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Superapproximation and Commutator Properties of Discrete Orthogonal Projections for Continuous Splines

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This paper builds upon the L_p-stability results for discrete orthogonal projections on the spaces S_h of continuous splines of order r obtained by R. D. Grigorieff and I. H. Sloan in (1998, *Bull. Austral. Math. Soc.* **58**, 307–332). Properties of such projections were proved with a minimum of assumptions on the mesh and on the quadrature rule defining the discrete inner product. The present results, which include superapproximation and commutator properties, are similar to those derived by I. H. Sloan and W. Wendland (1999, *J. Approx. Theory* **97**, 254–281) for smoothest splines on uniform meshes. They are expected to have applications (as in I. H. Sloan and W. Wendland, *Numer. Math.* (1999, **83**, 497–533)) to qualocation methods for non-constant-coefficient boundary integral equations, as well as to the wide range of other numerical methods in which quadrature is used to evaluate L_2-inner products. As a first application, we consider the most basic variable-coefficient boundary integral equation, in which the constant-coefficient operator is the identity. The results are also extended to the case of periodic boundary conditions, in order to allow appplication to boundary integral equations on closed curves.

Key Words: continuous splines; discrete projection; L_p-stability; superapproximation; commutator property; qualocation.

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1. INTRODUCTION

This paper develops superapproximation and commutator properties of discrete orthogonal projections on spaces of continuous splines on an interval, for general meshes. It builds upon stability and convergence results established in [3].

We start in Subsection 1.1 by recalling the general setting, and the notations and the results established in [3]. In Subsection 1.2 we will sketch the main results of this paper and discuss their possible use.

1.1. *General Setting and Previous Results*

For some $L>0$, let the interval $[0, L]$ be partitioned by π_h, defined by

$$\pi_h := \{0 = x_0 < x_1 < \cdots < x_n = L\}, \tag{1}$$

and on this partition define for some $r \geqslant 2$ the spline-space S_h of continuous piecewise polynomials of degree less than or equal to $r-1$ (or equivalently, of order r) relative to π_h, by

$$S_h := \{\psi \in C[0, L] : \psi|_{I_k} \in P_{r-1}, k = 0, ..., n-1\}, \tag{2}$$

where P_d is the space of polynomials of degree less than or equal to d and $I_k := [x_k, x_{k+1}]$ for $k \in \{0, ..., n-1\}$.

To define a discrete inner product $(\cdot,\cdot)_h$ on S_h, which is meant to approximate the standard inner product $(\cdot,\cdot)$ in some sense, we first define a J-point quadrature rule Q on $[0, 1]$ by

$$Qg := \sum_{j=1}^{J} w_j g(\xi_j) \sim \int_0^1 g(x)\, dx, \tag{3}$$

where the weights w_j are positive and the sample-points ξ_j are strictly increasing in $[0, 1]$. This gives rise to a composite quadrature rule on $C[0, L]$ relative to π_h. Explicitly, the composite quadrature rule is

$$Q_h g := \sum_{k=0}^{n-1} h_k \sum_{j=1}^{J} w_j g(x_{k,j}) \sim \int_0^L g(x)\, dx, \tag{4}$$

where $h_k := x_{k+1} - x_k$ denotes the length of I_k, and $x_{k,j} := x_k + h_k \xi_j$ is the position of the j-th sample point in that sub-interval.

The positive semidefinite Hermitian sesquilinear form

$$(f, g)_h := Q_h(f\bar{g}), \; f, g \in C[0, L], \tag{5}$$

is an inner product on S_h if and only if $J \geqslant r$, see [3], which we will from now on assume to be the case (except in Section 5, where we consider the periodic case). Since the quadrature rule Q_h samples only at discrete points, $(f, g)_h$ is also defined for $f, g \in l_h$, the space of complex-valued grid functions on the set of points $\{x_{k,j}\}$. Derived from this inner product is the discrete orthogonal projection $R_h: C[0, L] \to S_h$, or $R_h: l_h \to S_h$, defined as follows:

$$R_h f \in S_h, \qquad (R_h f, \psi_h)_h = (f, \psi_h)_h \qquad \text{for all} \quad \psi_h \in S_h. \tag{6}$$

Sufficient conditions are known for the family of projections $\{R_h\}$ to be *p-stable*, a property that involves the norms $|\cdot|_{h,p}$ on l_h, defined as discrete counterparts of the p-norms $\|\cdot\|_p$ on $L_p(0, L)$ by

$$|f|_{h,p} := Q_h(|f|^p)^{1/p} \qquad \text{for} \quad p \in [1, \infty),$$

and

$$|f|_{h,\infty} := \max_{k,j} \{|f(x_{k,j})|\}. \tag{7}$$

On $C[0, L]$ these are only semi-norms, but when restricted to S_h they are indeed norms equivalent to the p-norms (see Lemma 2.2 of this paper).

The p-stability of $\{R_h\}$, which, for example, is known (see [3]) to hold for arbitrary $p \in [1, \infty]$ if Q is symmetric or if $\xi_0 = 0$, $\xi_J = 1$ and $J = r$, is the property that

$$\|R_h f\|_p \leqslant C\, |f|_{h,p}. \tag{8}$$

As usual, C is a positive constant independent of parameters of interest such as $h := \max_j h_j$ (and in this particular case also of p), that can take different values in different expressions. Other sufficient conditions are given in Proposition 3.3 of [3].

Remark 1.1. In the special case that $\xi_0 = 0$, $\xi_J = 1$ and $J = r$, the operator R_h is an interpolatory operator, i.e. $R_h f(x_{k,j}) = f(x_{k,j})$ $\forall k, j$, simplifying most of the coming analysis considerably.

The importance of p-stability lies in the fact, following from (8) together with $R_h \psi_h = \psi_h$ for arbitrary $\psi_h \in S_h$, that

$$\begin{aligned} \|R_h f - f\|_p &\leqslant \|R_h(f - \psi_h)\|_p + \|f - \psi_h\|_p \\ &\leqslant C\, |f - \psi_h|_{h,p} + \|f - \psi_h\|_p. \end{aligned} \tag{9}$$

This transforms the question of finding *a priori* L_p-bounds for $R_h f$, seen as approximations to f, into one of approximation theory. The resulting theorem, which is a version of Theorem 5.1 in [3] slightly adapted for our needs, is as follows.

THEOREM 1.1. *Let* $p \in [1, \infty]$, $\ell \in \{1, ..., r\}$ *and* $f \in W_p^1(0, L)$ *such that* $f|_{I_k} \in W_p^\ell(I_k)$, $k = 0, ..., n-1$. *Suppose that* $\{R_h\}$ *is p-stable. Then we have, if* $p \in [1, \infty)$,

$$|R_h f - f|_{h,p} + \|R_h f - f\|_p \leqslant C \left(\sum_{k=0}^{n-1} h_k^{\ell p} \|f^{(\ell)}\|_{L_p(I_k)}^p \right)^{1/p},$$

and, if $p = \infty$,

$$|R_h f - f|_{h,\infty} + \|R_h f - f\|_\infty \leqslant C \max_k \{h_k^\ell \|f^{(\ell)}\|_{L_\infty(I_k)}\}. \tag{10}$$

Other major issues covered in [3] are the approximation properties that derive Theorem 1.1 from (9), and $W_p^1(0, L)$-stability of the discrete projection, as well as similar properties of discrete projection on the subspace of S_h of functions satisfying homogeneous Dirichlet boundary conditions.

1.2. *Outline of This Paper*

In the following, we shall constantly use p-stability of $\{R_h\}$. We shall prove properties for R_h that are known to be valid for its continuous counterpart, which is the L_2-orthogonal projection P_h on S_h. An operator of central importance in the coming results is the operator G of multiplication with a fixed function $g \in W_\infty^r(0, L)$ defined by

$$G: L_p(0, L) \to L_p(0, L): v \mapsto gv. \tag{11}$$

It should be noted that operators may be L_2-adjoints but not adjoints in the discrete sense, and *vice versa*. Thus P_h is self-adjoint in the L_2-sense but not in the discrete sense, since $(P_h f, v)_h \neq (f, P_h v)_h$ in general. In contrast, R_h is self-adjoint in the discrete sense, since $(R_h f, v)_h = (R_h f, R_h v)_h = (f, R_h v)_h$, but is not self-adjoint in the L_2-sense. The operator G above is an exceptional example that is self-adjoint in both senses. Similarly, integration by parts is impossible in the discrete case, whereas in the continuous case it is often carried out without thought. Note also the two statements in our main theorem, Theorem 1.2 below: it is not the case that one is the adjoint of the other.

THEOREM 1.2 (Superapproximation). *Let* $p \in [1, \infty]$ *and assume that* $\{R_h\}$ *is p-stable. Then, for all* $f \in C[0, L]$,

$$|(I - R_h) G R_h f|_{h,p} + \|(I - R_h) G R_h f\|_p \leqslant Ch \|g'\|_{r-1,\infty} |R_h f|_{h,p}. \tag{12}$$

If $\{R_h\}$ *is also q-stable, where* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$|R_h G(I-R_h) f|_{h,p} + \|R_h G(I-R_h) f\|_p \leqslant Ch \|g'\|_{r-1,\infty} |(I-R_h) f|_{h,p}. \tag{13}$$

Remark 1.2. The smoothness requirements on the multiplier g can be weakened in certain situations. Suppose $g \in W^1_\infty(0, L)$, and that $g \in W^r_\infty(0, M)$ and $g \in W^r_\infty(M, L)$ for some $M \in (0, L)$. Then it will become clear from the proof that the theorem still holds with $\|g'\|_{r-1,\infty}$ now understood in the appropriate piecewise sense (as too does the later Theorem 3.3), provided that the partition π_h is chosen so that M is always a point of π_h. Similar remarks hold if there is a finite set $M_1, ..., M_\ell$ where the smoothness of g fails.

Each of the two operators in the above theorem, $(I-R_h)\, GR_h$ and $R_h G(I-R_h)$, is the other's adjoint with respect to $(\cdot,\cdot)_h$, but not with respect to $(\cdot,\cdot)$. The name *superapproximation* is attached to the theorem because of the factor h on the right, which does not appear if we use merely *ad hoc* bounds like the submultiplicative property. We observe that if (12) holds, then it holds equally with $R_h f$ replaced by $\psi_h \in S_h$, since $R_h \psi_h = \psi_h$. The theorem is proved in Section 3.1.

Note that $R_h G(I-R_h)=0$ for the special case of an interpolatory operator R_h, because in this case

$$(GR_h f)(x_{k,j}) = g(x_{k,j})(R_h f)(x_{k,j}) = g(x_{k,j}) f(x_{k,j}) = (Gf)(x_{k,j})$$

for all $f \in C[0, L]$ and $0 \leqslant k \leqslant n-1$, $1 \leqslant j \leqslant J$. In general $R_h G(I-R_h) \neq 0$, but the second part of the theorem shows that this operator is nevertheless "supersmall".

Corollary 1.3 (Commutator Property). *Let* $p \in [1, \infty]$ *and assume that* $\{R_h\}$ *is p-stable and q-stable, where* $\frac{1}{p}+\frac{1}{q}=1$. *Then for all* $f \in C[0, L]$,

$$|(GR_h - R_h G) f|_{h,p} + \|(GR_h - R_h G) f\|_p \leqslant Ch \|g'\|_{r-1,\infty} |f|_{h,p}. \tag{14}$$

Moreover, this commutator property holds for fixed $p \in [1, \infty]$ *and all* $f \in C[0, L]$ *if and only if* (12) *and* (13) *hold.*

Proof. The commutator can be written as

$$GR_h - R_h G = (I-R_h)\, GR_h - R_h G(I-R_h), \tag{15}$$

where the two terms in the right-hand side are exactly the ones that appear in the left-hand sides of (12) and (13). After taking the proper norms and applying the triangle inequality, the commutator property follows from

Theorem 1.2 and the p-stability property (8), combined with Lemma 2.2 below.

Conversely, let the commutator property hold, i.e. assume that (14) holds for all $f \in C[0, L]$. Then in particular it holds for $R_h f$ as well as for $(I - R_h) f$, which are both in $C[0, L]$, proving (12) and (13) respectively. ∎

Remark 1.3. The functions $(I - R_h)\, GR_h f$ and $R_h G(I - R_h) f$ are mutually $(\cdot, \cdot)_h$-orthogonal for arbitrary $f \in C[0, L]$. Thus the representation above implies

$$|(GR_h - R_h G) f|_{h,2}^2 = |(I - R_h)\, GR_h f|_{h,2}^2 + |R_h G(I - R_h) f|_{h,2}^2. \quad (16)$$

A similar commutator property was proved in [7] for smoothest periodic splines on uniform grids for several Sobolev norms. It was then used as a powerful tool in the proof of stability of qualocation methods for elliptic boundary integral equations in [8], starting from the previously proved stability properties for constant-coefficient equations in [6]. In practice, though, one would often prefer to use splines of lower smoothness (like the continuous splines of this paper), and to allow arbitrary meshes, in order to have greater flexibility and to allow local refinement. For this reason, we suggest that the results of this paper may be useful in extending the existing theory.

After proving some basic properties of discrete inner products and projections in Section 2, we will prove Theorem 1.2 in Section 3. In that section we will also present, in Theorem 3.3, a superapproximation result that holds for the derivatives of the functions involved. Unfortunately, we are not able to prove the corresponding "dual" result, nor the commutator property in this setting.

In Section 4 we show an application of Theorem 1.2, by discretising the operator G using the qualocation method. The operator G can be thought of as the simplest possible boundary integral operator (namely the identity) multiplied by a variable coefficient. At the same time the theory can be interpreted as giving stability and convergence proofs for projection with respect to a discrete weighted inner product.

Section 5 extends the results of [3], and the superapproximation results in Theorem 1.2, to the case of periodic boundary conditions, a setting that arises naturally for boundary integral equations on closed curves.

2. FURTHER PROPERTIES OF DISCRETE PROJECTIONS

In this section we shall prove some basic properties of discrete projections in relation to the norms defined earlier. Some of them are trivial, and are stated without proof.

LEMMA 2.1. *For all* $f, g \in C[0, L]$,

(1) $|(f, g)_h| \leqslant |f|_{h,p} |g|_{h,q}$, *for all* $p, q \in [1, \infty]$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$

(2) $(R_h f, g)_h = (f, R_h g)_h$.

LEMMA 2.2. *Let* $p \in [1, \infty]$. *Then there exists a positive constant* C *independent of* h, *but which may depend on* r, *such that for all* $\psi_h \in S_h$,

$$C^{-1} \|\psi_h\|_p \leqslant |\psi_h|_{h,p} \leqslant C \|\psi_h\|_p . \tag{17}$$

Proof. Let $p \in [1, \infty)$, and let $\psi_h \in S_h$ be given. Since we assumed the number J of quadrature points to be greater than or equal to r, $f \mapsto Q(|f|^p)^{1/p}$ defines a norm on the space P_{r-1} of polynomials of degree less than or equal to $r-1$ on the unit interval. Since all norms on a finite dimensional space are equivalent, there exists a positive constant C such that for all $\phi \in P_{r-1}$,

$$C^{-1} \|\phi\|^p_{L_p(0,1)} \leqslant Q(|\phi|^p) \leqslant C \|\phi\|^p_{L_p(0,1)} . \tag{18}$$

For each k, substitute $\phi(x) := \psi_h(x_k + h_k x)$, $x \in [0, 1]$, and multiply the resulting inequality by h_k. Summing over k and taking the p-th root derives the estimate. A similar argument gives the proof for $p = \infty$. ▮

The basis of the second bound (13) in Theorem 1.2 lies in the discrete duality property in the following Theorem 2.3, and in particular in Corollary 2.4.

THEOREM 2.3 (Discrete duality property). *For* $p \in [1, \infty]$ *we have, with* $\frac{1}{p} + \frac{1}{q} = 1$, *that*

$$\forall \chi_h \in S_h, \qquad |\chi_h|_{h,p} \leqslant |R_h|_q \sup_{0 \neq \psi_h \in S_h} \frac{|(\chi_h, \psi_h)_h|}{|\psi_h|_{h,q}}, \tag{19}$$

where

$$|R_h|_q := \sup_{|f|_{h,q} \leqslant 1} |R_h f|_{h,q} . \tag{20}$$

Note that the norm in this definition necessarily exists as a finite number.

Proof. As in the standard duality theory of weighted l_p-spaces, it can easily be shown that

$$|\chi|_{h,p} = \sup_{0 \neq \psi \in l_h} \frac{|(\chi, \psi)_h|}{|\psi|_{h,q}} \qquad \forall \chi \in l_h . \tag{21}$$

Thus

$$\sup_{|\psi_h|_{h,q}\leqslant 1} |(\chi_h, \psi_h)_h| = \sup_{|R_h\psi|_{h,q}\leqslant 1} |(\chi_h, R_h\psi)_h| = \sup_{|R_h\psi|_{h,q}\leqslant 1} |(\chi_h, \psi)_h|$$

$$\geqslant \sup_{|R_h|_q\,|\psi|_{h,q}\leqslant 1} |(\chi_h, \psi)_h| = \frac{1}{|R_h|_q}\,|\chi_h|_{h,p}\,.$$

This proves the theorem. ∎

Corollary 2.4. *Let $p \in [1, \infty]$ and assume that $\{R_h\}$ is q-stable, where $\frac{1}{p}+\frac{1}{q}=1$. Then there exists a positive constant C (independent of h) such that*

$$\forall \chi_h \in S_h,\ C\,|\chi_h|_{h,p} \leqslant \sup_{0\neq\psi_h\in S_h} \frac{|(\chi_h, \psi_h)_h|}{|\psi_h|_{h,q}}. \tag{22}$$

Proof. This follows from Theorem 2.3 and the q-stability of $\{R_h\}$, together with Lemma 2.2. ∎

3. MAIN RESULTS

In this section we will present our main results. First, in Section 3.1 we will prove the Superapproximation Theorem stated as Theorem 1.2 in the Introduction. As remarked, this also proves the commutator property in Corollary 1.3. Second, in Section 3.2 we will present some additional results in $W_p^1(0, L)$. Indeed, since we are working with continuous splines, their weak derivatives are well-defined, and bounds on those derivatives could turn out useful in practical applications.

3.1. *Proof of the Superapproximation Theorem*

We shall now prove Theorem 1.2. The first statement (12) is the easier to prove, and follows the model of the continuous case. It does not need the application of Corollary 2.4.

Proof of Theorem 1.2. Let $p \in [1, \infty)$. Taking $\ell = r$ in Theorem 1.1, we obtain

$$|(I-R_h)\,GR_h f|_{h,p} + \|(I-R_h)\,GR_h f\|_p$$

$$\leqslant C\left(\sum_{k=0}^{n-1} h_k^{rp}\,\|(gR_h f)^{(r)}\|_{L_p(I_k)}^p\right)^{1/p}. \tag{23}$$

Applying Leibniz's rule for differentiation of products, and the inverse inequality $\|\psi_h^{(\ell)}\|_{L_p(I_k)} \leqslant C h_k^{-\ell} \|\psi_h\|_{L_p(I_k)}$, where $\psi_h = R_h f$, we obtain the extra factor h by the fact that locally $\psi_h^{(r)} = 0$, as follows:

$$\begin{aligned}
\|(gR_h f)^{(r)}\|_{L_p(I_k)} &\leqslant C \sum_{m=1}^{r} \|g^{(m)}\|_{L_\infty(I_k)} \|(R_h f)^{(r-m)}\|_{L_p(I_k)} \\
&\leqslant C \|R_h f\|_{L_p(I_k)} \sum_{m=1}^{r} h_k^{m-r} \|g^{(m)}\|_{L_\infty(I_k)} \\
&\leqslant C h_k^{-r+1} \|R_h f\|_{L_p(I_k)} \|g'\|_{W_\infty^{r-1}(I_k)} \sum_{m=0}^{r-1} h_k^m \\
&\leqslant C h_k^{-r+1} \|R_h f\|_{L_p(I_k)} \|g'\|_{W_\infty^{r-1}(I_k)}. \qquad (24)
\end{aligned}$$

Substitution of this last term into (23) gives

$$|(I-R_h)\, GR_h f|_{h,p} + \|(I-R_h)\, GR_h f\|_p \leqslant Ch \|R_h f\|_p \|g'\|_{r-1,\infty}. \qquad (25)$$

This with Lemma 2.2 proves, for $p \in [1, \infty)$, the first statement of the theorem. Since (24) also holds for $p = \infty$, substituting (24) into (10) with $\ell = r$, gives, together with Lemma 2.2, the proof for $p = \infty$.

For the second part we will make use of Corollary 2.4. Starting with an arbitrary $\psi_h \in S_h$, we use $(\cdot, \cdot)_h$ self-adjointness of R_h and G as well as part (1) of Lemma 2.1 and finally the first part of this theorem to find

$$\begin{aligned}
|(R_h G(I-R_h) f, \psi_h)_h| &= |(G(I-R_h) f, \psi_h)_h| = |((I-R_h) f, G\psi_h)_h| \\
&= |((I-R_h) f, (I-R_h)\, G\psi_h)_h| \\
&\leqslant |(I-R_h) f|_{h,p}\, |(I-R_h)\, GR_h \psi_h|_{h,q} \\
&\leqslant Ch \|g'\|_{r-1,\infty}\, |(I-R_h) f|_{h,p}\, |\psi_h|_{h,q}. \qquad (26)
\end{aligned}$$

Application of Corollary 2.4 gives

$$|R_h G(I-R_h) f|_{h,p} \leqslant Ch \|g'\|_{r-1,\infty}\, |(I-R_h) f|_{h,p},$$

and the result now follows with Lemma 2.2. ▮

3.2. *Bounds for the Derivatives*

We will now prove a bound similar to the first bound in Theorem 1.1, but for the first derivative.

In this subsection we need to evaluate $|g|_{h,p}$, and hence $Q_h g$ from (4), for piecewise continuous functions g with possible jumps at the breakpoints x_k of π_h. There arises some ambiguity if $\xi_1 = 0$ and $\xi_J = 1$ because in this case $x_{k,J}$ and $x_{k+1,1}$ coincide geometrically. We use the convention that

$x_{k,J}$ is considered logically different from $x_{k+1,1}$ and define for $k=0, ..., n-2$

$$g(x_{k,J}) := g(x_{k+1}-0) \qquad \text{and} \qquad g(x_{k+1,1}) := g(x_{k+1}+0). \tag{27}$$

First notice that the proof in Lemma 2.2 of the equivalence of the norms $\|\cdot\|_p$ and $|\cdot|_{h,p}$ for the space of continuous piecewise polynomials S_h holds equally for the space of their derivatives, which are discontinuous splines of order $r-1$. The following extension of Lemma 2.2 can be stated.

LEMMA 3.1. *Let $p \in [1, \infty]$ and $m \in \{0, 1\}$. Then there exists a positive constant C such that for all $\psi_h \in S_h$,*

$$C^{-1} \|\psi_h^{(m)}\|_p \leqslant |\psi_h^{(m)}|_{h,p} \leqslant C \|\psi_h^{(m)}\|_p. \tag{28}$$

In the following we need, as in [3], the quantity σ defined by

$$\sigma := \min(Q_1^2, Q_0^2) - |\tau|, \tag{29}$$

where

$$Q_1 := Q(\phi_1^2)^{1/2}, \qquad Q_0 := Q(\phi_0^2)^{1/2} \qquad \text{and} \qquad \tau := Q(\phi_1 \phi_0). \tag{30}$$

Here $\phi_1, \phi_0 \in P_{r-1}$ are the unique polynomials satisfying

$$Q(\phi_j \phi) = 0 \ \ \forall \phi \in P^0_{r-1}, \qquad j = 1, 2, \tag{31}$$

$$\phi_0(0) = 1, \qquad \phi_0(1) = 0, \qquad \phi_1(0) = 0, \qquad \phi_1(1) = 1, \tag{32}$$

where P^0_d is the set of polynomials of degree d or less that vanish at $x=0$ and $x=1$.

The condition $\sigma > 0$ is the only sufficient condition for p-stability that we know of (see [3]), and it is satisfied for the two cases mentioned in the Introduction, i.e. Q either symmetric or as in Remark 1.1.

Under some restrictions on the mesh, we are able to prove the analogue of Theorem 1.1 in Theorem 3.3 below. In that theorem we use for convenience the convention $h_{-1} := h_n := 0$.

DEFINITION 3.2. Let $\gamma \in [0, 1]$ be the infimum over the family of meshes $\{\pi_h\}$ of all numbers $\gamma(\pi_h)$, where each $\gamma(\pi_h) \in [0, 1]$ is the largest number such that

$$\gamma(\pi_h) \leqslant \frac{h_{k-1}}{h_k} \leqslant \frac{1}{\gamma(\pi_h)}, \qquad k = 0, ..., n-1, \tag{33}$$

on the mesh π_h. If $\gamma=0$ then we call the family $\{\pi_h\}$ **unrestricted**. If $\gamma=1$ the family is called **uniform**, while in all other cases it is called **locally quasi-uniform**. The number $\gamma\in[0,1]$ is the corresponding **mesh parameter**.

THEOREM 3.3. *Let $p\in[1,\infty]$. Assume $\sigma>0$, and if $p>1$, assume also that $\{\pi_h\}$ is locally quasi-uniform. Moreover, if $p>1$ and $\tau\neq 0$, define $\rho:=1+\sigma/|\tau|$ and assume that for some positive $\delta<\rho^{p/(p-1)}$ and all $k, j\in\{0,1,...,n\}$,*

$$\frac{h_{k-1}+h_k}{h_{j-1}+h_j}\leqslant C\delta^{|k-j|}. \tag{34}$$

If $f\in W_p^1(0,L)$ such that $f|_{I_k}\in W_p^\ell(I_k)$, $k=0,...,n-1$, for some $\ell\in\{2,...,r\}$ then we have for $p\in[1,\infty)$

$$|(R_h f-f)'|_{h,p}+\|(R_h f-f)'\|_p\leqslant C\left(\sum_{k=0}^{n-1} h_k^{(\ell-1)p}\,\|f^{(\ell)}\|_{L_p(I_k)}^p\right)^{1/p} \tag{35}$$

and for $p=\infty$

$$|(R_h f-f)'|_{h,\infty}+\|(R_h f-f)'\|_\infty\leqslant C\max_k\{h_k^{\ell-1}\,\|f^{(\ell)}\|_{L_\infty(I_k)}\}. \tag{36}$$

Before giving the proof of this theorem, we state two lemmas. The proof of the first can be found in [3], Theorem 6.1, while the proof of the second follows very similar lines to those in the Appendix of [3] and therefore we omit it. In the first result $D_h f(x_{k,j}):=h_k^{-1}f(x_{k,j})$ for $k=0,...,n-1$, $j=1,...,J$.

LEMMA 3.4 [3]. *Under the conditions of Theorem 3.3 we have, for all $f\in l_h$,*

$$\|(R_h f)'\|_p\leqslant C\,|D_h f|_{h,p}. \tag{37}$$

LEMMA 3.5. *Let $p\in[1,\infty]$, $\ell\in\{2,...,r\}$ and $f\in W_p^1(0,L)$ such that $f|_{I_k}\in W_p^\ell(I_k)$, $k=0,...,n-1$. Then there exists a spline $\Pi f\in S_h$ that satisfies, for each $k=0,...,n-1$ and each $m\in\{0,1\}$,*

$$\|(f-\Pi f)^{(m)}\|_{L_\infty(I_k)}\leqslant Ch_k^{\ell-m-1/p}\,\|f^{(\ell)}\|_{L_p(I_k)}. \tag{38}$$

Proof of Theorem 3.3. Let $\Pi f\in S_h$ be as in Lemma 3.5, then by the triangle inequality and Lemma 3.4,

$$\begin{aligned}\|(R_h f-f)'\|_p&\leqslant\|[R_h(f-\Pi f)]'\|_p+\|(f-\Pi f)'\|_p\\&\leqslant C\,|D_h(f-\Pi f)|_{h,p}+\|(f-\Pi f)'\|_p.\end{aligned} \tag{39}$$

The first term in the left-hand side of (35) we can treat similarly, using Lemmas 3.1 and 3.4 to obtain

$$\begin{aligned} |(R_h f-f)'|_{h,\,p} &\leqslant |[R_h(f-\Pi f)]'|_{h,\,p}+|(f-\Pi f)'|_{h,\,p} \\ &\leqslant C\,|D_h(f-\Pi f)|_{h,\,p}+|(f-\Pi f)'|_{h,\,p}. \end{aligned} \tag{40}$$

Therefore, summing gives, restricting ourselves to $p\in[1,\infty)$,

$$\begin{aligned} &|(R_h f-f)'|_{h,\,p}+\|(R_h f-f)'\|_p \\ &\quad\leqslant C\,|D_h(f-\Pi f)|_{h,\,p}+\|(f-\Pi f)'\|_p+|(f-\Pi f)'|_{h,\,p} \\ &\quad= C\left(\sum_{k=0}^{n-1} h_k \sum_{j=1}^{J} w_j h_k^{-p}\,|(f-\Pi f)(x_{k,\,j})|^p\right)^{1/p} \\ &\qquad+\left(\sum_{k=0}^{n-1} \|(f-\Pi f)'\|^p_{L_p(I_k)}\right)^{1/p}+\left(\sum_{k=0}^{n-1} h_k \sum_{j=1}^{J} w_j|(f-\Pi f)'\,(x_{k,\,j})|^p\right)^{1/p} \\ &\quad\leqslant C\left(\sum_{k=0}^{n-1} h_k^{1-p}\,\max_j\,|(f-\Pi f)(x_{k,\,j})|^p\right)^{1/p} \\ &\qquad+\left(\sum_{k=0}^{n-1} h_k\,\|(f-\Pi f)'\|^p_{L_\infty(I_k)}\right)^{1/p}+C\left(\sum_{k=0}^{n-1} h_k\,\max_j\,|(f-\Pi f)'\,(x_{k,\,j})|^p\right)^{1/p} \\ &\quad\leqslant C\left(\sum_{k=0}^{n-1} h_k^{1-p}\,\|f-\Pi f\|^p_{L_\infty(I_k)}\right)^{1/p}+C\left(\sum_{k=0}^{n-1} h_k\,\|(f-\Pi f)'\|^p_{L_\infty(I_k)}\right)^{1/p} \\ &\quad\leqslant C\left(\sum_{k=0}^{n-1} h_k^{(\ell-1)p}\,\|f^{(\ell)}\|^p_{L_p(I_k)}\right)^{1/p}, \end{aligned} \tag{41}$$

where, in the last inequality, we have used Lemma 3.5. For $p=\infty$, the proof is similar. ∎

We are now able to prove the first of the superapproximation results in Theorem 1.2 for the derivatives as well. We omit the proof since it follows the same lines as the first part of the proof of Theorem 1.2 given in Section 3.1.

THEOREM 3.6 (Superapproximation). *Under the assumptions of Theorem* 3.3, *we have for all* $f\in C[0,L]$ *and with* $g\in W^r_\infty(0,L)$,

$$\begin{aligned} &|[(I-R_h)\,GR_h f]'|_{h,\,p}+\|[(I-R_h)\,GR_h f]'\|_p \\ &\qquad\leqslant Ch\,\|g'\|_{r-1,\,\infty}(|R_h f|_{h,\,p}+|(R_h f)'|_{h,\,p}). \end{aligned}$$

An analysis similar to that of the previous section, in order to obtain bounds for the derivative of $R_hG(I-R_h)f$, does not seem to be available.

4. QUALOCATION DISCRETISATION OF THE OPERATOR G

Many spline spaces, among them S_h, have the property that

$$\|(I-P_h)\, GP_h\| \to 0 \qquad \text{for} \quad h \to 0. \tag{42}$$

Here, P_h is the L_2-orthogonal projection on the spline-space, and G is the operator defined in (11). Siegfried Prössdorf remarks in [5] that a very simple but important consequence of this property is that the Galerkin method which discretises the action of multiplication with a function g (with $g(x) \neq 0$ for all x in the domain) is convergent. Here we will show that the same holds if qualocation is used for the approximation, using the superapproximation properties of the discrete projection R_h that we proved in the previous section.

Let $f \in L_2(0, L)$ be given and let $g \in W^r_\infty(0, L)$ be such that there exist constants g_0 and g_1 satisfying

$$0 < g_0 \leqslant g(x) \leqslant g_1. \tag{43}$$

We are interested in studying the qualocation discretisation of the simple equation:

$$\text{Find } u_h \in L_2(0, L) \text{ such that } Gu = f. \tag{44}$$

The standard Galerkin discretisation in the space S_h of continuous piecewise polynomials of degree $r-1$ would be the following:

$$\text{Find } u_h \in S_h \text{ such that } \forall \psi_h \in S_h : (Gu_h, \psi_h) = (f, \psi_h). \tag{45}$$

Unique solvability for the discretisation (45) above follows easily since $a(\cdot, \cdot) := (G\cdot, \cdot)$ defines an inner product on $L_2(0, L)$. We will not pursue the Galerkin formulation (45) further, but turn to its qualocation variant.

The qualocation method for the simple problem (44) is as follows, where we assume now $f \in C[0, L]$ and replace the $L_2(0, L)$-inner product by its discrete counterpart:

$$\text{Find } u_h \in S_h \text{ such that } \forall \psi_h \in S_h : (Gu_h, \psi_h)_h = (f, \psi_h)_h. \tag{46}$$

A unique solution exists since $a_h(\cdot, \cdot) := (G\cdot, \cdot)_h$ defines a weighted discrete inner product on S_h. In the case that $g(x) = g_0$ is a constant function it is clear that $u_h = R_h f/g_0 = R_h u$, and p-stability and convergence follow from the corresponding results for R_h summarised in Theorem 1.1. We will now proceed to prove that the introduction of a non-constant g influences stability and convergence only as a higher order perturbation.

Proposition 4.1. *Let $f \in C[0, L]$ and let $g \in W^r_\infty(0, L)$ satisfy* (43). *For some $p \in [1, \infty]$ assume $\{R_h\}$ to be p-stable and q-stable, where $\frac{1}{p} + \frac{1}{q} = 1$. If $u_h \in S_h$ is the solution of* (46), *then*

$$(1 - Ch \|g\|_\infty \|(g^{-1})'\|_{r-1,\infty}) |u_h - R_h u|_{h,p} \leqslant Ch \|g^{-1}\|_\infty \|g'\|_{r-1,\infty} |u - R_h u|_{h,p} \tag{47}$$

and hence there exists a number $H_0 > 0$ such that for all $h < H_0$

$$|u_h - R_h u|_{h,p} \leqslant Ch |u - R_h u|_{h,p}. \tag{48}$$

Proof. Splitting the term of interest into parts to which the Superapproximation Theorem 1.2 can be applied, we obtain for each $\psi_h \in S_h$

$$\begin{aligned} |(u_h - R_h u, \psi_h)_h| &= |(G(u_h - R_h u), G^{-1}\psi_h)_h| \\ &\leqslant |(G(u_h - R_h u), (I - R_h) G^{-1}\psi_h)_h| \\ &\quad + |(G(u_h - R_h u), R_h G^{-1}\psi_h)_h|. \end{aligned} \tag{49}$$

The first of these two terms can be bounded directly by applying the discrete Hölder inequality, which is part (1) of Lemma 2.1, and then the first superapproximation result (12) in Theorem 1.2, giving

$$\begin{aligned} &|(G(u_h - R_h u), (I - R_h) G^{-1}\psi_h)_h| \\ &\quad \leqslant |G(u_h - R_h u)|_{h,p} |(I - R_h) G^{-1}\psi_h|_{h,q} \\ &\quad \leqslant \|g\|_\infty |u_h - R_h u|_{h,p} \cdot Ch \|(g^{-1})'\|_{r-1,\infty} |\psi_h|_{h,q}. \end{aligned} \tag{50}$$

For the second term in (49) we first use part (2) of Lemma 2.1, then the qualocation orthogonality $R_h G(u - u_h) = 0$ given by (46), then the discrete Hölder inequality again, and finally the second superapproximation result (13) in Theorem 1.2, to obtain,

$$\begin{aligned} &|(G(u_h - R_h u), R_h G^{-1}\psi_h)_h| \\ &\quad = |(R_h G(I - R_h) u, G^{-1}\psi_h)_h| \\ &\quad \leqslant |R_h G(I - R_h) u|_{h,p} |G^{-1}\psi_h|_{h,q} \\ &\quad \leqslant Ch \|g'\|_{r-1,\infty} |(I - R_h) u|_{h,p} \|g^{-1}\|_\infty |\psi_h|_{h,q}. \end{aligned} \tag{51}$$

Combining Corollary 2.4 and (49), (50) and (51) leads to

$$\begin{aligned} |u_h - R_h u|_{h,p} &\leqslant C \sup_{0 \neq \psi_h \in S_h} \frac{|(u_h - R_h u, \psi_h)_h|}{|\psi_h|_{h,q}} \\ &\leqslant Ch \, \|g\|_\infty \, \|(g^{-1})'\|_{r-1,\infty} \, |u_h - R_h u|_{h,p} \\ &\quad + Ch \, \|g'\|_{r-1,\infty} \, \|g^{-1}\|_\infty \, |(I - R_h)\, u|_{h,p}. \end{aligned} \tag{52}$$

The claim now follows easily. ∎

COROLLARY 4.2. *Let the assumptions of Proposition* 4.1 *be satisfied. Then the qualocation method for discretisation of the operator G is convergent, since*

$$|u - u_h|_{h,p} \leqslant (1 + Ch)\, |(I - R_h)\, u|_{h,p}. \tag{53}$$

Finally, we point out that our results can be interpreted as stability and convergence results for projection in the discrete weighted $a_h(\cdot, \cdot)$ inner product. This could be of use in the study of qualocation methods with the spline-space S_h for non-constant coefficient boundary integral equations.

5. PERIODIC BOUNDARY CONDITIONS

For completeness, we consider the theory for the case in which periodic boundary conditions are imposed on both the continuous and discrete function spaces. This setting, encountered when discretising boundary integral equations on closed curves, i.e. when the curve is parametrised onto the unit interval, is not considered in [3]. Since this is a frequent practical application, it is important to analyse this case.

5.1. *The Discrete Inner Product Revisited*

First, we need to reconsider the theory of discrete inner products and discrete projections. It turns out that the periodicity has some consequences for these topics.

Let the periodic counterparts of the spaces $C[0, L]$ and S_h be defined by

$$C^\pi[0,L] := \{f \in C[0, L] : f(0) = f(L)\}, \quad \text{and} \quad S_h^\pi = S_h \cap C^\pi[0, L]. \tag{54}$$

We start with the problem of characterising $(\cdot, \cdot)_h$ as an inner product on S_h^π. In the following ϕ is the polynomial of degree $r-1$ defined by

$$\phi(\xi) := \prod_{j=1}^{r-1} (\xi - \xi_j). \tag{55}$$

Proposition 5.1. *The positive semidefinite Hermitian sesquilinear form $(\cdot, \cdot)_h$ is an inner product on S_h^π if and only if either $J \geqslant r$, or $J = r-1$ and one of the following conditions is satisfied*:

(1) $|\phi(0)| \neq |\phi(1)|$

(2) $\phi(0) = -\phi(1) \neq 0$ *and n is odd.*

Proof. Since $S_h^\pi \subset S_h$, it follows as before that $(\cdot, \cdot)_h$ is an inner product on S_h^π if $J \geqslant r$. Consider now the case $J = r-1$. Clearly, $|\psi_h|_{h,2} = 0$ for $\psi_h \in S_h^\pi$ if and only if

$$\psi_h(x_{k,j}) = 0, \qquad k = 0, ..., n-1, \quad j = 1, ..., r-1, \tag{56}$$

which implies that for each k the function $\psi_h|_{I_k}$ is a scaled multiple of the polynomial ϕ. If $\phi(0) = \phi(1)$, or if $\phi(0) = -\phi(1)$ and n is even, then a non-trivial $\psi_h \in S_h^\pi$ that is zero on all quadrature points can be constructed by glueing continuously together some scaled copies of ϕ. On the other hand, if $|\phi(0)| \neq |\phi(1)|$ the nodal values of scaled copies of ϕ that are being glued together have strictly monotone absolute values, which cannot lead to a ψ_h for which $\psi_h(0) = \psi_h(L)$. A similar argument applies in the case $\phi(0) = -\phi(1) \neq 0$ and odd n. If $J \leqslant r-2$, there exists a polynomial $\chi \in P_{r-1}$ that is zero on the quadrature points and for which $\chi(0) = \chi(1) = \alpha \neq 0$ (namely an interpolant on the two values α and the J values zero), which implies that $(\cdot, \cdot)_h$ is not an inner product by arguments similar to those above. ∎

From the characterisation in Proposition 5.1 one can easily derive conditions for special cases. Some of them are collected in the following corollary.

Corollary 5.2. *Let $J = r-1$. Then $(\cdot, \cdot)_h$ is an inner product if Q*

(1) *is unsymmetric and exactly one of the endpoints $\{0, 1\}$ is a quadrature point*

(2) *is unsymmetric, but symmetric around $\frac{1}{2}(\xi_1 + \xi_J)$*

(3) *is symmetric, with $\xi_1 \neq 0$ and $n(r-1)$ odd.*

It is not an inner product if Q

(4) *is a symmetric rule and r is odd*

(5) *is a symmetric rule and both n and r are even*

(6) *is a rule with $\xi_1 = 0$ and $\xi_J = 1$.*

For $r=2$ and $r=3$, condition (1) in Proposition 5.1 can easily be characterised in terms of the quadrature points ξ_j as in the following proposition. For greater values of r the characterisations are less trivial.

PROPOSITION 5.3. *For $r=2$ and $r=3$, the polynomial ϕ in* (55) *satisfies $|\phi(0)| = |\phi(1)|$ if and only if*

(1) *for $r=2$: $\xi_1 = \frac{1}{2}$*

(2) *for $r=3$: $\xi_1 \in [0, \frac{1}{2})$ and $\xi_1 + \xi_2 = 1$.*

5.2. *The p-Stability in the Periodic Case*

In this subsection we assume that $(\cdot, \cdot)_h$ is an inner product on S_h^π, and define the periodic counterparts R_h^π of R_h by

$$R_h^\pi f \in S_h^\pi, \qquad (R_h^\pi f, \psi_h)_h = (f, \psi_h)_h \qquad \text{for all} \quad \psi_h \in S_h^\pi. \tag{57}$$

Throughout this subsection we assume that the domain of definition of R_h^π is the space of periodic functions $C^\pi[0, L]$, which simplifies some arguments in the sequel. The family of projections $\{R_h^\pi\}$ has similar stability properties to $\{R_h\}$, but there are also some differences. Let us define the periodic spline functions $\{\psi_k\}$, using the functions ϕ_0 and ϕ_1 from (31) and (32), such that

$$\psi_0(x) := \left\{ \begin{array}{ll} \phi_0\left(\dfrac{x - x_0}{h_0}\right), & x \in [x_0, x_1] \\ \phi_1\left(\dfrac{x - x_{n-1}}{h_{n-1}}\right), & x \in [x_{n-1}, x_n] \end{array} \right\}, \tag{58}$$

$$\psi_k(x) := \left\{ \begin{array}{ll} \phi_1\left(\dfrac{x - x_{n-1}}{h_{n-1}}\right), & x \in [x_{k-1}, x_k] \\ \phi_0\left(\dfrac{x - x_k}{h_k}\right), & x \in [x_k, x_{k+1}] \end{array} \right\}, \qquad k = 1, ..., n-1, \tag{59}$$

and zero elsewhere on $[0, 1]$. (Note that ϕ_0 and ϕ_1 are well-defined if $(\cdot, \cdot)_h$ is an inner product, because Proposition 5.1 ensures that in this case there are at least $r-2$ quadrature points in the interior). As in the former case, p-stability can be proved by showing that the scaled Gram matrix $A_h^\pi = (A_{k\ell}^\pi)$, defined by

$$A_{k\ell}^\pi := \frac{(\psi_k, \psi_\ell)_h}{h_{k-1} + h_k}, \qquad k, \ell = 0, ..., n-1, \tag{60}$$

(with h_{-1} now given by $h_{-1} := h_{n-1}$) is uniformly in h strictly row diagonally dominant. By strict row diagonal dominance we mean that there exists $\sigma_0 > 0$ such that

$$\sigma^{\pi}_{h,k} := A^{\pi}_{kk} - \sum_{\ell=0,\, \ell \neq k}^{n-1} |A^{\pi}_{k\ell}| \geqslant \sigma_0, \qquad k = 0, ..., n-1. \tag{61}$$

As in [3], it follows from this property (see in particular Theorem 4.4 of [3]) that R^{π}_h is p-stable for arbitrary $p \in [1, \infty]$.

On observing the essentially tridiagonal nature of the matrix A^{π}_h, it follows that

$$\begin{aligned} \sigma^{\pi}_{h,k} &= \alpha_k Q_1^2 + (1 - \alpha_k)\, Q_0^2 - |\tau| \\ &= \tfrac{1}{2} Q(|\phi_1 - \phi_0 \operatorname{sgn} \tau|^2) + (\alpha_k - \tfrac{1}{2})(Q_1^2 - Q_0^2), \end{aligned} \tag{62}$$

where Q_1, Q_0 and τ are as in (30), and

$$\alpha_k := \frac{h_{k-1}}{h_{k-1} + h_k}, \qquad k = 0, ..., n-1. \tag{63}$$

(As in [3] we define $\operatorname{sgn} t := 1$ or -1 for $t \geqslant 0$ or $t < 0$, respectively.)

If the family of meshes $\{\pi_h\}$ is unrestricted, the α_k can attain any value between zero and one. In this case it follows from (61) and (29), rewritten as

$$\sigma = \min(Q_1^2, Q_0^2) - |\tau| = \tfrac{1}{2} Q(|\phi_1 - \phi_0 \operatorname{sgn} \tau|^2) - \tfrac{1}{2}\, |Q_1^2 - Q_0^2|, \tag{64}$$

that σ is a sharp lower bound for the $\sigma^{\pi}_{h,k}$. Therefore, for the case of unrestricted meshes the condition $\sigma > 0$ we used for the non-periodic case is still the proper sufficient condition to use in the periodic case. The uniform lower bound for the $\sigma^{\pi}_{h,k}$ can, however, be made larger than σ if we impose a mesh condition (see Theorem 5.4 below). This is an important difference from the non-periodic case: in that case the minimum of the corresponding numbers $\sigma_{h,k}$ is always equal to σ because of the fact that $\alpha_0 = 0$ and $\alpha_n = 1$ (the reader might wish to interpret the non-periodic case as a special case of the periodic case in which an interval of length zero forms the connection between the endpoints of the interval). In the periodic case this will sometimes allow us to prove stability for suitably restricted mesh sequences even when $\sigma \leqslant 0$.

THEOREM 5.4. *Let $p \in [1, \infty]$. Let $\gamma \in [0, 1]$ be the mesh parameter of a given family $\{\pi_h\}$ of meshes, and define the number σ_γ by*

$$\sigma_\gamma := \sigma + \frac{\gamma}{1+\gamma} \, |Q_1^2 - Q_0^2|$$

$$= \frac{1}{2} \left[Q(|\phi_1 - \phi_0 \operatorname{sgn} \tau|^2) - \frac{1-\gamma}{1+\gamma} \, |Q_1^2 - Q_0^2| \right]. \tag{65}$$

If $\sigma_\gamma > 0$ then property (61) *holds with $\sigma_0 := \sigma_\gamma$.*

Proof. The assumption on the mesh ratios (33) implies that for all meshes and all k,

$$\frac{\gamma}{1+\gamma} \leqslant \alpha_k \leqslant \frac{1}{1+\gamma}. \tag{66}$$

Using (61) and (64), it holds therefore that

$$\sigma_{h,k}^\pi - \sigma \geqslant \left(\frac{1}{2} - \left| \alpha_k - \frac{1}{2} \right| \right) |Q_1^2 - Q_0^2| \geqslant \frac{\gamma}{1+\gamma} \, |Q_1^2 - Q_0^2|, \tag{67}$$

So σ_γ is indeed a lower bound for the $\sigma_{h,k}^\pi$. The second expression for σ_γ can easily be derived using (64). ∎

COROLLARY 5.5. *The family of projections $\{R_h^\pi\}$ is p-stable for* **unrestricted** *meshes if any one of the following conditions holds*:

(1) *$J \geqslant r$ and $\tau = 0$*

(2) *$J \geqslant r$ and $Q_1 = Q_0$*

(3) *$J \geqslant r$ and Q is symmetric*

(4) *$J \geqslant r$ and Q is a quadrature rule with algebraic degree of precision at least $2r - 2$*

(5) *$J = r$ and $\xi_1 = 0$, $\xi_J = 1$.*

The family $\{R_h^\pi\}$ is p-stable for **locally quasi-uniform** *meshes if any one of the following conditions holds*:

(6) *$J = r - 1$ and* **either** *$\xi_1 = 0$ and $\xi_J < 1$,* **or** *$\xi_1 > 0$ and $\xi_J = 1$*

(7) $J=1$, $r=2$ (*the piecewise linear case*) *and* $0<\min\{\frac{\xi_1}{1-\xi_1}, \frac{1-\xi_1}{\xi_1}\}<\gamma$, $\xi_1 \neq \frac{1}{2}$, *where* γ *is the mesh parameter.*

The family $\{R_h^\pi\}$ *is p-stable for* **uniform** *meshes if any one of the following conditions holds*:

(8) $J \geqslant r$

(9) $J=r-1$ *and the polynomial in* (55) *satisfies* $|\phi(0)| \neq |\phi(1)|$.

Proof. In cases (1)–(5) we show that $\sigma>0$, from which follows, by Theorem 5.4, that (61) is satisfied with $\sigma_0=\sigma$, i.e. with no restriction on the mesh sequence.

First, assume (1) to hold. Then it follows from (64) that

$$\sigma=\min(Q_1^2, Q_0^2)>0,$$

because $\phi \mapsto \sqrt{Q(\phi^2)}$ defines a norm on P_{r-1} (cf. the proof of Lemma 2.2).

Keeping this in mind, assume (2) holds. From the second expression in (64) this implies

$$\sigma=\tfrac{1}{2}Q(|\phi_1-\phi_0 \operatorname{sgn} \tau|^2)=: \sigma_2, \tag{68}$$

which is positive because $\phi_1-\phi_0 \operatorname{sgn} \tau$ is not the zero function since it has the value 1 at the argument 1.

If (3) holds, i.e. if Q is symmetric, then $\phi_1(x)=\phi_0(1-x)$, from which it follows that $Q_1=Q_0$, so that this case is covered by case (2).

If (4) holds then Q integrates exactly $\phi_i\phi$ for $i=1, 2$ and $\phi \in P_{r-1}$, from which it follows that the defining condition for ϕ_i, that $Q(\phi_i\phi)=0$ $\forall \phi \in P_{r-1}^0$, is satisfied with the quadrature sum replaced by an integral. From this and the conditions on ϕ_0 and ϕ_1 at 0 and 1 it follows that $\phi_1(x)=\phi_0(1-x)$, and hence

$$\int_0^1 \phi_1(x)^2\,dx=\int_0^1 \phi_0(x)^2\,dx,$$

implying $Q_1=Q_0$, so that again this case is covered by (2).

In case (5) there are exactly $r-2$ interior quadrature nodes $\xi_2, ..., \xi_{r-1}$. Let $\{\lambda_j: j=2, ..., r-1\} \in P_{r-3}$ be the fundamental Lagrange polynomials for the $r-2$ interior nodes, that is

$$\lambda_j \in P_{r-3},\ \lambda_j(\xi_{j'})=\delta_{jj'},\ j,\ j'=2, ..., r-1.$$

It follows that $x(1-x)\,\lambda_j(x) \in P^0_{r-1}$, thus from the defining condition for ϕ_i we have

$$Q(x(1-x)\,\lambda_j \phi_i)=0, \qquad j=2, ..., r-1, \qquad i=0, 1.$$

Remembering that λ_j is a fundamental Lagrange polynomial, we see that $x(1-x)\,\lambda_j \phi_i$ vanishes at all interior points except ξ_j as well as at both endpoints, so that the last result implies (since $w_j>0$)

$$\phi_i(\xi_j)=0,\ j=2, ..., r-1,\ i=0, 1,$$

thus in this case ϕ_i must vanish at each interior node ξ_j. Since the product $\phi_1\phi_0$ also vanishes at $x=0$ and $x=1$, it follows that $\tau=0$. The result is now obtained from case (1).

For the proof of (6), supposing the first case to hold (i.e. $\xi_1=0$, $\xi_{r-1}<1$) we introduce a related new r-point quadrature rule

$$\tilde{Q}g := w_1(g(0)+g(1)) + \sum_{j=2}^{r-1} w_j\, g(\xi_j). \tag{69}$$

This means that the additional quadrature point $\xi_r := 1$ with weight $w_r := w_1$ is adjoined. With the aid of $\tilde{Q}$ a new discrete inner product $[\,\cdot, \cdot\,]_h$ and related norm $[\,\cdot\,]_{h,p}$ is defined on S^π_h. By virtue of the choice of the weights and the mesh assumption there exists a positive constant C such that the equivalence of norms

$$C^{-1}\,|f|_{h,p} \leqslant [f]_{h,p} \leqslant C\,|f|_{h,p}, \qquad f\in S^\pi_h, \tag{70}$$

holds. Let $\tilde{R}^\pi_h$ denote the discrete projection corresponding to $[\,\cdot, \cdot\,]_h$. It is easy to verify that $\tilde{R}^\pi_h = R^\pi_h$ since both mappings determine the interpolation in S^π_h. The family $\{\tilde{R}^\pi_h\}$ is p-stable according to case (5) and the desired stability property then can be inferred with the aid of (70). The proof of the other case in (6) is similar.

(Interestingly, it is easy to see for case (6) that mesh sequences exist for which the property (61) is **not** achieved for any $\sigma_0>0$. However, the condition in (61) is not necessary, only sufficient, so there is no contradiction).

For case (7) we have $r=2$ and $J=1$, in which case it is easy to see that $\phi_0(x)=1-x$, $\phi_1(x)=x$. Let $\xi := \min\{\xi_1, 1-\xi_1\}$. Then it follows from (64) (if we choose $w_1=1$), that

$$\sigma=\xi^2-\xi(1-\xi)=\xi(2\xi-1)\leqslant 0. \tag{71}$$

Applying Theorem 5.4 gives

$$\begin{aligned}\sigma_\gamma &= \sigma + \frac{\gamma}{1+\gamma}\,|Q_1^2 - Q_0^2| \\ &= \xi(2\xi-1) + \frac{\gamma}{1+\gamma}\,(1-2\xi) \\ &= (1-2\xi)\left(\frac{\gamma}{1+\gamma} - \xi\right). \end{aligned} \tag{72}$$

Clearly, $\sigma_\gamma > 0$ if and only if $\gamma > \frac{\xi}{1-\xi}$. The condition $\xi_1 \neq \frac{1}{2}$ is needed for $(\cdot, \cdot)_h$ to be an inner product on S_h^π.

Now assume (8) holds. For uniform partitions we have $\gamma = 1$, and therefore, combining (64) and (68), we find that $\sigma_\gamma = \sigma_2$, which we have already shown above to be positive if $J \geqslant r$.

In the case (9), in which $J = r-1$ and the partition is uniform, again we have $\gamma = 1$ and $\sigma_\gamma = \sigma_2$. This time the proof that $\sigma_2 > 0$ is indirect, since $\phi \mapsto \sqrt{Q(\phi^2)}$ does not now define a norm on P_{r-1}. Assume $\sigma_2 = 0$. Then by (68) we see that $\chi(\xi) := \phi_1(\xi) - \phi_0(\xi)\,\operatorname{sgn}\ \tau$ satisfies

$$\chi(\xi_j) = 0,\ j = 1, ..., r-1. \tag{73}$$

Since $\chi \in P_{r-1}$, it follows that χ is a nontrivial multiple of ϕ in (55). Moreover, due to (32) χ satisfies $|\chi(0)| = |\chi(1)| = 1$. This contradicts the assumption. ∎

Remark 5.1. All assertions in Corollary 5.5 still hold if as domain of definition of R_h^π the space l_h of grid functions is taken. The proof for this case requires only a slight change in the argument showing part (6).

In the case of $J = r-1$ quadrature points the norm equivalence stated in Lemma 2.2 in general does not hold anymore. Since Lemma 2.2 is used at several places in the course of the paper it is important to have the following

COROLLARY 5.6. *If the family $\{R_h^\pi\}$ is p-stable then there exists a positive constant C such that for all $\psi_h \in S_h^\pi$*

$$C^{-1}\,\|\psi_h\|_p \leqslant |\psi_h|_{h,p} \leqslant C\,\|\psi_h\|_p. \tag{74}$$

Proof. The first inequality is obtained by applying the stability inequality for R_h^π to elements in S_h^π. The second inequality is implied by the corresponding one in (17) because the norm with $r-1$ quadrature points is not larger than one with one additional point. ∎

Remark 5.2. S_h^π is a space of periodic splines with multiple knots. Collocation methods for this kind of spline on uniform grids have been studied in [4]. In our case the multiplicity is $M=r-1$. Explicit conditions for stability are given in [4] for double knots, which corresponds to $r=3$ in our case. Assuming the stability conditions for $\{R_h^\pi\}$ in Theorem 5.4 to hold with $J=r-1$, we have in effect derived the stability of collocation for the operator G in Section for general r and $M=r-1$. In the case $r=2$ and $J=1$ the excluding condition in (1) in Proposition 5.3 is well known.

Remark 5.3. If condition (3) in Corollary 5.2 is assumed to hold then $(\cdot,\cdot)_h$ is an inner product and R_h^π is well-defined. However, the example in [3], Proposition 7.4, can be used to prove that in this case for all $p\in[1,\infty]$ the family $\{R_h^\pi\}$ is not p-stable.

The results in Section 3 are essentially based on the p- and q-stability of $\{R_h\}$. The proofs go through with R_h replaced by R_h^π as long as $(\cdot,\cdot)_h$ is an inner product on S_h^π, which was assumed to be the case in this subsection. Consequently, the superapproximation result (12), the commutator property (14) and Theorem 3.6 are also valid in the periodic case. For completeness we will explicitly state their periodic versions. The multiplier function g in these results is assumed to satisfy the condition

$$g\in W_\infty^r(0,L)\cap C^\pi[0,L]. \tag{75}$$

The periodicity of the function g is needed here, because in the proofs the counterpart of Theorem 1.1 with R_h^π in place of R_h is used which requires the assumption $f\in W_p^1(0,L)\cap C^\pi[0,L]$ to hold.

THEOREM 5.7. *Let $p\in[1,\infty]$ and assume that $\{R_h^\pi\}$ is p-stable. Then, for all $f\in C^\pi[0,L]$,*

$$|(I-R_h^\pi)\,GR_h^\pi f|_{h,p}+\|(I-R_h^\pi)\,GR_h^\pi f\|_p\leqslant Ch\,\|g'\|_{r-1,\infty}\,|R_h^\pi f|_{h,p}. \tag{76}$$

If $\{R_h^\pi\}$ is also q-stable, where $\frac{1}{p}+\frac{1}{q}=1$, then

$$\begin{aligned}|R_h^\pi\,G(I-R_h^\pi)f|_{h,p}&+\|R_h^\pi\,G(I-R_h^\pi)f\|_p\\&\leqslant Ch\,\|g'\|_{r-1,\infty}|(I-R_h^\pi)f|_{h,p}.\end{aligned} \tag{77}$$

COROLLARY 5.8. *Let $p\in[1,\infty]$ and assume that $\{R_h^\pi\}$ is p-stable and q-stable, where $\frac{1}{p}+\frac{1}{q}=1$. Then for all $f\in C^\pi[0,L]$,*

$$\begin{aligned}|(GR_h^\pi-R_h^\pi\,G)f|_{h,p}&+\|(GR_h^\pi-R_h^\pi\,G)f\|_p\\&\leqslant Ch\,\|g'\|_{r\text{-}1,\infty}\,|f|_{h,p}.\end{aligned} \tag{78}$$

THEOREM 5.9. *Let $p \in [1, \infty]$. Let γ be the mesh parameter of the family $\{\pi_h\}$ of meshes and assume that $\sigma_\gamma > 0$. If $p > 1$ assume $\gamma > 0$, and if also $\tau \neq 0$ define $\rho := 1 + \sigma_\gamma / |\tau|$ and assume that for some positive $\delta < \rho^{p/(p-1)}$ and all $k, j \in \{0, 1, ..., n\}$,*

$$\frac{h_{k-1} + h_k}{h_{j-1} + h_j} \leqslant C\delta^{|k-j|}. \tag{79}$$

Then for all $f \in C^\pi[0, L]$ we have

$$|[(I - R_h^\pi)\, GR_h^\pi f]'|_{h,p} + \|[(I - R_h^\pi)\, GR_h^\pi f]'\|_p \leqslant Ch\, \|g'\|_{r-1,\infty}\, (|R_h^\pi f|_{h,p} + |(R_h^\pi f)'|_{h,p}).$$

Remark 5.4. In Theorem 5.7, Corollary 5.8 and in Theorem 5.9 the assumption that $f \in C^\pi[0, L]$ can be relaxed to $f \in C[0, L]$.

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REFERENCES

1. M. Crouzeix and V. Thomée, The stability in L_p and W_p^1 of the L_2-projection onto finite element function spaces, *Math. Comp.* **48** (1987), 521–532.
2. G. A. Chandler and I. H. Sloan, Spline qualocation methods for boundary integral equations, *Numer. Math.* **58** (1990), 537–567.
3. R. D. Grigorieff and I. H. Sloan, Stability of discrete orthogonal projections for continuous splines, *Bull. Austral. Math. Soc.* **58** (1998), 307–332.
4. W. McLean and S. Prössdorf, Boundary element collocation methods using splines with multiple knots, *Numer. Math.* **74** (1996), 419–451.
5. S. Prössdorf, Ein Lokalisierungsprinzip in der Theorie der Spline-Approximationen und einige Anwendungen, *Math. Nachr.* **119** (1984), 239–255.
6. I. H. Sloan and W. Wendland, Qualocation methods for elliptic boundary integral equations, *Numer. Math.* **79** (1998), 451–483.
7. I. H. Sloan and W. Wendland, Commutator properties for periodic splines, *J. Approx. Theory* **97** (1999), 254–281.
8. I. H. Sloan and W. Wendland, Spline qualocation methods for variable-coefficient elliptic equations on curves, *Numer. Math.* **83** (1999), 497–533.