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Classification of C^*-Algebras of Real Rank Zero and Unsuspended E-Equivalence Types

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In this article, examples are given to prove that the graded scaled ordered K-group is *not* the complete invariant for a C^*-algebra in the class of unital separable nuclear C^*-algebras of real rank zero and stable rank one, even for a

$X_{n,i}$ are two-dimensional finite CW complexes and $[n, i]$ are positive integers. (In the case of simple such C^*-algebras, it has been proved that the above invariant is the complete invariant by George Elliott and the author.) These examples prove that the classification conjecture of Elliott for the case of non simple real rank zero C^*-algebras should be revised—one needs extra invariants. The obstruction preventing two such C^*-algebras with the same graded scaled ordered K-group from being isomorphic is that they have different unsuspended E-equivalence types (a refinement of KK-equivalence type of C^*-algebras due to Connes and Higson). In this article, it is proved that for the above class of inductive limit C^*-algebras, the obstruction of unsuspended E-equivalence type is the only obstruction (i.e., if two C^*-algebras in the class are unsuspended E-equivalence, then they are isomorphic). It is a surprise that in the case of simple such C^*-algebras, or even the case of C^*-algebras with finitely many ideals, the obstruction will disappear (see Section 4).

Key Words: classification; ordered K-theory and KK-theory; real rank zero; E-theory.

1. INTRODUCTION

The establishment of Brown Douglas Fillmore theory [BDF, BDF1] and Kasparov KK-theory [K] have significant impact in the study of operator algebras, differential geometry, global analysis (see [BC, BD, BDT, C, C1, CM, CS, Do1, Do2, DHK, DHK1, DW, K1, K2], etc.).

K-theory and K-homology of C^*-algebras with some extra structures become powerful tools for the classification of C^*-algebras (see [Cu, Ell3, PV, Rf]). (For a C^*-algebra A, we call K^*A the K-homology of A, since

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$K^*(C(X)) = K_*(X)$ which is the K-homology of the space X.) We refer the reader to [Bl1] and [Do] for basic theory of K-theory and K-homology.

Comparing with the celebration classification of amenable von-Neumann algebras of Connes, Haagerup, Krieger and Takesaki, Elliott initiated a project with the ambitious goal to be a classification theory for all separable nuclear C^*-algebras. The theory is rapidly growing. Right now there are many beautiful classification results for two special cases: 1, the C^*-algebras are of real rank zero, and, 2, the C^*-algebras are simple. (A survey will appear in Elliott's Lecture on 1994 International congress of mathematicians.)

In [Zh], Zhang proved that the ideal lattice of a C^*-algebra of real rank zero and stable rank one is reflected by the ideal lattice of $K_0(A)$ as an ordered group, hence any ordered isomorphism of K_0 groups of such two C^*-algebras gives an isomorphism of the ideal lattices of the two C^*-algebras. Because of Zhang's result, for quite some time, people believed that the graded ordered K-group with dimension range $(K_*(A), K_*(A)_+, \Sigma_*(A))$ of a C^*-algebra is the complete invariant for the C^*-algebra in the class of separable nuclear C^*-algebras of real rank zero and stable rank one, at least for the C^*-algebra in the class which consists of C^*-algebras being expressed as an inductive limit of $\bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i}))$, where $X_{n,i}$ are finite CW complexes with uniformly bounded dimensions for all n and i. (See [Bl, BK, DN, Ell, Ell1, Ell2, EE, EG, EG1, EGLP, EGLP1, DN, GL, Lin, Lin1, P, Ph, Su], etc.) In particular, if $\dim(X_{n,i}) \leqslant 3$ for all n and i, then it is proved (see [EG1]) that the above-mentioned invariant is the complete invariant for the following two important cases:

(1) the limit algebra A is simple, or

(2) $H^*(X_{n,i})$ are torsion free for all $X_{n,i}$.

In this article, we will construct two non-isomorphic unital C^*-algebras A and B of real rank zero (and therefore of stable rank one, see [EG1]) which are inductive limits of $\bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i}))$ and $\bigoplus_{i=1}^{l_n} M_{\{n,i\}}(C(Y_{n,i}))$, respectively, with $(K_*(A), K_*(A)_+, \mathbf{1}_A) \cong (K_*(B), K_*(B)_+, \mathbf{1}_B)$, where $X_{n,i}$ and $Y_{n,i}$ are 2-dimensional finite CW complexes, $[n,i]$ and $\{n,i\}$ are positive integers. This disproves the conjecture of Elliott (see [Ell] and [Ell1]): the graded ordered K-group with dimensional range is the complete invariant for a separable nuclear C^*-algebra of real rank zero and stable rank one. (This means Elliott's classification conjecture for non simple real rank zero case should be revised—one needs extra invariant.) The obstruction preventing them from being isomorphic is that they have different unsuspended E-equivalence types (the notion of unsuspended E-equivalence type which is called asymptotic isomorphism type in [D] will be introduced in Section 2, also see [CH, CH1, D and DL]). In this article, we also prove the following: If A and B are C^*-algebras of real rank zero which can be expressed as inductive limits of $\bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i}))$ and $\bigoplus_{i=1}^{l_n} M_{\{n,i\}}(C(Y_{n,i}))$, respectively, with $X_{n,i}$, $Y_{n,i}$ 2-dimensional finite CW complexes, then A is isomorphic to B if and only if A is unsuspended E-equivalent to B (i.e., the above obstruction is the only obstruction). As pointed out in 4.24 of [EG1], in the above result, one can replace $\bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i}))$ by $\bigoplus_{i=1}^{k_n} P_{n,i} M_{[n,i]}(C(X_{n,i}))\, P_{n,i}$, and $\bigoplus_{i=1}^{l_n} M_{\{n,i\}}(C(Y_{n,i}))$ by $\bigoplus_{i=1}^{l_n} Q_{n,i} M_{\{n,i\}}(C(X_{n,i}))\, Q_{n,i}$, $P_{n,i}$ and $Q_{n,i}$ are projections in the corresponding C^*-algebras.

Surprisingly, if we further suppose that the algebra A has at most finitely many ideals (or equivalently, the ordered K_0-group has at most finitely many ideals), then the unsuspended E-equivalence type of A is completely determined by its graded scaled ordered K-group. This is also true for some other cases such as when $K_*(A)$ are torsion free. All these results are proved in Section 4. The results in Section 4 have several interesting applications. They can be used to construct examples with certain special properties (See 4.17 and 4.18). Also, applying the result in Section 4, we know that for above non isomorphic C^*-algebras A and B,

$$A \otimes M \cong B \otimes M,$$

for a certain UHF algebras M.

In our counterexample, we have constructed two inductive limit systems $A = \lim(A_n, \phi_{n,m})$ and $B = \lim(B_n, \psi_{n,m})$, with $A_n = B_n$ and $K_* \phi_{n,m} = K_* \psi_{n,m}$, and therefore A and B have same graded scaled ordered K-group. On the other hand, in the construction, one will see that $K^*\phi_{n,m} \neq K^*\psi_{n,m}$. And we will carefully use the differences between $K^*\phi_{n,m}$ and $K^*\psi_{n,m}$ to make different unsuspended E-equivalence types for A and B. (One needs to notice that $K_*(A) = K_*(B)$ implies that $K^*(A) = K^*(B)$, so the limit algebras have same K-homology group too.) It is mysterious why such construction can not be carried out for the case of simple inductive limit C^*-algebras, even for the case of C^*-algebras with finitely many ideals, for which it will be proved that the scaled ordered K-group is the complete invariant (see Section 4)—at these cases, $K^*\phi_{n,m}$ will not give any thing to the limit algebras more than those informations stored in $K_*\phi_{n,m}$, even though at each finite stage, $K^*\phi_{n,m}$ can not be recovered from $K_*\phi_{n,m}$.)

The unsuspended E-equivalence type does not like the invariant in the classical sense (i.e., associate a C^*-algebra with a group, or semi-group, or a number), even though one can regard it as an invariant in abstract sense. We leave the following problem open: find suitable invariant for real rank zero, stable rank one, separable, nuclear C^*-algebras including our examples. For real rank zero inductive limit algebras (with $\dim(X_{n,i}) \leqslant 3$), if we further suppose that A is simple or $K_*(A)$ is torsion free, then the scaled ordered K-group is the complete invariant. Therefore the new invariant should involve ideals of A and the torsion part of $K_*(A)$, and perhaps some extra structure on K-homology group $K^*(A)$. Also the new invariant should be reduced to the graded scaled ordered K-group at the case that the ordered K_0-group of A has finitely many ideals or the case that K_*A are torsion free. We will present a possible invariant at the end of the paper.

In this article, all the CW complexes are assumed to be connected. And also, we call $\bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i}))$ a direct sum of matrices over $X_{n,i}$.

The materials are organized as follows. In Section 2, we will prove the above-mentioned classification result (i.e., classification by using unsuspended E-equivalence type) and give several equivalent conditions. In Section 3, we will give the example to prove that the unsuspended E-equivalence type is *not* completely determined by the graded ordered K-group with dimension range for the above class of C^*-algebras. In Section 4 we will prove that, in several special cases, the unsuspended E-equivalence type of a C^*-algebra is completely determined by its graded scaled ordered K-group (or graded ordered K-group with dimension rage in the non-unital case). And therefore in those special cases, the graded scaled ordered K-group for a C^*-algebra is the complete invariant. The results in Section 4 generalize several main results in [EG1].

We will assume that the readers of this article are familiar with the materials in [EG1]. So we would often refer to [EG1] to avoid the repetition.

2. UNSUSPENDED E-EQUIVALENCE, SHAPE EQUIVALENCE, AND ISOMORPHISM

First we would like to review the construction of the asymptotic homotopy category due to Connes and Higson [CH, CH1], (the notion was given by Dadarlat [D]). We will quote some notation from [CH, CH1, D and DL] (for more details, please see [D]).

Definition 2.1. Let A and B be separable C^*-algebras. An asymptotic homomorphism from A to B is a family of maps $\{\phi_t\}_{t\in[1,\infty)}: A \to B$ such that

(1) For all $a \in A$, the maps $t \to \phi_t(a)$ are continuous, and

(2) For $a \in A$, $b \in A$, and $\lambda \in \mathbb{C}$,

$$\lim_{t \to \infty} \|\phi_t(a + \lambda b) - \phi_t(a) - \lambda \phi_t(b)\| = 0;$$

$$\lim_{t \to \infty} \|\phi_t(ab) - \phi_t(a) \cdot \phi_t(b)\| = 0;$$

$$\lim_{t \to \infty} \|\phi_t(a^*) - (\phi_t(a))^*\| = 0.$$

DEFINITION 2.2. Two asymptotic homomorphisms $\{\phi_t\}$ and $\{\phi'_t\}$ are asymptotically equivalent if

$$\lim_{t \to \infty} \|\phi_t(a) - \phi'_t(a)\| = 0$$

for each $a \in A$.

2.3. Let B be a C^*-algebra. Denote by $C_b([1, +\infty), B)$ the C^*-algebra of all continuous bounded functions from $[1, +\infty)$ to B. Let $C_0([1, +\infty), B)$ be the closed ideal of $C_b([1, +\infty), B)$ which consists of functions vanishing at infinity. Denote

$$C_b([1, +\infty), B)/C_0([1, +\infty), B) \triangleq B_\infty.$$

By using $\limsup \|\phi_t(a)\| \leqslant \|a\|$ ([CH, CH1]), one can prove that an asymptotic homomorphism from A to B induces a homomorphism from A to B_∞, and that two asymptotic homomorphisms from A to B induce the same homomorphisms from A to B_∞ if and only if they are asymptotically equivalent (see [CH, CH1]). The following lemma is useful (see [CH, CH1]).

LEMMA 2.4. *Let $\{\phi_t\}$ and $\{\psi_t\}$ be two asymptotic homomorphisms such that* $\lim_{t \to \infty} \|\phi_t(a) - \psi_t(a)\| = 0$ *for every a in a dense subset of A, then $\{\phi_t\}$ and $\{\psi_t\}$ are asymptotically equivalent.*

2.5. For a C^*-algebra B, let $B[0, 1]$ denote the C^*-algebra $C([0, 1], B) \cong B \otimes C[0, 1]$. Two asymptotic homomorphisms $\{\phi_t\}, \{\psi_t\}: A \to B$ are said to be homotopy equivalent, write as $\{\phi_t\} \sim \{\psi_t\}$, if there is an asymptotic homomorphism $\{\Phi_t\}: A \to B[0, 1]$, such that the restrictions of Φ_t at 0 and 1 are equal to $\{\phi_t\}$ and $\{\psi_t\}$, respectively. Notice that asymptotic equivalence implies homotopy equivalence.

The set of homotopy equivalence classes of asymptotic homomorphisms from A to B is denoted by $[\![A, B]\!]$. The homotopy equivalence class of an asymptotic homomorphism $\{\phi_t\}: A \to B$ is denoted by $[\![\phi_t]\!]$ or $[\![\phi]\!]$. We reserve the notation $[A, B]$ for the set of homotopy equivalence classes of $*$-homomorphisms from A to B.

2.6. Connes and Higson defined the composition of homotopy equivalence classes of asymptotic homomorphisms: $[\![A, B]\!] \times [\![B, C]\!] \to [\![A, C]\!]$, i.e., for any asymptotic homomorphisms $\{\phi_t\}: A \to B$ and $\{\psi_t\}: B \to C$, one can associate a $\{\theta_t\}: A \to C$, and the homotopy equivalence class $[\![\theta_t]\!] (\triangleq [\![\psi_t]\!] \circ [\![\phi_t]\!])$ depends only on the homotopy equivalence classes $[\![\phi_t]\!]$ and $[\![\psi]\!]$. Furthermore, they proved the associativity of the composition.

The following definition of asymptotic homotopy category can be found in [D].

DEFINITION 2.7. The asymptotic homotopy category is defined to be the category whose objects are all the separable C^*-algebras and whose maps are homotopy equivalence classes of asymptotic homomorphisms. Two C^*-algebras A and B are equivalent in the asymptotic homotopy category if there are asymptotic homomorphisms $\{\phi_t\}: A \to B$ and $\{\psi_t\}: B \to A$ such that

$$[\![\psi_t]\!] \circ [\![\phi_t]\!] = [\![id_A]\!] \in [\![A, A]\!] \qquad \text{and} \qquad [\![\phi_t]\!] \circ [\![\psi_t]\!] = [\![id_B]\!] \in [\![B, B]\!].$$

In this circumstance, we also say that A and B are unsuspended E-equivalent to each other. (More precisely, one may prefer to call them unsuspended unstabilized E-equivalent.)

It is evident that the unsuspended E-equivalence type of a C^*-algebra is an isomorphic invariant and a homotopic invariant of the C^*-algebra. An interesting result in [D] shows that the unsuspended E-equivalence type is also a shape equivalent invariant of the C^*-algebra.

2.8. In [CH, CHE1], Connes and Higson defined that $E(A, B) = [\![SA \otimes \mathscr{K}, SB \otimes \mathscr{K}]\!]$, where SA and SB are suspensions of A and B, respectively, and $\mathscr{K}$ is the algebra of compact operators on a separable infinite dimensional Hilbert space. And they proved that $E(A, B) \cong KK(A, B)$, provided that both A and B are K-nuclear C^*-algebras. Using obvious map: $[\![A, B]\!] \to [\![SA \otimes \mathscr{K}, SB \otimes \mathscr{K}]\!] = E(A, B)$, one can prove that unsuspended E-equivalence implies E-equivalence, and therefore implies KK-equivalence for K-nuclear C^*-algebras.

Any asymptotic homomorphism $\{\phi_t\}: A \to B$ gives an element (denoted by $[\![\phi_t]\!]_{KK}$) in $KK(A, B)$. Also $\{\phi_t\}$ induces a group homomorphism $[\![\phi_t]\!]_*: K_*(A) \to K_*(B)$. It can be described as follows. We only describe $[\![\phi_t]\!]_*: K_0(A) \to K_0(B)$ (it is similar to describe the map for K_1). Suppose that A and B are unital. $\{\phi_t\}$ induces asymptotic homomorphisms from $M_n(A)$ to $M_n(B)$ for all positive integers n (still denote them by $\{\phi_t\}$). For any $[p] \in K_0(A)$, represented by a projection $p \in M_n(A)$, one knows that, $\|\phi_t(p)^2 - \phi_t(p)\|$ and $\|\phi_t(p) - \phi_t(p^*)\|$ are very small when t is large enough. Therefore there is a projection $q \in M_n(B)$ with $\|q - \phi_t(p)\|$ small. One can define $[\![\phi_t]\!]_*([p]) = [q] \in K_0(B)$. This definition can be extended easily to all the elements in $K_0(A)$. For the non-unital case, let A^+, B^+ be the C^*-algebras by adjoining the units to A and B, respectively. Then $\{\phi_t\}$ induces a unital asymptotic homomorphism $\{\phi_t^+\}: A^+ \to B^+$. And $\{\phi_t^+\}$ defines $[\![\phi_t^+]\!]_*: K_0(A^+) \to K_0(B^+)$. It is obvious that $[\![\phi_t^+]\!]_*(K_0(A)) \subseteq K_0(B)$, when one regards $K_0(A)$ and $K_0(B)$ as the subsets of $K_0(A^+)$ and $K_0(B^+)$, respectively. $[\![\phi_t]\!]_*$ is defined to be $[\![\phi_t^+]\!]_*|_{K_0(A)}$. It is easy to see that $[\![\phi_t]\!]_*$ takes $K_*(A)_+$ to $K_*(B)_+$, and $\Sigma_*(A)$ to $\Sigma_*(B)$. That is $[\![\phi_t]\!]_{KK} \in KK(A, B)_{+, \Sigma}$. (The notations of $K_*(A)_+$, $\Sigma_*(A)$ and $KK(A, B)_{+, \Sigma}$ can be found in Section 1 of [EG1] which were quoted from [DN] and [Ell].) The following lemma is evident.

LEMMA 2.9. *If an asymptotic homomorphism $\{\phi_t\}: A \to B$ induces an unsuspended E-equivalence between A and B, then its KK-element $[\![\phi_t]\!]_{KK}$ induces an isomorphism between $(K_*(A), K_*(A)_+, \Sigma_*(A))$ and $(K_*(B), K_*(B)_+, \Sigma_*(B))$. Also, any stable unsuspended E-equivalence between $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$ induces an isomorphism between $(K_*(A), K_*(A)_+)$ and $(K_*(B), K_*(B)_+)$.*

LEMMA 2.10. *Suppose that A and B are separable nuclear C^*-algebras of real rank zero and stable rank one. If $\{\phi_t\}: A \to B$ is an asymptotic homomorphism with $[\![\phi_t]\!]_*: K_0(A) \to K_0(B)$ being the zero map, then $\{\phi_t\}$ is asymptotically equivalent to the zero asymptotic homomorphism.*

Proof. Since A and B have the cancellation of projections, $[\![\phi_t]\!]_* = 0$ on $K_0(A)$ implies that $\lim_{t \to \infty} \|\phi_t(p)\| = 0$ for any projection $p \in A$. The lemma follows from the fact that the set of finite linear combinations of projections is a dense subset of A. (Notice that A is of real rank zero.) Q.E.D

2.11. If A is of real rank zero and stable rank one, we know that $(K_*(A), K_*(A)_+)$ forms a graded ordered group. (For the definition of $K_*(A)_+$ we refer the readers to [Ell] and [DN].) We call an ordered subgroup $H (\subset G)$ of an ordered group (G, G_+) an ideal if it satisfies the following: $x \geqslant y \geqslant 0$ and $x \in H$ implies $y \in H$. As pointed out in [Ell] and [EG1], the ideal structure of $(K_*(A), K_*(A)_+)$ are completely determined by the ideal structure of $(K_0(A), K_0(A)_+)$. Also, the ideals of A are one-to-one corresponding to the ideals of $(K_0(A), K_0(A)_+)$. The correspondence is defined by sending $I \subset A$ to $K_0(I) \subset K_0(A)$. (Notice that the ideal I can be recovered from $K_0(I) (\subset K_0(A))$ as it is generated by projections $p \in A$ with $[p] \in K_0(I)$.) Based on the above, one knows that if I is an ideal of A and J is an ideal of I, then J is an ideal of A. We will make use of this fact several times.

In the rest of this section, we will always suppose that the C^*-algebras are separable, nuclear, of real rank zero, and of stable rank one (except in Remark 2.22).

2.12. Assume that $(K_*(A), K_*(A)_+, \Sigma_*(A))$ and $(K_*(B), K_*(B)_+, \Sigma_*(B))$ are isomorphic to each other. Then for any isomorphism between them, there is a KK-equivalence $\alpha \in KK(A, B)_{+, \Sigma}$ with inverse $\beta \in KK(B, A)_{+, \Sigma}$ which induces the isomorphism. As in 2.11, any ideal I of A corresponds to an ideal $K_0(I)$ of $K_0(A)$. Similarly, $\alpha_*(K_0(I)) \subset K_0(B)$ is an ideal of $K_0(B)$ which corresponds to an ideal $J \subset B$. In such a way, every ideal I of A corresponds to an ideal J of B. We call such a pair (I, J) a corresponding pair of ideals (under α or α_*). We say that the KK-equivalence α (or β) keeps the ideals if

$$i_{(A, I)} \times \alpha \times \pi_{(B, J)} = 0 \in KK(I, B/J)$$

and

$$i_{(B, J)} \times \beta \times \pi_{(A, I)} = 0 \in KK(J, A/I)$$

for any corresponding pair of ideals (I, J), where $i_{(A, I)} \in KK(I, A)$ and $i_{(B, J)} \in KK(J, B)$ are induced by the inclusion maps, and $\pi_{(A, I)} \in KK(A, A/I)$ and $\pi_{(B, J)} \in KK(B, B/J)$ are induced by quotient maps. We will use the notations $i_{(A, I)}$ and $\pi_{(A, I)}$ throughout this paper.

The following lemma follows from 2.10.

Lemma 2.13. *Suppose that A and B are separable nuclear C^*-algebras of real rank zero and stable rank one. Then any unsuspended E-equivalence (asymptotic homomorphism) $\{\phi_t\}: A \to B$ (with inverse $\{\psi_t\}: B \to A$) induces a KK-equivalence $[\![\phi_t]\!]_{KK} \in KK(A, B)_{+, \Sigma}$ (with inverse $[\![\psi_t]\!]_{KK} \in KK(B, A)_{+, \Sigma}$) which keeps the ideals*

We need the following definition (see [EfK]).

Definition 2.14. Two inductive limit systems $A = \lim (A_n, \phi_{n, m})$ $B = \lim (B_n, \psi_{n, m})$ are said to be shape equivalent, if there are subsequences $\{k_i\}$, $\{l_i\}$ and $\xi_i: A_{k_i} \to B_{l_i}$ and $\eta_i: B_{l_i} \to A_{k_{i+1}}$ such that

$$\eta_i \circ \xi_i \sim_h \phi_{k_i, k_{i+1}}: A_{k_i} \to A_{k_{i+1}}$$

and

$$\xi_{i+1} \circ \eta_i \sim_h \psi_{l_i, l_{i+1}}: B_{l_i} \to B_{l_{i+1}},$$

where $\sim_h$ means homotopy equivalence between homomorphisms.

This definition of shape equivalence depends on the inductive limit sequences. Actually, the fact that the limit C^*-algebras are isomorphic does not imply that the inductive limit systems are shape equivalent.

2.15. In this section, we will classify the real rank zero inductive limit C^*-algebras of $(A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ by the unsuspended E-equivalence types of the C^*-algebras, where $X_{n,i}$ are two-dimensional finite CW complexes. (As pointed out in Section 4 of [EG1], once the result has been proved, one can generalize it to the case of $A_n = \bigoplus_{i=1}^{k_n} P_{n,i} M_{[n,i]}(C(X_{n,i}))\, P_{n,i}$.) In the rest of this section, we will suppose that A and B are the C^*-algebras of the above form. Without loss of generality, we will also assume that all connecting homomorphisms $\phi_{n,m}$ are unital, and therefore the limit algebras are unital (see 1.2.4 of [EG1]). Furthermore, we can suppose that for any $p \neq 0 \in A_n$, $\phi_{n,m}(p) \neq 0 \in A_m$. (Otherwise, $\phi_{n,m}$ takes the block where p lives on, to zero in A_m, so we can simply delete this block.)

As in [EG1], we will first deal with the case that each space $X_{n,i}$ has finite (or torsion) cohomology group $H^2(X_{n,i})$. That is, we first prove the following theorem.

THEOREM 2.16. *Suppose that A and B are unital real rank zero inductive limits of $(A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ and $(B_n = \bigoplus_{i=1}^{l_n} M_{\{n,i\}}(C(Y_{n,i})), \psi_{n,m})$, respectively, and that $X_{n,i}$ and $Y_{n,i}$ are two-dimensional finite CW complexes with $H^2(X_{n,i})$ and $H^2(Y_{n,i})$ finite. The following are equivalent*:

(1) *A is unsuspended E-equivalent to B*;

(2) *$(K_*(A), K_*(A)_+, \mathbf{1}_A)$ is isomorphic to $(K_*(B), K_*(B)_+, \mathbf{1}_B)$, and there is a KK-equivalence $\alpha \in KK(A, B)_{+,\Sigma,\mathbf{1}}$ with inverse $\beta \in KK(B, A)_{+,\Sigma,\mathbf{1}}$ (inducing the isomorphism between $(K_*(A), K_*(A)_+, \mathbf{1}_A)$ and $(K_*(B), K_*(B)_+, \mathbf{1}_B)$) which keeps the ideals in the sense of* 2.12;

(3) *The inductive limit systems $(A_n, \phi_{n,m})$ and $(B_n, \psi_{n,m})$ are shape equivalent*;

(4) *A is isomorphic to B.*

Proof. (1) ⇒ (2) is Lemma 2.13, (3) ⇒ (4) is a special case of Theorem 2.2 in [EG1], and (4) ⇒ (1) is evident. So we need only prove (2) ⇒ (3).

Suppose that α and β are as in (2). By Theorem 2.39 of [EG1] (notice that $\tilde{K}_0(X_{n,i}) = H^2(X_{n,i})$ and $\tilde{K}_0(Y_{n,i}) = H^2(Y_{n,i})$ are finite groups), passing to subsequences $\{k_n\}$, $\{l_n\}$, there is a KK-theory intertwining (see the proof of Theorem 4.7 in [EG1]): $\alpha_n \in KK(A_{k_n}, B_{l_n})_{+,\Sigma,\mathbf{1}}$ and $\beta_n \in KK(B_{l_n}, A_{k_{n+1}})_{+,\Sigma,\mathbf{1}}$ such that the following diagram

$$\begin{array}{ccccccccc}
A_{k_1} & \xrightarrow{\phi_{k_1,k_2}} & A_{k_2} & \xrightarrow{\phi_{k_2,k_3}} & A_{k_3} & \longrightarrow & \cdots & \longrightarrow & A \\
\downarrow{\scriptstyle \alpha_1} & \nearrow{\scriptstyle \beta_1} & \downarrow{\scriptstyle \alpha_2} & \nearrow{\scriptstyle \beta_2} & & & & & {\scriptstyle \alpha}\downarrow\uparrow{\scriptstyle \beta} \\
B_{l_1} & \xrightarrow{\psi_{l_1,l_2}} & B_{l_2} & \xrightarrow{\psi_{l_2,l_3}} & B_{l_3} & \longrightarrow & \cdots & \longrightarrow & B
\end{array}$$

commutes at the level of KK, i.e.,

$$\alpha_n \times \beta_n = [\phi_{k_n, k_{n+1}}] \in KK(A_{k_n}, A_{k_{n+1}}),$$

$$\beta_n \times \alpha_{n+1} = [\psi_{l_n, l_{n+1}}] \in KK(A_{l_n}, A_{l_{n+1}}),$$

$$\alpha_n \times [\psi_{l_n, \infty}] = [\phi_{k_n, \infty}] \times \alpha \in KK(A_{k_n}, B)$$

and

$$\beta_n \times [\phi_{k_{n+1}, \infty}] = [\phi_{l_n, \infty}] \times \beta \in KK(B_{l_n}, A).$$

To save the notations, we suppose $k_n = n$, $l_n = n$. That is, the following diagram commutes at the level of KK:

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{\phi_{1,2}} & A_2 & \xrightarrow{\phi_{2,3}} & A_3 & \longrightarrow & \cdots & \longrightarrow & A \\
\downarrow{\scriptstyle \alpha_1} & \nearrow{\scriptstyle \beta_1} & \downarrow{\scriptstyle \alpha_2} & \nearrow{\scriptstyle \beta_2} & & & & & {\scriptstyle \alpha}\downarrow\uparrow{\scriptstyle \beta} \\
B_1 & \xrightarrow{\psi_{1,2}} & B_2 & \xrightarrow{\psi_{2,3}} & B_3 & \longrightarrow & \cdots & \longrightarrow & B
\end{array}$$

For each ideal $I \subset A$, by $A_n \cap I$ (or $I \cap A_n$), we denote the ideal of A_n generated by those projections (in A_n) whose images under $\phi_{n,\infty}$ are in $I \subset A$. It is easy to see that $A_n \cap I$ consists of several whole blocks of A_n.

(Warning: our definition of $A_n \cap I$ is different from the ordinary one, by which, $A_n \cap I$ was defined to be the collection of all the elements in A_n whose images under $\phi_{n,\infty}$ are in I.)

For each ideal $I \subset A$, since I is generated by the projections in I, we have two inductive limit sequences:

$$A_1 \cap I \xrightarrow{\phi_{1,2,I}} A_2 \cap I \xrightarrow{\phi_{2,3,I}} A_3 \cap I \longrightarrow \cdots \longrightarrow A \cap I (= I)$$

and

$$A_1/A_1 \cap I \xrightarrow{\phi^{\pi}_{1,2,I}} A_2/A_2 \cap I \xrightarrow{\phi^{\pi}_{2,3,I}} A_3/A_3 \cap I \longrightarrow \cdots \longrightarrow A/A \cap I,$$

where $\phi_{n,m,I}$ are the restrictions of $\phi_{n,m}$ on $A_n \cap I$, and $\phi^{\pi}_{n,m,I}$ are the quotient maps of $\phi_{n,m}: A_n \to A_m$ (modulo $A_n \cap I$).

The above notations are also used for B and the ideals J of B (with B in place of A, B_n in place of A_n, J in place of I, and $\psi_{n,m}$ in place of $\phi_{n,m}$).

We are going to prove the following assertion. Notice that $KK(A_n, B_m) = \bigoplus_i \bigoplus_j KK(A_n^i, B_m^j)$. We will use $(\ )^{i,j}$ to denote the component in $KK(A_n^i, B_m^j)$ of the given element in $KK(A_n, B_m)$.

Assertion. For each fixed block A_n^i of A_n, if $m \geqslant n$ large enough, then $\alpha_n \times [\phi_{n,m}] \in KK(A_n, B_m)$ satisfies that for any block B_m^j of B_n, one of the following is true:

(i) $(\alpha_n \times [\psi_{n,m}])^{i,j} \in KK(A_n^i, B_m^j)$ takes $\mathbf{1}_{A_n^i}$ to a strictly positive element in $K_0(B_m^j)$,

(ii) $(\alpha_n \times [\psi_{n,m}])^{i,j} = 0 \in KK(A_n^i, B_m^j)$.

For each block A_n^i of A_n, let $I \subset A$ be the ideal generated by A_n^i (i.e., generated by the images of all elements of A_n^i under $\phi_{n,\infty}$). Then $A_n^i \subseteq A_n \cap I$ (warning: they may not be equal). Let J be the corresponding ideal in B (or equivalently, $K_0(J)$ is the corresponding ideal of $K_0(I)$ under α).

We use $i_{(A_n, I)} \in KK(A_n \cap I, A_n)$ and $i_{(B_n, J)} \in KK(B_n \cap J, B_n)$ to denote the KK elements induced by the inclusions, and $\pi_{(A_n, I)} \in KK(A_n, A_n/A_n \cap I)$, $\pi_{(B_n, j)} \in KK(B_n, B_n/B_n \cap J)$ to denote the KK elements induced by quotient maps. Also let $i_{(A_n^i)} \in KK(A_n^i, A_n)$ and $i_{(B_n^j)} \in KK(B_n^j, B_n)$ denote the inclusions. We need to use the following equations:

$$i_{(A_n, I)} \times [\phi_{n,\infty}] = [\phi_{n,\infty,I}] \times i_{(A,I)} \in KK(A_n \cap I, A)$$

and

$$\pi_{(B_n, J)} \times [\psi^{\pi}_{n,\infty,J}] = [\psi_{n,\infty}] \times \pi_{(B,J)} \in KK(B_n, B/J).$$

By $\alpha_n \times [\psi_{n,\infty}] = [\phi_{n,\infty}] \times \alpha$, we know that

$$\begin{aligned} i_{(A_n, I)} &\times \alpha_n \times \pi_{(B_n, J)} \times [\psi^{\pi}_{n,\infty,J}] \\ &= i_{(A_n, I)} \times \alpha_n \times [\psi_{n,\infty}] \times \pi_{B,J} \\ &= i_{(A_n, I)} \times [\phi_{n,\infty}] \times \alpha \times \pi_{(B,J)} \\ &= [\phi_{n,\infty,I}] \times i_{(A,I)} \times \alpha \times \pi_{(B,J)} \\ &= [\phi_{n,\infty,I}] \times 0 = 0 \in KK(A_n \cap I, B/J). \end{aligned}$$

(Notice that $i_{(A,I)} \times \alpha \times \pi_{(B,J)} = 0$, since α keeps the ideals in the sense of 2.12.)

Therefore there is an m_1 such that

$$i_{(A_n, I)} \times \alpha_n \times \pi_{(B_n, J)} \times [\psi^{\pi}_{n,m_1,J}] = 0.$$

Hence

$$i_{(A_n, I)} \times \alpha_n \times [\psi_{n, m_1}] \times \pi_{(B_{m_1}, J)} = 0. \quad (*)$$

The rest of the proof (proof of the assertion and the theorem) will depend only on the above equation (and the corresponding equation for β_n). The above equation implies that

$$i_{(A_n^i)} \times \alpha_n \times [\psi_{n, m_1}] \times \pi_{(B_{m_1}, J)} = 0.$$

Since $B_{m_1} \cap J$ consists of several whole blocks of B_{m_1}, we can suppose that $B_{m_1} = B^{(1)} \oplus B^{(2)}$, where $B^{(1)} = B_{m_1} \cap J$ and $B^{(2)} = B_{m_1} / B_{m_1} \cap J$. If one chooses m to be m_1, it is clear that for any block $B_{m_1}^j$ inside $B^{(2)}$, (ii) $(\alpha_n \times [\psi_{n, m_1}])^{i, j} = 0$ holds (from the above equation). But it is not clear (even not true) that for any block in $B^{(1)}$, (i) in the assertion holds. We need to choose a larger m ($m \geqslant m_1$) to guarantee (i) of the assertion holds.

Since $\phi_{n, \infty}(A_n^i)$ generates I, we know that $[\phi_{n, \infty}]_* (\mathbf{1}_{A_n^i}) \triangleq x \in K_0(I)$ satisfies the condition that for any $y \in K_0(I)$, there is a positive integer $t \geqslant 0$ with $tx \geqslant y$. Hence, for any $z \in K_0(J)$, there is an integer t with $t\alpha_*(x) \geqslant z$ in $K_0(J)$, where $\alpha_*: K_*(A) \to K_*(B)$ is induced by $\alpha \in KK(A, B)$.

Since $[\psi_{m_1, \infty}]_* (\mathbf{1}_{B^{(1)}}) \in K_0(J)$ and $[\phi_{n, \infty}] \times \alpha = \alpha_n \times [\psi_{n, \infty}]$, there are an $m \geqslant m_1$ and a positive integer t_1 with

$$t_1((\alpha_n \times [\psi_{n, m}])_* (\mathbf{1}_{A_n^i})) \geqslant [\psi_{m_1, m}]_* (\mathbf{1}_{B^{(1)}}).$$

One can write $B_m = B^{(3)} \oplus B^{(4)}$, where $B^{(3)}$ consists of all the blocks B_m^j with the property that there is a $B_{m_1}^{j_1} \subset B^{(1)}$ with $\phi_{m_1, m}^{j_1, j} \neq 0$ (i.e., there is a block in $B^{(1)}$ whose image under $\phi_{m_1, m}$ has non-zero intersection with B_m^j), and $B^{(4)}$ consists of all the other blocks. Then it is obvious that there is a t_2 with

$$t_2([\psi_{m_1, m}])_* (\mathbf{1}_{B^{(1)}}) \geqslant \mathbf{1}_{B^{(3)}}.$$

Hence

$$t_2 \cdot t_1((\alpha_n \times [\psi_{n, m}])_* (\mathbf{1}_{A_n^i})) \geqslant \mathbf{1}_{B^{(3)}}.$$

That is, for each block in $B^{(3)}$, (i) of the assertion holds. Since for each block $B_{m_1}^j$ of $B^{(2)}$, $(\alpha_n \times [\psi_{n, m_1}])^{i, j} = 0$, one can see easily, from the definition of $B^{(4)}$, that for each block in $B^{(4)}$, (ii) of the assertion holds. This proves the assertion.

One needs to notice that, once m satisfies the condition in the assertion, any number larger than m also satisfies the condition (see 1.6.5 and 1.6.6 in [EG1]). So one can choose a common m for all blocks A_n^i of A_n.

By passing to subsequences $\{A_{k_n}\}$, $\{B_{l_n}\}$ (we still denote them by $\{A_n\}$ and $\{B_n\}$, i.e., suppose that $k_n=n$, $l_n=n$), we can assume that $\alpha_n \in KK(A_n, B_n)$ satisfies that for each block $A_n^i(\subset A_n)$ and $B_n^i(\subset B_n)$, either

(i) $\alpha_n^{i,j} \in KK(A_n^i, B_n^j)$ take $\mathbf{1}_{A_n^i}$ to a strictly positive element in $K_0(B_n^j)$, or

(ii) $\alpha_n^{i,j}=0$.

Also we can suppose that β_n satisfies the same condition. Notice that the above condition is exactly the condition (3) of Theorem 3.23 in [EG1]. Since it only involves 2-dimensional finite CW complexes here, the condition (1) and (2) of Theorem 3.23 in [EG1] automatically hold. Applying the theorem (passing to subsequence again), α_n and β_n can be realized by homomorphisms ξ_n, $A_n \to B_n$ and $\eta_n\colon B_n \to A_{n+1}$. $(2)\Rightarrow(3)$ follows from Theorem 3.25 in [EG1]. Q.E.D

One needs to notice that we use the following strategy. Once α_n (or β_n) satisfies ($*$), then it can be composed with $[\psi_{n,m}]$ (or $[\phi_{n+1,m}]$) to be realized by a homomorphism for large enough m. This strategy will be also used in Section 4.

2.17. Our next task is to remove the restriction that $H^2(X_{n,i})$ and $H^2(Y_{n,i})$ are finite. In this circumstance, one only has $(1)\Leftrightarrow(2)\Leftrightarrow(4)$. That is, in general, the particular inductive limit systems may not be shape equivalent to each other, even though the limit algebras are isomorphic. An example was given in the introduction of [EG] (see p. 264–265), where two non shape equivalent inductive limit systems were constructed for the C^*-algebra of the tensor product of a Bunce-Dedden algebra with itself.

The idea of the proof is inspired by Section 5 of [EG1]. First, as in 5.9 and 5.15 of [EG1], we can suppose that each $X_{n,i}$ (and $Y_{n,i}$) has one of the following special forms.

(0) $X=\{pt\}$, we call it type 0;

(1) $X=S^1$ or $[0,1]$, we call it type I;

(2) $H^1(X)=0$, $H^2(X)$ finite, we call it type II;

(3) $X=S^2$, we call it type III.

(Since it only involves 2-dimensional spaces, we do not have two other types of spaces (types III and V in [EG1]). Our type III is the type IV in [EG1].)

2.18. Let $A=\lim(A_n, \phi_{n,m})$, where $A_n=\bigoplus A_n^i=\bigoplus M_{[n,i]}(C(X_{n,i}))$, and $X_{n,i}$ are the spaces of type 0, I, II, III in 2.17. We say that A_n^i is of type

0, I, II or III according to $X_{n,i}$ being of type 0, I, II or III. As in [EG1], we denote $A_n^0 = \bigoplus (A_n^i)^0 \triangleq \bigoplus M_{[n,i]}(C_0(X_{n,i}))$, and $rA_n \triangleq A_n/A_n^0$, where $C_0(X_{n,i})$ is the set of continuous functions on $X_{n,i}$ which vanish at a certain fixed base point of $X_{n,i}$.

Suppose that X is of type 0, I, or II, (not of type III), one can check that

$$KK(C_0(X), C_0(S^2)) = 0,$$

using 23.1.1 of [Bl1]. (Note that $K_1(C(X))$ is free.) If A_n^i is not of type III (i.e., $X_{n,i} \neq S^2$), and A_{n+1}^j is of type III (i.e. $X_{n+1,j} = S^2$), then $\phi_{n,n+1}^{i,j}$ (the partial map of $\phi_{n,n+1}$) induces an element $[\phi_{n,n+1}^{i,j}] \in KK(A_n^i, A_{n+1}^j)$ with zero component in $KK((A_n^i)^0, (A_{n+1}^j)^0) = 0$ (see Section 1.6 of [EG1]). Suppose that $\phi_{n,n+1}^{i,j}(\mathbf{1}_{A_n^i}) = P \in A_{n+1}^j$ and that $\phi_{n,n+1}^{i,j}(e_{11}) = p \in A_{n+1}^j$, where e_{11} is a matrix unit of $M_{[n,i]}(\mathbb{C}) \subseteq A_n^i$. Then one can identify $PA_{n+1}^j P$ with $M_{[n,i]}(pA_{n+1}^j p)$. Define the homomorphism $(\phi_{n,n+1}^{i,j})'$: $A_n^i \to PA_{n+1}^j P \subset A_{n+1}^j$ by

$$(\phi_{n,n+1}^{i,j})'(f) = f(x_0)P$$

$$\triangleq \begin{pmatrix} f_{11}(x_0)p, & f_{12}(x_0)p, & ..., & f_{1[n,i]}(x_0)p \\ \vdots & & & \vdots \\ f_{[n,i]1}(x_0)p, & f_{[n,i]2}(x_0)p, & ..., & f_{[n,i][n,i]}(x_0)p \end{pmatrix} \in PA_{n+1}^j P,$$

where $x_0 \in X_{n,i}$ is the base point. Then

$$[\phi_{n,n+1}^{i,j}] = [(\phi_{n,n+1}^{i,j})'] \in KK(A_n^i, A_{n+1}^j).$$

We can define $\psi_{n,n+1}: A_n \to A_{n+1}$ by

$$\psi_{n,n+1}^{i,j} = \begin{cases} (\phi_{n,n+1}^{i,j})' & \text{if } A_n^i \text{ is not of type III and } A_{n+1}^i \text{ is of type III,} \\ \phi_{n,n+1}^{i,j} & \text{otherwise.} \end{cases}$$

By Corollary 2.25 of [EG1], $\lim(A_n, \psi_{n,m})$ is also of real rank zero. Using Theorem 3.25 of [EG1], one can prove that $\lim(A_n, \phi_{n,m})$ and $\lim(A_n, \psi_{n,m})$ are shape equivalent. Hence $\lim(A_n, \phi_{n,m}) \cong \lim(A_n, \psi_{n,m})$.

Notice that $\psi_{n,n+1}$ satisfies the following condition.

(**): for each block A_n^i of non type III and block A_{n+1}^j of type III, $\psi_{n,n+1}^{i,j}(A_n^i)(\subset A_{n+1}^j)$ is of finite dimensional.

The above condition (**) is an analogy of the property (SH) in [EG1]. One can verify that $\psi_{n,m}$ also satisfies the condition (**) for arbitrary $m > n$.

Based on the above argument, we can always suppose that in any inductive limit $A = \lim(A_n, \phi_{n,m})$, $\phi_{n,m}$ satisfies the condition (**) for each n and m.

The following result is an analogy of Theorem 5.23 in [EG1] (but not a direct consequence of it, since $\phi_{n,m}$ does not have property (SH) with respect to the block A_n^i of type II, and A_m^j of type I). However the proof is also a complete analogy of that of Theorem 5.23 in [EG1]. Instead of giving the complete proof of it, we will point out the only difference.

LEMMA 2.19. *Let $A = \lim(A_n, \phi_{n,m})$ be a real rank zero inductive limit algebra as in 2.18 (i.e., $X_{n,i}$ are of the special forms and $\phi_{n,m}$ satisfy $(**)$). For any finite set $F \subset A_n$ and $\varepsilon > 0$, there are an A_m, $\phi'_{n,m}\colon A_n \to A_m$ and a sub-algebra $B \subset A_m$, satisfying the following conditions*:

(1) $\operatorname{dist}(\phi_{n,m}(f), \phi'_{n,m}(f)) \leqslant 70\varepsilon$ *for any* $f \in F$;

(2) $\operatorname{dist}(\phi'_{n,m}(f), B) \leqslant 2\varepsilon$ *for any* $f \in F$;

(3) $\phi'_{n,m}(\mathbf{1}_{A_n^i}) \in B$ *for each block* A_n^i *of* A_n;

(4) *If* A_n^i *is of types* 0, *I*, *II*, *then* $\phi'_{n,m}(A_n^i) \subset B$;

(5) *If* A_m^j *is of types* 0, *I*, *II*, *then* $A_m^j \subset B$;

(6) *B is a direct sum of matrices over* 2-*dimensional finite CW complexes of special forms of types* 0, *I*, *II* (*without III*, *i.e.*, *without* S^2).

The only difference from the proof of Theorem 5.23 of [EG1] is that we need to rearrange the index set J_1^k, J_2^k $(k = 0, 1, 2, 3)$. What we need to do is to group those indices with blocks of types 0, I, II into J_1^k, and those indices with blocks of type III into J_2^k. For example.

$$J_1^0 = \{i \mid A_n^i \text{ if of types } 0, I, II\}$$

and

$$J_2^0 = \{i \mid A_n^i \text{ if of type } III\}.$$

The main point of grouping the blocks is to ensure that the image of the homomorphism from a block with index in J_1^k $(k = 0, 1, 2)$ to a block with index in J_2^{k+1} is a finite dimensional C^*-algebra. With this in mind, the proof is a complete repeat of that of Theorem 5.23 of [EG1] (see [EG1] for details).

By using the above lemma, one can prove the following.

COROLLARY 2.20. *Suppose that A is a real rank zero inductive limit of direct sums of matrices over arbitrary* 2-*dimensional CW complexes. Then A can be rewritten as a real rank zero inductive limit of direct sums of matrices over* 2-*dimensional CW complexes $X_{n,i}$ with $H^2(X_{n,i})$ being finite.*

The following is the main result of this section.

THEOREM 2.21. *Suppose that A and B are unital real rank zero inductive limits of $(A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ and $(B_n = \bigoplus_{i=1}^{\ell_n} M_{\{n,i\}}(C(Y_{n,i})), \psi_{n,m})$, respectively, where $X_{n,i}$, $Y_{n,i}$ are arbitrary 2-dimensional finite CW complexes. The following are equivalent.*

(1) *A is unsuspended E-equivalent to B.*

(2) *$(K_*(A), K_*(A)_+, \mathbf{1}_A)$ is isomorphic to $(K_*(B), K_*(B)_+, \mathbf{1}_B)$, and there is a KK-equivalence on $\alpha \in KK(A, B)_{+,\Sigma,\mathbf{1}}$ with inverse $\beta \in KK(B, A)_{+,\Sigma,\mathbf{1}}$ (inducing the isomorphism between $(K_*(A), K_*(A)_+, \mathbf{1}_A)$ and $(K_*(B), K_*(B)_+, \mathbf{1}_B)$) which keeps the ideals in the sense of* 2.12.

(3) *A is isomorphic to B.*

Notice that the systems $(A_n, \phi_{n,m})$ and $(B_n, \psi_{n,m})$, may not be shape equivalent to each other in the above result.

Remark 2.22. Instead of the algebras of real rank zero, one may consider those algebras A and B which satisfy that $K_0(A)$ and $K_0(B)$ have large denominators (see [N]) in the sense that for any nonzero projection $p \in A$ (or B) and positive integer n, there are a projection $q \in A$ (or B) and an integer m with $n[q] < [p] < m[q]$. One can prove that $(K_*(A), K_*(A)_+)$ and $(K_*(B), K_*(B)_+)$ are still ordered groups. Under this circumstance, we will only consider such special ideals of A (or B) that are generated by the projections inside themselves, in 2.12. One knows that any KK-equivalence $\alpha \in KK(A, B)_{+,\Sigma,\mathbf{1}}$ (with inverse $\beta \in KK(B, A)_{+,\Sigma,\mathbf{1}}$) induces a one-to-one correspondence between those special ideals of A and of B. We say that α (with inverse β) keeps the ideals, if the two equations in (2.12) hold for those special ideal pairs. One can prove that (1) $\Leftrightarrow$ (2) in Theorem 2.21 still holds in this case (see Proposition 5.33 of [EG1]). But in general, they do not imply (3), since the C^*-algebras may not be of real rank zero. (Notice that, for inductive limit A of direct sums of matrices over finite CW complexes with uniformly bounded dimension, if $K_0(A)$ has large denominations, then it is unsuspended E-equivalent to a real rank zero such inductive limit, see Proposition 5.33 of [EG1].)

3. NON ISOMORPHIC C^*-ALGEBRAS WITH THE SAME K-THEORY

In this section, we will construct two C^*-algebras A and B (in the class introduced in Section 2) which have the same graded scaled ordered K-group

$$(K_*(A), K_*(A)_+, \mathbf{1}_A) \cong (K_*(B), K_*(B)_+, \mathbf{1}_B),$$

with no KK-equivalence $\alpha \in KK(A, B)_{+, \Sigma, \mathbf{1}}$ keeping the ideals in the sense of 2.12. Hence A and B are not isomorphic and not unsuspended E-equivalent to each other. This proves that the condition (2) in Theorem 2.16 and 2.21 is strictly stronger than that

$$(K_*(A), K_*(A)_+, \mathbf{1}_A) \cong (K_*(B), K_*(B)_+, \mathbf{1}_B).$$

3.1. Let A, B and C be C^*-algebras for which the universal coefficient theorem holds. One has

$$0 \to \operatorname{Ext}^1(K_*(A), K_*(B)) \to KK(A, B) \xrightarrow{\gamma} \operatorname{Hom}^0(K_*(A), K_*(B)) \to 0,$$

where

$$\operatorname{Ext}^1(K_*(A), K_*(B)) = \operatorname{Ext}(K_1(A), K_0(B)) \oplus \operatorname{Ext}(K_0(A), K_1(B))$$

and

$$\operatorname{Hom}^0(K_*(A), K_*(B)) = \operatorname{Hom}(K_0(A), K_0(B)) \oplus \operatorname{Hom}(K_1(A), K_1(B)).$$

First, if

$$\alpha \in \operatorname{Ext}^1(K_*(A), K_*(B)) \subseteq KK(A, B)$$

and

$$\beta \in \operatorname{Ext}^1(K_*(B), K_*(C)) \subseteq KK(B, C),$$

then $\alpha \times \beta = 0 \in KK(A, C)$.

Second, if

$$\alpha \in \operatorname{Ext}(K_i(A), K_{i+1}(B)) \subseteq KK(A, B) \qquad (i = 0, 1 \ (\mathrm{mod}\ 2))$$

and $\beta \in KK(B, C)$, then

$$\alpha \times \beta \in \operatorname{Ext}(K_i(A), K_{i+1}(C)) \subseteq KK(A, C)$$

and $\alpha \times \beta$ depends only on α and the component of

$$\gamma(\beta) \in \operatorname{Hom}^0(K_*(B), K_*(C))$$

in $\mathrm{Hom}(K_{i+1}(B), K_{i+1}(C))$. Actually $\alpha\times\beta\in\mathrm{Ext}(K_i(A), K_{i+1}(C))$ can be described as follows (as in the theory of homological algebra). Since $\mathrm{Ext}(\cdot, \cdot)$ is a covariant factor of the second variable, the group homomorphism $\gamma(\beta)$ (considered as an element in $\mathrm{Hom}(K_{i+1}(B), K_{i+1}(C))$ induces a map

$$\mathrm{Ext}(K_i(A), K_{i+1}(B))\to\mathrm{Ext}(K_i(A), K_{i+1}(C)).$$

And $\alpha\times\beta$ is the image of α under the above map.

Finally, if

$$\beta\in\mathrm{Ext}(K_i(B), K_{i+1}(C))\subseteq KK(B, C)$$

and $\alpha\in KK(A, B)$, then $\alpha\times\beta$ depends only on β and the component of

$$\gamma(\alpha)\in\mathrm{Hom}^0(K_*(A), K_*(B))$$

in $\mathrm{Hom}(K_i(A), K_i(B))$. Notice that $\mathrm{Ext}(\cdot, \cdot)$ is a contravariant factor of the first variable.

The above facts were used in [EG1].

3.2. Let $\mathbb{P}^2$ be the real projective space defined by identifying all the pairs of antipodal points of S^2. It is well known that $H^2(\mathbb{P}^2)=\mathbb{Z}/2\mathbb{Z}=\mathbb{Z}_2$ and that $H^1(\mathbb{P}^2)=0$. From [DN], we know that

$$KK(C_0(\mathbb{P}^2), C_0(S^1))=kk(S^1, \mathbb{P}^2)=\mathbb{Z},$$

where $kk(Y, X)$ is the set of homotopy classes of homomorphisms from $C_0(X)$ to $C_0(Y)\otimes\mathcal{K}$.

Also we know that $\pi_1(\mathbb{P}^2)=\mathbb{Z}_2$. Let $\alpha\colon S^1\to\mathbb{P}^2$ be the generator of $\pi_1(\mathbb{P}^2)$ (which keeps the base point). Then α induces a homomorphism $\alpha^*\colon C_0(\mathbb{P}^2)\to C_0(S^1)$. And α is the generator of $KK(C_0(\mathbb{P}^2), C_0(S^1))=kk(S^1, \mathbb{P}^2)=\mathbb{Z}_2$. (We will not prove this fact, since we are not going to use it.) Also from [DN], one knows that

$$KK(C_0(\mathbb{P}^2), C_0(\mathbb{P}^2))=kk(\mathbb{P}^2, \mathbb{P}^2)=\mathbb{Z}_2,$$

$$KK(C_0(S^1), C_0(S^1))=kk(S^1, S^1)=\mathbb{Z},$$

and

$$KK(C_0(S^1), C_0(\mathbb{P}^2))=kk(\mathbb{P}^2, S^1)=0.$$

3.3. In this section, we will only make use of one space $X = \mathbb{P}^2 \vee S^1$, the wedge of X and S^1. One knows that

$$\begin{aligned} KK(C_0(X), C_0(X)) &= KK(C_0(\mathbb{P}^2), C_0(\mathbb{P}^2)) \oplus KK(C_0(S^1), C_0(S^1)) \\ &\quad \oplus KK(C_0(\mathbb{P}^2), C_0(S^1)) \oplus KK(C_0(S^1), C_0(\mathbb{P}^2)) \\ &= \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \oplus 0. \end{aligned}$$

As in 4.3 and 4.4 of [EG1], we can write

$$KK(C_0(X), C_0(X)) = KK_{\text{hom}}(C_0(X), C_0(X)) \oplus KK_{\text{ext}}(C_0(X), C_0(X)),$$

where $KK_{\text{ext}}(C_0(X), C_0(X)) = \text{Ext}^1(K_*(C_0(X)), K_*(C_0(X))) \subseteq KK(C_0(X), C_0(X))$.

Using the decomposition in Section 1.6 of [EG1], we know that

$$\begin{aligned} KK(C(X), C(X)) &= KK(C_0(X), C_0(X)) \oplus KK(\mathbb{C}, \mathbb{C}) \\ &\quad \oplus KK(\mathbb{C}, C_0(X)) \oplus KK(C_0(X), \mathbb{C}) \\ &= KK(C_0(X), C_0(X)) \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \oplus 0. \end{aligned}$$

(Notice that $KK(C_0(X), \mathbb{C}) = 0$.)

For our convenience, we will write

$$\begin{aligned} KK(C(X), C(X)) &= KK(\mathbb{C}, \mathbb{C}) \oplus KK(C_0(\mathbb{P}^2), C_0(\mathbb{P}^2)) \\ &\quad \oplus KK(\mathbb{C}, C_0(\mathbb{P}^2)) \oplus KK(C_0(S^1), C_0(S^1)) \\ &\quad \oplus KK(C_0(\mathbb{P}^2), C_0(S^1)) \\ &= \text{Hom}(K_0(\mathbb{C}), K_0(\mathbb{C})) \\ &\quad \oplus \text{Hom}(K_0(C_0(\mathbb{P}^2)), K_0(C_0(\mathbb{P}^2))) \\ &\quad \oplus \text{Hom}(K_0(\mathbb{C}), K_0(C_0(\mathbb{P}^2))) \\ &\quad \oplus \text{Hom}(K_1(C_0(S^1)), K_1(C_0(S^1))) \\ &\quad \oplus \text{Ext}(K_0(C_0(\mathbb{P}^2)), K_1(C_0(S^1))) \\ &= \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_2. \end{aligned}$$

Using the above decomposition, we can write each $\alpha \in KK(C(X), C(X))$ as the following:

$$\alpha = \alpha_{(1)} + \alpha_{(2)} + \alpha_{(3)} + \alpha_{(4)} + \alpha_{(5)} \in \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_2.$$

Comparing the natural decomposition in Section 4 of [EG1]:

$$KK(C(X), C(X)) = KK_{\text{hom}}(C(X), C(X)) \oplus KK_{\text{ext}}(C(X), C(X)),$$

one knows that $\alpha_{(5)} \in KK_{\text{ext}}$ and all the other four terms $\alpha_{(1)}, \alpha_{(2)}, \alpha_{(3)}$ and $\alpha_{(4)}$ are in KK_{hom}. We will use the above decomposition through this section. That is, for each $\alpha \in KK(C(X), C(X))$, we will write

$$\alpha = (\alpha_{(1)}, \alpha_{(2)}, \alpha_{(3)}, \alpha_{(4)}, \alpha_{(5)}).$$

One needs to notice that $\alpha \in KK(C(X), C(X))_+$ if and only if either $\alpha_{(1)} > 0$ or $(\alpha_{(1)}, \alpha_{(2)}, \alpha_{(3)}, \alpha_{(4)}) = 0$, but $\alpha_{(5)}$ may not be zero.

If $\alpha \in KK(C(X), C(X))$ and $\beta \in KK(C(X), C(X))$, then $\alpha \times \beta$ can be described as follows

$$(\alpha \times \beta)_{(1)} = \alpha_{(1)} \times \beta_{(1)},$$

$$(\alpha \times \beta)_{(2)} = \alpha_{(2)} \times \beta_{(2)},$$

$$(\alpha \times \beta)_{(3)} = \alpha_{(1)} \times \beta_{(3)} + \alpha_{(3)} \times \beta_{(2)},$$

$$(\alpha \times \beta)_{(4)} = \alpha_{(4)} \times \beta_{(4)}$$

and

$$(\alpha \times \beta)_{(5)} = \alpha_{(5)} \times \beta_{(4)} + \alpha_{(2)} \times \beta_{(5)}.$$

(Please see 3.1.)

If $\alpha = (\alpha_{(1)}, \alpha_{(2)}, \alpha_{(3)}, \alpha_{(4)}, \alpha_{(5)}) \in KK(M_{k_1}(C(X)), M_{k_2}(C(X)))_{+, \Sigma, \mathbf{1}}$ then it is automatically true that $\alpha_{(1)} = k_2/k_1 \in \mathbb{Z}$.

3.4. We would like to introduce two unital real rank zero inductive limit C^*-algebras $A = \lim(A_n, \phi_{n,m})$ and $B = \lim(B_n, \psi_{n,m})$ in which $A_n = B_n$ for all n.

We will choose A_1 to have one block, A_2 to have two blocks, and in general A_n to have n blocks. That is, $A_1 = A_1^1$, $A_2 = A_2^1 \oplus A_2^2$, ..., $A_n = A_n^1 \oplus A_n^2 \oplus \cdots \oplus A_n^n$. Similarly, B_n has n blocks, i.e., $B_n = B_n^1 \oplus B_n^2 \oplus \cdots \oplus B_n^n$. Also, we assume that $A_n^i = B_n^i = M_{[n,i]}(C(X))$, ($X$ as in 3.3), where $[n, i]$ are certain positive integers to be determined later on.

3.5. Before we construct our C^*-algebras, we would like to recall a result in [EG1]. We say that $\alpha \in KK(C(X), C(X))$ is L-large (where $L > 0$) if the element $\alpha_{(1)} \in KK(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$ (induced by α) satisfies $\alpha_{(1)} \geqslant L$. By using 3.23 and 3.27 of [EG1], one knows that for any $\varepsilon > 0$, there is an l such that any l-large KK-element $\alpha = (\alpha_{(1)}, \alpha_{(2)}, 0, \alpha_{(4)}, \alpha_{(5)}) \in KK(M_{k_1}(C(X)), M_{k_2}(C(X)))_{+, \Sigma, \mathbf{1}}$ (i.e., $k_2/k_1 \geqslant l$) can be realized by unital homomorphism

$\phi: M_{k_1}(C(X)) \to M_{k_2}(C(X))$ with $\mathrm{SPV}(\phi) < \varepsilon$ (see Section 1.4 of [EG1] for definition of SPV). (We suppose that $\alpha_{(3)} = 0$, for simplicity, i.e., we suppose that α takes a trivial projection to a trivial projection.)

3.6. We will specify what $[n, i]$ should be. First, we will give several properties of $\phi^{i,j}_{n,n+1}$ and $\psi^{i,j}_{n,n+1}$, and those properties will determine $\phi_{n,n+1}$, $\psi_{n,n+1}$ up to inner equivalence and homotopy equivalence (also, the numbers $[n, i]$ are determined by those properties too, certainly with unital property of $\phi_{n,n+1}, \psi_{n,n+1}$).

We require that the KK-elements $[\phi^{i,j}_{n,n+1}] \in KK(A^i_n, A^j_{n+1})$ and $[\psi^{i,j}_{n,n+1}] \in KK(B^i_n, B^j_{n+1})$ have the following components:

(1) $[\phi^{i,j}_{n,n+1}]_{(1)} = [\psi^{i,j}_{n,n+1}]_{(1)}$

$$= \begin{cases} l_n (>0) \in KK(\mathbb{C}, \mathbb{C}) (= \mathbb{Z}) & \text{if } j \geq i \\ 0 & \text{if } j < i \end{cases};$$

(2) $[\phi^{i,j}_{n,n+1}]_{(2)} = [\psi^{i,j}_{n,n+1}]_{(2)}$

$$= \begin{cases} 1 \in KK(C_0(\mathbb{P}^2), C_0(\mathbb{P}^2)) = \mathbb{Z} & \text{if } j = i+1 \\ 0 & \text{if } j \neq i+1 \end{cases};$$

(3) $[\phi^{i,j}_{n,n+1}]_{(3)} = [\psi^{i,j}_{n,n+1}]_{(3)} = 0 \in KK(\mathbb{C}, C_0(\mathbb{P}^2)) = \mathbb{Z}_2$, for all i, j.

(4) $[\phi^{i,j}_{n,n+1}]_{(4)} = [\psi^{i,j}_{n,n+1}]_{(4)}$

$$= \begin{cases} 1 \in KK(C_0(S^1), C_0(S^1)) = \mathbb{Z} & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases};$$

(5) $$[\phi^{i,j}_{n,n+1}]_{(5)} = \begin{cases} 1 \in KK(C_0(\mathbb{P}^2), C_0(S^1)) & \text{if } j = i \\ 0 & \text{if } j \neq \mathrm{i} \end{cases},$$

$[\psi^{i,j}_{n,n+1}]_{(5)} = 0$ for all i, j.

From the above, we have the following properties:

(6) $[\phi^{i,j}_{n,n+1}]_{(1)} = 0$ ($[\psi^{i,j}_{n,n+1}]_{(1)} = 0$, resp.) implies that $[\phi^{i,j}_{n,n+1}] = 0$ ($[\psi^{i,j}_{n,n+1}] = 0$, resp.);

(7) $[\phi_{n,n+1}]_{\mathrm{hom}} = [\psi_{n,n+1}]_{\mathrm{hom}}$.

By 3.5, if l_n is large enough, then one can choose unital homomorphisms

$$\phi^{i,j}_{n,n+1}, \psi^{i,j}_{n,n+1}: M_{[n,i]}(C(X)) \to M_{l_n \cdot [n,i]}(C(X))$$

with the above KK components and with $\mathrm{SPV}(\phi^{i,j}_{n,n+1})$ and $\mathrm{SPV}(\phi^{i,j}_{n,n+1})$ being as small as one wishes. To make $\phi_{n,n+1}$ and $\psi_{n,n+1}$ unital, we let

$$[n+1, i] = l_n([n,1]+[n,2]+[n,3]+\cdots+[n,i]).$$

(This makes $[\phi_{n,n+1}] \in KK(A_n, A_{n+1})_{+,\Sigma,\mathbf{1}}$ and $[\psi_{n,n+1}] \in KK(B_n, B_{n+1})_{+,\Sigma,\mathbf{1}}$.) By making suitable choice of l_n (large enough), one can define $\phi_{n,n+1}$, $\psi_{n,n+1}$ to satisfy the above conditions (1)–(5), and $\mathrm{SPV}(\phi^{i,j}_{n,n+1}) < \varepsilon_n$, $\mathrm{SPV}(\psi^{i,j}_{n,n+1}) < \varepsilon_n$ for given small numbers ε_n. Hence we can make $A = \lim(A_n, \phi_{n,m})$ and $B = \lim(B_n, \psi_{n,m})$ to be of real rank zero.

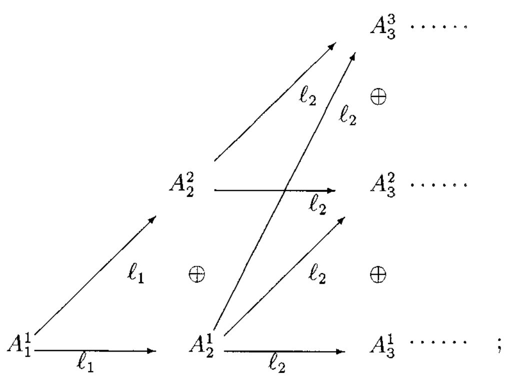
Since $[\phi_{n,n+1}]_{\mathrm{hom}} = [\psi_{n,n+1}]_{\mathrm{hom}}$, one knows that

$$(K_*(A), K_*(A)_+, \mathbf{1}_A) \cong (K_*(B), K_*(B)_+, \mathbf{1}_B).$$

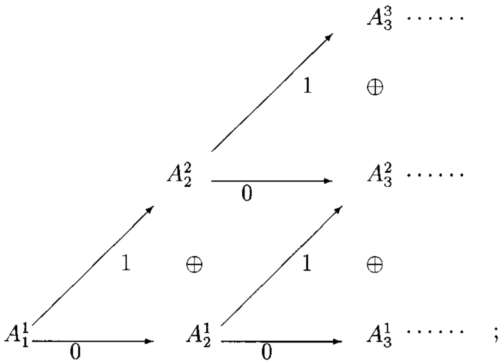
We will prove that the condition (2) of Theorem 2.21 does not hold, and therefore $A \not\cong B$.

3.7. Before we give the proof, we would like to explain the homomorphisms $\phi_{n,n+1}$, $\psi_{n,n+1}$ by the following pictures (actually, the following diagram will explain (1)–(5) of 3.6). (If no arrow is indicated between the given blocks, it is understood that the map is zero.)

(1) $[\phi^{i,j}_{n,n+1}]_{(1)} = [\psi^{i,j}_{n,n+1}]_{(1)} \in KK(\mathbb{C}, \mathbb{C}) (= KK(rA^i_n, rA^j_{n+1}) = \mathbb{Z})$ are given by

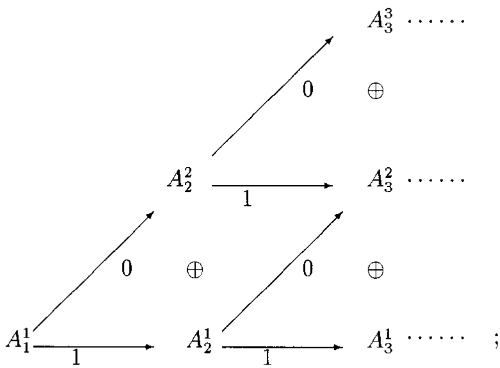


(2) $[\phi^{i,j}_{n,n+1}]_{(2)} = [\psi^{i,j}_{n,n+1}]_{(2)} \in KK(C_0(\mathbb{P}^2), C_0(\mathbb{P}^2)) (= \mathbb{Z}_2) \subseteq KK((A^i_n)^0, (A^j_{n+1})^0)$ are given by

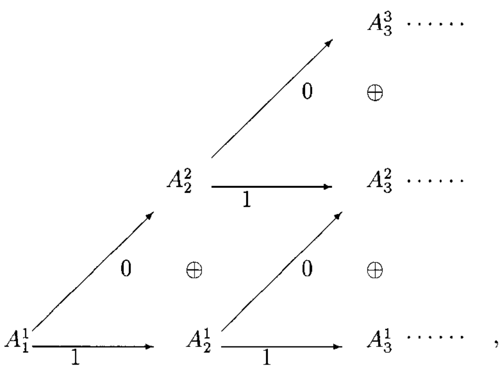


(3) $[\phi^{i,j}_{n,n+1}]_{(3)} = [\psi^{i,j}_{n,n+1}]_{(3)} = 0 \in KK(\mathbb{C}, C_0(\mathbb{P}^2))(=KK(rA^i_n, (A^j_{n+1})^0)$;

(4) $[\phi^{i,j}_{n,n+1}]_{(4)} = [\psi^{i,j}_{n,n+1}]_{(4)} \in KK(C_0(S^1), \ C_0(S^1))(=\mathbb{Z}) \subseteq KK(A^i_n)^0$, $(A^j_{n+1})^0)$ are given by



(5) $[\phi^{i,j}_{n,n+1}]_{(5)} \in \mathrm{Ext}(K_0(C_0(\mathbb{P}^2)), K_1(C_0(S^1)))$ $(=\mathbb{Z}_2)$ $\subseteq KK(A^i_n)^0$, $(A^j_{n+1})^0)$ are given by



and $[\psi_{n,n+1}]_{(5)} = 0$ for any n.

LEMMA 3.8. *The following are true*:

$$(8)\quad [\phi^{i,j}_{n,m}]_{(2)} = [\psi^{i,j}_{n,m}]_{(2)} = \begin{cases} 1 \in \mathbb{Z}_2, & \textit{if} \quad j-i = m-n \\ 0 \in \mathbb{Z}_2, & \textit{if} \quad j-i \neq m-n \end{cases};$$

$$(9)\quad [\phi^{i,j}_{n,m}]_{(4)} = [\psi^{i,j}_{n,m}]_{(4)} = \begin{cases} 1 \in \mathbb{Z}, & \textit{if} \quad i=j \\ 0 \in \mathbb{Z}, & \textit{if} \quad i \neq j \end{cases};$$

$$(10)\quad [\phi^{i,j}_{n,m}]_{(5)} = \begin{cases} 1 \in \mathbb{Z}_2, & \textit{if} \quad i \leqslant j \leqslant m-n+i-1 \\ 0 \in \mathbb{Z}_2, & \textit{otherwise} \end{cases}, \textit{ in particular},$$

$$(10')\quad [\phi^{1,j}_{1,m}]_{(5)} = \begin{cases} 1 & \textit{if} \quad j \leqslant m-1 \\ 0 & \textit{if} \quad j = m. \end{cases}$$

Proof. It is a direct calculation to prove the above results by using 3.3 and (1)–(5) of 3.6. We will only prove (10) which is the only one that seems not completely trivial. By (1) and (6), we know that if $j < i$, then $[\phi^{i,j}_{n,m}] = 0$, and therefore $[\phi^{i,j}_{n,m}]_{(5)} = 0$. If $m = n+1$, (10) is true by definition (see (5)). Suppose that (10) holds for $m_1 > n$, we are going to prove (10) for $m = m_1 + 1$. One has that

$$[\phi^{i,j}_{n,m_1+1}]_{(5)} = \sum_t [\phi^{i,t}_{n,m_1}]_{(2)} \times [\phi^{t,j}_{m_1,m_1+1}]_{(5)}$$
$$+ \sum_t [\phi^{i,t}_{n,m_1}]_{(5)} \times [\phi^{t,j}_{m_1,m_1+1}]_{(4)}.$$

By (9) and the induction assumption,

$$\text{the second term} = [\phi^{i,j}_{n,m_1}]_{(5)} \times [\phi^{j,j}_{m_1,m_1+1}]_{(4)}$$
$$= [\phi^{i,j}_{n,m_1}]_{(5)} \times 1 = \begin{cases} 1 & \text{if} \quad i \leqslant j \leqslant m_1 - n + i - 1 \\ 0 & \text{otherwise.} \end{cases}$$

By (8) and (5), the first summation includes only one non zero item which is for $t - i = m_1 - n$ and $j = t$. This item is

$$[\phi^{i,j}_{n,m_1+1}]_{(5)} = [\phi^{i,j}_{n,m_1}]_{(2)} \times [\phi^{j,j}_{m_1,m_1+1}]_{(5)}$$
$$= 1 \times 1 = 1 \in \mathbb{Z}_2,$$

where $j = m_1 - n + i = (m_1 + 1) - n + i - 1$. This ends the proof. Q.E.D

LEMMA 3.9. *A and B have exactly countably many ideals $A = I_1 \supset I_2 \supset I_3 \ldots$ and $B = J_1 \supset J_2 \supset J_3 \ldots$, respectively. And correspondingly, $(K_*(A), K_*(A)_+)$ and $(K_*(B), K_*(B)_+)$ have countably many ideals $K_*(A) = K_*(I_1) \supset K_*(I_2) \ldots$ and $K_*(B) = K_*(J_1) \supset K_*(J_2) \supset \ldots$, respectively.*

Proof. Suppose I_i is the ideal generated by the image of A_i^i (or equivalently by the image of A_n^i or $A_n^i \oplus A_n^{i+1} \oplus \cdots \oplus A_n^n$ for any $n \geqslant i$). It is obvious that $A = I_1 \supset I_2 \supset I_3 \ldots$ is a sequence of ideals of A. Since A is of real rank zero, any ideal of A is generated by the projections in the ideal. Suppose I is an ideal of A. For any projection $p \in I$, there are an A_n and a projection $q \in A_n$ with $[p] = [q]$. One can write $q = q_1 \oplus q_2 \oplus \cdots \oplus q_n$, where $q_i \in A_n^i$. Let i be the minimum integer with $q_i \neq 0$. Then q_i generates A_n^i and therefore q generates I_i. Hence for such i, $I_i \subset I$. If we take $i_0 \geqslant 1$ to be the minimum of all the above i (for all projections $p \in I$), then one can prove that $I_{i_0} \supset I$. Hence $I_{i_0} = I$. Q.E.D

3.10. One can prove that $K_1(A)$ and $K_1(B)$ are countable direct sums of $\mathbb{Z}$. We can write

$$K_1(A) = G_1 \oplus G_2 \oplus \cdots$$

and

$$K_1(B) = H_1 \oplus H_2 \oplus \cdots,$$

where each of G_i and H_i is equal to $\mathbb{Z}$, and

$$G_i = [\phi_{i,\infty}]_*(K_1(A_i^i)) = [\phi_{n,\infty}]_*(K_1(A_n^i)) \qquad (\text{for} \quad n \geqslant i)$$

and

$$H_i = [\psi_{i,\infty}]_*(K_1(B_i^i)) = [\psi_{n,\infty}]_*(K_1(B_n^i)) \qquad (\text{for} \quad n \geqslant i).$$

One can also prove that

$$G_i \subset K_*(I_i) \qquad \text{and} \qquad G_i \not\subset K_*(I_{i+1}).$$

Notice that,

$$[\phi_{n,\infty}]_*(K_1(A_n)) = G_1 \oplus G_2 \oplus \cdots \oplus G_n$$

and

$$[\psi_{n,\infty}]_*(K_1(B_n)) = H_1 \oplus H_2 \oplus \cdots \oplus H_n,$$

for any n.

3.11. Suppose that $\alpha \in KK(A, B)_{+,\Sigma,\mathbf{1}}$ (with inverse $\beta \in KK(B, A)_{+,\Sigma,\mathbf{1}}$) keeps the ideals, i.e., the condition (2) of Theorem 2.16 holds. First from the ideal structure of A and B (Lemma 3.9), one knows that

$$\alpha_*(K_*(I_i)) = K_*(J_i) \qquad \text{and} \qquad \beta_*(K_*(J_i)) = K_*(I_i).$$

By the definition of that α (with inverse β) keeps the ideals, one has that

$$i_{I_i} \times \alpha \times \pi_{J_i} = 0 \qquad \text{and} \qquad i_{J_i} \times \beta \times \pi_{I_i} = 0,$$

where $i_{I_i} \in KK(I_i, A)$ and $i_{J_i} \in KK(J_i, B)$ are induced by the inclusion maps, and $\pi_{I_i} \in KK(A, A/I_i)$ and $\pi_{J_i} \in KK(B, B/J_i)$ are induced by the quotient maps. We will also use the notation

$$i_{(A_n, I_i)} \in KK(A_n^i \oplus A_n^{i+1} \oplus \cdots \oplus A_n^n, A_n)$$

to denote the inclusion, and the notation

$$\pi_{(A_n, I_i)} \in KK(A_n, A_n/A_n^i \oplus A_n^{i+1} \oplus \cdots \oplus A_n^n) = KK(A_n, A_n^1 \oplus \cdots \oplus A_n^{i-1})$$

to denote the quotient map. The notation $i_{(B_n, J_i)}$ and $\pi_{(B_n, J_i)}$ are defined similarly.

As in the proof of Theorem 2.16, one can prove that, there is a KK-theory intertwining

$$\begin{array}{ccccccccc}
A_{k_1} & \xrightarrow{\phi_{k_1,k_2}} & A_{k_2} & \xrightarrow{\phi_{k_2,k_3}} & A_{k_3} & \longrightarrow & \cdots & \longrightarrow & A \\
\downarrow{\scriptstyle \alpha_1} & \nearrow{\scriptstyle \beta_1} & \downarrow{\scriptstyle \alpha_2} & \nearrow{\scriptstyle \beta_2} & & & & & {\scriptstyle \alpha}\downarrow\uparrow{\scriptstyle \beta} \\
B_{l_1} & \xrightarrow{\psi_{l_1,l_2}} & B_{l_2} & \xrightarrow{\psi_{l_2,l_3}} & B_{l_3} & \longrightarrow & \cdots & \longrightarrow & B.
\end{array}$$

That is $\alpha_n \times \beta_n = [\phi_{k_n, k_{n+1}}]$, $\beta_n \times \alpha_{n+1} = [\psi_{l_n, l_{n+1}}]$, $[\phi_{k_n, \infty}] \times \alpha = \alpha_n \times [\psi_{l_n, \infty}]$ and $[\psi_{l_n, \infty}] \times \beta = \beta_n \times [\phi_{k_{n+1}, \infty}]$. We will finally introduce a contradiction based on the above assumption, and hence prove that such α and β do not exist.

Lemma 3.12. *For any $\beta_n \in KK(B_{l_n}, A_{k_{n+1}})$ as above, there is an $m \geqslant n+1$ such that $\beta_n \times [\phi_{k_{n+1}, k_m}] \triangleq \tilde{\beta}$ satisfies that $\tilde{\beta}^{i,j} = 0 \in KK(B_{l_n}^i, A_{k_m}^j)$ whenever $j < i$.*

Proof. Using $[\psi_{l_n, \infty}] \times \beta = \beta_n \times [\phi_{k_{n+1}, \infty}]$, and $\beta_*(K_*(J_i)) = K_*(I_i)$, one can verify that $(\beta_n)_*(K_*(B_{l_n}^i)) \subseteq K_*(\bigoplus_{t \geqslant i} A_{k_{n+1}}^t)$ (notice that $\beta_n \in KK(B_{l_n}, A_{k_{n+1}})_{+, \Sigma, \mathbf{1}}$). Hence $(\beta_n^{i,j})_* = 0$ whenever $j < i$. Then for any m, $(\tilde{\beta}^{i,j})_* = 0$ whenever $j < i$. By the proof of Theorem 2.16, one can prove that $(\tilde{\beta}^{i,j}) = 0$ for m large enough and $j < i$. (Please see the assertion in the proof of Theorem 2.16 and one needs to notice that α (with inverse β) keeps the ideals.) Q.E.D

In what follows, we will assume that both α_n and β_n satisfy that $\alpha_n^{i,j} = 0$ and $\beta_n^{i,j} = 0$, respectively, whenever $j < i$.

3.13. Consider $([\psi_{l_1,\infty}]\times\beta)_*(K_1(B_{l_1}))$. This is a finitely generated subgroup of $K_1(A)=G_1\oplus G_2\oplus\cdots$. There is an M such that

$$([\psi_{l_1,\infty}]\times\beta)_*(K_1(B_{l_1}))\subseteq G_1\oplus G_2\oplus\cdots\oplus G_M.$$

LEMMA 3.14. *Suppose that* $L\geqslant M+l_1+1$, *where* M *is as in* 3.13. *If* $\eta\in KK(B_L,A_R)$ (*where* $R>L$) *satisfies that*

$$\eta\times[\phi_{R,\infty}]=[\psi_{L,\infty}]\times\beta\in KK(B_L,A),$$

and that $\eta^{i,j}=0$ *whenever* $j<i$, *then*

$$[\phi_{1,R}]\neq\alpha_1\times[\psi_{l_1,L}]\times\eta\in KK(A_1,A_R).$$

Proof. Denote $\xi=\alpha_1\times[\psi_{l_1,L}]\times\eta\in KK(A_1,A_R)$. We are going to prove that the partial KK element $\xi^{1,L-l_1}\in KK(A_1^1,A_R^{L-l_1})$ satisfies that

$$(\xi^{1,L-l_1})_{(5)}=0\in \operatorname{Ext}(K_0(C_0(\mathbb{P}^2)),K_1(C_0(S^1))).$$

Hence $\xi^{1,L-l_1}\neq\phi_{1,R}^{1,L-l_1}$, by (10′) in Lemma 3.8.

One knows that

$$\xi^{1,L-l_1}=\sum_{i,i}\alpha_1^{1,i}\times[\psi_{l_1,L}^{i,j}]\times\eta^{j,L-l_1}.$$

Hence, by (3.1) and (3.3),

$$\begin{aligned}(\xi^{1,L-l_1})_{(5)}&=\sum_{i,j}(\alpha_1^{1,i})_{(5)}\times[\psi_{l_1,L}^{i,j}]_{(4)}\times(\eta^{j,L-l_1})_{(4)}\\&\quad+\sum_{i,j}(\alpha_1^{1,i})_{(2)}\times[\psi_{l_1,L}^{i,j}]_{(5)}\times(\eta^{j,L-l_1})_{(4)}\\&\quad+\sum_{i,j}(\alpha_1^{1,i})_{(2)}\times[\psi_{l_1,L}^{i,j}]_{(2)}\times(\eta^{j,L-l_1})_{(5)}.\end{aligned}$$

We will prove that each of the above terms is 0.

1*st term*: From the assumptions, we know that

$$[\psi_{l_1,L}]\times\eta\times[\phi_{R,\infty}]=[\psi_{l_1,\infty}]\times\beta.$$

From (3.13), we have

$$([\psi_{l_1,\infty}]\times\beta)_*K_1(B_{l_1})\subseteq G_1\oplus G_2\oplus\cdots\oplus G_M.$$

Combining the above two facts, one can prove that

$$([\psi_{l_1, L}] \times \eta)_* K_1(B_{l_1}) \subseteq K_1(A_R^1) \oplus K_1(A_R^2) \oplus \cdots \oplus K_1(A_R^M).$$

(Notice that if $x \in K_1(A_R)$ and $x \notin K_1(A_R^1) \oplus \cdots \oplus K_1(A_R^M)$, then by (9) of 3.8, and 3.10, one has

$$[\phi_{R,\infty}]_*(x) \notin G_1 \oplus G_2 \oplus \cdots \oplus G_M.)$$

Hence for any i, $([\psi_{l_1, L}] \times \eta)_{(4)}^{i, L-l_1} = 0$ since $L - l_1 > M$. This proves that the first term is zero.

2nd term: By definition (see (5) of 3.6), $[\psi_{l_1, L}^{(i,j)}]_{(5)} = 0$, for any i, j. Hence the 2nd term is zero.

3rd term: We will prove that $[\psi_{l_1, L}^{i,j}]_{(2)} \times (\eta^{j, L-l_1})_{(5)} = 0$. By (8) of 3.8, we know that if $j \leqslant i + L - l_1 - 1$, then $[\psi_{l_1, L}^{i,j}]_{(2)} = 0$. On the other hand, if $j > i + L - l_1 - 1 \geqslant L - l_1$, then from assumption for η, we have

$$\eta^{j, L-l_1} = 0.$$

Hence $(\eta^{j, L-l_1})_{(5)} = 0$. This proves that term (3) is zero. Q.E.D

3.15. From *KK*-theory intertwining diagram in 3.11, we can choose $L = l_m > M + l_1 + 1$, and $\eta = \beta_m \in KK(B_{l_m}, A_{k_{m+1}})$ with $R = k_{m+1}$. Then

$$\eta \times [\phi_{R,\infty}] = [\psi_{L,\infty}] \times \beta$$

$$\eta^{i,j} = 0 \qquad \text{whenever } j < i \qquad \text{(by 3.12), and}$$

$$[\phi_{1,R}] = \alpha_1 \times [\psi_{l_1, L}] \times \eta \quad \text{(see the diagram in 3.11).}$$

This is a contradiction of 3.14. It proves the following main result.

THEOREM 3.16. *There are two unital C*-algebras A and B of real rank zero, which are inductive limits of direct sums of matrices over* 2*-dimensional finite CW complexes* (*i.e., they are in the class of Section* 2), *with the following properties*:

(1) $(K_*(A), K_*(A)_+, \mathbf{1}_A) \cong (K_*(B), K_*(B)_+, \mathbf{1}_B)$ *and*

(2) *there is no such* $\alpha \in KK(A, B)_{+,\Sigma,\mathbf{1}}$ (*with inverse* $\beta \in KK(B, A)_{+,\Sigma,\mathbf{1}}$) *that it induces isomorphism between the K-group and keeps the ideals. And therefore* $A \not\cong B$.

Remark 3.17. Similarly, one can prove that $A \otimes \mathscr{K}$ is not isomorphic to $B \otimes \mathscr{K}$. That is, A and B are not stably isomorphic.

Remark 3.18. In [D1], Dadarlat proved that if A and B are inductive limits of $\bigoplus_{i=1}^{k_n} M_{[n,i]}(C_0(X_{n,i}))$ and $\bigoplus_{i=1}^{l_n} M_{\{n,i\}}(C_0(Y_{n,i}))$ with $\sup\{\dim(X_{n,i}), \dim(Y_{n,i})\} < +\infty$, then $A \otimes \mathscr{K}$ and $B \otimes \mathscr{K}$ are unsuspended E-equivalent to each other if and only if $K_*(A) = K_*(B)$. (Notice that, in this case $K_*(A)_+ = K_*(B)_+ = \{0\}$.) One can compare this result with our example.

Remark 3.19. In comparison with the classification of (separable, nuclear) C^*-algebras of real rank zero and stable rank one, one can consider the classification of (separable, nuclear) purely infinite C^*-algebras of real rank zero ([Ro, Ro1, BEEK, ER, and LP]). For simplicity, let us consider only stable such C^*-algebras. In the case of $K_1(A) = 0$, the invariant, denoted by $P_0(A)$ (which plays the role of $\Sigma_0(A)$), is the semi-group of Murray-von Neumann equivalence (or unitary equivalence) classes of A (see [Ro]). (It includes $K_0(A)$ as the sub-invariant. And if A is simple, then $K_0(A)$ and $P_0(A)$ ($= K_0(A) \coprod \{0\}$) contain the same information.) In the case of $K_1(A) \neq 0$, one needs to consider the graded semigroup $P_*(A)$ which consists of partial unitaries modulo the following equivalence relation: $u \sim v$ if and only if $u^*u = uu^*$ is Murray-von Neumann equivalent to $v^*v = vv^*$ and $[u \oplus (\mathbf{1} - uu^*)] = [v \oplus (\mathbf{1} - vv^*)] \in K_1(A)$, where $\mathbf{1}$ is the unit of A^+, the C^*-algebra A adjoining a unit. (The author is indebted to Professor G. Elliott for explaining the above formulation of invariant.)

By using our construction in this section, one can construct two non-isomorphic separable nuclear purely infinite C^*-algebras C and D of real rank zero which have the same invariant (i.e., $P_*(C) = P_*(D)$) as follows. Let E be any simple purely infinite stable C^*-algebra with $K_0(E) = \mathbb{Z}$ and $K_1(E) = 0$. Then $P_*(E) = P_0(E) = K_0(E) \coprod \{0\}$. Let

$$C = A \otimes E \qquad \text{and} \qquad D = B \otimes E,$$

where A and B are the C^*-algebras in (3.6). Then

$$P_*(C) = P_*(D).$$

However, there is no isomorphism between C and D. Otherwise, the isomorphism will induce a KK-element which keeps the ideals. Repeating the procedure of this section, one can prove that this is impossible.

Remark 3.20. Using the spirit of this article, one can give examples with the property in 3.19 within the class of purely infinite real rank zero C^*-algebras to be expressed as the inductive limit of C^*-algebras of form $\bigoplus_i M_{k_i}(C(S^1)) \otimes O_{n_i}$, where O_{n_i} are Cuntz algebras with n_i even (one can always let $n_i = 4$). The construction and the proof are even simpler. The detail will appear elsewhere.

Remark 3.21. In the construction of $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$, one sees that the number of blocks of A_n goes to infinity as n goes to infinity. This must be the case for such examples. We will prove that if the limit C^*-algebra has finitely many ideals, then the unsuspended E-equivalence type is completely determined by the graded scaled ordered K-group (see Section 4).

Remark 3.22. In our example, $K_*(A)$ (and $K_*(B)$) has torsion. This also must be the case. We will prove (in Section 4) that if $K_*(A)$ is torsion free, then the unsuspended E-equivalence type of A is completely determined by the graded scaled ordered K group. In particular, $A \otimes M_{2^\infty} \cong B \otimes M_{2^\infty}$, for our examples A and B, where M_{2^∞} is the UHF algebra with $K_0(M_{2^\infty}) = \{n/2^m, n, m \text{ are integers}\}$

Remark 3.23. We have seen that, since A and B do not satisfy condition (2) of Theorem 2.21, they are not isomorphic. The inherent cause of being not isomorphic is that they have different unsuspended E-equivalence types. When we consider only the inductive limit C^*-algebras of direct sums of matrices over 2-dimensional finite CW complexes, and suppose the algebras to be of real rank zero, then Theorem 2.21 tells us that the isomorphic type of a C^*-algebra is determined by the unsuspended E-equivalence type of the C^*-algebra, completely. But if we do not assume that the C^*algebras are of real rank zero, then the condition that two C^*-algebras have the same unsuspended E-equivalence type is much weaker than the condition that they gave the same isomorphic type, and the former condition is easier to be satisfied (see Remark 2.22). In the next section, we also suppose that the algebras are of real rank zero, and we will prove that, in several cases (with a restriction on the ordered group $K_*(A)$ for each case), the isomorphic type (and unsuspended E-equivalence type) is completely determined by graded scaled ordered K-group. However with the restriction on $K_*(A)$ (in each theorem) and without the condition of real rank zero, it is still true that the unsuspended E-equivalence type is completely determined by the graded scaled ordered K-group, provided that $K_0(A)$ has a large denominator. Notice that for non real rank zero inductive limit C^*-algebras (to be isomorphic to each other), there are other invariants such as ideal spaces (or spectrum) and tracial data besides the unsuspended E-equivalence types. But, unlike the unsuspended E-equivalence type, those invariants are determined by graded scaled ordered K-groups if the C^*-algebras are of real rank zero.

4. RELATED CLASSIFICATION RESULTS

4.1. Suppose that A is a unital real rank zero inductive limit of $A_n = \bigoplus M_{[n,i]}(C(X_{n,i}))$, the direct sums of matrices over 2-dimensional finite CW complexes $X_{n,i}$. In this section, we will prove that the isomorphism type (and unsuspended E-equivalence type) of A is decided by its graded scaled ordered K-group completely, at the following two cases (and some other cases):

(1) A has at most finitely many ideals, or equivalently, $(K_*(A), K_*(A)_+)$ has at most finitely many ideals;

(2) $K_*(A)$ is torsion free.

It is worth while to point out that two related cases have been classified in [EG1] as the main results:

(1′) A is simple; (Theorem 5.8 of [EG1].)

(2′) $K_*(A_n)$ is torsion free (or $H^*(X_{n,i})$ is torsion free for each n, i). (Theorem 5.28 of [EG1].)

In [EG1], the C^*-algebra A was allowed to be an inductive limit of $A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i}))$ with $X_{n,i}$ being 3-dimensional finite CW complexes. In this article, for the case (2), we also allow $X_{n,i}$ to be 3-dimensional finite CW complexes. This is a generalization of Theorem 5.28 of [EG1]. Notice that, it is more natural to put conditions on the limit algebra A than to put that on A_n. In particular, after this generalization, one knows that $A \otimes M_{2^\infty} = B \otimes M_{2^\infty}$ for the non isomorphic C^*-algebras A and B we constructed in Section 3. However, for the case (1), we only prove the result for $X_{n,i}$ being 2-dimensional. So we did not obtain the full generalization of Theorem 5.8 of [EG1]. Case (1) includes all the inductive limits of $\bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i}))$ (dim $X_{n,i} \leqslant 2$) with $\{k_n\}$ uniformly bounded (notice that the limit algebra may not be simple even if one supposes that $k_n = 2$ for all n). This is the reason that the example in Section 3 must involve unbounded numbers of blocks of A_n.

The proofs of our generalizations are certainly inspired by the original proofs of that of the theorems in [EG1]. It also involves some new ideas. For instance, we need to use some techniques from homological algebra. We will not repeat the parts of proofs which have already appeared in [EG1]. Instead, we will refer to [EG1] for those parts and emphasize on the differences (or new ideas).

4.2. Let us deal with case (1) now—suppose that the inductive limits have finitely many ideals. As in [EG1], we will start with the case that there is an additional restriction on the inductive limit sequences—each space $X_{n,i}$ has the property that $H^*(X_{n,i})$ is finite.

THEOREM 4.3. *Suppose that A and B are unital real rank zero inductive limits of $(\bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ and $(\bigoplus_{i=1}^{l_n} M_{\{n,i\}}(C(Y_{n,i})), \psi_{n,m})$*

respectively, where $X_{n,i}$, $Y_{n,i}$ are 2-dimensional finite CW complexes with $H^2(X_{n,i})$, $H^2(Y_{n,i})$ finite. And further suppose that A and B have at most finitely many ideals (or equivalently, $(K_*(A), K_*(A)_+)$ and $(K_*(B), K_*(B)_+)$ have at most finitely many ideals). Then A is isomorphic to B if and only if

$$(K_*(A), K_*(A)_+, \mathbf{1}_A) \cong (K_*(B), K_*(B)_+, \mathbf{1}_B).$$

4.4. Suppose that τ: $(K_*(A), K_*(A)_+, \mathbf{1}_A) \to (K_*(B), K_*(B)_+, \mathbf{1}_B)$ is an isomorphism. Then τ induces a correspondence between the ordered ideals of $(K_*(A), K_*(A)_+, \mathbf{1}_A)$ and that of $(K_*(B), K_*(B)_+, \mathbf{1}_B)$. And therefore it induces a correspondence between the set of the ideals of A and the set of ideals of B. We denote the ideals of A by $\{I_a\}_{a\in\Omega}$, and the ideals of B by $\{J_a\}_{a\in\Omega}$, where Ω is an index set of finitely many elements. That is, under the correspondence, I_a goes to J_a for each a.

It is difficult to prove the property (2) in Theorem 2.16, directly. We will introduce two other sequences $\tilde{A} = \lim(A_n, \tilde{\phi}_{n,m})$ and $\tilde{B} = \lim(B_n, \tilde{\psi}_{n,m})$, and prove a weaker analogy of property (2) for $\tilde{A}$ and $\tilde{B}$.

4.5. As in [EG1], we denote $A_n^0 = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C_0(X_{n,i})))$, and $B_n^0 = \bigoplus_{i=1}^{l_n} M_{\{n,i\}}(C_0(Y_{n,i}))$, where $C_0(X_{n,i})$ and $C_0(Y_{n,i})$ are sets of continuous functions on $X_{n,i}$ and $Y_{n,i}$, respectively, vanishing at given base points of $X_{n,i}$ and $Y_{n,i}$, respectively. And denote $rA_n = A_n/A_n^0$ and $rB_n = B_n/B_n^0$.

One needs to repeat the construction of 5.3 and 5.4 of [EG1], to obtain $\tilde{A} = \lim(A_n, \tilde{\phi}_{n,m})$ and $\tilde{B} = \lim(B_n, \tilde{\psi}_{n,m})$ with the properties that: $\tilde{\phi}_{n,m}$ is homotopic to $\phi_{n,m}$, $\tilde{\psi}_{n,m}$ is homotopic to $\psi_{n,m}$, $\tilde{\phi}_{n,m}(A_n^0) \subset A_m^0$, and $\tilde{\psi}_{n,m}(B_n^0) \subset B_m^0$.

By Theorem 2.2 of [EG1], we need only to prove that $\tilde{A} = \lim(A_n, \tilde{\phi}_{n,m})$ and $\tilde{B} = \lim(B_n, \tilde{\psi}_{n,m})$ are shape equivalent to each other.

By the following identification

$$\begin{array}{ccccccc}
K_*(A_1) & \xrightarrow{(\tilde{\phi}_{1,2})_*} & K_*(A_1) & \longrightarrow & \cdots & \longrightarrow & K_*(\tilde{A}) \\
\Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle id} & & & & \\
K_*(A_1) & \xrightarrow{(\phi_{1,2})} & K_*(A_2) & \longrightarrow & \cdots & \longrightarrow & K_*(A),
\end{array}$$

we can identify $(K_*(\tilde{A}), K_*(\tilde{A})_+, \mathbf{1}_{\tilde{A}})$ with $(K_*(A), K_*(A)_+, \mathbf{1}_A)$. In this way, the ideals of $(K_*(A), K_*(A)_+, \mathbf{1}_A)$ are one-to-one corresponding to the ideals of $(K_*(\tilde{A}, K_*(\tilde{A})_+, \mathbf{1}_{\tilde{A}})$. Hence $\{I_a\}_{a\in\Omega}$ are one-to-one corresponding to those ideals of $\tilde{A}$ which are generated by the projections in them. We denote such ideals by $\{\tilde{I}_a\}_{a\in\Omega}$. (Notice that this is not a complete list of ideals of $\tilde{A}$, since $\tilde{A}$ is not of real rank zero.) We understand that $\{\tilde{J}_a\}_{a\in\Omega}$ are the ideals of $\tilde{B}$ which are corresponding to $\{J_a\}_{a\in\Omega}$. We can endow an order structure on Ω by the following: $a \leqslant b$ if $\tilde{I}_a \subset \tilde{I}_b$

(or, equivalently, $\tilde{J}_a \subset \tilde{J}_b$, $I_a \subset I_b$, or $J_a \subset J_b$). Notice that the isomorphism $\tau: (K_*(A), K_*(A)_+, \mathbf{1}_A) \to (K_*(B), K_*(B)_+, \mathbf{1}_B)$ induces an isomorphism $\tilde{\tau}: (K_*(\tilde{A}), K_*(\tilde{A})_+, \mathbf{1}_{\tilde{A}}) \to (K_*(\tilde{B}), K_*(\tilde{B})_+, \mathbf{1}_{\tilde{B}})$ with the property $\tilde{\tau}(K_*(\tilde{I}_a)) = K_*(\tilde{J}_a)$.

4.6. As in 5.4 of [EG1], one has an ideal $A^0 \subset \tilde{A}$ which is an inductive limit

$$A_1^0 \xrightarrow{\tilde{\phi}_{1,2}} A_2^0 \xrightarrow{\tilde{\phi}_{2,3}} A_3^0 \longrightarrow \cdots \longrightarrow A_n^0 \longrightarrow \cdots A^0,$$

and an ideal $B^0 \subset \tilde{B}$ which is an inductive limit

$$B_1^0 \xrightarrow{\tilde{\psi}_{1,2}} B_2^0 \xrightarrow{\tilde{\psi}_{2,3}} B_3^0 \longrightarrow \cdots \longrightarrow B_n^0 \longrightarrow \cdots B^0,$$

Similarly, $\tilde{\tau}: K_*(\tilde{A}) \to K_*(\tilde{B})$ induces an isomorphism from $K_*(A^0)$ to $K_*(B^0)$, i.e., $\tilde{\tau}(K_*(A^0)) = K_*(B^0)$. Let $\tau^0 = \tilde{\tau}|_{K_*(A^0)}$. It needs to be noticed that $K_*(A^0) = (\text{tor } K_0(\tilde{A})) \oplus K_1(\tilde{A})$, where tor G denotes the torsion part of G for any group G.

Denote $I_a^0 = A^0 \cap \tilde{I}_a$ and $J_a^0 = B^0 \cap \tilde{J}_a$.

As in the proof of Theorem 2.16, for any fixed ideal $\tilde{I}_a \subset \tilde{A}$, we define $A_n \cap \tilde{I}_a$ to be the ideal of A_n which is generated by those projections in A_n whose images (under $\tilde{\phi}_{n,\infty}$) are in $\tilde{I}_a \subset \tilde{A}$. Obviously $A_n \cap \tilde{I}_a$ consists of several whole blocks of A_n. We use $A_n^0 \cap \tilde{I}_a$ to denote $(A_n \cap \tilde{I}_a)^0$. Then $A_n^0 \cap \tilde{I}_a$ $(\triangleq (A_n \cap I_a)^0)$ consists of several whole blocks of A_n^0.

Also, we have two inductive limit sequences:

$$A_1 \cap \tilde{I}_a \xrightarrow{\tilde{\phi}_{1,2}} A_2 \cap \tilde{I}_a \xrightarrow{\tilde{\phi}_{2,3}} A_3 \cap \tilde{I}_a \longrightarrow \cdots \longrightarrow \tilde{A} \cap \tilde{I}_a (= \tilde{I}_a)$$

and

$$A_1^0 \cap \tilde{I}_a \xrightarrow{\tilde{\phi}_{1,2}} A_2^0 \cap \tilde{I}_a \xrightarrow{\tilde{\phi}_{2,3}} A_3^0 \cap \tilde{I}_a \longrightarrow \cdots \longrightarrow A^0 \cap \tilde{I}_a (= I_a^0).$$

The above are true since $\tilde{I}_a$ is generated by the projections inside the ideal. It is evident that

$$K_*(I_a^0) = K_*(\tilde{I}_a) \cap K_*(A^0) = (\text{tor } K_0(\tilde{I}_a)) \oplus K_1(\tilde{I}_a),$$

and that

$$K_*(A_n^0 \cap \tilde{I}_a) = K_*(A_n^0) \cap K_*(A_n \cap \tilde{I}_a) = (\text{tor } K_0(A_n \cap \tilde{I}_a)) \oplus K_1(A_n \cap \tilde{I}_a).$$

For each ideal $\tilde{J}_a \subset \tilde{B}$, one can define $B_n \cap \tilde{J}_a$, J_a^0, and $B_n^0 \cap \tilde{J}_a$ similarly.

CONVENTION 4.7. *If D is an ideal of C, we use $i_{(C,D)}$ to denote the inclusion $D \hookrightarrow C$, and $\pi_{(C,D)}$ to denote the quotient map $C \to C/D$. Also we use them to denote the corresponding KK-theory elements.*

LEMMA 4.8. *There is an $\alpha^0 \in KK(A^0, B^0)$ with inverse $\beta^0 \in KK(B^0, A^0)$ such that*

(i) *α^0 induces τ^0: $K_*(A^0) \to K_*(B^0)$ and that*

(ii) *for each $a \in \Omega$*

$$i_{(A^0, I_a^0)} \times \alpha^0 \times \pi_{(B^0, J_a^0)} \in \mathrm{Ext}(K_1(I_a^0), K_0(B^0/J_a^0)) \subseteq KK(I_a^0, B^0/J_a^0)$$

and

$$i_{(B^0, J_a^0)} \times \alpha^0 \times \pi_{(A^0, I_a^0)} \in \mathrm{Ext}(K_1(J_a^0), K_0(A^0/I_a^0)) \subseteq KK(J_a^0, A^0/I_a^0).$$

4.9. One needs to notice that if the condition (i) holds, then

$$\begin{aligned} i_{(A^0, I_a^0)} \times \alpha^0 \times \pi_{(B^0, J_a^0)} &\in \mathrm{Ext}^1(K_*(I_a^0), K_*(B^0/J_a^0)) \\ &= \mathrm{Ext}(K_0(I_a^0), K_1(B^0/J_a^0)) \oplus \mathrm{Ext}(K_1(I_a^0), K_0(B^0/J_a^0)). \end{aligned}$$

Our condition (ii) says that the first component (i.e., the component in $\mathrm{Ext}(K_0(I_a^0), K_1(B^0/J_a^0))$) of $i_{(A^0, I_a^0)} \times \alpha^0 \times \pi_{(B^0, J_a^0)}$ is zero. This is weaker than the condition of keeping the ideal pair (I_a^0, J_a^0) (which means both components are zero) defined in 2.12 (and used in Theorem 2.16). However this weaker condition will be enough for the proof of Theorem 4.3.

Suppose that Lemma 4.8 holds. The proof of Theorem 4.3 is similar to that of 2.16. We discuss it here briefly. (The reader can fill in the details.)

First, by the proof of 5.5 in [EG1], there is an $\alpha \in KK(\tilde{A}, \tilde{B})_{+, \Sigma, \mathbf{1}}$ with inverse $\beta \in KK(\tilde{B}, \tilde{A})_{+, \Sigma, \mathbf{1}}$ such that

(1) α induces $\tilde{\tau}$: $K_*(\tilde{A}) \to K_*(\tilde{B})$;

(2) the following diagrams commute (at level of KK),

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\alpha} & \tilde{B} \\ {\scriptstyle i_{A^0}}\uparrow & & \uparrow{\scriptstyle i_{B^0}} \\ A^0 & \xrightarrow{\alpha^0} & B^0 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \tilde{A} & \xleftarrow{\beta} & \tilde{B} \\ {\scriptstyle i_{A^0}}\uparrow & & \uparrow{\scriptstyle i_{B^0}} \\ A^0 & \xleftarrow{\beta^0} & B^0 \end{array}$$

where $i_{A^0} = i_{(\tilde{A}, A^0)} \in KK(A^0, \tilde{A})$ and $i_{B^0} = i_{(\tilde{B}, B^0)} \in KK(B^0, \tilde{B})$.

One can lift α and β into the following commuting (at level of KK) diagram

$$\begin{array}{ccccccccc} A_{k_1} & \xrightarrow{\tilde{\phi}_{k_1,k_2}} & A_{k_2} & \xrightarrow{\tilde{\phi}_{k_2,k_3}} & A_{k_3} & \longrightarrow & \cdots & \longrightarrow & \tilde{A} \\ {\scriptstyle \alpha_1}\downarrow & \overset{\beta_1}{\nearrow} & {\scriptstyle \alpha_2}\downarrow & \overset{\beta_2}{\nearrow} & & & & & {\scriptstyle \alpha}\downarrow\uparrow{\scriptstyle \beta} \\ B_{l_1} & \xrightarrow{\tilde{\psi}_{l_1,l_2}} & B_{l_2} & \xrightarrow{\tilde{\psi}_{l_2,l_3}} & B_{l_3} & \longrightarrow & \cdots & \longrightarrow & \tilde{B} \end{array}$$

where $\alpha_n \in KK(A_{k_n}, B_{l_n})_{+,\Sigma,\mathbf{1}}$ and $\beta_n \in KK(B_{l_n}, A_{k_{n+1}})_{+,\Sigma,\mathbf{1}}$. For each ideal $\tilde{I}_a$, using

$$i_{(A^0, I_a^0)} \times \alpha^0 \times \pi_{(B^0, J_a^0)} \in \operatorname{Ext}(K_1(I_a^0), K_0(B^0/J_a^0))$$

and $i_{A^0} \times \alpha = \alpha^0 \times i_{B^0} \in KK(A^0, \tilde{B})$, one knows that

$$\begin{aligned} & i_{(\tilde{A}, I_a^0)} \times \alpha \times \pi_{(\tilde{B}, \tilde{J}_a)} \\ & \quad = i_{(A^0, I_a^0)} \times i_{A^0} \times \alpha \times \pi_{(\tilde{B}, \tilde{J}_a)} \\ & \quad = i_{(A^0, I_a^0)} \times \alpha^0 \times i_{B^0} \times \pi_{(\tilde{B}, \tilde{J}_a)} \\ & \quad = i_{(A^0, I_a^0)} \times \alpha^0 \times \pi_{(B^0, J_a^0)} \times i_{(\tilde{B}/\tilde{J}_a, B^0/J_a^0)} \\ & \qquad \in \operatorname{Ext}(K_1(I_a^0), K_0(B^0/J_a^0) \times KK(B^0/J_a^0, \tilde{B}/\tilde{J}_a)) \\ & \qquad \subset \operatorname{Ext}(K_1(I_a^0), K_0(\tilde{B}/\tilde{J}_a)) \qquad \text{(by 3.1)}. \end{aligned}$$

Using $\alpha_n \times [\psi_{l_n,\infty}] = [\phi_{k_n,\infty}] \times \alpha$ and 3.1, one can prove that for each n, there is an m such that (see the proof of $(*)$ in 2.16)

$$\begin{aligned} & i_{(A_{k_n}, A_{k_n}^0 \cap \tilde{I}_a)} \times \alpha_n \times [\psi_{l_n, l_m}] \times \pi_{(B_{l_m}, B_{l_m} \cap \tilde{J}_a)} \\ & \qquad \in \operatorname{Ext}(K_1(A_{k_n}^0 \cap \tilde{I}_a), K_0(B_{l_m}/B_{l_m} \cap \tilde{J}_a)). \end{aligned}$$

Since $K_1(A_{k_n}^0 \cap \tilde{I}_a)$ is a free group, we know

$$\operatorname{Ext}(K_1(A_{k_n}^0 \cap \tilde{I}_a), K_0(B_{l_m}/B_{l_m} \cap \tilde{J}_a)) = 0.$$

Hence $i_{(A_{k_n}, A_{k_n}^0 \cap \tilde{I}_a)} \times \alpha_n \times [\psi_{l_n, l_m}] \times \pi_{(B_{l_m}, B_{l_m} \cap \tilde{J}_a)} = 0$. Without loss of generality, we can suppose that

$$i_{(A_{k_n}, A_{k_n}^0 \cap \tilde{I}_a)} \times \alpha_n \times \pi_{(B_{l_n}, B_{l_n} \cap \tilde{J}_a)} = 0.$$

(That is, use $\alpha_n \times [\psi_{l_n, l_m}]$ to replace α_n.)

(One needs to notice that, we can not obtain $i_{(\tilde{A}, I_a^0)} \times \alpha \times \pi_{(\tilde{B}, \tilde{J}_a)} = 0$, since $K_1(I_a^0)$ is not a free group (only torsion free) and therefore

$\mathrm{Ext}(K_1(I_a^0), K_0(\widetilde{B}/\widetilde{J}_a))$ may not be trivial. However, after passing to finite stages, we obtain the above required equation.)

Since α induces $\tilde{\tau}\colon K_*(\widetilde{A}) \to K_*(\widetilde{B})$, one knows that $i_{(\widetilde{A}, \widetilde{I}_a)} \times \alpha \times \pi_{(\widetilde{B}, \widetilde{J}_a)}$ induces zero map from $K_*(\widetilde{I}_a)$ to $K_*(\widetilde{B}/\widetilde{J}_a)$. Passing to subsequence, one can suppose that $i_{(A_{k_n}, A_{k_n} \cap \widetilde{I}_a)} \times \alpha_n \times \pi_{(B_{l_n}, B_{l_n} \cap \widetilde{J}_a)}$ induces zero map from $K_*(A_{k_n} \cap \widetilde{I}_a)$ to $K_*(B_{l_n}/B_{l_n} \cap \widetilde{J}_a)$. Combining this fact, the above equation, and the decomposition

$$\begin{aligned} &KK(A_{k_n} \cap \widetilde{I}_a, B_{l_n}/B_{l_n} \cap \widetilde{J}_a) \\ &\quad = KK(A_{k_n}^0 \cap \widetilde{I}_a, B_{l_n}/B_{l_n} \cap \widetilde{J}_a) \oplus KK(r(A_{k_n} \cap \widetilde{I}_a), B_{l_n}/B_{l_n} \cap \widetilde{J}_a), \end{aligned}$$

one can prove that

$$i_{(A_{k_n}, A_{k_n} \cap \widetilde{I}_a)} \times \alpha_n \times \pi_{(B_{l_n}, B_{l_n} \cap \widetilde{J}_a)} = 0.$$

(Here we use the facts that $A_{k_n}^0 \cap \widetilde{I}_a = (A_{k_n} \cap \widetilde{I}_a)^0$ and that $K_*(r(A_{k_n} \cap \widetilde{I}_a))$ is a free group.)

The above equation is exactly the equation $(*)$ in the proof of 2.16. The rest of the proof is a repetition of the corresponding part of that of 2.16.

Therefore, we know that, to prove Theorem 4.3, one needs only to prove Lemma 4.8. It is obvious that Lemma 4.8 follows from the next lemma.

LEMMA 4.10. *There is a system of KK-equivalences* $\{\alpha_a\}_{a \in \Omega}$, $\alpha_a \in KK(I_a^0, J_a^0)$ *for all* $a \in \Omega$ *such that*

(1) α *induces* $\tilde{\tau}|_{K_*(I_a^0)}\colon K_*(I_a^0) \to K_*(J_a^0)$, *and that*

(2) *if* $a < b$ (*and therefore* $I_a^0 \subset I_b^0$ *and* $J_a^0 \subset J_b^0$), *then the diagram*

$$\begin{array}{ccc} I_b^0 & \xrightarrow{\alpha_b} & J_b^0 \\ {\scriptstyle i_{(I_b^0, I_a^0)}}\uparrow & & \uparrow{\scriptstyle i_{(J_b^0, J_a^0)}} \\ I_a^0 & \xrightarrow{\alpha_a} & J_a^0 \end{array} \qquad \text{(D)}$$

commutes up to modulo $\mathrm{Ext}(K_1, K_0)$, *i.e.*,

$$i_{(I_b^0, I_a^0)} \times \alpha_b - \alpha_a \times i_{(J_b^0, J_a^0)} \in \mathrm{Ext}(K_1(I_a^0), K_0(J_b^0)).$$

(*Lemma* 4.8 *can be obtained by taking* $I_b^0 = A^0$ *and* $\alpha^0 = \alpha_b$).

Proof. We will prove it by induction strategy.

For any $a_0 \in \Omega$, denote $\Omega_{a_0} = \{a < a_0, a \in \Omega\}$ and $\bar{\Omega}_{a_0} = \{a \leqslant a_0, a \in \Omega\} = \Omega_{a_0} \cup \{a_0\}$. Suppose that $a_0 \in \Omega$ and suppose that there is a system of KK-equivalences $\alpha_a \in KK(I_a^0, J_a^0)\}_{a \in \Omega_{a_0}}$, such that for each

$a<b<a_0$, the diagram (D) commutes up to modulo $\mathrm{Ext}(K_1, K_0)$ and that α_a induces $\tau|_{K_*(I_a^0)}$. We are going to prove that there is an $\alpha_{a_0} \in KK(I^0_{a_0}, J^0_{a_0})$ (induces $\tilde{\tau}|_{K_*(I^0_{a_0})}$) such that for each $a<b \leqslant a_0$, the diagram (D) commutes up to modulo $\mathrm{Ext}(K_1, K_0)$. We will use (3.1) frequently.

We divide the proof into two situations:

(1) There is an $a_1 < a_0$ such that $a \in \Omega_{a_0}$ implies $a \leqslant a_1$;

(2) There is no such a_1.

Suppose that $a<b<c$. Consider the following diagram

$$\begin{array}{ccc} I^0_c & \longrightarrow & J^0_c \\ \uparrow & & \uparrow \\ I^0_b & \longrightarrow & J^0_b \\ \uparrow & & \uparrow \\ I^0_a & \longrightarrow & J^0_a. \end{array}$$

By (3.1), one can prove that if two small squares commute up to modulo $\mathrm{Ext}(K_1, K_0)$, then the large rectangle commutes up to modulo $\mathrm{Ext}(K_1, K_0)$.

1st Case. To guarantee that the diagram (D) commutes for any $a<b=a_0$ up to modulo $\mathrm{Ext}(K_1, K_0)$, one needs only the following diagram

$$\begin{array}{ccc} I^0_{a_0} & \xrightarrow{\alpha_{a_0}} & J^0_{a_0} \\ \uparrow & & \uparrow \\ I^0_{a_1} & \xrightarrow[\alpha_{a_1}]{} & J^0_{a_1} \end{array}$$

commutes up to modulo $\mathrm{Ext}(K_1, K_0)$. The existence of the above α_{a_0} can be proved as the proof of 5.5 of [EG1] (considering $I^0_{a_1}$, $J^0_{a_1}$ in places of A^0 and B^0, respectively, and $I^0_{a_0}$, $J^0_{a_0}$ in place of $\tilde{A}$ an $\tilde{B}$, respectively). (Actually, one can obtain an exactly commutative diagram in above.)

2nd Case. Suppose that $\{a_1, a_2, ..., a_k\}$ is the set of all maximum elements of Ω_{a_0}. Hence for any $i \neq j$, $\tilde{I}_{a_i}$ and $\tilde{I}_{a_j}$ do not contain each other. So

$$\tilde{I}_{a_i} \subsetneqq \tilde{I}_{a_i} + \tilde{I}_{a_j} \subseteq \tilde{I}_{a_0},$$

where $\tilde{I}_{a_i} + \tilde{I}_{a_j}$ is the ideal generated by $\tilde{I}_{a_i} \cup \tilde{I}_{a_i}$. This proves that $\tilde{I}_{a_i} + \tilde{I}_{a_j} = \tilde{I}_{a_0}$. Therefore

$$I^0_{a_i} + I^0_{a_j} = I^0_{a_0}.$$

Denote $G_i = K_0(I^0_{a_i})$ for $0 \leqslant i \leqslant k$, and $H = K_1(J^0_{a_0})$. Using the above equation, one can prove that if the map

$$\theta\colon G_i \oplus G_j \to G_0$$

is induced by inclusions (i.e., $\theta|_{G_i}\colon G_i \to G_0$ and $\theta|_{G_j}\colon G_j \to G_0$ are inclusions), then it is a surjection (see the proof of the next lemma).

Choose any $\alpha'_{a_0} \in KK(I^0_{a_0}, J^0_{a_0})$ which induces the isomorphism

$$\tilde{\tau}|_{K_*(I^0_{a_0})}\colon K_*(I^0_{a_0}) \to K_*(J^0_{a_0}).$$

Let $\gamma_i = (i_{(I^0_{a_0}, I^0_{a_i})} \times \alpha'_{a_0} - \alpha_{a_i} \times i_{(J^0_{a_0}, J^0_{a_i})} \in KK(I^0_{a_i}, J^0_{a_0})$. Since α'_{a_0} induces $\tau_{a_0} = \tau|_{K_*(I^0_{a_0})}$, one knows that

$$\begin{aligned}\gamma_i &\in \mathrm{Ext}^1(K_*(I^0_{a_i}), K_*(J^0_{a_0}))\\ &= \mathrm{Ext}(K_0(I^0_{a_i}), K_1(J^0_{a_0})) \oplus \mathrm{Ext}(K_1(I^0_{a_i}), K_0(J^0_{a_0})).\end{aligned}$$

Write $\gamma_i = \gamma_i^1 + \gamma_i^2$, where $\gamma_i^1 \in \mathrm{Ext}(K_0(I^0_{a_i}), K_1(J^0_{a_0}))$ and $\gamma_i^2 \in \mathrm{Ext}(K_1(I^0_{a_i}), K_0(J^0_{a_0}))$. We are going to prove that there is a $\gamma^1 \in \mathrm{Ext}(K_0(I^0_{a_0}), K_1(J^0_{a_0}))$ such that

$$i_{(I^0_{a_0}, I^0_{a_i})} \times \gamma^1 - \gamma_i^1 = 0 \qquad \text{for each } i.$$

Consider the following complex

$$0 \leftarrow G_0 \xleftarrow{\mathscr{K}_1} \bigoplus_{i=1}^{k} G_i \xleftarrow{\mathscr{K}_2} \bigoplus_{1 \leqslant i < j \leqslant k} G_i \cap G_j,$$

where $\mathscr{K}_1$ is induced by inclusions, i.e., $\mathscr{K}_1|_{G_i}$ are the inclusions from G_i to G_0, and $\mathscr{K}_2$ is also induced by inclusion but with certain sign corrections, more precisely, for any $g \in G_i \cap G_j$ $(i < j)$

$$\mathscr{K}_2(g) = g \oplus (-g) \in G_i \oplus G_j \subset \bigoplus_{i=1}^{k} G_i.$$

Claim 1. The above sequence is exact (in particular, $\ker \mathscr{K}_1 = \operatorname{im} \mathscr{K}_2$).

We are going to use this claim to prove the lemma and postpone the proof of the claim to the next lemma. (One needs to notice that the claim is not true for arbitrary Abelian group G_0 with finitely many subgroups $\{G_i\}$ whose union generates G_0. More precisely, it may not be true that $\ker \mathscr{K}_1 = \operatorname{im} \mathscr{K}_2$. We need to use certain special properties of those groups, see next lemma.)

Let $\tilde{G}=(\bigoplus_{1\leqslant i<j\leqslant k} G_i\cap G_j)/\ker \mathscr{K}_2$. Then we have the following short exact sequence

$$0\leftarrow G_0 \xleftarrow{\mathscr{K}_1} \bigoplus_{i=1}^{k} G_i \xleftarrow{\tilde{\mathscr{K}}_2} \tilde{G}\leftarrow 0,$$

where $\tilde{\mathscr{K}}_2$ is induced by $\mathscr{K}_2$. It induces a long exact sequence of $\mathrm{Ext}(\cdot, H)$ (see [CE]):

$$0\to \mathrm{Hom}(G_0, H)\to \mathrm{Hom}\left(\bigoplus_{i=1}^{k} G_i, H\right)\to \mathrm{Hom}(\tilde{G}, H)$$

$$\to \mathrm{Ext}(G_0, H)\xrightarrow{\mathscr{K}_1^*} \mathrm{Ext}\left(\bigoplus_{i=1}^{k} G_i, H\right)\xrightarrow{\tilde{\mathscr{K}}_2^*} \mathrm{Ext}(\tilde{G}, H)\to 0,$$

where $\mathscr{K}^*\colon \mathrm{Ext}(X, H)\to \mathrm{Ext}(Y, H)$ is induced by group homomorphism $\mathscr{K}\colon Y\to X$.

On the other hand, since the diagram (D), for each $a<b=a_i$, is commutative up to modulo $\mathrm{Ext}(K_1, K_0)$ (the induction assumption), one knows that

$$\mathscr{K}_2^*\left(\bigoplus_{i=1}^{k} \gamma_i^1\right)=0\in \mathrm{Ext}\left(\bigoplus_{1\leqslant i<j\leqslant k} G_i\cap G_j, H\right).$$

(See (3.1) also.) We argue that

$$\tilde{\mathscr{K}}_2^*\left(\bigoplus_{i=1}^{k} \gamma_i^1\right)=0\in \mathrm{Ext}(\tilde{G}, H)$$

as follows. Consider another short exact sequence

$$0\leftarrow \tilde{G}\leftarrow \bigoplus_{1\leqslant i<j\leqslant k} G_i\cap G_j\leftarrow \ker \mathscr{K}_2\leftarrow 0.$$

One has the exact sequence

$$\mathrm{Hom}(\ker \mathscr{K}_2, H)\to \mathrm{Ext}(\tilde{G}, H)\to \mathrm{Ext}\left(\bigoplus_{1\leqslant i<j\leqslant k} G_i\cap G_j, H\right).$$

Since ker $\mathscr{K}_2$, a subgroup of $\bigoplus_{1\leqslant i<j\leqslant k} G_i\cap G_j$, is a torsion group (notice that $G_i=K_0(I^0_{a_i})$ is a torsion group), and H is a torsion free group, we know that $\mathrm{Hom}(\ker\mathscr{K}_2, H)=0$. This implies that

$$\mathrm{Ext}(\tilde{G}, H)\to \mathrm{Ext}\left(\bigoplus_{1\leqslant i<j\leqslant k} G_i\cap G_j, H\right)$$

is injective. Hence that $\mathscr{K}_2^*(\bigoplus \gamma_i^1)=0$ implies that $\tilde{\mathscr{K}}_2^*(\bigoplus \gamma_i^1)=0$. Therefore

$$\bigoplus \gamma_i^1 \in \mathrm{image}(\mathscr{K}_1^*).$$

That is, there exists

$$\gamma^1\in \mathrm{Ext}(G_0, H)=\mathrm{Ext}(K_0(I^0_{a_0}), K_1(J^0_{a_0}))\subset KK(I^0_{a_0}, J^0_{a_0})$$

such that

$$i_{(I^0_{a_0}, I^0_{a_i})}\times\gamma^1-\gamma_i^1=0$$

for each $1\leqslant i\leqslant k$ (see 3.1). Let

$$\alpha_{a_0}=\alpha'_{a_0}-\gamma^1\in KK(I^0_{a_0}, J^0_{a_0}).$$

We know that the diagram (D) for α_{a_i} and $\alpha_{a_0}(a_i<a_0)$ is commutative up to modulo $\mathrm{Ext}(K_1, K_0)$ for each $1\leqslant i\leqslant k$. Therefore it is also true that the diagram (D) for any $a<a_0$ is commutative up to modulo $\mathrm{Ext}(K_1, K_0)$. The lemma follows routinely from the induction strategy (one can start with all minimum ideals). Q.E.D

LEMMA 4.11. *Claim* (1) *in the proof of Lemma* 4.10 *is true.*

Proof. Using $I^0_{a_i}+I^0_{a_j}=I^0_{a_0}$ and $\tilde{I}_{a_i}+\tilde{I}_{a_j}=\tilde{I}_{a_0}$, we can prove that

$$\theta\colon G_i\oplus G_j\to G_0$$

is surjective as follows, where θ is induced by inclusions. Denote $\tilde{G}_i=K_0(\tilde{I}_{a_i})$, for $0\leqslant i\leqslant k$. Suppose that $x\in G_0$. Then there is an n such that $x\in K_0(A_n\cap I^0_{a_0})$. One can write $x=\bigoplus x_t$, where $x_t\in K_0(A_n^t)$ and t runs over the set

$$\{t\mid (A_n^t)^0\subseteq A_n\cap I^0_{a_0}\}.$$

Since $\widetilde{I}^0_{a_i}+\widetilde{I}^0_{a_i}=\widetilde{I}^0_{a_j}$, $(A^t_n)^0 \subseteq A_n \cap I^0_{a_0}$ implies either $(A^t_n)^0 \subseteq A_n \cap I^0_{a_i}$ or $(A^t_n)^0 \subseteq A_n \cap I^0_{a_j}$. Therefore x can be written as x^1+x^2, where $x^1 \in K_0(I^0_{a_i})=G_i$ and $x^2 \in K_0(I^0_{a_j})=G_j$. This proves that θ is surjective. (This fact was mentioned in the proof of 4.10.) Hence $\mathscr{K}_1$ is surjective.

We have to prove $\operatorname{im} \mathscr{K}_2=\ker \mathscr{K}_1$. We need only to prove that $\ker \mathscr{K}_1 \subset \operatorname{im} \mathscr{K}_2$. (Notice that $\ker \mathscr{K}_1 \supset \operatorname{im} \mathscr{K}_2$ is obvious and that this fact makes the sequence be complex.)

Let $G_{i,j}=G_i \cap G_j \triangleq K_0(I^0_{a_i}) \cap K_0(I^0_{a_j})$.

We observe that for each $i, j_1 \neq j_2$, $\widetilde{I}_{a_i}=\widetilde{I}_{a_i} \cap \widetilde{I}_{a_{j_1}}+\widetilde{I}_{a_i} \cap \widetilde{I}_{a_{j_2}}$, since $\widetilde{I}_{a_0}=\widetilde{I}_{a_{j_1}}+\widetilde{I}_{a_{j_2}}$. As in the case above, we know that the map

$$\theta\colon G_{i,j_1} \oplus G_{i,j_2} \to G_i$$

(induced by inclusions) is surjective.

Suppose that $z_i \in G_i$ with $\sum_{i=1}^k z_i=0$ (i.e., $\mathscr{K}_1(\bigoplus z_i)=0$). We are going to construct $z_{i,j} \in G_i \cap G_j (i<j)$ such that

$$\mathscr{K}_2\left(\bigoplus_{1 \leqslant i<j \leqslant k} z_{i,j}\right)=\bigoplus z_i.$$

(One needs to notice that in the construction below, we always have $z_{i,j}=0$ for $|i-j| \geqslant 3$.)

By using the surjectivity of θ for $i=1$, $j_1=2$, $j_2=3$, there are $z_{1,2} \in G_{1,2}$, $z_{1,3} \in G_{1,3}$ such that

$$z_1=z_{1,2}+z_{1,3}. \tag{1}$$

Consider $z_2'=z_2+z_{1,2} \in G_2$. By surjectivity of θ again (for $i=2$, $j_1=3$, and $j_2=4$), there are $z_{2,3} \in G_{2,3}$, $z_{2,4} \in G_{2,4}$, such that $z_2'=z_{2,3}+z_{2,4}$. Therefore

$$z_2=-z_{1,2}+z_{2,3}+z_{2,4}. \tag{2}$$

Similarly, consider $z_3'=z_3+z_{1,3}+z_{2,3} \in G_3$, there exist $z_{3,4} \in G_{3,4}$, $z_{3,5} \in G_{3,5}$ with $z_3'=z_{3,4}+z_{3,5}$. And so

$$z_3=-z_{1,3}-z_{2,3}+z_{3,4}+z_{3,5}. \tag{3}$$

In general, suppose we have constructed $z_{i,j}$ for $i<l$ $(j \leqslant l+1)$. We can consider $z_l'=z_l+z_{l-2,l}+z_{l-1,l}$. (Notice that $z_{i,l}=0$ if $i<l-2$). There are $z_{l,l+1} \in G_{l,l+1}$, $z_{l,l+2} \in G_{l,l+2}$ such that $z_l'=z_{l,l+1}+z_{l,1+2}$. Hence

$$z_l=-z_{l-2,l}-z_{l-1,l}+z_{l,l+1}+z_{l,1+2}. \tag{l}$$

This procedure can be carried out until $l=k-2$. That is we can construct $z_{i,j}\in G_{i,j}$ for all $j=i+1$ and $j=i+2$ except $z_{k-1,k}\in G_{k-1,k}$, with the equation (l) for each $1\leqslant l\leqslant k-2$. We need to construct $z_{k-1,k}$ now. Adding all the above equations (1), (2), (3), ..., (l), ..., $(k-2)$, one has

$$z_1+z_2+\cdots+z_{k-2}=z_{k-3,k-1}+z_{k-2,k-1}+z_{k-2,k}.$$

Since $\sum_{i=1}^k z_i=0$, we know

$$z_{k-1}+z_k=-z_{k-3,k-1}-z_{k-2,k-1}-z_{k-2,k}.$$

That is

$$z_{k-1}+z_{k-2,k-1}+z_{k-3,k-1}=-z_{k-2,k}-z_k.$$

Notice that the left-hand side of the above is in G_{k-1} and the right-hand side of the above is in G_k. Let

$$z_{k-1,k}=-z_{k-2,k}-z_k=z_{k-1}+z_{k-2,k-1}+z_{k-3,k-1}\in G_{k-1,k}.$$

That is,

$$z_{k-1}=-z_{k-2,k-1}-z_{k-3,k-1}+z_{k-1,k}, \qquad (k-1)$$

$$z_k=-z_{k-2,k}-z_{k-1,k}, \qquad (k)$$

This ends the proof (see (1), (2), ... (l), ... $(k-1)$, (k)). Q.E.D

Remark 4.12. Notice that, in (4.10), $\alpha_a\in KK(I_a^0, J_a^0)$ and $\alpha_b\in KK(I_b^0, J_b^0)$ $(a<b)$ are only compatible up to modulo $\mathrm{Ext}(K_1, K_0)$. But as in (4.9), after we lift α to the finite stage α_n, we know that $\alpha_n|_{A_n\cap\tilde{I}_a}$ (or $\alpha_n|_{A_n\cap I_a^0}$) and $\alpha_n|_{A_n\cap\tilde{I}_b}$ (or $\alpha_n|_{A_n\cap I_b^0}$) are exactly compatible.

One needs to notice the following fact (which may clear some confusion). Suppose that $\alpha_n\in KK(A_n, B)$, $\alpha\in KK(A, B)$ and $\alpha'\in KK(A, B)$ satisfy

$$\alpha_n=[\phi_{n,\infty}]\times\alpha \qquad \text{and} \qquad \alpha_n=[\phi_{n,\infty}]\times\alpha'$$

for each n. It is not automatically true that

$$\alpha=\alpha'.$$

One has only

$$\alpha-\alpha'\in\lim{}^1 KK^1(A_n, B)\subseteq KK(A, B),$$

where $\lim^1$ is Milnor's $\lim^1$ (see 21.3 of [Bl1]).

The following main theorem of this section follows from 2.20 and 4.3.

THEOREM 4.13. *Suppose that A and B are real rank zero unital inductive limits of direct sums of matrices over arbitrary 2-dimensional finite CW complexes. And suppose that A and B have at most finitely many ideals (or equivalently, $(K_*(A), K_*(A)_+)$ and $(K_*(B), K_*(B)_+)$ have at most finitely many ideals). Then A is isomorphic to B if and only if*

$$(K_*(A), K_*(A)_+, \mathbf{1}_A) \cong (K_*(B), K_*(B)_+, \mathbf{1}_B).$$

The rest of the article is devoted to the proof of the following theorem.

THEOREM 4.14. *Suppose that A and B are real rank zero unital inductive limits of direct sums of matrices over 3-dimensional finite CW complexes $(A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$, and $(B_n = \bigoplus_{i=1}^{l_n} M_{\{n,i\}}(C(Y_{n,i})), \psi_{n,m})$, respectively. And suppose that $K_*(A)$ and $K_*(B)$ are torsion free. Then A is isomorphic to B if and only if*

$$(K_*(A), K_*(A)_+, \mathbf{1}_A) \cong (K_*(B), K_*(B)_+, \mathbf{1}_B).$$

4.15. By 5.15 of [EG1], we can suppose $X_{n,i}$ to have the special form as in 5.9 of [EG1]. Therefore, there is a natural splitting

$$KK(A_n, A_m) = KK_{\mathrm{hom}}(A_n, A_m) \oplus KK_{\mathrm{ext}}(A_n, A_m).$$

Similar to 5.19 of [EG1], passing to subsequence, one can construct another real rank zero inductive limit system $(A_n, \phi^1_{n,m})$ with the following properties:

(1) $[\phi^1_{n,m}] \in KK_{\mathrm{hom}}(A_n, A_m)$;

(2) $[\phi^1_{n,m}]$ and $[\phi_{n,m}]$ have the same components in $KK_{\mathrm{hom}}(A_n, A_m)$.

Furthermore, one can suppose that $\phi^1_{n,m}$ satisfy the condition (SH) in 5.18 of [EG1]. Theorem 4.14 follows from Remark 5.27 of [EG1] and the following lemma.

LEMMA 4.16. $\lim(A_n, \phi_{n,m})$ *and* $\lim(A_n, \phi^1_{n,m})$ *are shape equivalent to each other.*

Proof. Notice that $K_*(A_n) = \bigoplus_{i=1}^{k_n} \bigoplus_{j=0}^{3} H^j(X_{n,i})$. We will construct an intertwining in the level of homotopy, i.e., construct subsequences $\{k_n\}$,

$\{l_n\}$ with $k_n < l_n < k_{n+1}$ and homomorphisms $\xi_n\colon A_{k_n} \to A_{l_n}$, and $\eta_n\colon A_{l_n} \to A_{k_{n+1}}$ such that the following diagram

$$\begin{array}{ccccccc}
A_{k_1} & \xrightarrow{\phi_{k_1, k_2}} & A_{k_2} & \xrightarrow{\phi_{k_2, k_3}} & A_{k_3} & \longrightarrow & \cdots \\
\xi_1 \downarrow & \eta_1 \nearrow & \xi_2 \downarrow & \eta_2 \nearrow & \downarrow & & \\
A_{l_1} & \xrightarrow[\phi^1_{l_1, l_2}]{} & A_{l_2} & \xrightarrow[\phi^1_{l_2, l_3}]{} & A_{l_3} & \longrightarrow & \cdots
\end{array}$$

commutes at the level of homotopy. From Theorem 3.25 of [EG1], we need only the above diagram to be commutative at the level of *KK*.

Let $k_1 = 1$. By $K_*(A)$ torsion free, one can choose l_1 such that

$$[\phi_{k_1, l_1}]_* (\text{tor } K_*(A_1)) = 0.$$

And define

$$\xi_1 = \phi_{k_1, l_1}.$$

For the above l_1, one can choose k_2 such that

$$[\phi^1_{l_1, k_2}]_* (\text{tor } K_*(A_{l_1})) = [\phi_{l_1, k_2}]_* (\text{tor } K_*(A_{l_1})) = 0.$$

And define

$$\eta_1 = \phi^1_{l_1, k_2}.$$

In general, suppose that A_{k_n} (or A_{l_n}, resp.) has been constructed, we need to choose l_n (or k_{n+1}, resp.) such that

$$[\phi_{k_n, l_n}]_* (\text{tor } K_*(A_{k_n})) = 0 \qquad (\text{or } [\phi^1_{l_n, k_{n+1}}]_* (\text{tor } K_*(A_{l_n})) = 0 \text{ resp.})$$

and define $\xi_n\colon A_{k_n} \to A_{l_n}$ (or $\eta_n\colon A_{l_n} \to A_{k_{n+1}}$ resp.) by

$$\xi_n = \phi_{k_n, l_n} \qquad (\text{or } \eta_n = \phi^1_{l_n, k_{n+1}} \text{ resp.}).$$

In this way, we have constructed the above diagram. We need to prove that each triangle in the diagram commutes at the level of *KK*. We need only prove that the first triangle commutes at the level of *KK*, the proofs for other triangles are the same.

First,

$$[\phi_{n, m}]_{\text{hom}} = [\phi^1_{n, m}]_{\text{hom}} \qquad \text{implies}$$

$$[\phi_{k_1, k_2}]_{\text{hom}} = [\xi_1 \times \eta_1]_{\text{hom}}.$$

Notice that

$$[\phi_{k_1,k_2}]_{\mathrm{ext}} = [\phi_{k_1,l_1}]_{\mathrm{ext}} \times [\phi_{l_1,k_2}]_{\mathrm{hom}} + [\phi_{k_1,l_1}]_{\mathrm{hom}} \times [\phi_{l_1,k_2}]_{\mathrm{ext}}$$

and that

$$\begin{aligned}[\xi_1 \times \eta_1]_{\mathrm{ext}} &= ([\phi_{k_1,l_1}] \times [\phi^1_{l_1,k_2}])_{\mathrm{ext}} \\ &= [\phi_{k_1,l_1}]_{\mathrm{ext}} \times [\phi^1_{l_1,k_2}]_{\mathrm{hom}} + [\phi_{k_1,l_1}]_{\mathrm{hom}} \times [\phi^1_{l_1,k_2}]_{\mathrm{ext}}.\end{aligned}$$

We know that

$$[\phi_{k_1,l_1}]_{\mathrm{ext}} \times [\phi_{l_1,k_2}]_{\mathrm{hom}} = [\phi_{k_1,l_1}]_{\mathrm{ext}} \times [\phi^1_{l_1,k_2}]_{\mathrm{hom}}.$$

Suppose that

$$\begin{aligned}E\colon \mathrm{Ext}^1(\mathrm{tor}\, K_*(A_{l_1}), K_*(A_{k_2}))(&= \mathrm{Ext}^1(K_*(A_{l_1}), K_*(A_{k_2}))) \\ &\to \mathrm{Ext}^1(\mathrm{tor}\, K_*(A_{k_1}), K_*(A_{k_2}))\end{aligned}$$

is the map induced by the homomorphism

$$[\phi_{k_1,l_1}]_*\colon \mathrm{tor}\, K_*(A_{k_1}) \to \mathrm{tor}\, K_*(A_{l_1}).$$

(Here we use the fact that $K_*(A_{l_1})$ is finitely generated to guarantee that $\mathrm{Ext}(K_*(A_{l_1}), K_*(A_{k_2})) = \mathrm{Ext}^1(\mathrm{tor}\, K_*(A_{l_1}), K_*(A_{k_2}))$.) Then $[\phi_{k_1,l_1}]_{\mathrm{hom}} \times [\phi_{l_1,k_2}]_{\mathrm{ext}}$ is the image of $[\phi_{l_1,k_2}]_{\mathrm{ext}}$ under the map E (see 3.1). But $[\phi_{k_1,l_1}]_* = 0$ on $\mathrm{tor}\, K_*(A_{k_1})$ and hence $E = 0$. This proves that

$$[\phi_{k_1,l_1}]_{\mathrm{hom}} \times [\phi_{l_1,k_2}]_{\mathrm{ext}} = 0 = [\phi_{k_1,l_1}]_{\mathrm{hom}} \times [\phi^1_{l_1,k_2}]_{\mathrm{ext}}.$$

This ends the proof. Q.E.D

The following results can be proved similarly (see 5.29 and 5.30 of [EG1].) We omit the proofs.

PROPOSITION 4.17. *Suppose that A and B are real rank zero unital inductive limits of direct sums of matrices over 3-dimensional finite CW complexes. And suppose that there is no infinitesimal in $K_0(A)$ and $K_0(B)$, when we regard $K_*(A)$ and $K_*(B)$ as ordered groups (i.e., for any $x \neq 0 \in K_0(A)$ (or $K_0(B)$), there is a trace τ on A (or B) with $\tau(x) \neq 0$). Then A is isomorphic to B if and only if*

$$(K_*(A), K_*(A)_+, \mathbf{1}_A) = (K_*(B), K_*(B)_+, \mathbf{1}_B).$$

Recently, this proposition was used by Dadarlat and the author to prove that a certain real rank zero AD-algebra is not an AH-algebra.

PROPOSITION 4.18. *Suppose that A and B are real rank zero unital inductive limits of direct sums of matrices over 3-dimensional finite CW complexes* $X_{n,i}$, *and* $Y_{m,j}$ *respectively with* $H^2(X_{n,i})$ *and* $H^2(Y_{m,j})$ *finite. And suppose that* $K_0(A)$ *and* $K_0(B)$ *are torsion free. Then A is isomorphic to B if and only if*

$$(K_*(A), K_*(A)_+, \mathbf{1}_A) = (K_*(B), K_*(B)_+, \mathbf{1}_B).$$

Recently, this proposition was used in [G] to find an example of non-simple real rank zero inductive limit of direct sums of matrices over 3-dimensional finite CW complexes, which can not be expressed as an inductive limit of direct sum of matrices over 3-dimensional finite CW complexes with finite H^2-groups.

PROPOSITION 4.19. *Suppose that A and B are real rank zero unital inductive limits of direct sums of matrices over 3-dimensional finite CW complexes. And suppose that* $K_1(A) = K_1(B) = 0$. *Then A is isomorphic to B if and only if*

$$(K_*(A), K_*(A)_+, \mathbf{1}_A) \cong (K_*(B), K_*(B)_+, \mathbf{1}_B).$$

(Notice that Proposition 4.18 is (but Proposition 4.17 is not) a generalization of 5.29 of [EG1], and that Proposition 4.19 is a generalization of 5.30 of [EG1].)

Remark 4.20. As pointed out in the introduction, all results in this article hold for non unital inductive limits and for inductive limits of direct sums of homogeneous algebras over finite CW complexes $\bigoplus P_{n,i} M_{[n,i]}(C(X_{n,i})) P_{n,i}$. Also, all results hold for $X_{n,i}$ to be compact metrizable spaces of corresponding dimension (i.e., dimension to be 2 or 3), since by [Bl] (Proposition 2.3 and its proof), in any inductive limit system, the compact metrizable spaces can always be replaced by finite CW complexes of corresponding dimension.

Remark 4.21. In this article, the C^*-algebras are classified by using unsuspended E-equivalence types which can be considered to be a refinement of KK-equivalence type. So it is K-theoretical in nature. But this is not an invariant in the classical sense and it is difficult to tell whether two C^*-algebras have the same unsuspended E-equivalence type (except for the special cases of this section). We propose the following invariant as a possible replacement: the couple $(K_*(A), \Sigma_*(A))$ and the triple $(KK(A, A), [\![(A, A \otimes \mathscr{K}]\!], \alpha)$ together with module structure of $K_*(A)$ regarded as a module over $KK(A, A)$, where $[\![A, A \otimes \mathscr{K}]\!]$ is the set of equivalence class of asymptotic homomorphisms from A to $A \otimes \mathscr{K}$ and $\alpha\colon [\![A, A \otimes \mathscr{K}]\!] \to KK(A, A)$ is the canonical map. This invariant involves not only the order for K-theory but also the order for K-homology. It is not clear yet that, to what extent, the unsuspended E-equivalence type can be recovered from the above invariant.

Note added in proof. After this paper has been submitted, Dadarlat and the author complete a paper [DG], in which we proved a more general classification result in terms of more sophisticated invariants. The invariants are constructed in a sequence of papers of Dadarlat–Loring, Eilers, and Dadarlat–Gong, which solve the problem (proposed in the introduction) of finding suitable invariants for the class of nonsimple C^*-algebras of real rank zero considered in this paper. Theorem 4.14 and Proposition 4.19 are consequences of the results in [DG]. But Proposition 4.17 and Proposition 4.18 cannot be recovered from [DG]. Very recently, based on the methods used in 4.10 and 4.11 of this paper, Eilers developed techniques which can be used to recover Theorem 4.13 of this paper from [DG].

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