Nonoscillation of First Order Impulse Differential Equations with Delay*

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Oscillation properties of impulse functional-differential equations are studied for equations of the type

\[ \dot{x}(t) = \sum_{i=1}^{m} p_i(t)x(t - \tau_i(t)) = 0, \quad t \in [a, b], \]

\[ x(\xi) = 0, \quad \xi \in [a, b], \]

\[ x(t_j) = \beta_j x(t_j - 0), \quad j = 1, \ldots, k, \]

\[ a < t_1 < t_2 < \cdots < t_k < b. \]

The proven test for oscillation generalizes the known ones and allows consideration of the solvability of boundary value problems for the corresponding nonhomogeneous impulse equations. In particular, for the scalar impulse equation

\[ \dot{x}(t) + p(t)x(t - \tau(t)) = 0, \quad t \in [0, \infty), \]

\[ x(\xi) = 0 \quad \text{for } \xi < 0, \]

\[ x(t_j) = \beta_j x(t_j - 0), \quad \beta_j > 0, \quad j = 1, 2, \ldots, \]

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nonoscillation of impulse equations

Let
\[ \frac{1 + \ln B(t)}{e} \geq \int_{r(t)}^{t} p_+ (s) \, ds \]
where \( r(t) = \max(t - \tau(t), 0), \ t > 0. \)

Then the nontrivial solution of this equation has no zeros on \([0, \infty)\).

1. Introduction

In the present paper we investigate the following scalar equation with impulses

\[
(\mathcal{L}x)(t) = x'(t) + \sum_{i=1}^{m} p_i(t) x(t - \tau_i(t)) = f(t), \quad t \in [0, \infty),
\]

\[
x(\xi) = 0 \quad \text{for } \xi < 0, \]

\[
x(t_j) = \beta_j x(t_j - 0), \quad \beta_j > 0, \ j = 1, 2, \ldots,
\]

where

\[ p_i, f : [0, +\infty) \to (-\infty, +\infty) \]

are locally summable functions

and

\[ \tau_i : [0, +\infty) \to (0, +\infty) \]

are measurable functions, \( i = 1, \ldots, m. \)

Equations with impulses are intensively studied by many authors, see for example the recent monographs by Lakshmikantham et al. [1], Pandit and Deo [2], Samoilenko and Perestyuk [3], and the bibliography there.

Various comparison theorems for solutions of the Cauchy and periodic problems for ordinary differential impulse equations have been obtained in [4, 5, 6]. On the basis of the comparison theorems, tests for stability are constructed (see, for example, [6]).

An extension of the results of this type to the impulse equations with delay meets essential difficulties, implied by the following reasons. In the case of ordinary impulse equations \( \tau_i = 0 \) for \( i = 1, \ldots, m \) the graph of the solution consists of the graphs of solution of nonimpulsive ordinary differential equation on the intervals \((t_i, t_{i+1})\). Evidently, the solution \( x(t) \) of the homogeneous equation in this case preserves its sign. This fact leads to the conclusion about invariance of sign of the Green’s function \( G(t, s) \) for different problems, since the cross-sections of \( G(t, s) \) (for fixed \( s \)) are just the solution of the homogeneous equation.
The solutions of the equations with deflecting argument in contrast to
the ordinary ones may change their sign. This makes their properties
especially different. The oscillation properties of such equations have
interested many researchers. Let us recall in this connection the recent
monographs by Ladde et al. [7] and Gyori and Ladas [8]. The “addition” of
impulses in those equations makes the properties of the solutions even
more complicated.

The problem of the existence of a nonoscillatory solution for the
impulse equation with delay,

\begin{align*}
x'(t) + ax(t - \tau) &= 0, \quad t \in [0, +\infty), \quad (1.4) \\
x(t_j) &= \beta_j x(t_j - 0), \quad j = 1, 2, \ldots, \\
x(\xi) &= \varphi(\xi) \quad \text{for} \ \xi < 0, \quad (1.5)
\end{align*}

where \(a, \tau\) are positive constants, was studied in the well known paper by
Gopalsamy and Zhang [9]. They have obtained the following result: there
exists a nonoscillatory solution for (1.4) if the two conditions

\begin{align*}
a\tau &< \frac{1}{e}, \quad (1.6) \\
\sum_{j=1}^{\infty} |1 - \beta_j| &< \infty \quad (1.7)
\end{align*}

hold.

It is evident that for this statement it is at least required that \(\beta_j \to 1\) for
\(j \to \infty\), i.e., there is “vanishing” of impulses. We manage to eliminate this
essential restriction (see, for example, our test for nonoscillation in the
abstract). Moreover, the proposed approach does not assume that the
coefficients \(p_j, \tau_i\) are positive constants.

Let us point out that in [9], as well as in many other investigations, the
equations (1.1), (1.5), (1.3) are considered. It does not lead to any addi-
tional generality in comparison with the equations (1.1), (1.2), (1.3), since
the first can be presented in our form, where the right-hand side \(f^\alpha\) is
defined as

\[f^\alpha(t) = f(t) - \sum_{i=1}^{m} p_i(t) \varphi_i(t),\]

where

\[
\varphi_i(t) = \begin{cases} 
\varphi(t - \tau_i(t)), & t - \tau_i(t) < 0, \\
0, & t - \tau_i(t) \geq 0.
\end{cases}
\]
The idea of considering just the condition (1.2) is that the space of the solutions of the homogeneous equations (1.8), (1.2), (1.3), where

\[(L x)(t) = 0, \quad t \in [0, +\infty), \tag{1.8}\]

is one-dimensional. This fact is essentially used in Section 2, where the following is established: the general solution of (1.1), (1.2), (1.3) has the representation

\[x(t) = X(t)x(0) + \int_0^t C(t, s)f(s)\, ds,\]

where \(X(t)\) is the solution of the homogeneous equations (1.8), (1.2), (1.3), satisfying the initial condition \(x(0) = 1\), and \(C(t, s)\) is the Cauchy function of the equations (1.1), (1.2), (1.3).

In Section 3 we propose a theorem on equivalence of several statements in the case \(p_i \geq 0, \ i = 1, \ldots, m\). Some of them are nonoscillation of solutions of the homogeneous equation, positivity of the Cauchy function \(C(t, s)\), positivity of Green’s function of the periodic problem and existence of a positive function \(v\) satisfying inequality \(L v \leq 0\) and conditions (1.2), (1.3). Choosing a function \(v\), we get effective tests for nonoscillation and positivity of different Green functions.

In conclusion let us point out that the positivity of Green’s function could be a basis for constructing special monotonic techniques for obtaining comparison results for nonlinear impulse equations with delay.

2. ON A REPRESENTATION OF GENERAL SOLUTIONS FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH IMPULSES

Let \(M_0\) be a linear continuous Volterra operator acting from the space \(D_0\) of absolutely continuous functions \(x: [0, b] \rightarrow \mathbb{R}^n\) to the space \(L\) of summable functions \(y: [0, b] \rightarrow \mathbb{R}^n\). This operator defines some linear functional-differential equation

\[M_0 x = f. \tag{2.1}\]

The space \(D_0\) is isomorphic to the topological product \(L \times \mathbb{R}^n\), where the isomorphism \(J_0: L \times \mathbb{R}^n \rightarrow D_0\) can be defined, for example, by the equality

\[J_0(z, \alpha)(t) = \int_0^t z(s)\, ds + \alpha. \tag{2.2}\]
Define some other space $D$, isomorphic to the topological product $L \times R^n$, by the equality

$$J(z, \alpha)(t) = \int_0^t \Omega(t, s) z(s) \, ds + \omega(t) \alpha,$$

(2.3)

where

$$\omega(t) = \sum_{i=1}^{k+1} x_{[t_{i-1}, t_i]}(t) \prod_{j=1}^i \beta_{i-j},$$

(2.4)

$$\Omega(t, s) = \sum_{i=1}^{k+1} x_{[t_{i-1}, t_i]}(t) x_{[t_{i-1}, t_i]}(s) \beta_0 + \sum_{i=2}^{k+1} \sum_{r=1}^{i-1} x_{[t_{i-1}, t_i]}(t) x_{[t_{i-1}, t_i]}(s) \prod_{j=1}^{i-r} \beta_{i-j}.$$ 

Here $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = b$, $\beta_i$ ($i = 1, \ldots, k$) are $n \times n$ matrices such that $\det \beta_i \neq 0$, $\beta_0 = E$ (unit matrix), and $x_{[t_{i-1}, t_i]}$ is a characteristic function of the set $[t_{i-1}, t_i]$. It is clear that $D$ is the space of piecewise continuous functions $x: [0, b] \to R^n$. At the points $t_i$ ($i = 1, \ldots, k$) these functions are continuous from the right and have a discontinuity of the first kind satisfying the equality

$$x(t_i) = \beta_i x(t_i - 0), \quad i = 1, 2, \ldots, k.$$

(2.5)

Obviously, $D$ and $D_0$ coincide if $\beta_i = E$ for $i = 1, 2, \ldots, k$.

The operator $M_0: D_0 \to L$ can be extended by continuity to the space $D$. Denote this new operator as $M: D \to L$. Such an assumption gives a possibility to consider the impulse equation

$$M_0 x = f, \quad x(t_i) = \beta_i x(t_i - 0), \quad i = 1, 2, \ldots, k.$$ 

(2.6)

A solution for equation (2.6) is a function from the space $D$, satisfying the equality $(M_0 x)(t) = f(t)$ for almost all $t \in [0, b]$.

In our notation we can write Eq. (2.6) in the form

$$Mx = f.$$ 

(2.7)

Here, we emphasize that the definition of the solution differs from the one in [9], where validity of the aforementioned equality is assumed for all $t \in [0, b], t \neq t_i, i = 1, \ldots, k$.

The following lemma establishes a formula of representation of the general solution for Eq. (2.7):
Lemma 2.1 (cf. Theorem 1 in [10]). Let the Cauchy problem

\[ Mx = f, \quad x(0) = \alpha, \]  

be uniquely solvable for every pair \((f, \alpha) \in L \times \mathbb{R}^n\). Then the general solution of (2.7) has the representation

\[ x(t) = X(t)\alpha + \int_0^t C(t, s)f(s)\, ds, \]  

where the columns of the \(n \times n\) matrix \(X(t)\), the fundamental matrix of (2.7), constitute a basis of the linear manifold of the solutions of the homogeneous equation \(Mx = 0\) and \(C(t, s)\) is the Cauchy matrix of (2.7).

Proof. Isomorphism (2.3) implies the equality

\[ x(t) = \int_0^t \Omega(t, s)\dot{x}(s)\, ds + \omega(t)\alpha, \quad t \in [0, b]. \]

The operator \(M: D \to L\) has the representation

\[ (Mx)(t) = MJ(x, x(0))(t) = (Q\dot{x})(t) + A(t)x(0), \]  

where \(Q: L \to L\) is a linear bounded Volterra operator and the columns of the \(n \times n\) matrix \(A\) belong to \(L\). Equality (2.10) implies that the derivative of the solution of (2.8) satisfies the equation

\[ (Q\dot{x})(t) = f(t) - A(t)\alpha, \quad t \in [0, b]. \]

The conditions of the lemma imply that there exists a bounded inverse operator \(Q^{-1}: L \to L\). Moreover, \(Q^{-1}\) is a Volterra operator. Such an operator has the representation

\[ (Q^{-1}z)(t) = \frac{d}{dt} \int_0^t K(t, s)z(s)\, ds. \]

The solution \(x\) of (2.8) is of the form

\[ x(t) = J(Q^{-1}(f - A\alpha), \alpha)(t) \]
\[ = \int_0^t \Omega(t, s) \left[ \frac{d}{ds} \int_0^t K(s, \tau)(f(\tau) - A(\tau)\alpha)\, d\tau \right] ds + \omega(t)\alpha \]
\[ = \int_0^t C(t, s)f(s)\, ds + \left[ E\omega(t) - \int_0^t C(t, s)A(s)\, ds \right] \alpha, \]
where
\[ C(t, s) = K(t, s) + \sum_{i=2}^{k+1} \sum_{r=1}^{i-1} \chi_{[t_{i-1}, t_i)}(t) \chi_{[0, t_i)}(s) \prod_{j=1}^{i-r} \beta_{i-j} K(t_i, s) \]
\[ - \sum_{i=2}^{k+1} \chi_{[t_{i-1}, t_i)}(t) \chi_{[0, t_{i-1})}(s) K(t_{i-1}, s) \]
\[ - \sum_{i=3}^{k+1} \sum_{r=1}^{i-2} \chi_{[t_{i-2}, t_i)}(t) \chi_{[0, t_i)}(s) \prod_{j=1}^{i-r-1} \beta_{i-j} K(t_i, s). \]

**Remark 2.1.** If the operator \( Q : L \rightarrow L \) (see the equality (2.10)) is of the form \( Q = I - \Gamma \), where \( I \) is the unit operator and \( \Gamma \) is a weakly completely continuous Volterra operator,
\[ (\Gamma z)(t) = \int_0^t \Gamma(t, s) z(s) \, ds, \]
then the general solution of (2.7) has the representation
\[ x(t) = C(t, 0) x(0) + \int_0^t C(t, s) \, ds. \tag{2.12} \]

This representation is called the Cauchy formula. We do not go into a proof of formula (2.12), because it can be constructed by analogy with the scheme of the work by Maksimov [11] without using new methods and ideas.

### 3. NONOSCILLATION PROPERTIES OF SCALAR IMPULSIVE EQUATION OF THE FIRST ORDER

Consider problem (1.1), (1.2), (1.3). Its general solution has the representation
\[ x(t) = C(t, 0) x(0) + \int_0^t C(t, s) f(s), \tag{3.1} \]
where \( C(s, t) \) is the Cauchy function of (1.1), (1.2), (1.3). For a fixed \( s \in [0, t] \) the function \( C(\cdot, s) \) is a solution to the homogeneous “\( s \)-truncated” equation
\[ (\mathcal{L} x)(t) = x'(t) + \sum_{i=1}^{m} p_i(t) x(t - \tau_i(t)) = 0, \quad t \geq s, \tag{3.2} \]
\[ x(t_j) = \beta_j x(t_j - 0), \quad j = k_s, k_s + 1, \ldots, \tag{3.3} \]
\[ x(\xi) = 0 \quad \text{for } \xi < s. \tag{3.4} \]
where \( k_j \) are integers such that

\[
t_{k_j} > s > t_{k_j - 1}, \quad \beta_j > 0, \quad j = k_s, k_s + 1, \ldots,
\]

while \( C(s, s) = 1 \).

If the boundary value problem (1.1), (1.2), (1.3), and

\[
x(b) = 0
\]

is uniquely solvable in the space \( D \) for every function \( f \in L \), then its solution can be represented in the form

\[
x(t) = \int_0^b G(t, s) f(s) \, ds,
\]

where Green's function \( G(t, s) \) of this problem is

\[
G(t, s) = C(t, s) - \frac{C(b, s)C(t, 0)}{C(b, 0)}
\]

and \( C(t, s) = 0 \) for \( t < s \).

If the periodic problem (1.1), (1.2), (1.3), and

\[
x(0) - x(b) = 0
\]

is uniquely solvable in the space \( D \) for every \( f \in L \), then its solution can be represented in the form

\[
x(t) = \int_0^b P(t, s) f(s) \, ds
\]

where

\[
P(t, s) = C(t, s) - \frac{C(b, s)C(t, 0)}{1 - C(b, 0)}.
\]

The boundary value problem

\[
\begin{align*}
x'(t) &= f(t), \quad t \in [0, b], \\
x(t_j) &= \beta_j x(t_j - 0), \quad j = 1, 2, \ldots, k, \\
x(b) &= 0,
\end{align*}
\]

has the unique solution for every summable \( f \). Denote by \( G_0(t, s) \) the Green function of problem (3.9). Note that \( G_0(t, s) = 0 \) for \( 0 \leq s \leq t \leq b \) and \( G_0(t, s) < 0 \) for \( 0 \leq t < s \leq b \).
Using (3.6), we can find, for example, the Green’s function of problem (3.9) in the case $k = 3$.

Define an operator $K: D \mapsto D$ by the equality

$$(Kx)(t) = -\int_0^b G_0(t, s) \sum_{i=1}^m p_i(s) x(s - \tau_i(s)) \chi_{[0, b]}(s - \tau_i(s)) \, ds.$$  

(3.10)

**Theorem 3.1.** Let $p_i \geq 0$ for $i = 1, \ldots, m$. Then the following assertions are equivalent:

1. the Cauchy function $C(t, s)$ of (1.1)–(1.3) is positive for $0 \leq s \leq t \leq b$,
2. a nontrivial solution of the homogeneous equation (1.8), (1.2), (1.3) has no zeros on $[0, b]$,
3. the spectral radius of the operator $K$ is less than one,
(4) problems (1.1), (1.2), (1.3), (3.5) are uniquely solvable for every  
\( f \in L \) and its Green function \( G(t, s) \) is negative for \( 0 \leq t < s \leq b \) and  
nonpositive for \( 0 \leq s \leq t \leq b \),

(5) (only in the case \( \beta_1 < 1, \ldots, \beta_k < 1 \)) the periodic problems, (1.1),  
(1.2), (1.3), (3.7) are uniquely solvable and its Green function \( P(t, s) \) is  
positive for \( t, s \in [0, b] \),

(6) there exists a nonnegative function \( v \in D \) such that

\[
(\mathcal{L}v)(t) \leq 0, \quad v(b) - \int_b^t (\mathcal{L}v)(s) \, ds > 0, \quad t \in [0, b].
\]

We prove Theorem 3.1 according to the following scheme:

(6) \( \Rightarrow \) (3) \( \Rightarrow \) (4) \( \Rightarrow \) (6), \quad (3) \( \Rightarrow \) (1) \( \Rightarrow \) (2) \( \Rightarrow \) (6).

(6) \( \Rightarrow \) (3). The function \( v \) satisfies the integral equation

\[
v(t) - (Kv)(t) = \psi(t), \quad t \in [0, b],
\]

where

\[
\psi(t) = v(b) - \int_t^b G_0(t, s)(\mathcal{L}v)(s) \, ds.
\]

Since \( \psi(t) > 0 \), the spectral radius \( \rho(K) \) of the operator \( K \) is less than 1  
[12].

(3) \( \Rightarrow \) (4). The equation \( x = Kx + g \), where

\[
g(t) = -\int_b^t G_0(t, s)f(s) \, ds,
\]

is equivalent to problem (1.1), (1.2), (1.3), (3.5). The condition \( \rho(K) < 1 \) implies that this problem is uniquely solvable and its solution can be  
represented in the form

\[
x(t) = g(t) + \int_0^t \left[ G(t, s) - G_0(t, s) \right] f(s) \, ds.
\]

If \( f \leq 0 \), then \( 0 \leq g \leq x \). Consequently, \( G(t, s) \leq G_0(t, s) \).

(4) \( \Rightarrow \) (6). In order to prove this assertion we set

\[
v(t) = -\int_0^b G(t, s) \, ds.
\]
(3) $\Rightarrow$ (1). Define an operator $K_{[\nu, \mu]} : D_{[\nu, \mu]} \to D_{[\nu, \mu]}$, where $[\nu, \mu] \subseteq [0, b]$, by the equality

$$(K_{[\nu, \mu]} x)(t) = - \int_0^t G^{[\nu, \mu]}(t, s) \sum_{i=1}^m p_i(s) x(s - \tau_i(s)) \chi_{[\nu, \mu]}(s - \tau_i(s)) \, ds,$$

where $G^{[\nu, \mu]}$ is the Green's function of the problem

$$x'(t) = f(t), \quad t \in [\nu, \mu],$$

$$x(t_j) = \beta_j x(t_j - 0), \quad j = k_\nu, \ldots, k_\mu - 1,$$

$$x(\mu) = 0.$$

Here $D_{[\nu, \mu]}$ is the space of bounded functions continuous on each interval $[t_{i-1}, t_i]$.

The proof is based on the following assertion.

**Lemma 3.1.** If $\rho(K) < 1$, then $\rho(K_{[\nu, \mu]}) < 1$ for $[\nu, \mu] \subseteq [0, b]$.

**Proof.** By virtue of (3) $\Rightarrow$ (4), problems (1.1), (1.2), (1.3), (3.5) are uniquely solvable and the Green function $G(t, s)$ is nonpositive for $0 \leq s \leq t \leq b$ and negative for $0 \leq t < s \leq b$.

The function

$$v(t) = - \int_0^b G(t, s) \, ds$$

is a positive solution of the boundary value problem

$$(Lx)(t) = -1, \quad t \in [0, b],$$

$$x(t_j) = \beta_j x(t_j - 0), \quad j = 1, 2, \ldots, k,$$

$$x(\xi) = 0 \quad \text{for} \quad \xi < 0,$$

$$x(b) = 0.$$

It is clear that

$$v'(t) + \sum_{i=1}^m v(t - \tau_i(t)) \chi(\nu, \mu)(t - \tau_i(t))$$

$$= -1 - \sum_{i=1}^m v(t - \tau_i(t)) [1 - \chi(\nu, \mu)(t - \tau_i(t))], \quad t \in [\nu, \mu].$$

Now, following the proof of the assertion (6) $\Rightarrow$ (3), we obtain that the spectral radius $\rho(K_{[\nu, \mu]})$ is less than one.

Lemma 3.1 is proved. $\blacksquare$
We continue the proof of the assertion (3) \(\Rightarrow\) (1).
Let us assume the contrary. Then there exist \(\nu\) and \(\mu\) \((\nu < \mu)\) such that \(C(\nu, \mu) = 0\). In this case \(u(t) = C(t, \nu)\) is a characteristic function of the operator \(K_{\nu\mu}\) and Lemma 3.1 implies that \(\rho(K_{\nu\mu}) < 1\).

(1) \(\Rightarrow\) (5). The periodic problems (1.1), (1.2), (1.3), (3.7), are uniquely solvable if and only if \(C(b, 0) \neq 1\). Since \(C(t, s) > 0\) for \(0 \leq s \leq t \leq b\), then obviously \(C(b, 0)\) is also positive. Since \(\beta_1 < 1, \ldots, \beta_k < 1\), \(C(\cdot, 0)\) is nonincreasing and \(C(b, 0) < C(0, 0) = 1\). Now (3.8) implies positivity of \(P(t, s)\).

(5) \(\Rightarrow\) (2). Setting \(t < s\) in (3.8), we obtain that the function \(C(t, 0)\) cannot have a zero on \([0, b]\).

In order to prove (2) \(\Rightarrow\) (6) we set \(v(t) = C(t, 0)\).
Note that the assertion (1) \(\Rightarrow\) (2) is obvious. Theorem 3.1 is proved.

The condition \(p_i \geq 0\) for \(i = 1, \ldots, m\) is essential, as the following examples shows.

**Example 3.1.** For the equation

\[ x'(t) - x(0) = f(t), \quad t \in [0, 1], \]

in the case \(\beta_j = 1, j = 1, \ldots, k\), the assertion (1) does not imply the assertions (3), (4).

**Example 3.2.** For the equation

\[ x'(t) + p(t)x(h(t)) = f(t), \quad t \in [0, \frac{2}{3}], \]
\[ x(\xi) = 0 \quad \text{for} \ \xi < 0, \]
\[ h(t) = \begin{cases} t - 1, & t \in [0, 2], \\ 1, & t \in (2, \frac{2}{3}], \end{cases} \]
\[ p(t) = \begin{cases} -1, & t \in [0, 2], \\ 1, & t \in (2, \frac{2}{3}]. \end{cases} \]

In the case \(\beta_j = 1\) for \(j = 1, \ldots, k\) the assertions (2) and (6) do not imply the assertion (1).

Denote

\[ p_i^+ (t) = \max\{p_i(t), 0\}, \quad p_i^- (t) = \max\{0, -p_i(t)\}. \]
Let $C^+(t,s)$ be the Cauchy function of the equation

$$x'(t) + \sum_{i=1}^{m} p^i(t) x(t - \tau_i(t)) = f(t), \quad t \in [0, +\infty), \quad (3.11)$$

$$x(t_j) = \beta_j x(t_j - 0), \quad j = 1, 2, \ldots, k,$$

$$x(\xi) = 0 \quad \text{for} \ \xi < 0.$$

**Theorem 3.2.** If $C^+(t,s) > 0$ for $0 \leq s \leq t \leq b$, then $C(t,s) \geq C^+(t,s) > 0$ for $0 \leq s \leq t \leq b$.

**Proof.** Using the substitution

$$x(t) = \int_{0}^{t} C^+(t,s) z(s) \, ds,$$

we obtain the following equation in the space $L$,

$$z(t) - (Hz)(t) = f(t), \quad t \in [0, b],$$

where the operator $H: L \rightarrow L$ is defined by the equality

$$(Hz)(t) = \sum_{i=1}^{m} p^i(t) \chi(t, t)(t - \tau_i(t)) \int_{0}^{t} C^+(t, t - \tau_i(t)) z(s) \, ds.$$

The spectral radius of the Volterra operator $H: L \rightarrow L$ is equal to zero (see, for example, [10]). Using the Neumann series, we obtain

$$z = (I - H)^{-1} f = (I + H + H^2 + \cdots) f$$

and

$$x(t) = \int_{0}^{t} C^+(t,s) [(I + H + H^2 + \cdots) f] z(s) \, ds.$$

The positivity of $C^+(t,s)$ and of the operator $H$ implies that $C(t,s) \geq C^+(t,s) > 0$.

**Remark 3.1.** Obviously, letting $b$ tend to infinity, we can obtain, with the help of Theorem 3.1 and Theorem 3.2, the properties of $C(t,s)$ and nonoscillation of the solutions on the infinite interval $[0, +\infty)$.

Denote

$$d_-(t) = \min\{j: t_j \in (t - \tau_i(t), t]\},$$

$$d_+(t) = \max\{j: t_j \in (t - \tau_i(t), t]\},$$

$$B(t) = \prod_{j=d_-(t)}^{d_+(t)} \beta_j.$$
**Corollary 3.1.** Let \( m = 1 \) and
\[
\int_{t - \tau(t)}^{t} p^+_1(s) \, ds \leq \frac{1 + \ln B(t)}{e},
\]  
(3.12)
for almost all \( t \in [0, +\infty) \). Then the nontrivial solutions of the homogeneous equations (1.8), (1.2), (1.3) have no zero on \([0, +\infty)\).

Note that inequality (3.12) cannot be improved [7, 8, 13].

**Proof.** In order to prove the inequality \( C^+(t, s) > 0 \) for \( 0 \leq s \leq t < +\infty \), we set
\[
v(t) = \begin{cases}
\exp(-e\int_{0}^{t} p^+(s) \, ds), & 0 \leq t \leq t_1, \\
\beta_1 \exp(-e\int_{0}^{t} p^+(s) \, ds), & t_1 \leq t \leq t_2, \\
\vdots & \\
\beta_1 \beta_2 \cdots \beta_k \exp(-e\int_{0}^{t} p^+(s) \, ds), & t_k \leq t \leq t_{k+1}, \\
\vdots
\end{cases}
\]  
(3.13)

Theorem 3.2 implies that \( C(t, s) \geq C^+(t, s) > 0 \) for \( 0 \leq s \leq t < +\infty \). Obviously, that nontrivial solution \( x(t) = C(t, 0) \) is positive on \([0, +\infty)\). The corollary has been proved, since all nontrivial solutions to (1.8), (1.2), (1.3) are proportional.

**Remark 3.2.** From the proof of Corollary 3.1 it is clear that inequality (3.12) implies existence of a nonoscillating solution to Eqs. (1.4), (1.3), (1.5), considered by Gopalsamy and Zhang [9]. On the other hand, their result (see inequalities (1.6), (1.7)) obtains new applications. For example, inequalities (1.6), (1.7) guarantee that the Cauchy function \( C(t, s) \) is positive.

**Remark 3.3.** The case \( \beta_j > 1 \) for all \( j \) impulses can improve nonoscillation properties of solutions.

Denote
\[
\Delta = \sup_j (t_{j+1} - t_j), \quad \beta = \inf_j \beta_j, \quad h = \ln \beta.
\]

**Corollary 3.2.** Let \( m = 1 \) and let there exist a natural \( k \) such that
\[
\tau_1 > k\Delta, \quad \int_{t - \tau(t)}^{t} p^+_1(s) \, ds \leq \frac{1 + kh}{e}, \quad t \in [0, +\infty).
\]
Then the homogeneous equation (1.8), (1.2), (1.3) has a nonoscillating solution.
Consider the equations (1.1), (1.2), and
\[ x(t_j) = \beta_j x(t_j - 0) + \alpha_j, \quad j = 1, 2, \ldots \] (3.14)

**Theorem 3.3.** Let \( m = 1, \alpha_j \geq 0 \) for \( j = 1, 2, \ldots \), and equality (3.12) be fulfilled. Then the inequalities \( f(t) \geq 0 \) for \( t \in [0, +\infty) \) and \( x(0) > 0 \) imply that \( x(t) > 0 \) for \( t \in [0, +\infty) \).

**Proof.** The proof of this theorem is based on the following idea:
Let a function \( x \) be a solution of the equations (1.1), (1.2), (3.14), and let \( b_j \) be the first zero of \( x \), i.e., \( x(t) > 0, t \in [0, b_j) \), \( x(b_j) = 0 \). Then there exist \( \gamma_j, j = 1, 2, \ldots \), such that \( x \) is also a solution of the equations (1.1), (1.2), and
\[ x(t_j) = \gamma_j x(t_j - 0), \quad j = 1, 2, \ldots, \] (3.15)
for the interval \([0, b_j)\).

The inequalities \( \alpha_j \geq 0, j = 1, 2, \ldots \), imply that \( \gamma_j \geq \beta_j > 0 \) for \( j = 1, 2, \ldots \).

Obviously, for the equations (1.1), (1.2), (3.15) the conditions of Corollary 3.1 are fulfilled for the interval \([0, b_j)\). Corollary 3.1, Theorem 3.1, and Theorem 3.2 imply that the Cauchy function \( C_\gamma(t, s) \) of the equations (1.1), (1.2), (3.15) is positive for \( 0 \leq s \leq t < +\infty \).

The solution \( x \) of the equations (1.1), (1.2), (3.14) can be represented in the form
\[ x(t) = \int_0^t C_\gamma(t, s) f(s) \, ds + C_\gamma(t, 0). \] (3.16)

Now the inequalities \( C_\gamma(t, s) > 0, f(t) \geq 0, x(0) > 0 \) imply that \( x(t) > 0 \) for \( t \in [0, b_j) \). This contradicts the assumption \( x(b_j) = 0 \).

**Remark 3.4.** It is clear from the proof of Theorem 3.3 that nontrivial solutions of the homogeneous equation (1.8), (1.2), (3.14) have no zero on \([0, +\infty)\).

**REFERENCES**


