have to show that the domain $\mathcal{D}(\mathbf{A})$ of the operator $\mathbf{A}$ is dense in $\mathcal{K}$ and that $\mathbf{A}$ and its adjoint $\mathbf{A}^*$ are dissipative operators. Clearly, $\mathbf{A}$ is a closed linear operator and, since the trace of an L^2 function is not well-defined, $\mathcal{D}(\mathbf{A})$ is dense in $\mathcal{K}$. We now prove that $\mathbf{A}$ is dissipative, i.e.,

$$\big\langle \mathbf{A}u(t), u(t)\big\rangle_{\mathcal{K}} \leqslant 0, \tag{2.8}$$

where u stands for the triplet $(\mathbf{E}, \mathbf{H}, \bar{\mathbf{H}}^t)$.

$$\begin{aligned}
\big\langle \mathbf{A}u(t), u(t)\big\rangle_{\mathcal{K}} &= \int_{\Omega} \big[\nabla \times \mathbf{H}(t) \cdot \mathbf{E}(t) - \nabla \times \mathbf{E}(t) \cdot \mathbf{H}(t)\big]\, dx \\
&\quad + \int_0^{\infty} \int_{\partial\Omega} \left[-\eta'(s) \left(\mathbf{H}(t) \times \mathbf{n} - \frac{\partial \bar{\mathbf{H}}^t}{\partial s}(s) \times \mathbf{n}\right) \cdot \bar{\mathbf{H}}^t(s) \times \mathbf{n}\right] d\sigma\, ds \\
&= \int_{\partial\Omega} \mathbf{H}(t) \times \mathbf{E}(t) \cdot \mathbf{n}\, d\sigma - \frac{1}{2} \int_0^{\infty} \int_{\partial\Omega} \eta''(s) \big|\bar{\mathbf{H}}^t(s) \times \mathbf{n}\big|^2\, d\sigma\, ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty\int_{\partial\Omega}\mathbf{H}(t)\times\mathbf{n}\cdot\big[-\eta'(s)\bar{\mathbf{H}}^t(s)\times\mathbf{n}\big]\,d\sigma\,ds\\
&=\int_{\partial\Omega}\left[-\mathbf{E}(t)\cdot\mathbf{H}(t)\times\mathbf{n}+\mathbf{H}(t)\times\mathbf{n}\cdot\int_0^\infty-\eta'(s)\bar{\mathbf{H}}^t(s)\times\mathbf{n}\,ds\right]d\sigma\\
&\quad-\frac{1}{2}\int_0^\infty\int_{\partial\Omega}\eta''(s)\big|\bar{\mathbf{H}}^t(s)\times\mathbf{n}\big|^2\,d\sigma\,ds. \qquad (2.9)
\end{aligned}
$$

Taking into account that the boundary condition assumes the form (2.1), we find

$$
\big\langle(\mathbf{A}u(t),u(t)\big\rangle_{\mathcal{K}}=-\int_{\partial\Omega}\eta_0\big|\mathbf{H}(t)\times\mathbf{n}\big|^2\,d\sigma-\frac{1}{2}\int_0^\infty\int_{\partial\Omega}\eta''(s)\big|\bar{\mathbf{H}}^t(s)\times\mathbf{n}\big|^2\,d\sigma\,ds\leqslant 0,
$$

since $\eta''\geqslant 0$ and $\eta_0\geqslant 0$. Condition (2.8) is therefore proved.

We next consider the adjoint operator $\mathbf{A}^*$. Its domain $\mathcal{D}(\mathbf{A}^*)$ is defined as the set of the triples $v\in\mathcal{K}$ such that a $w\in\mathcal{K}$ exists satisfying

$$
\langle\mathbf{A}u,v\rangle_{\mathcal{K}}=\langle u,w\rangle_{\mathcal{K}},\quad \forall u\in\mathcal{D}(\mathbf{A}).
$$

Let $u(t)=(\mathbf{E}(t),\mathbf{H}(t),\bar{\mathbf{H}}^t)\in\mathcal{D}(\mathbf{A})$ and $v(t)=(\mathbf{e}(t),\mathbf{h}(t),\bar{\mathbf{h}}^t)\in\mathcal{K}$, then

$$
\begin{aligned}
\big\langle(\mathbf{A}u(t),v(t)\big\rangle_{\mathcal{K}}&=\int_\Omega\big[\nabla\times\mathbf{H}(t)\cdot\mathbf{e}(t)-\nabla\times\mathbf{E}(t)\cdot\mathbf{h}(t)\big]\,dx\\
&\quad+\int_0^\infty\int_{\partial\Omega}\left[-\eta'(s)\left(\mathbf{H}(t)\times\mathbf{n}-\frac{\partial\bar{\mathbf{H}}^t}{\partial s}(s)\times\mathbf{n}\right)\cdot\bar{\mathbf{h}}^t(s)\times\mathbf{n}\right]d\sigma\,ds\\
&=-\int_\Omega\big[\nabla\times\mathbf{h}(t)\cdot\mathbf{E}(t)-\nabla\times\mathbf{e}(t)\cdot\mathbf{H}(t)\big]\,dx\\
&\quad-\int_{\partial\Omega}\big[\mathbf{e}(t)\cdot\mathbf{H}(t)\times\mathbf{n}+\mathbf{E}(t)\cdot\mathbf{h}(t)\times\mathbf{n}\big]\,d\sigma\\
&\quad+\int_0^\infty\int_{\partial\Omega}\mathbf{H}(t)\times\mathbf{n}\cdot\big[-\eta'(s)\bar{\mathbf{h}}^t(s)\times\mathbf{n}\big]\,d\sigma\,ds\\
&\quad+\int_0^\infty\int_{\partial\Omega}\eta'(s)\bar{\mathbf{h}}^t(s)\times\mathbf{n}\cdot\frac{\partial}{\partial s}\bar{\mathbf{H}}^t(s)\times\mathbf{n}\,d\sigma\,ds.
\end{aligned}
$$

By using condition (2.1), we find

$$
\big\langle\mathbf{A}u(t),v(t)\big\rangle_{\mathcal{K}}=-\int_\Omega\big[\nabla\times\mathbf{h}(t)\cdot\mathbf{E}(t)-\nabla\times\mathbf{e}(t)\cdot\mathbf{H}(t)\big]\,dx
$$

$$
\begin{aligned}
&- \int_{\partial\Omega} \int_0^\infty -\eta'(s)\left[\mathbf{h}(t) \times \mathbf{n} - \frac{\partial}{\partial s}\bar{\mathbf{h}}^t(s) \times \mathbf{n}\right] \cdot \bar{\mathbf{H}}^t(s) \times \mathbf{n}\, ds\, d\sigma \\
&- \int_{\partial\Omega} \int_0^\infty \eta''(s)\bar{\mathbf{h}}^t(s) \times \mathbf{n} \cdot \bar{\mathbf{H}}^t(s) \times \mathbf{n}\, ds\, d\sigma \\
&- \int_{\partial\Omega} \mathbf{H}(t) \times \mathbf{n} \cdot \left(\mathbf{e}(t) + \eta_0 \mathbf{h}(t) \times \mathbf{n} + \int_0^\infty \eta'(s)\bar{\mathbf{h}}^t(s) \times \mathbf{n}\, ds\right) d\sigma \\
= {} & -\langle u(t), \mathbf{A}v(t)\rangle_{\mathcal{K}} - \int_{\partial\Omega} \int_0^\infty \eta''(s)\bar{\mathbf{h}}^t(s) \times \mathbf{n} \cdot \bar{\mathbf{H}}^t(s) \times \mathbf{n}\, ds\, d\sigma \\
&- \int_{\partial\Omega} \mathbf{H}(t) \times \mathbf{n} \cdot \left(\mathbf{e}(t) + \eta_0 \mathbf{h}(t) \times \mathbf{n} + \int_0^\infty \eta'(s)\bar{\mathbf{h}}^t(s) \times \mathbf{n}\, ds\right) d\sigma.
\end{aligned}
\tag{2.10}
$$

Hence every $v \equiv (\mathbf{e}, \mathbf{h}, \bar{\mathbf{h}}^t) \in \mathcal{D}(\mathbf{A}^*)$ must satisfy the "dual boundary condition"

$$
\mathbf{e}_\tau(t) = -\eta_0 \mathbf{h}(t) \times \mathbf{n} + \int_0^\infty -\eta'(s)\bar{\mathbf{h}}^t(s) \times \mathbf{n}\, ds
$$

and $w = \mathbf{A}^* v$ is defined as

$$
\mathbf{A}^*\big(\mathbf{e}, \mathbf{h}, \bar{\mathbf{h}}^t\big) = -\left(\frac{1}{\varepsilon}\nabla \times \mathbf{h}, -\frac{1}{\mu}\nabla \times \mathbf{e}, \mathbf{h}(t) - \frac{\partial}{\partial s}\bar{\mathbf{h}}^t(s) - \frac{\eta''(s)}{\eta'(s)}\bar{\mathbf{h}}^t(s) \times \mathbf{n}\right).
$$

We finally show that $\mathbf{A}^*$ is dissipative. From (2.10) and (2.9) we immediately get:

$$
\begin{aligned}
\langle \mathbf{A}^* u(t), u(t)\rangle_{\mathcal{K}} &= -\langle \mathbf{A}u(t), u(t)\rangle_{\mathcal{K}} - \int_{\partial\Omega} \int_0^\infty \eta''(s)\big|\bar{\mathbf{H}}^t(s) \times \mathbf{n}\big|^2\, ds\, d\sigma \\
&= -\int_{\partial\Omega} \left[-\mathbf{E}(t) \cdot \mathbf{H}(t) \times \mathbf{n} + \mathbf{H}(t) \times \mathbf{n} \cdot \int_0^\infty -\eta'(s)\bar{\mathbf{H}}^t(s) \times \mathbf{n}\, ds\right] d\sigma \\
&\quad - \frac{1}{2}\int_0^\infty \int_{\partial\Omega} \eta''(s)\big|\bar{\mathbf{H}}^t(s) \times \mathbf{n}\big|^2\, d\sigma\, ds \\
&= -\int_{\partial\Omega} \left[\eta_0 \big|\mathbf{H}(t) \times \mathbf{n}\big|^2 + \frac{1}{2}\int_0^\infty \eta''(s)\big|\bar{\mathbf{H}}^t(s) \times \mathbf{n}\big|^2\, ds\right] d\sigma \leqslant 0,
\end{aligned}
$$

for any $u \in \mathcal{D}(\mathbf{A}^*)$. This completes the proof. □

Remark 2.2. Suppose that η satisfy condition (1.10) and $\eta_0 \geqslant 0$. If $(\mathbf{E}(t), \mathbf{H}(t), \bar{\mathbf{H}}^t)$ is a solution of the Maxwell equations (1.3)–(1.4) with a vanishing source, then $\dot{\Psi}(t) \leqslant 0$, since $\dot{\Psi}(t) = \langle \mathbf{A}u(t), u(t)\rangle_{\mathcal{K}}$, which is non-positive, by (2.8).

We end this section with an existence result. Quoting Da Prato and Sinestrari [3], we say that a function $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^+, \mathcal{K}) \cap L^p_{\mathrm{loc}}(\mathbb{R}^+, \mathcal{D}(\mathbf{A}))$ is a *strict solution* of the abstract Cauchy problem $\dot{u} = \mathbf{A}u + f; u(0) = u_0$ (with $u_0 \in \mathcal{D}(\mathbf{A})$, $f \in L^p_{\mathrm{loc}}(\mathbb{R}^+, \mathcal{K})$, and $p \in [1, \infty[$) if $u(0) = u_0$ and $\dot{u}(t) = \mathbf{A}u(t) + f(t)$ holds for almost any positive t.

Theorem 2.3. *Suppose that η satisfy condition* (1.10) *and $\eta_0 \geqslant 0$. Let $\mathbf{E}_0, \mathbf{H}_0 \in L^2(\Omega, \mathbb{R}^3)$ be such that $\operatorname{div}\mathbf{E}_0 = 0$, $\operatorname{div}\mathbf{H}_0 = 0$, $\nabla \times \mathbf{E}_0, \nabla \times \mathbf{H}_0 \in L^2(\Omega, \mathbb{R}^3)$ and that $\mathbf{E}_0 \times \mathbf{n} = \eta_0\mathbf{H}_0 \times \mathbf{n}$ on $\partial\Omega$. Let $\mathbf{J}_f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^+, L^2(\Omega, \mathbb{R}^3))$. Then the problem* (1.3)–(1.5), *with initial conditions* $(\mathbf{E}_0, \mathbf{H}_0)$ *and null past history, has a unique strict solution $u \in C^1(\mathbb{R}^+, \mathcal{K}) \cap C(\mathbb{R}^+, \mathcal{D}(\mathbf{A}))$ such that $\dot{u} \in L^p_{\mathrm{loc}}(\mathbb{R}^+, \mathcal{K})$.*

Proof. Since $(\mathbf{A}, \mathcal{D}(\mathbf{A}))$ generates a strongly continuous semigroup in $\mathcal{K}$, we can apply [3, Theorem 8.1], with $u_0 = (\mathbf{E}_0, \mathbf{H}_0, \mathbf{0})$ and $f = (\mathbf{J}_f, \mathbf{0}, \mathbf{0})$. □

3. Maximal energy

In this section we show that the function $\phi_{\partial\Omega}$ defined in (1.9) is a boundary free energy density, in the sense of (1.7). Then,

$$\Phi(t) = \varepsilon\mu\Psi_\Omega(t) + \int_{\partial\Omega} \phi_{\partial\Omega}(t)\, d\sigma(x), \tag{3.1}$$

(here Ψ_Ω is the function defined in (2.4)) can be considered as the energy of the system and is a non-increasing function, when the source in the Maxwell equations (1.3)–(1.4) vanishes (see Remark 3.2).

In order to prove (1.7) we state some preliminary facts. We first note that the function $\frac{\partial^2\eta(|s_1-s_2|)}{\partial s_1\partial s_2}$ is defined in the distribution sense in terms of the Dirac measure as

$$\frac{\partial^2\eta(|s_1 - s_2|)}{\partial s_1\partial s_2} = -\eta''\big(|s_1 - s_2|\big) - 2\eta'(0)\delta(s_1 - s_2),$$

(the right-hand side is well-defined since $\eta \in H^2(\mathbb{R}^+)$). In the sequel, we will use the following elementary identities:

$$\frac{\partial\eta(|s_1 - s_2|)}{\partial s_1} = -\frac{\partial\eta(|s_1 - s_2|)}{\partial s_2}, \qquad \int_0^\infty \frac{\partial\eta(|s_1 - s_2|)}{\partial s_1}\, ds_1 = -\eta(s_2), \tag{3.2}$$

moreover, being $\mathbf{H}(s) = 0$ for any $s < 0$, $\bar{\mathbf{H}}^t$ is a continuous bounded function for every positive t. We next prove that

$$\int_0^\infty\int_0^\infty \frac{\partial^2\eta(|s_1-s_2|)}{\partial s_1\partial s_2}\big(\mathbf{H}(t)-\mathbf{H}^t(s_1)\big)\times\mathbf{n}\cdot\bar{\mathbf{H}}^t(s_2)\times\mathbf{n}\,ds_1\,ds_2$$
$$=\mathbf{H}(t)\times\mathbf{n}\cdot\int_0^\infty \eta(s_2)\mathbf{H}^t(s_2)\times\mathbf{n}\,ds_2. \tag{3.3}$$

To prove (3.3) it is sufficient to integrate by parts, use the second identity in (3.2) and (2.2):

$$\int_0^\infty\int_0^\infty \frac{\partial^2\eta(|s_1-s_2|)}{\partial s_1\partial s_2}\big(\mathbf{H}(t)-\mathbf{H}^t(s_1)\big)\times\mathbf{n}\cdot\bar{\mathbf{H}}^t(s_2)\times\mathbf{n}\,ds_1\,ds_2$$
$$\big(\text{since } \bar{\mathbf{H}}^t \text{ is bounded and } \eta'\in L^2(\mathbb{R}^+)\big)$$
$$=-\int_0^\infty\int_0^\infty \frac{\partial\eta(|s_1-s_2|)}{\partial s_1}\big(\mathbf{H}(t)-\mathbf{H}^t(s_1)\big)\times\mathbf{n}\cdot\frac{\partial\bar{\mathbf{H}}^t}{\partial s_2}(s_2)\times\mathbf{n}\,ds_1\,ds_2$$
$$=\mathbf{H}(t)\times\mathbf{n}\cdot\int_0^\infty \eta(s_2)\mathbf{H}^t(s_2)\times\mathbf{n}\,ds_2$$
$$+\int_0^\infty\int_0^\infty \frac{\partial\eta(|s_1-s_2|)}{\partial s_1}\mathbf{H}^t(s_1)\times\mathbf{n}\cdot\mathbf{H}^t(s_2)\times\mathbf{n}\,ds_1\,ds_2.$$

This proves (3.3), since last integral vanishes.

We are in position to show that $\phi_{\partial\Omega}$ is a boundary free energy density. Indeed, from (3.3) and (1.5) it follows that

$$\dot{\phi}_{\partial\Omega}(t)=\int_0^\infty\int_0^\infty \frac{\partial^2\eta(|s_1-s_2|)}{\partial s_1\partial s_2}\big(\mathbf{H}(t)-\mathbf{H}^t(s_1)\big)\times\mathbf{n}\cdot\bar{\mathbf{H}}^t(s_2)\times\mathbf{n}\,ds_1\,ds_2$$
$$=\mathbf{E}_\tau(t)\times\mathbf{H}(t)\cdot\mathbf{n}-\eta_0\big|\mathbf{H}(t)\times\mathbf{n}\big|^2.$$

This proves (1.7).

We next show that the operator $\mathbf{A}$ defined in (2.7) generates a strongly continuous semigroup of linear contractions in a Hilbert space $\mathcal{H}$ defined in terms of the 'Maximal' energy (3.1). To this aim, we define $\mathcal{H}$ as the space of the triplets $(\mathbf{E}(t),\mathbf{H}(t),\bar{\mathbf{H}}^t)$ with $\mathbf{E}(t),\mathbf{H}(t)\in L^2(\Omega)$ and $\bar{\mathbf{H}}^t:\partial\Omega\times\mathbb{R}^+\to\mathbb{R}^3$ such that

$$\int_{\partial\Omega}\int_0^\infty\int_0^\infty \frac{\partial^2\eta(|s_1-s_2|)}{\partial s_1\partial s_2}\bar{\mathbf{H}}^t(s_1)\times\mathbf{n}\cdot\bar{\mathbf{H}}^t(s_2)\times\mathbf{n}\,ds_1\,ds_2\,d\sigma<\infty,$$

for almost every $t\in\mathbb{R}^+$ and we set the inner product as

$$\big\langle\big(\mathbf{E}_1(t),\mathbf{H}_1(t),\bar{\mathbf{H}}_1^t\big),\big(\mathbf{E}_2(t),\mathbf{H}_2(t),\bar{\mathbf{H}}_2^t\big)\big\rangle_{\mathcal{H}}$$

$$= \int_\Omega \left[\varepsilon \mathbf{E}_1(t) \cdot \mathbf{E}_2(t) + \mu \mathbf{H}_1(t) \cdot \mathbf{H}_2(t)\right] dx$$

$$+ \int_{\partial\Omega} \int_0^\infty \int_0^\infty \frac{\partial^2 \eta(|s_1 - s_2|)}{\partial s_1 \partial s_2} \bar{\mathbf{H}}_1^t(s_1) \times \mathbf{n} \cdot \bar{\mathbf{H}}_2^t(s_2) \times \mathbf{n}\, ds_1\, ds_2\, d\sigma. \tag{3.4}$$

Theorem 3.1. *If η satisfy condition* (1.6)*, then* $(\mathbf{A}, \mathcal{D}(\mathbf{A}))$ *generates a strongly continuous semigroup of linear contractions in $\mathcal{H}$.*

Proof. As in Section 2, we have to show that the domain $\mathcal{D}(\mathbf{A})$ of the operator $\mathbf{A}$ is dense in $\mathcal{H}$ and that $\mathbf{A}$ and its adjoint $\mathbf{A}^*$ are dissipative operators, the claim follows from the Lumer–Phillips theorem.

The domain $\mathcal{D}(\mathbf{A}) \subset \mathcal{H}$ is the set of the triplets $(\mathbf{E}, \mathbf{H}, \bar{\mathbf{H}}^t)$ satisfying the boundary condition (1.5) and such that $\mathbf{A}(\mathbf{E}, \mathbf{H}, \bar{\mathbf{H}}^t) \in \mathcal{H}$, then it is a dense subset of $\mathcal{H}$. Note that, if $(\mathbf{E}, \mathbf{H}, \bar{\mathbf{H}}^t)) \in \mathcal{D}(\mathbf{A})$, then $\nabla \times \mathbf{E}, \nabla \times \mathbf{H} \in L^2(\Omega)$, as a consequence $\mathbf{H} \times \mathbf{n} \in L^2(\partial\Omega)$ and

$$\int_{\partial\Omega} \int_0^\infty \int_0^\infty \frac{\partial^2 \eta(|s_1 - s_2|)}{\partial s_1 \partial s_2} \mathbf{H}^t(s_1) \times \mathbf{n} \cdot \mathbf{H}^t(s_2) \times \mathbf{n}\, ds_1\, ds_2\, d\sigma < \infty,$$

for almost every $t \in \mathbb{R}^+$. We now prove that $\mathbf{A}$ is dissipative, i.e.,

$$\langle \mathbf{A}u(t), u(t)\rangle_{\mathcal{H}} \leqslant 0, \tag{3.5}$$

for every $u = (\mathbf{E}, \mathbf{H}, \bar{\mathbf{H}}^t) \in \mathcal{D}(\mathbf{A})$. By using (3.3), we find

$$\begin{aligned}
\langle \mathbf{A}u(t), u(t)\rangle_{\mathcal{H}} &= \int_\Omega \left[\nabla \times \mathbf{H}(t) \cdot \mathbf{E}(t) - \nabla \times \mathbf{E}(t) \cdot \mathbf{H}(t)\right] dx \\
&\quad + \int_{\partial\Omega} \int_0^\infty \int_0^\infty \frac{\partial^2 \eta(|s_1 - s_2|)}{\partial s_1 \partial s_2} \left(\mathbf{H}(t) - \frac{\partial \bar{\mathbf{H}}^t}{\partial s}(s_1)\right) \\
&\quad \times \mathbf{n} \cdot \bar{\mathbf{H}}_2^t(s_2) \times \mathbf{n}\, ds_1\, ds_2\, d\sigma \\
&= \int_{\partial\Omega} \left(\mathbf{H}(t) \times \mathbf{E}(t) \cdot \mathbf{n} + \mathbf{H}(t) \times \mathbf{n} \cdot \int_0^\infty \eta(s_2) \mathbf{H}^t(s_2) \times \mathbf{n}\, ds_2\right) d\sigma,
\end{aligned} \tag{3.6}$$

thus (1.5) gives

$$\langle \mathbf{A}u(t), u(t)\rangle_{\mathcal{H}} = -\int_{\partial\Omega} \eta_0 \left|\mathbf{H}(t) \times \mathbf{n}\right|^2 d\sigma \leqslant 0$$

and the proof of (3.5) is accomplished.

We next compute the adjoint operator $\mathbf{A}^*$. For any $u(t) = (\mathbf{E}(t), \mathbf{H}(t), \bar{\mathbf{H}}^t) \in \mathcal{D}(\mathbf{A})$ and $v(t) = (\mathbf{e}(t), \mathbf{h}(t), \bar{\mathbf{h}}^t) \in \mathcal{H}$ we find

$$\begin{aligned}
\big\langle \mathbf{A}u(t), v(t)\big\rangle_{\mathcal{H}} &= \int_{\Omega} \big[\nabla \times \mathbf{H}(t)\cdot \mathbf{e}(t) - \nabla \times \mathbf{E}(t)\cdot \mathbf{h}(t)\big]\,dx \\
&\quad + \int_{\partial\Omega}\int_0^\infty\int_0^\infty \frac{\partial^2 \eta(|s_1-s_2|)}{\partial s_1 \partial s_2}\left(\mathbf{H}(t)\times \mathbf{n} - \frac{\partial \bar{\mathbf{H}}^t}{\partial s_1}(s_1)\times \mathbf{n}\right) \\
&\quad \cdot \bar{\mathbf{h}}^t(s_2)\times \mathbf{n}\,ds_1\,ds_2\,d\sigma \\
&= -\int_{\Omega} \big[\nabla \times \mathbf{h}(t)\cdot \mathbf{E}(t) - \nabla \times \mathbf{e}(t)\cdot \mathbf{H}(t)\big]\,dx \\
&\quad - \int_{\partial\Omega} \big[\mathbf{e}(t)\cdot \mathbf{H}(t)\times \mathbf{n} + \mathbf{E}(t)\cdot \mathbf{h}(t)\times \mathbf{n}\big]\,d\sigma \\
&\quad - \int_{\partial\Omega}\int_0^\infty\int_0^\infty \frac{\partial \eta(|s_1-s_2|)}{\partial s_1}\left(\mathbf{H}(t)\times \mathbf{n} - \frac{\partial \bar{\mathbf{H}}^t}{\partial s_1}(s_1)\times \mathbf{n}\right) \\
&\quad \cdot \frac{\partial \bar{\mathbf{h}}^t}{\partial s_2}(s_2)\times \mathbf{n}\,ds_1\,ds_2\,d\sigma.
\end{aligned}$$

We now use condition (1.5), the identities (3.2), then we integrate again by parts. We find

$$\begin{aligned}
\big\langle \mathbf{A}u(t), v(t)\big\rangle_{\mathcal{H}} &= -\int_{\Omega} \big[\nabla \times \mathbf{h}(t)\cdot \mathbf{E}(t) - \nabla \times \mathbf{e}(t)\cdot \mathbf{H}(t)\big]\,dx \\
&\quad - \int_{\partial\Omega} \Bigg[\mathbf{e}(t)\cdot \mathbf{H}(t)\times \mathbf{n} \\
&\quad + \left(\eta_0 \mathbf{H}(t)\times \mathbf{n} + \int_0^\infty \eta(s)\mathbf{H}^t(s)\times \mathbf{n}\,ds\right)\cdot \mathbf{h}(t)\times \mathbf{n}\Bigg]\,d\sigma \\
&\quad \cdot \int_{\partial\Omega} \mathbf{H}(t)\times \mathbf{n}\cdot \int_0^\infty \eta(s_2)\mathbf{h}^t(s_2)\times \mathbf{n}\,ds_2\,d\sigma \\
&\quad + \int_{\partial\Omega}\int_0^\infty\int_0^\infty \frac{\partial^2 \eta(|s_1-s_2|)}{\partial s_1 \partial s_2}\,\frac{\partial \bar{\mathbf{h}}^t}{\partial s_2}(s_2)\times \mathbf{n}\cdot \bar{\mathbf{H}}^t(s_1)\times \mathbf{n}\,ds_1\,ds_2\,d\sigma.
\end{aligned}$$

By using again the second identity (3.2), we finally get

$$\begin{aligned}
\big\langle \mathbf{A}u(t), v(t)\big\rangle_{\mathcal{H}} &= -\int_{\Omega} \big[\nabla \times \mathbf{h}(t)\cdot \mathbf{E}(t) - \nabla \times \mathbf{e}(t)\cdot \mathbf{H}(t)\big]\,dx \\
&\quad - \int_{\partial\Omega}\int_0^\infty\int_0^\infty \frac{\partial^2 \eta(|s_1-s_2|)}{\partial s_1 \partial s_2}\left(\mathbf{h}(t)\times \mathbf{n} - \frac{\partial \bar{\mathbf{h}}^t}{\partial s_1}(s_1)\times \mathbf{n}\right)
\end{aligned}$$

$$\cdot \bar{\mathbf{H}}^t(s_2) \times \mathbf{n}\, ds_1\, ds_2\, d\sigma$$
$$- \int_{\partial\Omega} \mathbf{H}(t) \times \mathbf{n} \cdot \left(\mathbf{e}(t) + \eta_0 \mathbf{h}(t) \times \mathbf{n} - \int_0^\infty \eta(s) \mathbf{h}^t(s) \times \mathbf{n}\, ds \right) d\sigma.$$

Hence every $v \equiv (\mathbf{e}, \mathbf{h}, \bar{\mathbf{h}}^t) \in \mathcal{D}(\mathbf{A}^*)$ must satisfy the "dual boundary condition"

$$\mathbf{e}_\tau(t) = -\eta_0 \mathbf{h}(t) \times \mathbf{n} + \int_0^\infty \eta(s) \mathbf{h}^t(s) \times \mathbf{n}\, ds, \tag{3.7}$$

moreover, $\mathbf{A}^* = -\mathbf{A}$.

We finally show that $\mathbf{A}^*$ is dissipative. For any $u \in \mathcal{D}(\mathbf{A}^*)$ we use (3.6) and we find

$$\begin{aligned}\langle \mathbf{A}^* u(t), u(t) \rangle_{\mathcal{H}} &= -\langle \mathbf{A} u(t), u(t) \rangle_{\mathcal{H}} \\ &= -\int_{\partial\Omega} \left(\mathbf{H}(t) \times \mathbf{E}(t) \cdot \mathbf{n} + \mathbf{H}(t) \times \mathbf{n} \cdot \int_0^\infty \eta(s_2) \mathbf{H}^t(s_2) \times \mathbf{n}\, ds_2 \right) d\sigma,\end{aligned}$$

then, by using (3.7), we finally obtain

$$\langle \mathbf{A}^* u(t), u(t) \rangle_{\mathcal{H}} = -\int_{\partial\Omega} \eta_0 \left| \mathbf{H}(t) \times \mathbf{n} \right|^2 d\sigma \leqslant 0.$$

This completes the proof of Theorem 3.3. □

Remark 3.2. Suppose that η satisfy condition (1.6). If $(\mathbf{E}(t), \mathbf{H}(t), \bar{\mathbf{H}}^t)$ is a solution of the Maxwell equations (1.3)–(1.4) with a vanishing source, then $\dot{\Phi}(t) \leqslant 0$, since $\dot{\Phi}(t) = \langle \mathbf{A} u(t), u(t) \rangle_{\mathcal{H}}$, which is non-positive, by (3.5).

Theorem 3.3. *Suppose that η satisfy condition* (1.6)*. Let $\mathbf{J}_f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^+, L^2(\Omega, \mathbb{R}^3))$ and let $\mathbf{E}_0, \mathbf{H}_0 \in L^2(\Omega, \mathbb{R}^3)$ be such that* $\operatorname{div} \mathbf{E}_0 = 0$, $\operatorname{div} \mathbf{H}_0 = 0$, $\nabla \times \mathbf{E}_0, \nabla \times \mathbf{H}_0 \in L^2(\Omega, \mathbb{R}^3)$ *and that $\mathbf{E}_0 \times \mathbf{n} = \eta_0 \mathbf{H}_0 \times \mathbf{n}$ on $\partial\Omega$. Then the problem* (1.3)–(1.5)*, with initial conditions* $(\mathbf{E}_0, \mathbf{H}_0)$ *and null past history, has a unique strict solution $u \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C(\mathbb{R}^+, \mathcal{D}(\mathbf{A}))$ such that $\dot{u} \in L^p_{\mathrm{loc}}(\mathbb{R}^+, \mathcal{H})$.*

Proof. As in the proof of Theorem 2.3 we apply [3, Theorem 8.1]. □

4. H.U.M. estimates

In this section we give some preliminary results to prove the energy estimates of H.U.M. type.

Lemma 4.1. *If* $(\mathbf{E}, \mathbf{H})$ *is a solution of the Maxwell equations* (1.3)–(1.4) *with a vanishing source, then*

(i) *for any function* $\phi \in C^2(\bar{\Omega})$ *and any* $T > 0$, *we have*

$$\begin{aligned}
&\int_{\Sigma} \left[\frac{1}{\varepsilon} \big(\nabla\phi(x) \cdot \mathbf{H}(x,t) \big) \mathbf{H}(x,t) \cdot \mathbf{n} + \frac{1}{\mu} \big(\nabla\phi(x) \cdot \mathbf{E}(x,t) \big) \mathbf{E}(x,t) \cdot \mathbf{n} \right] d\sigma(x)\, dt \\
&\quad = \int_{\Omega} \mathbf{E}(x,T) \times \mathbf{H}(x,T) \cdot \nabla\phi(x)\, dx - \int_{\Omega} \mathbf{E}(x,0) \times \mathbf{H}(x,0) \cdot \nabla\phi(x)\, dx \\
&\qquad + \int_{Q} \sum_{i,j=1}^{3} \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) \left(\frac{1}{\varepsilon} H_i(x,t) H_j(x,t) + \frac{1}{\mu} E_i(x,t) E_j(x,t) \right) dx\, dt \\
&\qquad + \frac{1}{2} \int_{\Sigma} \left(\frac{1}{\varepsilon} |\mathbf{H}|^2 + \frac{1}{\mu} |\mathbf{E}|^2 \right) \nabla\phi(x) \cdot \mathbf{n}\, d\sigma(x)\, dt \\
&\qquad - \frac{1}{2} \int_{Q} \Delta\phi(x) \left(\frac{1}{\varepsilon} |\mathbf{H}|^2 + \frac{1}{\mu} |\mathbf{E}|^2 \right) dx\, dt;
\end{aligned}$$

(ii) *for any* $T > 0$ *and* $t_0 > 0$ *we have*

$$\begin{aligned}
&\frac{1}{2} \int_{Q} \left(\frac{1}{\varepsilon} \big|\mathbf{H}(x,t)\big|^2 + \frac{1}{\mu} \big|\mathbf{E}(x,t)\big|^2 \right) dx\, dt \\
&\quad = \frac{1}{2} \int_{\Omega} (T + t_0) \left(\frac{1}{\varepsilon} \big|\mathbf{H}(x,T)\big|^2 + \frac{1}{\mu} \big|\mathbf{E}(x,T)\big|^2 \right) dx \\
&\qquad - \frac{t_0}{2} \int_{\Omega} \left(\frac{1}{\varepsilon} \big|\mathbf{H}(x,0)\big|^2 + \frac{1}{\mu} \big|\mathbf{E}(x,0)\big|^2 \right) dx \\
&\qquad + \frac{1}{\varepsilon\mu} \int_{\Sigma} (t + t_0) \mathbf{E}(x,t) \times \mathbf{H}(x,t) \cdot \mathbf{n}(x)\, d\sigma(x)\, dt.
\end{aligned}$$

Proof. We first show that the following identities hold for any function $\phi \in C^2(\bar{\Omega})$ and for any $T > 0$,

$$\begin{aligned}
&\int_{\Omega} (\mathbf{E} \times \mathbf{H})(x,T) \cdot \nabla\phi(x)\, dx - \int_{\Omega} (\mathbf{E} \times \mathbf{H})(x,0) \cdot \nabla\phi(x)\, dx \\
&\qquad + \int_{Q} \left[\frac{1}{\varepsilon} (\nabla \times \mathbf{H}) \cdot (\nabla\phi \times \mathbf{H}) + \frac{1}{\mu} (\nabla \times \mathbf{E}) \cdot (\nabla\phi \times \mathbf{E}) \right] (x,t)\, dx\, dt = 0, \qquad (4.1)
\end{aligned}$$

$$\begin{aligned}
&\int_{\Sigma} \left[\frac{1}{\varepsilon} \big(\nabla\phi(x) \cdot \mathbf{H}(x,t) \big) \mathbf{H}(x,t) \cdot \mathbf{n}(x) \right. \\
&\qquad \left. + \frac{1}{\mu} \big(\nabla\phi(x) \cdot \mathbf{E}(x,t) \big) \mathbf{E}(x,t) \cdot \mathbf{n}(x) \right] d\sigma(x)\, dt
\end{aligned}$$

$$= \int_Q \left[\frac{1}{\varepsilon} \mathbf{H}(x,t) \cdot \nabla\big(\nabla\phi(x) \cdot \mathbf{H}(x,t)\big) + \frac{1}{\mu} \mathbf{E}(x,t) \cdot \nabla\big(\nabla\phi(x) \cdot \mathbf{E}(x,t)\big) \right] dx\, dt. \tag{4.2}$$

In order to prove (4.1), we note that, since $(\mathbf{E}, \mathbf{H})$ is a solution of the Maxwell equations with a vanishing source, we have

$$\int_Q \left(\dot{\mathbf{E}} - \frac{1}{\varepsilon} \nabla \times \mathbf{H} \right)(x,t) \cdot \big(\nabla\phi(x) \times \mathbf{H}(x,t)\big)\, dx\, dt = 0,$$
$$\int_Q \left(\dot{\mathbf{H}} + \frac{1}{\mu} \nabla \times \mathbf{E} \right)(x,t) \cdot \big(\nabla\phi(x) \times \mathbf{E}(x,t)\big)\, dx\, dt = 0,$$

for any function $\phi \in C^2(\bar{\Omega})$. By subtracting the first from the second of the previous two equations, we get

$$- \int_Q \big(\mathbf{E}(x,t) \times \dot{\mathbf{H}}(x,t) + \dot{\mathbf{E}}(x,t) \times \mathbf{H}(x,t)\big) \cdot \nabla\phi(x)\, dx$$
$$+ \int_Q \left(\frac{1}{\varepsilon} (\nabla \times \mathbf{H}) \cdot (\nabla\phi \times \mathbf{H}) + \frac{1}{\mu} (\nabla \times \mathbf{E}) \cdot (\nabla\phi \times \mathbf{E}) \right)(x,t)\, dx\, dt = 0,$$

and (4.1) follows immediately from this identity.

We next prove (4.2). Since $\mathbf{E}$ and $\mathbf{H}$ are divergence-free, then

$$\frac{1}{\mu} \int_Q \big(\nabla\phi(x) \cdot \mathbf{E}(x,t)\big) \operatorname{div} \mathbf{E}(x,t)\, dx\, dt = 0,$$
$$\frac{1}{\varepsilon} \int_Q \big(\nabla\phi(x) \cdot \mathbf{H}(x,t)\big) \operatorname{div} \mathbf{H}(x,t)\, dx\, dt = 0,$$

for any $\phi \in C^2(\bar{\Omega})$. The identity (4.2) can be obtained by summing the previous two equations and then applying the divergence theorem.

We are in position to prove (i). Thanks to identities (4.1) and (4.2) we get

$$\int_\Sigma \left[\frac{1}{\varepsilon} \big(\nabla\phi(x) \cdot \mathbf{H}(x,t)\big) \mathbf{H}(x,t) \cdot \mathbf{n}(x) + \frac{1}{\mu} \big(\nabla\phi(x) \cdot \mathbf{E}(x,t)\big) \mathbf{E}(x,t) \cdot \mathbf{n}(x) \right] d\sigma(x)\, dt$$
$$= \int_\Omega \mathbf{E}(x,T) \times \mathbf{H}(x,T) \cdot \nabla\phi(x)\, dx - \int_\Omega \mathbf{E}(x,0) \times \mathbf{H}(x,0) \cdot \nabla\phi(x)\, dx$$
$$+ \int_Q \left[\frac{1}{\varepsilon} \mathbf{H}(x,t) \cdot \nabla\big(\nabla\phi(x) \cdot \mathbf{H}(x,t)\big) + \frac{1}{\mu} \mathbf{E}(x,t) \cdot \nabla\big(\nabla\phi(x) \cdot \mathbf{E}(x,t)\big) \right] dx\, dt$$
$$+ \int_Q \left[\frac{1}{\varepsilon} \big(\nabla \times \mathbf{H}(x,t)\big) \cdot \big(\nabla\phi(x) \times \mathbf{H}(x,t)\big) \right.$$

$$+\frac{1}{\mu}\big(\nabla\times\mathbf{E}(x,t)\big)\cdot\big(\nabla\phi(x)\times\mathbf{E}(x,t)\big)\bigg]\,dx\,dt.$$

A direct calculation shows that

$$(\nabla\times\mathbf{E})\cdot(\nabla\phi\times\mathbf{E})+\mathbf{E}\cdot\nabla(\nabla\phi\cdot\mathbf{E})=\sum_{i,j=1}^{3}\frac{\partial^2\phi}{\partial x_i\partial x_j}E_iE_j+\frac{1}{2}\nabla\phi\cdot\nabla\big(|\mathbf{E}|^2\big),$$

$$(\nabla\times\mathbf{H})\cdot(\nabla\phi\times\mathbf{H})+\mathbf{H}\cdot\nabla(\nabla\phi\cdot\mathbf{H})=\sum_{i,j=1}^{3}\frac{\partial^2\phi}{\partial x_i\partial x_j}H_iH_j+\frac{1}{2}\nabla\phi\cdot\nabla\big(|\mathbf{H}|^2\big);$$

consequently we find

$$\begin{aligned}
&\int_{\Sigma}\bigg[\frac{1}{\varepsilon}\big(\nabla\phi(x)\cdot\mathbf{H}(x,t)\big)\mathbf{H}(x,t)\cdot\mathbf{n}(x)+\frac{1}{\mu}\big(\nabla\phi(x)\cdot\mathbf{E}(x,t)\big)\mathbf{E}(x,t)\cdot\mathbf{n}(x)\bigg]\,d\sigma(x)\,dt\\
&\quad=\int_{\Omega}\mathbf{E}(x,T)\times\mathbf{H}(x,T)\cdot\nabla\phi(x)\,dx-\int_{\Omega}\mathbf{E}(x,0)\times\mathbf{H}(x,0)\cdot\nabla\phi(x)\,dx\\
&\qquad+\int_{Q}\sum_{i,j=1}^{3}\frac{\partial^2\phi}{\partial x_i\partial x_j}(x)\bigg[\frac{1}{\varepsilon}H_i(x,t)H_j(x,t)+\frac{1}{\mu}E_i(x,t)E_j(x,t)\bigg]\,dx\,dt\\
&\qquad+\frac{1}{2}\int_{Q}\nabla\phi(x)\cdot\bigg[\frac{1}{\varepsilon}\nabla\big(\big|\mathbf{H}(x,t)\big|^2\big)+\frac{1}{\mu}\nabla\big(\big|\mathbf{E}(x,t)\big|^2\big)\bigg]\,dx\,dt.
\end{aligned}\tag{4.3}$$

Equation (i) follows from this identity and from the divergence theorem.

We next prove (ii). If $(\mathbf{E},\mathbf{H})$ is a solution of (1.3)–(1.4) with a vanishing source, the identities

$$\int_{Q}\bigg(\dot{\mathbf{E}}-\frac{1}{\varepsilon}\nabla\times\mathbf{H}\bigg)(x,t)\cdot\frac{1}{\mu}(t+t_0)\mathbf{E}(x,t)\,dx\,dt=0,$$

$$\int_{Q}\bigg(\dot{\mathbf{H}}+\frac{1}{\mu}\nabla\times\mathbf{E}\bigg)(x,t)\cdot\frac{1}{\varepsilon}(t+t_0)\mathbf{H}(x,t)\,dx\,dt=0,$$

hold for any $t_0>0$. If we sum the previous two relations, we obtain

$$\begin{aligned}
&\int_{Q}\frac{1}{2}(t+t_0)\frac{\partial}{\partial t}\bigg(\frac{1}{\mu}\big|\mathbf{E}(x,t)\big|^2+\frac{1}{\varepsilon}\big|\mathbf{H}(x,t)\big|^2\bigg)\,dx\,dt\\
&\quad-\frac{1}{\varepsilon\mu}\int_{\Sigma}(t+t_0)\mathbf{E}(x,t)\times\mathbf{H}(x,t)\cdot\mathbf{n}(x)\,d\sigma(x)\,dt,
\end{aligned}$$

from which (ii) follows immediately. $\square$

Corollary 4.2. *If* $(\mathbf{E},\mathbf{H})$ *is a solution of the Maxwell equations* (1.3)–(1.4) *with a vanishing source, then*

(i) *for any $T > 0$, and any $x_0 \in \Omega$ we have*

$$\int_\Sigma \left[\frac{H_n(x,t)}{\varepsilon}(x-x_0)\cdot \mathbf{H}(x,t) + \frac{E_n(x,t)}{\mu}(x-x_0)\cdot \mathbf{E}(x,t)\right] d\sigma(x)\,dt$$
$$= \int_\Omega \mathbf{E}(x,T)\times \mathbf{H}(x,T)\cdot (x-x_0)\,dx - \int_\Omega \mathbf{E}(x,0)\times \mathbf{H}(x,0)\cdot(x-x_0)\,dx$$
$$+ \frac{1}{2}\int_\Sigma \left(\frac{1}{\varepsilon}|\mathbf{H}|^2 + \frac{1}{\mu}|\mathbf{E}|^2\right)(x-x_0)\cdot \mathbf{n}(x)\,d\sigma(x)\,dt$$
$$- \frac{1}{2}\int_Q \left(\frac{1}{\varepsilon}|\mathbf{H}|^2 + \frac{1}{\mu}|\mathbf{E}|^2\right) dx\,dt,$$

where $E_n(x,t) = \mathbf{E}(x,t)\cdot \mathbf{n}(x)$ and $H_n(x,t) = \mathbf{H}(x,t)\cdot \mathbf{n}(x)$;

(ii) *for any $T > 0$, $t_0 > 0$ and any $x_0 \in \Omega$ we have*

$$\int_\Omega \left[(T+t_0)\left(\frac{1}{\varepsilon}\big|\mathbf{H}(x,T)\big|^2 + \frac{1}{\mu}\big|\mathbf{E}(x,T)\big|^2\right) - 2(\mathbf{E}\times\mathbf{H})(x,T)\cdot(x-x_0)\right] dx$$
$$- \int_\Omega \left[t_0\left(\frac{1}{\varepsilon}\big|\mathbf{H}(x,0)\big|^2 + \frac{1}{\mu}\big|\mathbf{E}(x,0)\big|^2\right) - 2(\mathbf{E}\times\mathbf{H})(x,0)\cdot(x-x_0)\right] dx$$
$$= \int_\Sigma \Bigg[\left(\frac{1}{\varepsilon}\big|\mathbf{H}(x,t)\big|^2 + \frac{1}{\mu}\big|\mathbf{E}(x,t)\big|^2\right)(x-x_0) - \frac{2}{\varepsilon\mu}(t+t_0)(\mathbf{E}\times\mathbf{H})(x,t)$$
$$- 2\left(\frac{1}{\varepsilon}\big((x-x_0)\cdot \mathbf{H}(x,t)\big)\mathbf{H}(x,t) + \frac{1}{\mu}\big((x-x_0)\cdot\mathbf{E}(x,t)\big)\mathbf{E}(x,t)\right)\Bigg]$$
$$\cdot\, \mathbf{n}(x)\,d\sigma\,dt.$$

Proof. Assertion (i) immediately follows from the statement (i) of Lemma 4.1, by choosing $\phi(x) = \frac{1}{2}|x-x_0|^2$. If we sum relation (i) of this corollary with identity (ii) of Lemma 4.1, we then get (ii). □

5. Exponential decay

In this section we will prove that, if $\eta_0 > 0$, η satisfy the conditions $\eta' < 0$, $\eta'' \geqslant 0$ and $\eta'' + \kappa\eta' > 0$, then the Graffi type energy Ψ defined in (2.5) decays exponentially. We follow the approach introduced by Kapitanov and Perla Menzala in [7].

Proposition 5.1. *Let Ω be a strongly star-shaped open set. Assume that η satisfy conditions* (1.10), (1.11) *and let* $(\mathbf{E},\mathbf{H})$ *be a solution to problem* (1.3)–(1.5) *with a vanishing source. Then it exist $t_0 > T_0 > 0$ such that*

$$(T+t_0-T_0)\Psi(T) \leqslant (t_0+T_0)\Psi(0) \tag{5.1}$$

for any $T > 0$.

Proof. From identity (ii) of Corollary 4.2, we have

$$\begin{aligned}
&2(T+t_0)\Psi_\Omega(T) - 2\int_\Omega (\mathbf{E}\times\mathbf{H})(x,T)\cdot(x-x_0)\,dx \\
&\quad - 2t_0\Psi_\Omega(0) + 2\int_\Omega (\mathbf{E}\times\mathbf{H})(x,0)\cdot(x-x_0)\,dx \\
&\quad = \int_\Sigma \Bigg[\bigg(\frac{1}{\varepsilon}\big|\mathbf{H}(x,t)\big|^2 + \frac{1}{\mu}\big|\mathbf{E}(x,t)\big|^2\bigg)(x-x_0) - \frac{2}{\varepsilon\mu}(t+t_0)(\mathbf{E}\times\mathbf{H})(x,t) \\
&\quad\quad - 2\bigg(\frac{1}{\varepsilon}\big((x-x_0)\cdot\mathbf{H}(x,t)\big)\mathbf{H}(x,t) + \frac{1}{\mu}\big((x-x_0)\cdot\mathbf{E}(x,t)\big)\mathbf{E}(x,t)\bigg)\Bigg] \\
&\quad\quad \cdot\mathbf{n}(x)\,d\sigma\,dt,
\end{aligned}$$

for any $T>0$, $t_0>0$, and $x_0\in\Omega$.

Now we want to estimate the right-hand side of the previous relation. To this end we first observe that on Σ we have the following decompositions:

$$\mathbf{H} = H_n\mathbf{n} + \mathcal{H}, \quad \mathbf{E} = E_n\mathbf{n} + \mathcal{E},$$

where $\mathcal{H} = (\mathbf{H} - H_n\mathbf{n})$ and $\mathcal{E} = (\mathbf{E} - E_n\mathbf{n})$, therefore

$$\mathbf{H}\cdot\mathbf{n} = H_n, \qquad |\mathcal{H}| = |\mathbf{H}\times\mathbf{n}|, \qquad \mathbf{E}\cdot\mathbf{n} = E_n, \qquad |\mathcal{E}| = |\mathbf{E}\times\mathbf{n}|.$$

Moreover, since Ω is strongly star-shaped with respect to x_0 and $\partial\Omega$ is smooth, for any $x\in\partial\Omega$ it exists $\gamma_0>0$ such that $(x-x_0)\cdot\mathbf{n}\geqslant\gamma_0|x-x_0|$. Consequently,

$$\begin{aligned}
&\int_\Sigma \Bigg[\bigg(\frac{1}{\varepsilon}\big|\mathbf{H}(x,t)\big|^2 + \frac{1}{\mu}\big|\mathbf{E}(x,t)\big|^2\bigg)(x-x_0)\cdot\mathbf{n}(x) \\
&\quad - 2\bigg(\frac{1}{\varepsilon}\big((x-x_0)\cdot\mathbf{H}(x,t)\big)\mathbf{H}(x,t) \\
&\quad + \frac{1}{\mu}\big((x-x_0)\cdot\mathbf{E}(x,t)\big)\mathbf{E}(x,t)\bigg)\cdot\mathbf{n}(x)\Bigg]\,d\sigma\,dt \\
&= \int_\Sigma \Bigg[\bigg(\frac{1}{\varepsilon}\big(\big|\mathcal{H}(x,t)\big|^2 - \big|H_n(x,t)\big|^2\big) + \frac{1}{\mu}\big(\big|\mathcal{E}(x,t)\big|^2 - \big|E_n(x,t)\big|^2\big)\bigg) \\
&\quad \times (x-x_0)\cdot\mathbf{n}(x) \\
&\quad - 2(x-x_0)\cdot\bigg(\frac{1}{\varepsilon}\mathcal{H}(x,t)H_n + \frac{1}{\mu}\mathcal{E}(x,t)E_n\bigg)\Bigg]\,d\sigma\,dt \\
&\leqslant \int_\Sigma \bigg[(x-x_0)\cdot\mathbf{n}(x) + \frac{1}{\gamma_0}|x-x_0|\bigg] \\
&\quad \times \bigg(\frac{1}{\varepsilon}\big|\mathbf{H}(x,t)\times\mathbf{n}(x)\big|^2 + \frac{1}{\mu}\big|\mathbf{E}(x,t)\times\mathbf{n}(x)\big|^2\bigg)\,d\sigma\,dt. \qquad (5.2)
\end{aligned}$$

If we take into account the boundary condition (1.5), for any $(x,t)\in\Sigma$ we have

$$
\begin{aligned}
\left|\mathbf{E}(x,t)\times\mathbf{n}(x)\right|^2 &= \left|\eta_0\mathbf{H}(x,t)\times\mathbf{n}(x) - \int_0^\infty \eta'(s)\bar{\mathbf{H}}^t(x,s)\times\mathbf{n}(x)\,ds\right|^2 \\
&\leqslant 2\eta_0^2\left|\mathbf{H}(x,t)\times\mathbf{n}(x)\right|^2 + 2\left|\int_0^\infty \eta'(s)\bar{\mathbf{H}}^t(x,s)\times\mathbf{n}(x)\,ds\right|^2 \\
&\leqslant 2\eta_0^2\left|\mathbf{H}(x,t)\times\mathbf{n}(x)\right|^2 \\
&\quad + \left(4\int_0^\infty -\eta'(s)\,ds\right)\left(\frac{1}{2}\int_0^\infty \bigl(-\eta'(s)\bigr)\left|\bar{\mathbf{H}}^t(x,s)\times\mathbf{n}(x)\right|^2 ds\right).
\end{aligned}
$$

The above inequality, together with (5.2) and (2.3), yields

$$
\begin{aligned}
&2(T+t_0)\Psi_\Omega(T) - 2\int_\Omega (\mathbf{E}\times\mathbf{H})(x,T)\cdot(x-x_0)\,dx \\
&\quad - 2t_0\Psi_\Omega(0) + 2\int_\Omega (\mathbf{E}\times\mathbf{H})(x,0)\cdot(x-x_0)\,dx \\
&\quad \leqslant \int_\Sigma \left[(x-x_0)\cdot\mathbf{n}(x) + \frac{1}{\gamma_0}|x-x_0|\right]\left[\left(\frac{1}{\varepsilon} + 2\frac{\eta_0^2}{\mu}\right)\left|\mathbf{H}(x,t)\times\mathbf{n}(x)\right|^2\right. \\
&\quad \left. + \left(4\int_0^\infty -\eta'(s)\,ds\right)\left(\frac{1}{2\mu}\int_0^\infty \bigl(-\eta'(s)\bigr)\left|\bar{\mathbf{H}}^t(x,s)\times\mathbf{n}(x)\right|^2 ds\right)\right] d\sigma\,dt \\
&\quad - \frac{2}{\varepsilon\mu}\int_\Sigma (t+t_0)\mathbf{H}(x,t)\times\mathbf{n}(x)\cdot\mathbf{E}\,d\sigma\,dt \\
&\quad \leqslant R\left(1+\frac{1}{\gamma_0}\right)\int_\Sigma \left[\left(\frac{1}{\varepsilon} + 2\frac{\eta_0^2}{\mu}\right)\left|\mathbf{H}(x,t)\times\mathbf{n}(x)\right|^2\right. \\
&\quad \left. + \left(\frac{4}{\mu}\int_0^\infty -\eta'(s)\,ds\right)\psi_{\partial\Omega}(t)\right] d\sigma\,dt - \frac{2}{\varepsilon\mu}\int_\Sigma (t+t_0)\frac{\partial}{\partial t}\psi_{\partial\Omega}(t)\,d\sigma\,dt \\
&\quad - \frac{1}{\varepsilon\mu}\int_\Sigma (t+t_0)\left[2\eta_0\left|\mathbf{H}(x,t)\times\mathbf{n}(x)\right|^2\right. \\
&\quad \left. + \int_0^\infty \eta''(s)\left|\bar{\mathbf{H}}^t(x,s)\times\mathbf{n}(x)\right|^2 ds\right] d\sigma\,dt,
\end{aligned}
$$

where $R = \sup_{\bar{\Omega}} |x-x_0|$. Thus we get

$$
2(T+t_0)\Psi(T) - 2t_0\Psi(0)
$$

$$
\begin{aligned}
&\leqslant 2R\sqrt{\varepsilon\mu}\big(\Psi(T)+\Psi(0)\big)+2\int_{\Sigma}\psi_{\partial\Omega}(t)\,d\sigma\,dt\\
&\quad-\int_{\Sigma}(t+t_0)\left[2\eta_0\big|\mathbf{H}(x,t)\times\mathbf{n}(x)\big|^2+\int_0^\infty\eta''(s)\big|\bar{\mathbf{H}}^t(x,s)\times\mathbf{n}(x)\big|^2\,ds\right]d\sigma\,dt\\
&\quad+\varepsilon\mu R\left(1+\frac{1}{\gamma_0}\right)\int_{\Sigma}\left[\left(\frac{1}{\varepsilon}+2\frac{\eta_0^2}{\mu}\right)\big|\mathbf{H}(x,t)\times\mathbf{n}(x)\big|^2\right.\\
&\quad\left.+\left(\frac{4}{\mu}\int_0^\infty-\eta'(s)\,ds\right)\psi_{\partial\Omega}(t)\right]d\sigma\,dt
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
&2(T+t_0-R\sqrt{\varepsilon\mu})\Psi(T)-2(t_0+R\sqrt{\varepsilon\mu})\Psi(0)\\
&\quad\leqslant\left(2+4\varepsilon R\left(1+\frac{1}{\gamma_0}\right)\int_0^\infty-\eta'(s)\,ds\right)\int_{\Sigma}\psi_{\partial\Omega}(t)\,d\sigma\,dt\\
&\quad\quad-2\eta_0\int_{\Sigma}\left[(t+t_0)-\frac{\varepsilon\mu R}{2\eta_0}\left(1+\frac{1}{\gamma_0}\right)\left(\frac{1}{\varepsilon}+2\frac{\eta_0^2}{\mu}\right)\right]\big|\mathbf{H}(x,t)\times\mathbf{n}(x)\big|^2\,d\sigma\,dt\\
&\quad\quad-\int_{\Sigma}(t+t_0)\int_0^\infty\eta''(s)\big|\bar{\mathbf{H}}^t(x,s)\times\mathbf{n}(x)\big|^2\,ds\,d\sigma\,dt.
\end{aligned}
$$

Thanks to condition (1.11), we get

$$
\begin{aligned}
&-\int_{\Sigma}(t+t_0)\int_0^\infty\eta''(s)\big|\bar{\mathbf{H}}^t(x,s)\times\mathbf{n}(x)\big|^2\,ds\,d\sigma\,dt\\
&\quad\leqslant\kappa\int_{\Sigma}(t+t_0)\int_0^\infty\eta'(s)\big|\bar{\mathbf{H}}^t(x,s)\times\mathbf{n}(x)\big|^2\,ds\,d\sigma\,dt\\
&\quad=-2\kappa\int_{\Sigma}(t+t_0)\psi_{\partial\Omega}(t)\,d\sigma\,dt,
\end{aligned}
$$

therefore

$$
\begin{aligned}
&(T+t_0-R\sqrt{\varepsilon\mu})\Psi(T)-(t_0+R\sqrt{\varepsilon\mu})\Psi(0)\\
&\quad\leqslant-\kappa\int_{\Sigma}\left[(t+t_0)-\frac{1}{\kappa}\left(1+2\varepsilon R\left(1+\frac{1}{\gamma_0}\right)\int_0^\infty-\eta'(s)\,ds\right)\right]\psi_{\partial\Omega}(t)\,d\sigma\,dt\\
&\quad\quad-\eta_0\int_{\Sigma}\left[(t+t_0)-\frac{\varepsilon\mu R}{2\eta_0}\left(1+\frac{1}{\gamma_0}\right)\left(\frac{1}{\varepsilon}+2\frac{\eta_0^2}{\mu}\right)\right]\big|\mathbf{H}(x,t)\times\mathbf{n}(x)\big|^2\,d\sigma\,dt.
\end{aligned}
$$

Let us define

$$T_0 = R\sqrt{\varepsilon\mu}, \qquad T_1 = \frac{1}{\kappa}\left(1 + 2\varepsilon R\left(1+\frac{1}{\gamma_0}\right)\int_0^\infty -\eta'(s)\,ds\right), \quad \text{and}$$

$$T_2 = \frac{\varepsilon\mu R}{2\eta_0}\left(1+\frac{1}{\gamma_0}\right)\left(\frac{1}{\varepsilon}+2\frac{\eta_0^2}{\mu}\right)$$

(recall that $\eta_0 > 0$), then the inequality above can be rewritten as

$$\begin{aligned}&(T+t_0-T_0)\Psi(T) - (t_0+T_0)\Psi(0)\\ &\quad\leqslant -\kappa\int_\Sigma (t+t_0-T_1)\psi_{\partial\Omega}(t)\,d\sigma\,dt - \eta_0\int_\Sigma (t+t_0-T_2)\left|\mathbf{H}(x,t)\times\mathbf{n}(x)\right|^2 d\sigma\,dt.\end{aligned}$$

If we choose $t_0 \geqslant \max\{T_0, T_1, T_2\} = T^*$, the positiveness of η_0 and κ yields the thesis. □

Theorem 5.2. *Let us assume that the domain Ω is strongly star-shaped and the memory kernel η satisfies the conditions* (1.10) *and* (1.11). *If* $(\mathbf{E}, \mathbf{H})$ *is a solution to problem* (1.3)–(1.5) *with a vanishing source, then there exist two positive constants C, α such that*

$$\Psi(t) \leqslant C\exp(-\alpha t)\Psi(0) \tag{5.3}$$

for any $t \geqslant 0$.

Proof. Thanks to Theorem 2.1 and Proposition 5.1 the energy norm $\|(\mathbf{E}, \mathbf{H}, \bar{\mathbf{H}}^t)\|_{\mathcal{K}}$ belongs to $L^p(\mathbb{R}^+)$ for any $p > 1$. The thesis follows therefore from a general theory of semigroups due to Datko and Pazy (see, for instance, [12, Theorem 4.1, p. 116]). □

Remark 5.3. As stated in the introduction, assumption (1.11) yields the exponential decay of η':

$$\left|\eta'(x,s)\right| = -\eta'(x,s) < c_0(x)e^{-\kappa(x)s}, \quad \forall s \in \mathbb{R}^+.$$

On the other hand, we may relax condition $\eta \in H^2(\mathbb{R}^+)$ by only requiring $\eta' \in L^1(\mathbb{R}^+)$ and $\eta'' \in L^2([\delta, +\infty[)$, for every positive δ. Under this assumption, that allows η' to have a weak singularity at the origin, the statement of Theorem 5.2 holds true.

6. A necessary condition for the exponential decay

We say that a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ *exponentially decays* if a positive constant α exists such that

$$\int_0^\infty e^{\alpha t}\left|f(t)\right| dt < \infty.$$

In this section we consider a positive scalar kernel η and we show that the $L^2(\partial\Omega)$ norm of a solution $(\mathbf{E}, \mathbf{H})$ to the system of Maxwell equations (1.3)–(1.4), with boundary condition (1.5) cannot exponentially decay, unless η does.

Proposition 6.1. *Let* $(\mathbf{E},\mathbf{H})$ *be a solution to problem* (1.3)–(1.5)*, with initial data* $\mathbf{E}_0$ *and* $\mathbf{H}_0$ *and vanishing source, such that*

$$\int_0^\infty e^{2\alpha t}\left(\int_{\partial\Omega}\left(\left|\mathbf{E}_\tau(t)\right|^2+\left|\mathbf{H}_\tau(t)\right|^2\right)d\sigma\right)dt<\infty, \tag{6.1}$$

for some positive constant α*. If* Ω *is star-shaped and* $\eta\geqslant 0$*, then* η *decays exponentially.*

The proof of the proposition relies on the use of the Laplace transform. Indeed, from condition (6.1) it follows that the Laplace transform of $\mathbf{E}_\tau$ and $\mathbf{H}_\tau$

$$\hat{\mathbf{E}}_\tau(x,z)=\int_0^\infty e^{-zt}\mathbf{E}_\tau(x,t)\,dt,\qquad \hat{\mathbf{H}}_\tau(x,z)=\int_0^\infty e^{-zt}\mathbf{H}_\tau(x,t)\,dt,$$

is defined for almost every $x\in\partial\Omega$ and for any $z\in D$, where

$$D=\{z\in\mathbb{C}\colon\ \operatorname{Re}z>-\alpha\}.$$

The proof of Proposition 6.1 is a plain consequence of the following known result (see [11, Theorem 2] or [5, Lemma 3.1]).

Lemma 6.2. *Let* $U\subset\mathbb{C}$ *be a neighborhood of* 0 *and let* $g:U\to\mathbb{C}$ *be a holomorphic function. If* $G\in L^1(\mathbb{R}^+)$ *is a non-negative function such that* $\hat{G}(z)=g(z)$ *for every* $z\in U\cap\left\{\operatorname{Re}z\geqslant 0\right\}$*, then G exponentially decays.*

We first prove a preliminary result, in the spirit of the H.U.M.

Lemma 6.3. *If* $(\mathbf{E},\mathbf{H})$ *is a solution to problem* (1.3)–(1.4) *with a vanishing source and* $\mathbf{E}(x,t)\times\mathbf{n}(x)=\mathbf{0}$ *for almost any* $(x,t)\in\Sigma$*, then*

(i) $\Psi_\Omega(t)=\Psi_\Omega(0)$*, for any* $t>0$*,*
(ii) *it exists a positive constant R, depending on the domain* Ω*, such that*

$$(T-2\sqrt{\varepsilon\mu}R)\Psi_\Omega(0)\leqslant\frac{R}{2\varepsilon}\int_{\partial\Omega\times[0,T]}|\mathbf{H}\times\mathbf{n}|^2(x,t)\,\mathrm{d}\sigma(x)\,\mathrm{d}t.$$

Proof. Let us first prove the assertion (i). If we consider the time-derivative of the function Ψ_Ω defined in (2.4), we obtain

$$\begin{aligned}\dot{\Psi}_\Omega(t)&=\int_\Omega\left(\frac{1}{\varepsilon}\mathbf{H}(x,t)\cdot\dot{\mathbf{H}}(x,t)+\frac{1}{\mu}\mathbf{E}(x,t)\cdot\dot{\mathbf{E}}(x,t)\right)dx\\&=\int_\Omega\frac{1}{\varepsilon\mu}(\mathbf{E}\cdot\nabla\times\mathbf{H}-\mathbf{H}\cdot\nabla\times\mathbf{E})(x,t)\,dx\\&=\int_{\partial\Omega}\frac{1}{\varepsilon\mu}(\mathbf{H}\times\mathbf{E})(x,t)\cdot\mathbf{n}(x)\,d\sigma(x)=0,\end{aligned}$$

for almost any $t \geqslant 0$, thanks to hypotheses of the lemma. This proves that Ψ_Ω is constant and, therefore, the assertion (i).

In order to prove (ii), we recall that we have assumed $\mathbf{E} \times \mathbf{n} = 0$ on Σ and $\mathbf{H}(x,0) \cdot \mathbf{n} = 0$ on $\partial\Omega$. It follows that[1]

$$\mathbf{E}(x,t) = E_n(x,t)\mathbf{n}(x), \qquad \left|\mathbf{E}(x,t)\right| = \left|\mathbf{E}_n(x,t)\right|,$$
$$\left|\mathbf{H}(x,t)\right| = \left|\mathbf{H}(x,t) \times \mathbf{n}(x)\right|,$$

for any $(x,t) \in \Sigma$. Then, from the identity (i) of Corollary 4.2, we get

$$\begin{aligned}
&\int_0^T \Psi_\Omega(t)\,\mathrm{d}t - \int_\Omega \mathbf{E}(x,T) \times \mathbf{H}(x,T) \cdot (x - x_0)\,dx \\
&\qquad + \int_\Omega \mathbf{E}(x,0) \times \mathbf{H}(x,0) \cdot (x - x_0)\,dx \\
&\quad = -\int_\Sigma \left(\frac{1}{\mu}\left|E_n(x,t)\right|^2 (x - x_0) \cdot \mathbf{n}(x)\right) d\sigma(x)\,dt \\
&\qquad + \frac{1}{2}\int_\Sigma \left(\frac{1}{\varepsilon}\left|\mathbf{H}(x,t) \times \mathbf{n}(x)\right|^2 + \frac{1}{\mu}\left|E_n(x,t)\right|^2\right)(x - x_0) \cdot \mathbf{n}(x)\,d\sigma(x)\,dt \\
&\quad \leqslant \frac{1}{2}\int_\Sigma \frac{1}{\varepsilon}\left|\mathbf{H}(x,t) \times \mathbf{n}(x)\right|^2 (x - x_0) \cdot \mathbf{n}(x)\,d\sigma(x)\,dt,
\end{aligned} \tag{6.2}$$

since Ω is star-shaped with respect to x_0 and, consequently, $(x - x_0) \cdot \mathbf{n}(x) \geqslant 0$ for any $x \in \partial\Omega$.

Let $R = \sup_\Omega |x - x_0|$. Then

$$\begin{aligned}
\left|\int_\Omega \mathbf{E}(x,t) \times \mathbf{H}(x,t) \cdot (x - x_0)\,dx\right| &\leqslant \int_\Omega \left|\mathbf{E}(x,t)\right|\left|\mathbf{H}(x,t)\right||x - x_0|\,dx \\
&\leqslant \frac{\sqrt{\varepsilon\mu}}{2} R \int_\Omega \left(\frac{1}{\varepsilon}|\mathbf{H}|^2(x,t) + \frac{1}{\mu}|\mathbf{E}|^2(x,t)\right) dx \\
&= \sqrt{\varepsilon\mu} R \Psi(t),
\end{aligned}$$

for any $t \in [0,T]$. Thanks to (i), the function Ψ_Ω is constant, therefore from (6.2) follows

$$(T - 2\sqrt{\varepsilon\mu}R)\Psi_\Omega(0) \leqslant \frac{R}{2\varepsilon} \int_{\partial\Omega \times [0,T]} |\mathbf{H} \times \mathbf{n}|^2(x,t)\,d\sigma(x)\,dt.$$

In this way we have concluded the proof of Lemma 6.3. □

[1] As a consequence of the boundary condition $\mathbf{E} \times \mathbf{n}_{|\partial\Omega} = \mathbf{0}$, if we multiply Eq. $(1.3)_2$ by $\nabla\phi$ with $\phi \in C^\infty(\Omega)$ and, after an integration over the domain Ω, take into account Eq. $(1.4)_2$, we get $\frac{\partial}{\partial t}(\mathbf{H} \cdot \mathbf{n})_{|\partial\Omega} = 0$, that is $\mathbf{H} \cdot \mathbf{n} = \mathbf{H}_0 \cdot \mathbf{n}$ on Σ.

As an immediate consequence of the previous lemma, we get the following corollary.

Corollary 6.4. *If* $(\mathbf{E},\mathbf{H})$ *is a solution to problem* (1.3)–(1.4) *with a vanishing source, such* $\mathbf{E}(x,t)\times\mathbf{n}(x)=\mathbf{0}$ *and* $\mathbf{H}(x,t)\times\mathbf{n}(x)=\mathbf{0}$ *for almost any* $(x,t)\in\Sigma$*, then* $(\mathbf{E},\mathbf{H})$ *is necessarily the trivial solution.*

Corollary 6.5. *If* $(\mathbf{E},\mathbf{H})$ *is a nontrivial solution of the problem* (1.3)–(1.5)*, satisfying condition (6.1), then a* $k\in\mathbb{N}\cup\{0\}$ *exists such that*

$$\int_{\partial\Omega}\left|\frac{\partial^k\hat{\mathbf{H}}}{\partial z^k}(x,0)\times\mathbf{n}\right|^2 d\sigma(x)\neq 0.$$

Proof. Assume, by contradiction, that the claim is false. Then

$$\frac{\partial^k\hat{\mathbf{H}}}{\partial z^k}(x,0)\times\mathbf{n}=\mathbf{0},$$

for almost any $x\in\partial\Omega$ and for all $k\in\mathbb{N}\cup\{0\}$. Note that $\hat{\mathbf{H}}(x,\cdot)$ is a holomorphic function in D, for almost every x, then the above identity yields $\hat{\mathbf{H}}(x,\cdot)\times\mathbf{n}\equiv\mathbf{0}$ and thus $\mathbf{H}(x,t)\times\mathbf{n}=\mathbf{0}$ for almost any $(x,t)\in\partial\Omega\times\mathbb{R}^+$. By Corollary 6.4, this implies that $(\mathbf{E},\mathbf{H})$ is the trivial solution. Last assertion contradicts the hypothesis of the corollary, then the proof is done. □

Proof of Proposition 6.1. Let $(\mathbf{E},\mathbf{H})$ be a nontrivial solution of the problem (1.3)–(1.5) satisfying condition (6.1). Being $\eta\in L^1(\mathbb{R}^+)$, its Laplace transform is defined in the set $\{z\in\mathbb{C}\colon \operatorname{Re} z\geqslant 0\}$ and the condition (1.5) yields

$$\hat{\mathbf{E}}_\tau(x,z)=\big(\eta_0+\hat{\eta}(z)\big)\hat{\mathbf{H}}(x,z)\times\mathbf{n}, \tag{6.3}$$

for almost any $x\in\partial\Omega$ and for every $z\in\mathbb{C}$ such that $\operatorname{Re} z\geqslant 0$. We next show that this equation implies the exponential decay of η.

Let k be the first integer such that the assertion of Corollary 6.5 holds. For every $z\in D$ we set

$$f(z)=\int_{\partial\Omega}\big(\hat{\mathbf{H}}(x,z)\times\mathbf{n}\big)\cdot\left(\frac{\partial^k\hat{\mathbf{H}}}{\partial z^k}(x,0)\times\mathbf{n}\right)d\sigma(x). \tag{6.4}$$

Clearly, f is holomorphic in the domain D; moreover, by our choice of k we have

$$f(0)=f'(0)=\cdots=f^{(k-1)}(0)=0,\qquad f^{(k)}(0)\neq 0.$$

Hence there exists a function g, holomorphic in D, such that $g(0)\neq 0$ and

$$f(z)=z^{k-1}g(z)\quad\text{for every } z\in D. \tag{6.5}$$

We next define a function F in the domain D as follows:

$$F(z)=\int_{\partial\Omega}\hat{\mathbf{E}}(x,z)\times\frac{\partial^k\hat{\mathbf{H}}(x,0)}{\partial z^k}\cdot\mathbf{n}\,d\sigma(x),$$

Thanks to our assumptions, F is holomorphic in the domain D; moreover it follows from (6.3) that

$$F(z) = -\big(\eta_0 + \hat{\eta}(z)\big) f(z),$$

for every $z \in \mathbb{C}$ such that $\operatorname{Re} z \geqslant 0$.

Since F is holomorphic in D, we get from the previous identity and from (6.5) that there exists a function G, holomorphic in D, such that

$$F(z) = z^{k-1} G(z).$$

We then have

$$G(z) = -\big(\eta_0 + \hat{\eta}(z)\big) g(z) \tag{6.6}$$

for every $z \in \mathbb{C}$ such that $\operatorname{Re} z \geqslant 0$. Recall that $g(0) \neq 0$, then the function $G(z)/g(z)$ is holomorphic in some neighborhood U of 0. By (6.6), we then have

$$\hat{\eta}(z) = -\frac{G(z)}{g(z)} - \eta_0$$

for any $z \in U$ such that $\operatorname{Re} z \geqslant 0$ thus, by Lemma 6.2, η exponentially decays. This completes the proof of Proposition 6.1. $\square$

Acknowledgments

We thank Mauro Fabrizio for suggesting us the problem and for his interest in our work, and Giovanni Dore for bringing our attention on reference [3]. We also thank the anonymous referee whose valuable remarks improved the readability of our paper.

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