Computers and Mathematics with Applications 59 (2010) 399–410

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A Weiszfeld algorithm for the solution of an asymmetric extension of the generalized Fermat location problem

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ARTICLE INFO

Article history:
Received 7 July 2009
Accepted 7 July 2009

Keywords:
Generalized Fermat location
Asymmetric distance
Weiszfeld algorithm

ABSTRACT

The Generalized Fermat Problem (in the plane) is: given $n \geq 3$ destination points find the point $\bar{x}^*$ which minimizes the sum of Euclidean distances from $\bar{x}^*$ to each of the destination points.The Weiszfeld iterative algorithm for this problem is globally convergent, independent of the initial guess. Also, a test is available, $\grave{a}$ priori, to determine when $\bar{x}^*$ a destination point. This paper generalizes earlier work by the first author by introducing an asymmetric Euclidean distance in which, at each destination, the x-component is weighted differently from the y-component. A Weiszfeld algorithm is studied to compute $\bar{x}^*$ and is shown to be a descent method which is globally convergent (except possibly for a denumerable number of starting points). Local convergence properties are characterized. When $\bar{x}^*$ is not a destination point the iteration matrix at $\bar{x}^*$ is shown to be convergent and local convergence is always linear. When $\bar{x}^*$ is a destination point, local convergence can be linear, sub-linear or super-linear, depending upon a computable criterion. A test, which does not require iteration, for $\bar{x}^*$ to be a destination, is derived. Comparisons are made between the symmetric and asymmetric problems. Numerical examples are given.

1. Introduction

A classical locational problem originally solved by Fermat is: given the location of three vertices of a triangle, find the point (source) which minimizes the sum of the distances to the vertices. Its generalization to n non-collinear points in the plane (destination points) and to distances which are weighted Euclidean distances, is sometimes called the Generalized Weber Problem. An efficient, reliable method for solving this problem is essential, for example, as a subroutine for solving location–allocation problems in which there is more than one source [1,2]. The Generalized Weber problem has been studied extensively both theoretically and for the purpose of applications [3–10]. [11] studies weighted (symmetric) Euclidean distances sorted in deceasing order. Extensions to weighted powers of L_p distances are studied in [12,13]. Further extensions to spaces more general than Euclidean are studied in [14] which also gives an excellent bibliography of relevant literature. These papers use a Weiszfeld algorithm. A Newton-type acceleration of the Weiszfeld algorithm is given in [15]. A Primal-Dual Interior-Point Method is given in [16]. In these studies the distance to a destination is symmetric, that is for each destination the x-component and y-component are weighted equally.

In this paper, which extends the results in [7], we consider only the classical Fermat Location problem with the generalization that the Euclidean distances can be asymmetric i.e. for each destination point $\bar{x}_j$ the weight of the x-component w_{x_j} and the weight of the y-component w_{y_j} may be unequal. An example in practice of such an asymmetric

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1 The work of this author was supported in part by the MIT Lincoln Laboratory Graduate Fellowship Program.

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doi:10.1016/j.camwa.2009.07.007

weighting might be the following. A plant is to be located at a point $\bar{x}$ which minimizes the cost of transportation to n warehouses, located at $\bar{x}_j$, $j = 1, \ldots, n$. However, the cost of transporting to $\bar{x}_j$ in the north direction y, may be higher than in east direction x. This may be due to winds from the north at $\bar{x}_j$, poor roads heading north, or mountainous country. It is natural then to include this asymmetry in the mathematical model.

In Section 2, we state the problem, and in Section 3 we formulate an extension of the Weiszfeld algorithm [17]. Section 4 shows that this iterative algorithm is a descent method which is globally convergent except for a denumerable set of initial points. Section 5 characterizes local convergence when the minimizer $\bar{x}^*$ is not a destination point. We compute the iteration matrix and prove that it is a convergent matrix, hence the rate of convergence is linear. Section 6 characterizes local convergence when $\bar{x}^*$ is a destination point, so the iteration matrix at $\bar{x}^*$ is not defined. Nevertheless, a convergent factor λ is computed. If $0 < \lambda < 1$ convergence is linear, if $\lambda = 1$ convergence is shown to be sub-linear, and if $\lambda = 1$ convergence is super-linear. Numerical examples are given. Finally, we compare some properties of the symmetric and asymmetric problems in Section 7.

2. Statement of problem

Given n points in the plane, $\bar{x}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$ not collinear, and non-negative weights w_{x_i}, w_{y_i}, $i = 1 \ldots n$. Let $\bar{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ be a point in the plane. Define the asymmetric Euclidean distance from $\bar{x}$ to $\bar{x}_i$

$$d_i(\bar{x}) \equiv \sqrt{w_{x_i}^2 (x - x_i)^2 + w_{y_i}^2 (y - y_i)^2}. \tag{1}$$

The asymmetric Fermat location problem is to find $\bar{x}$ which minimizes the sum of the distances from $\bar{x}$ to $\bar{x}_i$

$$f(\bar{x}) = \sum_{i=1}^{n} d_i(\bar{x}). \tag{2}$$

Each $d_i(\bar{x})$ is convex. The $\bar{x}_i$ are not collinear, so that $f(\bar{x})$ is a strictly convex function of $\bar{x}$. Therefore, the minimizing point of f is unique. We now formulate an algorithm to find

$$\min_{\bar{x}} f(\bar{x}). \tag{3}$$

3. The algorithm

The algorithm is derived formally by setting to zero the gradient of $f(\bar{x})$ when $\bar{x}$ is not one of the destination points $\bar{x}_i$. Note that

$$\frac{\partial d_i(\bar{x})}{\partial x} = \frac{w_{x_i}^2 (x - x_i)}{d_i(\bar{x})}, \qquad \frac{\partial d_i(\bar{x})}{\partial y} = \frac{w_{y_i}^2 (y - y_i)}{d_i(\bar{x})} \tag{4}$$

so that

$$\frac{\partial f}{\partial x} = \sum_i \frac{w_{x_i}^2 (x - x_i)}{d_i(\bar{x})} = 0, \qquad \frac{\partial f}{\partial y} = \sum_i \frac{w_{y_i}^2 (y - y_i)}{d_i(\bar{x})} = 0.$$

This leads to the iteration

$$x = \frac{\sum_i \frac{w_{x_i}^2 x_i}{d_i(\bar{x})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})}} \equiv H_x(\bar{x}), \qquad y = \frac{\sum_i \frac{w_{y_i}^2 y_i}{d_i(\bar{x})}}{\sum_i \frac{w_{y_i}^2}{d_i(\bar{x})}} \equiv H_y(\bar{x}). \tag{5}$$

$$\bar{x} = H(\bar{x}) \equiv \begin{bmatrix} H_x(\bar{x}) \\ H_y(\bar{x}) \end{bmatrix} \quad \text{for } \bar{x} \neq \bar{x}_i. \tag{6}$$

In order to define $H(\bar{x})$ at a destination point $\bar{x}_j$ rewrite (5) as

$$H_x(\bar{x}) = \frac{\frac{w_{x_j}^2 x_j}{d_j(\bar{x})} + \sum_{i \neq j} \frac{w_{x_i}^2 x_i}{d_i(\bar{x})}}{\frac{w_{x_j}^2}{d_j(\bar{x})} + \sum_{i \neq j} \frac{w_{x_i}^2}{d_i(\bar{x})}} = \frac{x_j + \frac{d_j(\bar{x})}{w_{x_j}^2} \sum_{i \neq j} \frac{w_{x_i}^2 x_i}{d_i(\bar{x})}}{1 + \frac{d_j(\bar{x})}{w_{x_j}^2} \sum_{i \neq j} \frac{w_{x_i}^2}{d_i(\bar{x})}} \to \bar{x}_j, \quad \text{as } \bar{x} \to \bar{x}_j. \tag{7}$$

Similarly,

$$H_y(\bar{x}) \to \bar{y}_j, \quad \text{as } \bar{x} \to \bar{x}_j. \tag{8}$$

Therefore, we define

$$H(\bar{x}_j) \equiv \bar{x}_j. \tag{9}$$

A non-destination point is the unique minimizer $\bar{x}_{\min}$ if and only if (6) is satisfied. Later, we derive a criterion for a destination point $\bar{x}_j$ to be the minimizer. This criterion is computable à priori, and does not require iteration.

The iteration algorithm we study is

$$\bar{x}^{k+1} = H(\bar{x}^k), \quad k = 0, 1, \ldots, \text{ where } \bar{x}^0 \text{ is arbitrary.} \tag{10}$$

We will show that $\bar{x}^k$ converges to $\bar{x}_{\min}$ for all $\bar{x}^0$ except for a set of measure zero. The iteration (7) is a descent method, that is

$$f(\bar{x}^{k+1}) \le f(\bar{x}^k) \tag{11}$$

and equality holds if and only if $\bar{x}^{k+1} = \bar{x}^k = \bar{x}_{\min}$. A convenient choice for $\bar{x}^0$ is obtained by minimizing

$$g(\bar{x}) = \sum_{i=1}^{n} d_i^2(\bar{x}). \tag{12}$$

Set the gradient of $g(\bar{x})$ to zero and use (4):

$$\frac{\partial g}{\partial x} = 2\sum_i d_i \frac{\partial d_i}{\partial x} = 2\sum_i w_{x_i}^2 (x - x_i) = 0$$

so that

$$x^0 = \frac{\sum_i w_{x_i}^2 x_i}{\sum_i w_{x_i}^2}. \tag{13}$$

Similarly

$$y^0 = \frac{\sum_i w_{y_i}^2 y_i}{\sum_i w_{y_i}^2}. \tag{14}$$

Now let $X_{\min} = \min_i x_i$, $X_{\max} = \max_i x_i$, $Y_{\min} = \min_i y_i$, $Y_{\max} = \max_i y_i$. Then from (5)

$$X_{\min} = \frac{\sum_i \frac{w_{x_i}^2 X_{\min}}{d_i(\bar{x})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})}} \le x \le \frac{\sum_i \frac{w_{x_i}^2 X_{\max}}{d_i(\bar{x})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})}} = X_{\max}$$

$$Y_{\min} = \frac{\sum_i \frac{w_{x_i}^2 Y_{\min}}{d_i(\bar{x})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})}} \le y \le \frac{\sum_i \frac{w_{x_i}^2 Y_{\max}}{d_i(\bar{x})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})}} = Y_{\max}. \tag{15}$$

Thus, for all $k \ge 1$, $\bar{x}^k$ and $\bar{x}_{\min}$ are in the rectangle $X_{\min} \le x \le X_{\max}$, $Y_{\min} \le y \le Y_{\max}$, that is, in the smallest rectangle parallel to the coordinate axes that contains all the destination points.

Now the iteration (5) can be written as

$$x^{k+1} = H_x(\bar{x}^k) = \frac{\sum_i \frac{w_{x_i}^2 x_i}{d_i(\bar{x}^k)}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}^k)}} = x^k - \left(x^k - \frac{\sum_i \frac{w_{x_i}^2 x_i}{d_i(\bar{x}^k)}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}^k)}} \right)$$

$$= x^k - \frac{\sum_i \frac{w_{x_i}^2 (x^k - x_i)}{d_i(\bar{x}^k)}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}^k)}} = \bar{x}^k - \frac{1}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}^k)}} \frac{\partial f}{\partial x}(\bar{x}^k). \tag{16}$$

Similarly,

$$\bar{y}^{k+1} = \bar{y}^k - \frac{1}{\sum_i \frac{w_{y_i}^2}{d_i(\bar{x}^k)}} \frac{\partial f}{\partial y}(\bar{x}^k). \tag{17}$$

So,

$$\bar{x}^{k+1} = \bar{x}^k - D(\bar{x}^k)\nabla f(\bar{x}^k) \tag{18}$$

where

$$D = \begin{pmatrix} \frac{1}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}^k)}} & 0 \\ 0 & \frac{1}{\sum_i \frac{w_{y_i}^2}{d_i(\bar{x}^k)}} \end{pmatrix}. \tag{19}$$

Since D is a diagonal matrix (not a full matrix), we call the iteration scheme a "gradient-like" method with pre-computed step size.

4. Descent method

We now show that the algorithm is a descent method.

Theorem. *The iteration $\bar{x}^{k+1} = H(\bar{x}^k)$ is a descent method for any $\bar{x}^0$, that is*

$$f(H(\bar{x}^k)) \le f(\bar{x}^k) \quad \text{for } k = 0, 1, 2, \ldots. \tag{20}$$

The equality holds if and only if $H(\bar{x}^*) = \bar{x}^*$, i.e. $\bar{x}^*$ is the unique minimizer of f.

Proof. Let

$$F(\gamma, \eta) = \sum_i [w_{x_i}^2(\gamma - x_i)^2 m_{x_i} + w_{y_i}^2(\eta - y_i)^2 m_{y_i}] \tag{21}$$

where m_{x_i} and m_{y_i} are positive functions of $w_{x_i}, w_{y_i}, x_i, y_i$, and $\bar{x}$. $F(\gamma, \eta)$ is a strictly convex function of γ and η with a unique minimum at γ^* and η^* computed from

$$\frac{\partial F}{\partial \gamma} = \sum_i 2w_{x_i}^2(\gamma - x_i)m_{x_i} = 0, \qquad \frac{\partial F}{\partial \eta} = \sum_i 2w_{y_i}^2(\eta - y_i)m_{y_i} = 0$$

$$\gamma^* = \frac{\sum_i w_{x_i}^2 m_{x_i} x_i}{\sum_i w_{x_i}^2 m_{x_i}}, \qquad \eta^* = \frac{\sum_i w_{y_i}^2 m_{y_i} y_i}{\sum_i w_{y_i}^2 m_{y_i}}. \tag{22}$$

For any $\bar{x}$, let $m_{x_i} = m_{y_i} = m_i = 1/d_i(\bar{x})$. Then

$$\gamma^* = \frac{\sum_i \frac{w_{x_i}^2 x_i}{d_i(\bar{x})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})}} = H_x(\bar{x}), \qquad \eta^* = \frac{\sum_i \frac{w_{y_i}^2 y_i}{d_i(\bar{x})}}{\sum_i \frac{w_{y_i}^2}{d_i(\bar{x})}} = H_y(\bar{x}) \tag{23}$$

and $F(\gamma^*, \eta^*) \le F(\gamma, \eta)$. The equality holds if and only if $(\gamma^*, \eta^*) = (\gamma, \eta)$. Let $(\gamma, \eta) = (x, y)$. Then

$$\begin{aligned} F(\gamma^*, \eta^*) &= \sum_i \left[(H_x(\bar{x}) - x_i)^2 \frac{w_{x_i}^2}{d_i(\bar{x})} + (H_y(\bar{x}) -_i)^2 \frac{w_{y_i}^2}{d_i(\bar{x})} \right] = \sum_i \frac{d_i^2(H(\bar{x}))}{d_i(\bar{x})} \\ &\le \sum_i \left[\frac{(x - x_i)^2 w_{x_i}^2}{d_i(\bar{x})} + \frac{(y - y_i)^2 w_{y_i}^2}{d_i(\bar{x})} \right] = \sum_i \frac{d_i^2(\bar{x})}{d_i(\bar{x})} = \sum_i d_i(\bar{x}) \\ &= f(\bar{x}). \end{aligned} \tag{24}$$

Now

$$\begin{aligned} d_i(H(\bar{x})) &= d_i(\bar{x}) + d_i(H(\bar{x})) - d_i(\bar{x}) \\ d_i^2(H(\bar{x})) &= d_i^2(\bar{x}) + 2d_i(\bar{x})[d_i(H(\bar{x})) - d_i(\bar{x})] + [d_i(H(\bar{x})) - d_i(\bar{x})]^2 \\ \frac{d_i^2(H(\bar{x}))}{d_i(\bar{x})} &= d_i(\bar{x}) + 2[d_i(H(\bar{x})) - d_i(\bar{x})] + \frac{[d_i(H(\bar{x})) - d_i(\bar{x})]^2}{d_i(\bar{x})} \end{aligned} \tag{25}$$

and from (24) we have

$$\sum_i \{d_i(\bar{x}) + 2[d_i(H(\bar{x})) - d_i(\bar{x})]\} + \frac{[d_i(H(\bar{x})) - d_i(\bar{x})]^2}{d_i(\bar{x})} \le d_i(\bar{x})$$

so

$$2\sum_i [d_i(H(\bar{x})) - d_i(\bar{x})] \le 0$$

$$f(H(\bar{x})) = \sum_i d_i(H(\bar{x})) \le \sum_i d_i(\bar{x}) = f(\bar{x}). \tag{26}$$

The equality holds if and only if $H(\bar{x}) = \bar{x}$.

It is possible for an iterate $H(\bar{x}^k)$ to equal $\bar{x}^k$ for finite k. If $\bar{x}^k$ is not a destination point then $\bar{x}^k$ is $\bar{x}_{\min}$, the unique minimizing point. However, if $\bar{x}^k$ is a destination point, say $\bar{x}_j$ then the iteration stops at $\bar{x}_j$ which is not necessarily the minimizer. It is shown in [9] that this can occur only for a denumerable number of starting points $\bar{x}^0$. Except for these starting points the iteration always converges to the unique minimizer $\bar{x}_{\min}$.

We study the rate of convergence to $\bar{x}_{\min}$. We show that when $\bar{x}_{\min}$ is not a destination point the rate of convergence is linear in the sense described later. When $\bar{x}_{\min}$ is a destination point the rate of convergence can be linear, sub-linear or super-linear. We also give a criterion, computable à priori from the given data,for $\bar{x}_{\min}$ to be a destination point. If this criterion is used initially then is never necessary to be concerned whether the iteration converges to a non-destination point.

5. $\bar{x}_{\min}$ is not a destination point

Local convergence of the iteration scheme $\bar{x}^{k+1} = H(\bar{x}^k)$ to $\bar{x}_{\min}$ is determined by the eigenvalues of the matrix $H'(\bar{x}_{\min})$. In Appendix we show that

$$H'(\bar{x}_{\min}) = \left.\begin{pmatrix} \dfrac{\sum_i \frac{w_{x_i}^4 (x-x_i)^2}{d_i^3(\bar{x})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})}} & \dfrac{\sum_i \frac{w_{x_i}^2 w_{y_i}^2 (x-x_i)(y-y_i)}{d_i^3(\bar{x})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})}} \\ \dfrac{\sum_i \frac{w_{x_i}^2 w_{y_i}^2 (x-x_i)(y-y_i)}{d_i^3(\bar{x})}}{\sum_i \frac{w_{y_i}^2}{d_i(\bar{x})}} & \dfrac{\sum_i \frac{w_{y_i}^4 (y-y_i)^2}{d_i^3(\bar{x})}}{\sum_i \frac{w_{y_i}^2}{d_i(\bar{x})}} \end{pmatrix}\right|_{\bar{x}=\bar{x}_{\min}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv A. \tag{27}$$

If $w_{x_i} \neq w_{y_i}$ for some i, the matrix $H'(\bar{x}_{\min}) = A$ is not symmetric. The eigenvalues of A are the roots of $\lambda^2 - (a+d)\lambda + (ad - bc) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$.

$$\begin{aligned} \lambda_{pm} &= \frac{1}{2}(a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}) \\ &= \frac{1}{2}(a + d \pm \sqrt{(a-d)^2 + 4bc}) \ge 0. \end{aligned} \tag{28}$$

A short computation shows that the larger eigenvalue $\lambda_+ < 1$ if and only if $\operatorname{tr} A - \det A < 1$, that is

$$\frac{\sum_i \frac{w_{x_i}^4 (x-x_i)^2}{d_i^3(\bar{x})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})}} + \frac{\sum_i \frac{w_{y_i}^4 (y-y_i)^2}{d_i^3(\bar{x})}}{\sum_i \frac{w_{y_i}^2}{d_i(\bar{x})}} - \left\{ \frac{\sum_i \frac{w_{x_i}^4 (x-x_i)^2}{d_i^3(\bar{x})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})}} \frac{\sum_i \frac{w_{y_i}^4 (y-y_i)^2}{d_i^3(\bar{x})}}{\sum_i \frac{w_{y_i}^2}{d_i(\bar{x})}} - \frac{\sum_i \left(\frac{w_{x_i}^2 w_{y_i}^2 (x-x_i)(y-y_i)}{d_i^3(\bar{x})}\right)^2}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})} \sum_i \frac{w_{y_i}^2}{d_i(\bar{x})}} \right\} < 1$$

or

$$\begin{aligned} &\left(\sum_i \frac{w_{x_i}^4 (x-x_i)^2}{d_i^3(\bar{x})}\right)\left(\sum_i \frac{w_{y_i}^2}{d_i(\bar{x})}\right) + \left(\sum_i \frac{w_{y_i}^4 (y-y_i)^2}{d_i^3(\bar{x})}\right)\left(\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})}\right) - \left\{\left(\sum_i \frac{w_{x_i}^4 (x-x_i)^2}{d_i^3(\bar{x})}\right)\left(\sum_i \frac{w_{y_i}^4 (y-y_i)^2}{d_i^3(\bar{x})}\right)\right. \\ &\quad \left. - \left(\sum_i \frac{w_{x_i}^2 w_{y_i}^2 (x-x_i)(y-y_i)}{d_i^3(\bar{x})}\right)^2\right\} < \left(\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})}\right)\left(\sum_i \frac{w_{y_i}^2}{d_i(\bar{x})}\right). \end{aligned}$$

This leads to

$$\left(\sum_i \frac{w_{x_i}^2 w_{y_i}^2 (x-x_i)(y-y_i)}{d_i^3(\bar{x})}\right)^2 < \left(\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})} - \frac{w_{x_i}^4 (x-x_i)^2}{d_i^3(\bar{x})}\right)\left(\sum_i \frac{w_{y_i}^2}{d_i(\bar{x})} - \frac{w_{y_i}^4 (y-y_i)^2}{d_i^3(\bar{x})}\right)$$
$$= \left(\sum_i \frac{w_{x_i}^2 (w_{x_i}^2 (x-x_i)^2 + w_{y_i}^2 (y-y_i)^2) - w_{x_i}^4 (x-x_i)^2}{d_i^3(\bar{x})}\right)$$
$$\times \left(\sum_i \frac{w_{y_i}^2 (w_{x_i}^2 (x-x_i)^2 + w_{y_i}^2 (y-y_i)^2) - w_{y_i}^4 (x-x_i)^2}{d_i^3(\bar{x})}\right)$$
$$= \left(\sum_i \frac{w_{x_i}^2 w_{y_i}^2 (y-y_i)^2}{d_i^3(\bar{x})}\right)\left(\sum_i \frac{w_{x_i}^2 w_{y_i}^2 (x-x_i)^2}{d_i^3(\bar{x})}\right).$$

Now let $a_i = \frac{w_{x_i} w_{y_i} (x-x_i)}{d_i^3(\bar{x})},\ b_i = \frac{w_{x_i} w_{y_i} (y-y_i)}{d_i^3(\bar{x})}$. By Schwarz's inequality $(\sum_i a_i b_i)^2 \le (\sum_i a_i^2)(\sum_i b_i^2)$. The equality holds if and only if $a_i = kb_i$ for some k and all i, that is, if $(x-x_i) = k(y-y_i)$ i.e. the destination points (x_i, y_i) are collinear, contrary to assumption. Therefore *strict* inequality holds. We have shown that $\lambda_+ \equiv \rho(A)$, the spectral radius of A is strictly less then 1, i.e. A is a convergent matrix.

Since A is not symmetric $\|A\|_2^2 \ne \rho^2(A)$. Instead, $\|A\|_2^2 = \rho(A^{\mathrm{T}}A)$. The explicit computation of $\rho(A^{\mathrm{T}}A)$ is cumbersome so we use the following property.

Let $A \in C^{n\times n}$ be a non-singular matrix. For any $\varepsilon > 0$ there is a constant M such that $\|A^k\|_2 \le M(\rho(A)+\varepsilon)^k$ for all non-negative integers k ([18], page 359, P 7.3.1.). Now A in (27) is non singular because

$$\det(A) = \left(1/\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})}\right)\left(1/\sum_i \frac{w_{y_i}^2}{d_i(\bar{x})}\right)\left(\sum_i \frac{w_{x_i}^4 (x-x_i)^2}{d_i^3(\bar{x})} \sum_i \frac{w_{y_i}^4 (y-y_i)^2}{d_i^3(\bar{x})} - \left(\sum_i \frac{w_{x_i}^2 w_{y_i}^2 (x-x_i)(y-y_i)}{d_i^3(\bar{x})}\right)^2\right).$$

By Schwarz's inequality, the last factor is non-negative and equals zero if and only if $(x-x_i) = k(y-y_i)$ for some k and all i. This is impossible because the destination points are assumed not to be collinear. Thus $\det(A)$ is positive and A is non singular. We have already shown that $\rho(A) < 1$. We can now choose $\varepsilon > 0$ small enough so that $\eta = (\rho(A)+\varepsilon) < 1$ so $\|A^k\|_2 \le M\eta^k$. Define the error vector at iteration k as $\bar{e}^k \equiv \bar{x}^k - \bar{x}_{\min}$. Then for $\bar{x}^k$ near $\bar{x}_{\min}\bar{e}^k = \bar{x}^k - \bar{x}_{\min} \approx H'(\bar{x}_{\min})(\bar{x}^{k-1} - \bar{x}_{\min}) = A\bar{e}^{k-1} \approx A^2\bar{e}^{k-2} \approx \cdots \approx A^k\bar{e}^0$ where $\bar{e}^0 = \bar{x}^0 - \bar{x}_{\min}$ is the error vector at the initial guess. Thus, $\|\bar{e}^k\|_2 = \|A^k\bar{e}^0\|_2 \le M\eta^k\|\bar{e}^0\|_2$ with $\eta < 1$, that is, at iteration k, M times the initial error is reduced in the 2-norm by a factor η^k, where η is arbitrarily close to $\rho(H'(\bar{x}_{\min}))$.

5.1. Numerical examples

In example 1, 300 destination points $\bar{x}_i$ (see Fig. 1(a)) and their respective x and y weights $w_{x_i}, w_{y_i}, i = 1, \ldots, 300$ were chosen from a random number generator. The initial guess was $x^0 = 10, y^0 = 0$ and the iteration was terminated with a convergence tolerance 10^{-8} which was at iteration number 43. Fig. 1(b) shows the ratio $\|\bar{x}_{i+1} - \bar{x}_{\min}\|_2 / \|\bar{x}_i - \bar{x}_{\min}\|_2$ which converges to the calculated convergence factor $\lambda_+ = .63323$. The predicted convergence factor is $\lambda_+ = \rho(H')(\bar{x}_{\min}) = .63331$ Fig. 1(c) shows the convergence of the objective function.

In example 2, 500 destination points and their respective x and y weights were again chosen at random (see Fig. 2(a) and (c)). Since $\|\bar{x}_{i+1} - \bar{x}_{\min}\|_2 \approx \lambda_+\|\bar{x}_i - \bar{x}_{\min}\|_2 \approx \lambda_+^i\|\bar{x}_1 - \bar{x}_{\min}\|_2$, it follows that $\log\|\bar{x}_i - \bar{x}_{\min}\|_2 \approx i\log\lambda_+ + \log\|\bar{x}_1 - \bar{x}_{\min}\|_2$. This is shown in Fig. 2(d), a semilog plot which is a straight line with slope $\approx \log\lambda_+$. Fig. 2(b) shows $\|\bar{x}_i - \bar{x}_{\min}\|_2$ as a function of the iteration number i.

6. $\bar{x}_{\min}$ is a destination point

6.1. A criterion for $\bar{x}_{\min}$ to be a destination point

We derive a criterion which can be computed à priori from the destination points and their weights, without any iteration, for the unique minimizing point to be a destination point. The objective function $f(\bar{x})$ is given in (2).

Given any destination point $\bar{x}_j$, rescale f with respect to the distance d_j associated with $\bar{x}_j$. Let $x' = w_{x_j}x,\ y' = w_{y_j}y$ and let $x_i' = w_{x_j}x_i,\ y_i' = w_{y_j}y_i$. Now the objective function given in terms of x' and y' is

$$f(\bar{x}') = \sqrt{(x'-x_j')^2 + (y'-y_i')^2} + \sum_{i\ne j}\sqrt{\frac{w_{x_i}^2}{w_{x_j}^2}(x'-x_i')^2 + \frac{w_{y_i}^2}{w_{y_j}^2}(y-y_i')^2}. \quad (29)$$

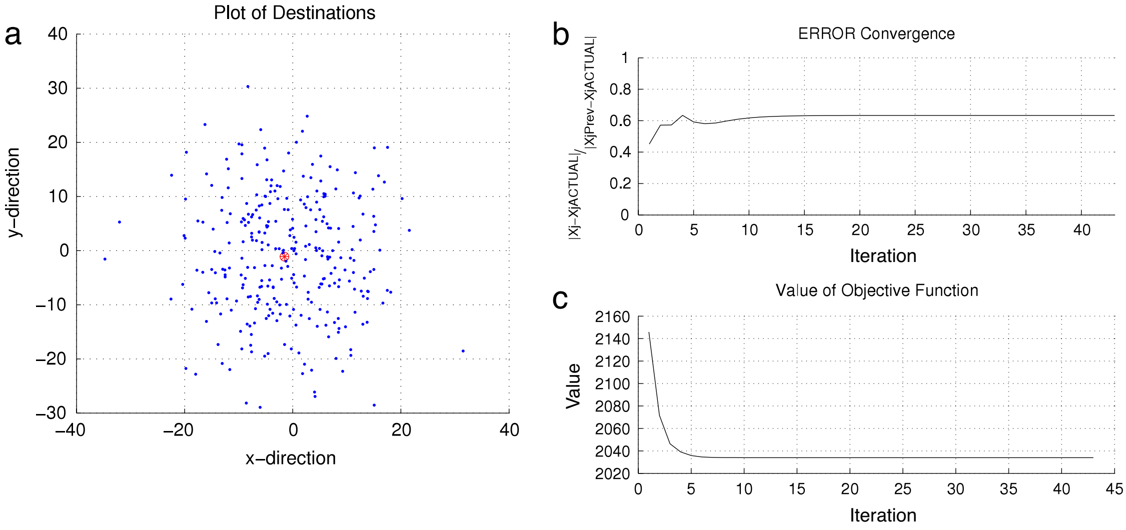
a
Plot of Destinations
y–direction
x–direction
b
ERROR Convergence
|Xj−XjACTUAL|/|XjPrev−XjACTUAL|
Iteration
c
Value of Objective Function
Value
Iteration

Fig. 1.

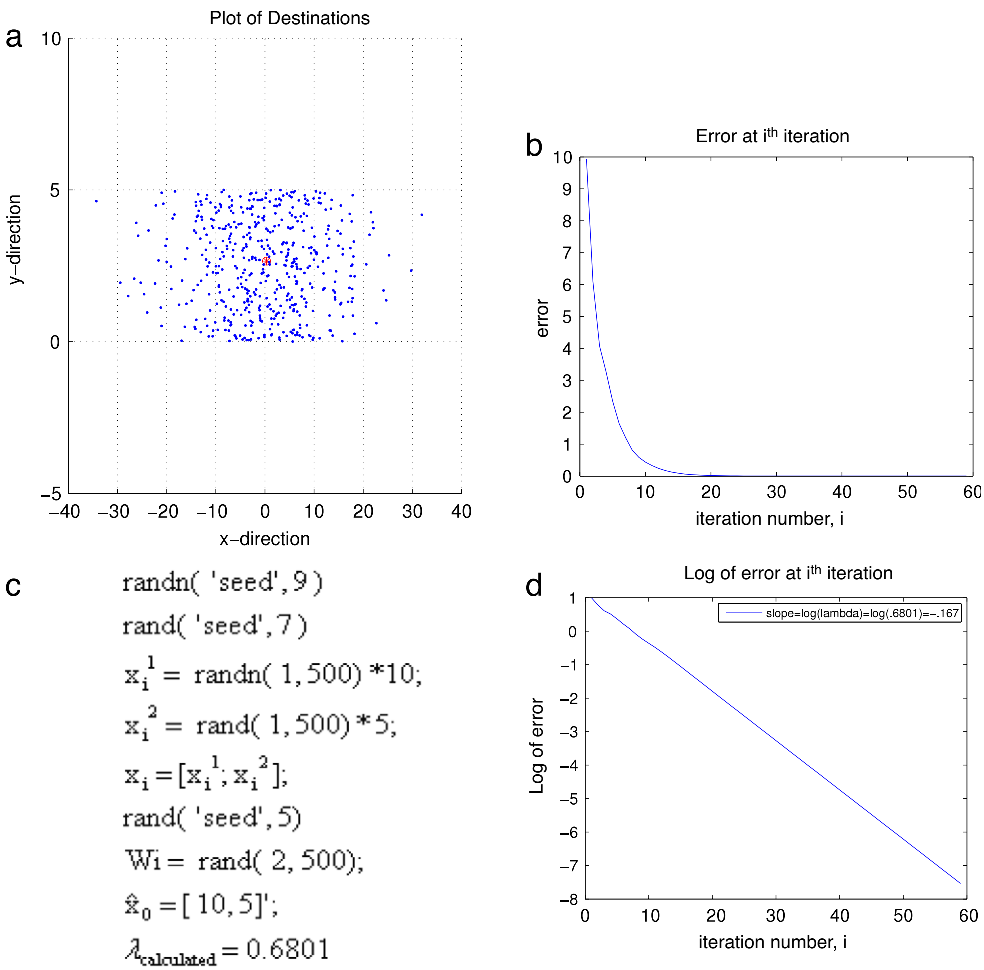
a
Plot of Destinations
y–direction
x–direction
b
Error at ith iteration
error
iteration number, i
c
randn('seed',9)
rand('seed',7)
x_i^1 = randn(1,500) *10;
x_i^2 = rand(1,500) *5;
x_i = [x_i^1; x_i^2];
rand('seed',5)
Wi = rand(2,500);
x̂_0 = [10,5]';
λ_calculated = 0.6801
d
Log of error at ith iteration
slope=log(lambda)=log(.6801)=−.167
Log of error
iteration number, i

Fig. 2.

We compute the rate of change of f at $\bar{x}_j'$ along a unit vector $\bar{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, $z_1^2 + z_2^2 = 1$.

$$f(\bar{x}' + t\bar{z}) = t + \sum_{i \neq j} \sqrt{\frac{w_{x_i}^2}{w_{x_j}^2}(x_j' + tz_1 - x_i')^2 + \frac{w_{y_i}^2}{w_{y_j}^2}(y_j' + tz_2 - y_i')^2}. \tag{30}$$

So

$$\frac{\partial f}{\partial t}(\bar{x}' + t\bar{z}) = 1 + \sum_{i \neq j} \frac{\frac{w_{x_i}^2}{w_{x_j}^2}(x_j' + tz_1 - x_i')z_1 + \frac{w_{y_i}^2}{w_{y_j}^2}(y_j' + tz_2 - y_i')z_2}{\sqrt{\frac{w_{x_i}^2}{w_{x_j}^2}(x_j' + tz_1 - x_i')^2 + \frac{w_{y_i}^2}{w_{y_j}^2}(y_j' + tz_2 - y_i')^2}}$$

$$\begin{aligned} \frac{\partial f}{\partial t}(\bar{x}' + t\bar{z})|_{t=0} &= 1 + \sum_{i \neq j} \frac{\frac{w_{x_i}^2}{w_{x_j}^2}(x_j' - x_i')z_1}{\sqrt{\frac{w_{x_i}^2}{w_{x_j}^2}(x_j' - x_i')^2 + \frac{w_{y_i}^2}{w_{y_j}^2}(y_j' - y_i')^2}} + \frac{\frac{w_{y_i}^2}{w_{y_j}^2}(y_j' - y_i')z_1}{\sqrt{\frac{w_{x_i}^2}{w_{x_j}^2}(x_j' - x_i')^2 + \frac{w_{y_i}^2}{w_{y_j}^2}(y_j' - y_i')^2}} \\ &= 1 - R_{x_j}'z_1 - R_{y_j}'z_2 \end{aligned} \tag{31}$$

where

$$R_{x_j}' = \frac{\frac{w_{x_i}^2}{w_{x_j}^2}(x_i' - x_j')}{\sqrt{\frac{w_{x_i}^2}{w_{x_j}^2}(x_j' - x_i')^2 + \frac{w_{y_i}^2}{w_{y_j}^2}(y_j' - y_i')^2}}, \qquad R_{y_j}' = \frac{\frac{w_{y_i}^2}{w_{y_j}^2}(y_i' - y_j')}{\sqrt{\frac{w_{x_i}^2}{w_{x_j}^2}(x_j' - x_i')^2 + \frac{w_{y_i}^2}{w_{y_j}^2}(y_j' - y_i')^2}}. \tag{32}$$

Define $\bar{R_j}' = \begin{bmatrix} R_{x_j}' \\ R_{y_j}' \end{bmatrix}$, $|\bar{R_j}'| = \sqrt{R_{x_j}'^2 + R_{y_j}'^2}$. The greatest descent at $\bar{x_j}'$ is in the direction $z_1 = \frac{R_{x_j}'}{|\bar{R_j}'|}$, $z_2 = \frac{R_{y_j}'}{|\bar{R_j}'|}$. Then in order for $\bar{x_j}'$ to be the minimizing point we must have $1 - \frac{\bar{R_j}' \cdot \bar{R_j}'}{|\bar{R_j}'|^2|} \geq 0$ or $1 - |\bar{R_j}'| \geq 0$, $1 \geq |\bar{R_j}'|$. In terms of x, y we have

$$R_{x_j}' = \frac{1}{w_{x_j}} \sum_{i \neq j} \frac{w_{x_i}^2(x_i - x_j)}{\sqrt{w_{x_i}^2(x_i - x_j)^2 + w_{y_j}^2(y_i - y_j)^2}} = \frac{R_{x_j}}{w_{x_j}}. \tag{33}$$

Similarly,

$$R_{y_j}' = \frac{R_{y_j}}{w_{y_j}} \quad \text{and} \quad |\bar{R_j}'|^2 = \bar{R}_{x_j}'^2 + \bar{R}_{y_j}'^2 = \frac{R_{x_j}^2}{w_{x_j}^2} + \frac{R_{y_j}^2}{w_{y_j}^2}. \tag{34}$$

Finally, the destination point $\bar{x}_j$ is the minimizing point if and only if

$$1 \geq \sqrt{\frac{R_{x_j}^2}{w_{x_j}^2} + \frac{R_{y_j}^2}{w_{y_j}^2}}. \tag{35}$$

Note that if $w_{x_j} = w_{y_j} = w_j$ then the criterion is $1 \geq \frac{1}{w_j}\sqrt{R_{x_j}^2 + R_{y_j}^2}$ or $w_j \geq |\bar{R}_j|$ as in [9].

6.2. Rate of convergence to a destination point

Define

$$\begin{aligned} G_{x,j}(\bar{x}) &\equiv \sum_{i \neq j} \frac{w_{x_i}^2 x_i}{d_i(\bar{x})}, \qquad g_{x,j}(\bar{x}) \sum_{i \neq j} \frac{w_{x_i}^2}{d_i(\bar{x})} \\ G_{y,j}(\bar{x}) &\equiv \sum_{i \neq j} \frac{w_{y_i}^2 y_i}{d_i(\bar{x})}, \qquad g_{y,j}(\bar{x}) \sum_{i \neq j} \frac{w_{y_i}^2}{d_i(\bar{x})}. \end{aligned} \tag{36}$$

By the definition of $H_x(\bar{x})$ in (5) it follows that

$$H_x(\bar{x}) = \frac{x_j + \frac{d_j(\bar{x})}{w_{x_j}^2} G_{x,j}(\bar{x})}{1 + \frac{d_j(\bar{x})}{w_{x_j}^2} g_{x,j}(\bar{x})}. \tag{37}$$

For $\bar{x}$ near $\bar{x}_j$, we have

$$\begin{aligned} H_x(\bar{x}) &= \left(x_j + \frac{d_j(\bar{x})}{w_{x_j}^2} G_{x,j}(\bar{x})\right)\left(1 - \frac{d_j(\bar{x})}{w_{x_j}^2} g_{x,j}(\bar{x}) + \frac{d_j^2(\bar{x})}{w_{x_j}^4} g_{x,j}^2(\bar{x})\right) + O(d_j^3(\bar{x})) \\ &= x_j + \frac{d_j(\bar{x})}{w_{x_j}^2}(G_{x,j}(\bar{x}) - x_j g_{x,j}(\bar{x})) - \frac{d_j^2(\bar{x})}{w_{x_j}^4} g_{x,j}(G_{x,j}(\bar{x})(\bar{x}) - x_j g_{x,j}(\bar{x})) + \cdots. \end{aligned} \tag{38}$$

Noting that from the definition of $R_{x,j}$ in (33) we have

$$G_{x,j}(\bar{x}) - x_j g_{x,j}(\bar{x}) = \sum_{i \neq j} \frac{w_{x_i}^2 x_i}{d_i(\bar{x})} - x_j \sum_{i \neq j} \frac{w_{x_i}^2}{d_i(\bar{x})} = \sum_{i \neq j} \frac{w_{x_i}^2 (x_i - x_j)}{d_i(\bar{x})} \equiv R_{x,j} \tag{39}$$

and (38) becomes

$$H_x(\bar{x}) = x_j + \frac{d_j(\bar{x})}{w_{x_j}^2} R_{x,j} - \frac{d_j^2(\bar{x})}{w_{x_j}^4} g_{x,j} R_{x,j} + \cdots. \tag{40}$$

Similarly,

$$H_y(\bar{x}) = y_j + \frac{d_j(\bar{x})}{w_{y_j}^2} R_{y,j} - \frac{d_j^2(\bar{x})}{w_{y_j}^4} g_{y,j} R_{y,j} + \cdots \tag{41}$$

and

$$\begin{aligned} H(\bar{x}) &= \bar{x}_j + d_j(\bar{x})\hat{R}_j - d_j^2(\bar{x})\tilde{g}_j(\bar{x})\hat{R}_j + \cdots \\ \hat{R}_j &= \begin{bmatrix} \hat{R}_{x,j} \\ \hat{R}_{y,j} \end{bmatrix} = \begin{bmatrix} \dfrac{R_{x,j}}{w_{x,j}^2} \\ \dfrac{R_{y,j}}{w_{y,j}^2} \end{bmatrix} \\ \hat{g}_j &= \begin{bmatrix} \hat{g}_{x,j} \\ \hat{g}_{y,j} \end{bmatrix} = \begin{bmatrix} \dfrac{g_{x,j}}{w_{x,j}^2} \\ \dfrac{g_{y,j}}{w_{y,j}^2} \end{bmatrix} \end{aligned} \tag{42}$$

and $\tilde{g}_j$ is the diagonal matrix

$$\tilde{g}_j = \begin{pmatrix} \hat{g}_{x,j} & 0 \\ 0 & \hat{g}_{y,j} \end{pmatrix}. \tag{43}$$

We now express the error at the $(r+1)$st iterate i.e. its distance to the minimizing destination point $\bar{x}_j$, in terms of in terms of the distance $d_j(\bar{x})$ associated with $\bar{x}_j$. Since $\bar{x}^{r+1} = H(\bar{x}^r)$, from (42) with $\bar{x} = \bar{x}^r$ we have

$$\begin{aligned} d_j^2(\bar{x}^{r+1}) &= w_{x_j}^2 (x^{r+1} - x_j)^2 + w_{y_j}(y^{r+1} - y_j)^2 \\ &= w_{x_j}^2 [d_j(\bar{x}^r)(\hat{R}_{x,j} - d_j(\bar{x}^r)\hat{g}_{x,j}(\bar{x}^r)\hat{R}_{x,j})]^2 + w_{y_j}^2 [d_j(\bar{x}^r)(\hat{R}_{y,j} - d_j(\bar{x}^r)\hat{g}_{y,j}(\bar{x}^r)\hat{R}_{y,j})]^2 + \cdots \\ &= d_j^2(\bar{x}^r)\{w_{x_j}^2 \hat{R}_{x,j}^2 (1 - 2d_j(\bar{x}^r)\hat{g}_{x,j}(\bar{x}^r) + \cdots)\} + d_j^2(\bar{x}^r)\{w_{y_j}^2 \hat{R}_{y,j}^2 (1 - 2d_j(\bar{x}^r)\hat{g}_{y,j}(\bar{x}^r))\} + \cdots. \end{aligned} \tag{44}$$

Therefore,

$$\frac{d_j^2(\bar{x^{r+1}})}{d_j^2(\bar{x^r})} = w_{x_j}^2 \hat{R}_{x,j}^2 + w_{y_j}^2 \hat{R}_{y,j}^2 - 2d_j(\bar{x^r})\{w_{x_j}^2 \hat{R}_{x,j}^2 \hat{g}_{x,j}(\bar{x}^r) + w_{y_j}^2 \hat{R}_{y,j}^2 \hat{g}_{y,j}(\bar{x}^r)\} + \cdots. \tag{45}$$

Table 1
Comparisons.

Property	Asymmetric (Eq. no.)	Symmetric
Solution domain	Enclosing rectangle (15)	Convex hull
Algorithm	"gradient-like" (18)	Gradient
Global convergence	Descent method (25)	Same
Local convergence		
Convergence matrix	Not symmetric (27)	Symmetric
Rate of convergence	Linear	
To a non-destination	$\lambda_+ = \rho(H'(\bar{x}_{\min}))$(28)	Same
Criterion for convergence	(35) and (46)	
to destination $\bar{x}_j$	$\lambda_j = \sqrt{R_{x_j}^2/w_{x_j}^2 + R_{y_j}^2/w_{y_j}^2} \le 1$	$\lambda_j = \lvert\bar{R}_j\rvert/w_j \le 1$
Rate of convergence	Superlinear if $\lambda_j = 0$	
to destination $\bar{x}_j$	Linear if $0 < \lambda_j < 1$	
	Sublinear if $\lambda_j = 1$	Same

Now define

$$\lambda_j \equiv \sqrt{w_{x_j}^2 \hat{R}_{x,j}^2 + w_{y_j}^2 \hat{R}_{y,j}^2} = \sqrt{\frac{R_{x,j}^2}{w_{x_j}^2} + \frac{R_{y,j}^2}{w_{y_j}^2}}. \tag{46}$$

Since $\bar{x}^r$ converges to the minimizing point $\bar{x} = \bar{x}_j$ if and only if $0 \le \lambda_j \le 1$, it follows from (46) that there are three possibilities:

1. $\lambda_j = 0$, that is $R_{x,j} = R_{x,j} = 0$. Then convergence to the minimizing point $\bar{x} = \bar{x}_j$ is *super-linear*.
2. $0 < \lambda_j < 1$. Then convergence is *linear*, with reduction factor λ_j at each iteration.
3. $\lambda_j = 1$. Then (46) becomes

$$\begin{aligned}
\frac{d_j^2(\overline{x^{r+1}})}{d_j^2(\bar{x^r})} &= 1 - 2d_j(\bar{x}^r)\{w_{x_j}^2 \hat{R}_{x,j}^2 \hat{g}_{x,j}(\bar{x}^r) + w_{y_j}^2 \hat{R}_{y,j}^2 \hat{g}_{y,j}(\bar{x}^r)\} + \cdots \\
&= 1 - 2d_j(\bar{x}^r)\left\{\frac{R_{x,j}^2}{w_{x_j}^2}\hat{g}_{x,j}(\bar{x}^r) + \frac{R_{y,j}^2}{w_{y_j}^2}\hat{g}_{y,j}(\bar{x}^r)\right\} + \cdots \\
&= 1 - d_j(\bar{x}^r)\theta + \cdots
\end{aligned} \tag{47}$$

where $\theta = 2\{\frac{R_{x,j}^2}{w_{x_j}^2}\hat{g}_{x,j}(\bar{x}^r) + \frac{R_{y,j}^2}{w_{y_j}^2}\hat{g}_{y,j}(\bar{x}^r)\}$. Note that $\theta > 0$ for all r. In this case, convergence is *sub-linear* with the reduction factor approaching 1 as $r \to \infty$.

6.3. A numerical example

Three destination points $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are at the vertices of a triangle (Fig. 3(a)). The weights $w_{x_1}, w_{x_2}, w_{x_3}, w_{y_1}, w_{y_2}, w_{y_3}$ are such that (35) is satisfied for $\bar{x}_1$, that is $\lambda_1 = .8533 < 1$ (Fig. 3(c)), so the destination point $\bar{x}_1$ is the minimizing point. Note that all the iterates are in the enclosing rectangle $-.5 \le x \le 1$, $-\sqrt{(3)}/2 \le y \le \sqrt{(3)}/2$ but none are in the convex hull of the destinations. Since from (45) $d_1(x^{i+1}) \approx \lambda_1 d_1(x^i) \approx \cdots \approx \lambda^i d_1(\bar{x}^1)$ (Fig. 3(b)), we have $\log(d_1(x^{i+1})) \approx i(\log\lambda_1) + d_1(x^1)$. This is as shown in Fig. 3(d) where the semilog plot is a straight line with slope $\log\lambda_1$.

7. Comparison

In Table 1, some salient features of the asymmetric and symmetric problems are compared. Common properties are: the algorithms for both methods are descent methods which are globally convergent (except for a denumerable number of starting points); when the solution is not at a destination the algorithm is locally convergent linearly; when the the solution is at a destination the algorithm's local convergence can be linear, sub-linear, or super-linear; there is a computable criterion for the solution to be at a destination. The differences are: in the symmetric case all iterates are in the convex hull of the destinations, in the asymmetric case they are in the smallest rectangle which contains all the destinations; in the symmetric case the algorithm is a gradient method, in the asymmetric case it is a "gradient-like" method.

Acknowledgements

Thanks to Dr. Liangjun Xie for his dedicated and capable assistance in the preparation of this paper.

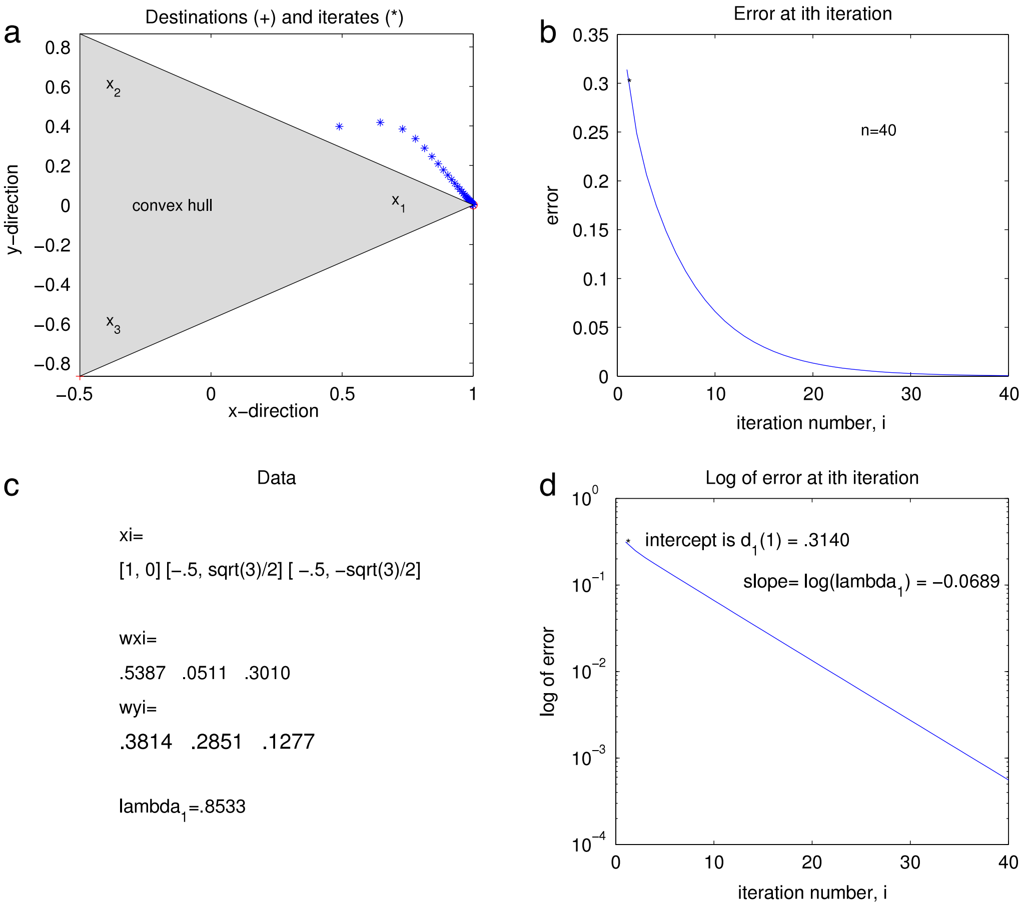

Fig. 3.

Appendix

In this Appendix, we derive the expression for $H'(\bar{x}_{\min}) \equiv \frac{\partial^2 H}{\partial \bar{x}^2}$ in (27). From (27) we have

$$\frac{\partial H_x}{\partial x} = \frac{\sum_i - \frac{w_{x_i}^2 x_i \frac{\partial d_i}{\partial x}(\bar{x}_{\min})}{d_i^2(\bar{x}_{\min})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})}} + \sum_i \frac{w_{x_i}^2 x_i}{d_i(\bar{x}_{\min})} \left\{ \frac{-1}{\left(\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})}\right)^2} \sum_i - \frac{w_{x_i}^2}{d_i^2(\bar{x}_{\min})} \frac{\partial d_i}{\partial x}(x_{\min}) \right\}$$

$$= \frac{\sum_i - \frac{w_{x_i}^2 x_i w_{x_i}^2 (x_{\min} - x_i)}{d_i^3(\bar{x}_{\min})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})}} + \sum_i \frac{w_{x_i}^2 x_i}{d_i(\bar{x}_{\min})} \left\{ \frac{1}{\left(\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})}\right)^2} \sum_i \frac{w_{x_i}^2}{d_i^3(\bar{x}_{\min})} w_{x_i}^2 (x_{\min} - x_i) \right\}.$$

From (5) with $\bar{x} = \bar{x}_{\min}$ we have

$$x_{\min} \sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})} = \sum_i \frac{w_{x_i}^2 x_i}{d_i(\bar{x}_{\min})}, \qquad y_{\min} \sum_i \frac{w_{y_i}^2}{d_i(\bar{x})_{\min}} = \sum_i \frac{w_{y_i}^2 y_i}{d_i(\bar{x}_{\min})}$$

so

$$\frac{\partial H_x}{\partial x} = \frac{\sum_i - \frac{w_{x_i}^2 x_i w_{x_i}^2 (x_{\min} - x_i)}{d_i^3(\bar{x}_{\min})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})}} + x \sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})} \left\{ \frac{1}{\left(\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})}\right)^2} \sum_i \frac{w_{x_i}^2}{d_i^3(\bar{x}_{\min})} w_{x_i}^2 (x_{\min} - x_i) \right\}$$

$$= \frac{\sum_i \frac{w_{x_i}^4 (x_{\min}-x_i)^2}{d_i^3(\bar{x}_{\min})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})}}.$$

Similarly for $\frac{\partial H_y}{\partial y}$.

$$\frac{\partial H_x}{\partial y} = \frac{\sum_i -\frac{w_{x_i}^2 x_i \frac{\partial d_i}{\partial y}(\bar{x}_{\min})}{d_i^2(\bar{x}_{\min})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})}} + \sum_i \frac{w_{x_i}^2 x_i}{d_i(\bar{x}_{\min})} \left\{ \frac{-1}{\left(\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})}\right)^2} \sum_i -\frac{w_{x_i}^2}{d_i(\bar{x}_{\min})} \frac{\partial d_i}{\partial y}(x_{\min}) \right\}$$

$$= \frac{\sum_i -\frac{w_{x_i}^2 x_i w_{y_i}^2 (y-y_i)}{d_i^3(\bar{x}_{\min})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})}} + \sum_i \frac{w_{x_i}^2 x_i}{d_i(\bar{x}_{\min})} \left\{ \frac{1}{\left(\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})}\right)^2} \sum_i \frac{w_{x_i}^2}{d_i^3(\bar{x}_{\min})} w_{y_i}^2 (y-y_i) \right\}.$$

Again from (5)

$$\frac{\partial H_x}{\partial y} = \frac{\sum_i -\frac{w_{x_i}^2 x_i w_{y_i}^2 (y_{\min}-y_i)}{d_i^3(\bar{x}_{\min})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})}} + x \sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})} \left\{ \frac{1}{\left(\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})}\right)^2} \sum_i \frac{w_{x_i}^2}{d_i^3(\bar{x}_{\min})} w_{y_i}^2 (y_{\min}-y_i) \right\}$$

$$= \frac{\sum_i \frac{w_{x_i}^2 w_{y_i}^2 (x_{\min}-x_i)(y_{\min}-y_i)}{d_i^3(\bar{x}_{\min})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x}_{\min})}}.$$

Similarly for $\frac{\partial H_y}{\partial x}$. This gives

$$H'(\bar{x}_{\min}) = \begin{pmatrix} \frac{\sum_i \frac{w_{x_i}^4 (x-x_i)^2}{d_i^3(\bar{x})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})}} & \frac{\sum_i \frac{w_{x_i}^2 w_{y_i}^2 (x-x_i)(y-y_i)}{d_i^3(\bar{x})}}{\sum_i \frac{w_{x_i}^2}{d_i(\bar{x})}} \\ \frac{\sum_i \frac{w_{x_i}^2 w_{y_i}^2 (x-x_i)(y-y_i)}{d_i^3(\bar{x})}}{\sum_i \frac{w_{y_i}^2}{d_i(\bar{x})}} & \frac{\sum_i \frac{w_{y_i}^4 (y-y_i)^2}{d_i^3(\bar{x})}}{\sum_i \frac{w_{y_i}^2}{d_i(\bar{x})}} \end{pmatrix}_{\bar{x}=\bar{x}_{\min}}$$

as in (27).

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