Henselian Valued Stable Fields

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INTRODUCTION

A field \( E \) is said to be stable, if the Schur index \( \text{ind}(D) \) of each central division \( E \)-algebra \( D \) of finite dimension is equal to the exponent \( \text{exp}(D) \), i.e., the order of the similarity class \([D]\) as an element of the Brauer group \( \text{Br}(E) \); we say that \( E \) is stable closed, if its finite extensions are stable fields. The class of stable closed fields is considerably smaller than the class of stable fields (see Proposition 4.1 and Corollaries 2.6, 4.4, and 4.5). Both classes are naturally singled out by the well-known general relations between indices and exponents of central simple algebras (cf. [P, Sects. 14.4 and 19.6]). Index-exponent relations are subject to additional restrictions when the centres of these algebras are global fields or other fields often used in algebra, number theory, and algebraic geometry (cf. [P, Sects. 17.10 and 18.6; Ar, Theorem (1.1); L, Theorems 5, 6, and 12; MS, (16.8)-(16.10)]). The corresponding results have been obtained as consequences of arithmetic, diophantine, topological, or other specific properties of the centres. In particular, the stability of global fields and local fields follows from the description of their Brauer groups by class field theory (cf. [W, Chap. XII, Sect. 2, and Chap. XIII, Sect. 6]). We refer the reader to [Ar, FSa], for examples of stable fields given by commutative algebra and the theory of algebraic surfaces. It is not known whether the mentioned results can be proved and interpreted in a unified manner. This raises the interest in the systematic study of stable fields from suitably chosen special classes.

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As a step in this direction, this paper deals with the problem of characterizing stable fields with Henselian valuations by the algebraic properties of their value groups and residue fields. The obtained results give a full answer to the discussed question for several important classes of the considered valued fields, including the classes of almost perfect equicharacteristic fields, discrete valued fields with perfect residue fields, and iterated Laurent formal power series fields in \( n \geq 2 \) indeterminates. They also show that the only obstruction to the general solution of the posed problem is laid by the relations between Schur indices and exponents of central division algebras of \( q \)-primary dimensions over Henselian valued fields with residue fields of characteristic \( q > 0 \). The paper gives a possibility of studying stable fields by applying various constructive methods. This enables one to understand better a number of properties of these fields, e.g., the behaviour of their Galois cohomological invariants and the structure of their Brauer groups (see Corollary 4.8 and [Ch₃, Ch₄]).

Throughout this paper, simple algebras are assumed to be associative with a unit and of finite dimensions over their centres. As usual, Brauer groups of fields are considered to be additively presented, Galois groups are regarded as profinite with respect to the Krull topology, and homomorphisms of profinite groups are supposed to be continuous. Our basic terminology and notations concerning valuation theory, simple algebras, Brauer groups, field extensions, Galois theory, profinite groups, and Galois cohomology are standard (as those used, for example, in [E, J, W, TY, P, L, 1, Se, K]). For convenience of the reader, we recall in Section 1 a part of the vocabulary of valuation theory, together with some preliminaries needed for the further discussion. The main results of the paper are presented in Sections 2 and 3: Theorem 2.1 describes the basic relations between value groups and residue fields of stable fields with Henselian valuations, and Theorem 3.1 gives a general sufficient condition for stability of Henselian valued fields. In Section 4, we discuss consequences of the main results. They show that the stability of a Henselian valued field allows much greater diversity in its structure than the property of being stable closed.

1. PRELIMINARIES AND THE FIRST NECESSARY CONDITION FOR STABILITY OF HENSELIAN VALUED FIELDS

Let \( K \) be a field with a Krull valuation \( \nu \), and let \( \nu(K), O_K, \) and \( \hat{K} \) be the value group, the valuation ring, and the residue field of \((K, \nu)\), respectively. We say that \( \nu(K) \) is a totally indivisible group, if it is \( p \)-indivisible (i.e., \( \nu(K) \neq p.\nu(K) \)), for every prime number \( p \). The valuation \( \nu \) is called Henselian, if \( \nu(K) \neq (0) \) and each of the following
equivalent conditions holds:

(1.1) (i) For any monic polynomial \( f \in \mathcal{O}_K[X] \), if the reduction \( \hat{f} \in \hat{K}[X] \) has a simple root \( \hat{a} \in \hat{K} \), then \( f \) has a simple root \( a \) in \( \mathcal{O}_K \) such that \( \hat{a} = \hat{a} \);
(ii) \( v \) extends to a uniquely determined valuation on each algebraic extension of \( K \);
(iii) Every division \( K \)-algebra of finite dimension has a unique valuation extending \( v \).

For a proof of the equivalence of the conditions in (1.1), we refer the reader to [R, Er, W]. It is clear from (1.1)(i) that \( K \) has the following properties when \( v \) is Henselian:

(1.2) Let \( p \) be a prime number not equal to \( \text{char} \; \hat{K} \). Then:

(i) \( 1 + \alpha \) is a \( p \)th power of an element of \( \mathcal{O}_K \) whenever \( v(\alpha) > 0 \);
(ii) The multiplicative group \( K^*/K^+ \) is isomorphic to the direct sum \( \hat{K}^*/\hat{K}^+ \oplus v(K)/pv(K) \).

Assume as above that \((K, v)\) is a Henselian valued field and let \( D \) be a division \( K \)-algebra of finite dimension \([D:K]\). The main algebraic structures associated with \( D \) are its residue division ring \( \hat{D} \) and value group \( \nu(D) \) (or \( \nu_D(D) \)) with respect to the valuation \( \nu_D \) of \( D \) extending \( v \). It is well known that \( \hat{D} \) is a \( \hat{K} \)-algebra of dimension \([\hat{D}:\hat{K}]\) dividing \([D:K]\), \( \nu(D) \) is an abelian group, and \( v(K) \) is a subgroup of \( \nu(D) \) of index \( e(\nu(D)/\nu(K)) \leq [D:K][\hat{D}:\hat{K}]^{-1} \); more precisely, by the Ostrowski–Draxl theorem, we have

(1.3) (i) \([D:K] = [\hat{D}:\hat{K}]e(\nu(D)/\nu(K))d(\nu(D)/\nu(K))\), for some natural number \( d(\nu(D)/\nu(K)) \) (called a defect of \( D \) over \( K \));
(ii) If \( d(\nu(D)/\nu(K)) \neq 1 \), then \( \text{char} \; \hat{K} = q > 0 \) and \( d(\nu(D)/\nu(K)) \) is a power of \( q \).

As shown in [TY], this leads to a complete description of the relations between defects and Schur indices of central division algebras over Henselian valued fields. The \( K \)-algebra \( D \) is called defectless (with respect to \( v \)) if \( d(\nu(D)/\nu(K)) = 1 \), and it is called immediate if \( d(\nu(D)/\nu(K)) = [D:K] \). For example, \( D \) is defectless over \( K \) provided that \([D:K] \) is not divisible by \( \text{char} \; \hat{K} \); this condition is fulfilled, if \( D \) is central over \( K \) and \( K \) is a perfect field with \( \text{char} \; K = \text{char} \; \hat{K} \) (cf. [P, Sect. 14.4; A, Chap. VII, Theorem 22]). The defectlessness of central division \( K \)-algebras is also guaranteed when \((K, v)\) is a Henselian discrete valued field or an \( n \)-discretely valued field, e.g., an iterated Laurent formal power series field in \( n \) indeterminates [TY, Propositions 2.1 and 2.2].
The $K$-algebra $D$ is said to be totally ramified, if $e(D/K) = [D:K]$. An 
algebraic extension $K'_1$ of $K$ is called totally ramified, if $e(K'_1/K) = \lbrack K'_1;K \rbrack$, for every finite extension $K'_1$ of $K$ in $K_1$. We say that $D$ is inertial over $K$, if $[D;K] = [D;K]$ and the centre $Z(D)$ of $D$ is a separable 
extension of $K$. By Theorem 2.8(a) of [JW], for every finite dimensional 
division $K'$-algebra $S$ with $Z(S)$ separable over $K$, there exists an inertial 
division $K'$-algebra $S$ such that $S$ is $K'$-isomorphic to $S$. The $K'$-algebra $S$ 
is uniquely determined by $S$ up to an isomorphism. This algebra is called an 
inertial lift of $S$ over $K$.

The main necessary condition for stability of Henselian valued fields 
obtained in [Ch] can be stated as follows:

**Proposition 1.1.** Let $K$ be a stable field with a Henselian valuation $v$. Then the residue field $K$ is also stable. Moreover, if $v(K)$ is a $p$-infinite group for some prime number $p$, then every cyclic $p$-extension $E$ of $K$ embeds as a subalgebra in each central division $K'$-algebra of index divisible by the degree $[E:K]$.

The description of the algebraic properties of stable fields with Henselian valuations is simplified by the notions of a $Z_2'$-extension, an almost perfect field, and a $p$-group of Demushkin type, and also, by some notions used in the theory of ordered fields. Let us note that a field $M$ is said to be formally real, if $-1$ is not presentable over $M$ as a finite sum of squares; $M$ is called a nonreal field, otherwise. We say that $M$ is Pythagorean, if it is formally real and the element $a^2 + b^2$ is a square in $M$, for each pair $(a, b) \in M \times M$. By a $Z_p'$-extension, we mean a Galois extension with a Galois group isomorphic to the additive group $Z_p'$ of $p$-adic integers. A field $E$ is said to be almost perfect, if every finite extension of $E$ is simple, i.e., generated by a primitive element. This condition can be restated by saying that either char $E = 0$ or char $E = q > 0$ and the degree of $E$ over the subfield $E' = \langle \alpha^q : \alpha \in E \rangle$ is equal to 1 or $q$ (cf. [Ch, (4.1)]). The described alternative is valid, for example, when $E$ is a rational function field or a formal Laurent power series field in one indeterminate over a perfect field $E$; in these cases, $E$ is perfect if and only if char $E = 0$. A field $F$ is called quasi-local, if the following statement is true, for every finite extension $F/F$:

If $F/F$ is a cyclic extension and $D$ is a central division $F_1$-algebra of index divisible by the degree $[F_1:F]$, then $F_1$ embeds in $D$ as an $F_1$-subalgebra.

It has been proved in [Ch] that the class of quasi-local fields contains the residue fields of Henselian valued stable closed fields with totally indivisible value groups. Note also that a Henselian discrete valued field $(K, v)$ lies in this class if and only if $K$ is perfect and the absolute Galois group of $K$ is metabelian of cohomological dimension $\leq 1$ [Ch, Corollary
2.5. This generalizes the well-known fact that local fields are quasi-local (see also [CF, Chap. VI, Appendix to Sect. 1]). Other examples of quasi-local fields and more information about them can be found in [Ch₂, Ch₃, Ch₄].

An infinite pro-$p$-group $P$ is said to be a $p$-group of Demushkin type, if the homomorphism $\varphi_p : H^1(P, F_p) \to H^1(P, F_p)$ mapping each $b \in H^1(P, F_p)$ into the cup-product $a \cup b$ is surjective, for every nonzero element $a \in H^1(P, F_p)$ (where $H^1(P, F_p)$ is the $j$th cohomology group of $P$ with coefficients in the prime field $F_p$ of characteristic $p$, see [Se, Chap. I, 2.2; K, 3.9]). This condition is satisfied when $P$ is a free pro-$p$-group (cf. [Se, Chap. I, 4.1 and 4.2]) or a Demushkin pro-$p$-group of finite or countable rank (cf. [D, Lab₁, Lab₂, MW]). It also holds when $P$ is isomorphic to the Galois group of the maximal $p$-extension $K(p) / K$, provided that $(K, v)$ is a Henselian valued stable nonreal field such that $v(K) \neq p.v(K)$, $K(p) \neq K$, and $K$ contains a primitive $p$th root of unity (see Proposition 1.1 and [Ch₂, Lemma 3.9; Se, Chap. II, Proposition 3.4]).

Groups of Demushkin type and quasi-local nonreal fields are related as follows (cf. [Ch₂, Theorem 4.1 and Lemmas 3.9 and 4.5]):

**Proposition 1.2.** Let $E$ be a nonreal field, $E_{sep}$ a separable closure of $E$, and $G_E := G(E_{sep} / E)$ the absolute Galois group of $E$.

(i) If $E$ is a quasi-local field and $p$ is a prime number for which the cohomological $p$-dimension $cd_p(G_E)$ is $> 0$, then each open subgroup of a Sylow pro-$p$-subgroup of $G_E$ is a $p$-group of Demushkin type;

(ii) If $E$ is perfect and the Sylow pro-$p$-subgroups of $G_E$ are of Demushkin type together with their open subgroups, whenever $cd_p(G_E) > 0$, then $E$ is quasi-local.

In addition to Proposition 1.2, the results of [Ch₂, Sect. 3] contain an essentially complete characterization of quasi-local formally real fields by their absolute Galois groups. They also give a classification of the profinite groups realizable as such Galois groups.

2. RELATIONS BETWEEN RESIDUE FIELDS AND VALUE GROUPS OF HENSELIAN VALUED STABLE FIELDS

Proposition 1.1 indicates that the study of a Henselian valued stable field $(K, v)$ should concentrate not only on the general properties of the residue field $K$, but also on the specific relations between $K$ and the quotient group $v(K) / p.v(K)$ arising when $p$ is a prime number for which $v(K) \neq p.v(K)$. The main result of this section proves that $v(K) / p.v(K)$
is a cyclic group, provided that either \(G(\hat{K}(p)/\hat{K})\) is a pro-\(p\)-group of rank \(\geq 2\) or \(\hat{K}\) is formally real and \(p = 2\). In addition, it turns out that \(v(K)/p.v(K)\) is of order \(\leq p^3\) whenever \(K\) contains a primitive \(p\)th root of unity.

**Theorem 2.1.** Let \(K\) be a stable field with a Henselian valuation \(v\) and let \(p\) be a prime number. Then, \(K, v\), the rank \(\tau(p)\) of the abelian \(p\)-group \(v(K)/p.v(K)\), the residue field \(\hat{K}\), and the maximal \(p\)-extension \(\hat{K}(p)/\hat{K}\) satisfy the following conditions:

1. If \(K\) contains a primitive \(p\)th root of unity or \(\text{char} K = p\), then \(\tau(p) \leq 3\); the equality \(\tau(p) = 3\) is possible only in the case of \(\text{char} K \neq p\).

   Moreover, if \(\text{char} K = p\) and \(\tau(p) \leq 2\), then \(K/K^p\) is a finite extension of degree \(p^\delta\), for some integer \(\delta \leq 2 - \tau(p)\).

   On the other hand, if \(\tau(p) \geq 3\) and \(K(p) \neq \hat{K}\), then \(\hat{K}\) does not contain a primitive \(p\)th root of unity.

2. If \(\tau(p) = 3\) and \(K\) contains a primitive \(p\)th root of unity, then it contains a primitive root of unity of each \(p\)-power degree and the Galois group \(G(K(p)/K)\) is isomorphic to the additive group \(Z_p^2\) of \(3\)-tuples of \(p\)-adic integers, provided with the natural topology. Moreover, if \(\text{char} K = p\), then \(\hat{K}\) is a perfect field and \(v(\pi) > t.v(p)\), for all rational integers \(t\) and every \(\pi \in K^*, \text{ such that } v(\pi) > 0\) and \(v(\pi) \notin p.v(K)\).

3. If \(\tau(p) > 2\), then \(\hat{K}(p)/\hat{K}\) is a \(Z_p\)-extension unless \(\hat{K}(p) = \hat{K}\).

4. If \(\hat{K}\) is formally real and \(v(K) \neq 2.v(K)\), then \(\hat{K}\) is Pythagorean, \(\hat{K}(2)/\hat{K}\) is a quadratic extension, and \(\tau(2) = 1\).

**Remark 2.2.** Note that if \((L,w)\) is a valued finite extension of a Henselian valued field \((K,v)\), then the value group \(w(L)\) is totally indivisible if and only if so is \(v(K)\). More precisely, it follows from (1.3) that the \(p\)-groups \(v(K)/p.v(K)\) and \(w(L)/p.w(L)\) are of one and the same rank \(\tau(p) \leq +\infty\), for each prime number \(p\). Furthermore, if \(v(K) \neq p.v(K)\), the degree \([L:K]\) is not divisible by \(p\) and \(\Sigma\) is a subset of \(K^*\), then the co-set \(\{v(\lambda) + p.v(K) : \lambda \in \Sigma\}\) is a minimal set of generators of the quotient group \(v(K)/p.v(K)\) if and only if \(\{w(\lambda) + p.w(L) : \lambda \in \Sigma\}\) is such a set for \(w(L)/p.w(L)\). Therefore, Proposition 1.2 and Theorem 2.1 can easily be applied to the study of the basic types of Henselian valued stable closed fields.

**Proof.** We show that the conditions of Theorem 2.1 are necessary to exclude the existence of any central division algebra over \(K\) which is a tensor product of two cyclic \(K\)-algebras of index \(p\). Our argument is grounded on the following lemma.
Lemma 2.3. Let $E_1$ and $E_2$ be cyclic extensions of a field $E$ of prime degree $p$. Denote by $D_i$ the cyclic $E$-algebras $(E_i, \sigma_i, b_i)$, for some elements $b_i \in E^*$, a fixed $E$-automorphism $\sigma_i$ of the field $E_i$ of order $p$, and $i = 1, 2$. Then $D_1 \otimes_E D_2$ is a central division $E$-algebra if and only if $D' := D_1 \otimes_E E_2$ is a division algebra, such that the product $\bar{d} = \prod_{i=0}^{p-1} \sigma_i^p(d)$ is not equal to $b_2$, for any $d \in D'$, $\sigma_i^p$ being the unique $E$-automorphism of $D'$ inducing $\sigma_2$ on $E_2$ and acting on $D_1$ as the identity.

Lemma 2.3 is obtained in a straightforward way from Lemma 1.2 and [A, Chap. XI, Theorems 11 and 12].

Suppose now that the field $K$ has an extension $K_3$ representable as a compositum of two distinct cyclic extensions $K_1$ and $K_2$ of $K$ of degree $p$, and let $D_1, D_2, D'$, $\sigma_1, \sigma_2, \sigma'_1, b_1$, and $b_2$ be defined as in Lemma 2.3, for $E = K$, $E_1 = K_3$, and $E_2 = K_2$. It is known (cf. [P, Sect. 13.4, Corollary] or [Ch, Lemma 3.5]) that $D'$ is a division algebra if and only if $K_2$ does not embed in $D_1$ as a $K$-subalgebra. When $D'$ is such an algebra, its unique valuation $v'$ extending $v$ has the property that $v'(\sigma_1^p(d)) = v'(d)$ and the value of the product $\bar{d}$ defined in Lemma 2.3 is an element of $p.v'(D')$, for every $d \in D'$. Note also that $\sigma_1^p$ induces in this case an automorphism of the residue division $K$-algebra of $(D', v')$. Considering $K_1$, $K_2$, and $K_3$ with their valuation extending $v$, one obtains from formula (1.3), Lemma 2.3, and these observations that $D_1 \otimes_K D_2$ is a division $K$-algebra whenever the subgroup of $v(K)/p.v(K)$ generated by the co-classes of $v(b_1)$ and $v(b_2)$ is of order $p^2$ and one of the following three conditions is fulfilled:

(a) $\hat{K}_3/\hat{K}$ is a Galois extension of degree $p^2$;
(b) char $\hat{K} = p$ and $\hat{K}_3/\hat{K}$ is a purely inseparable simple extension of degree $p^2$;
(c) char $\hat{K} = p$, $\hat{K}_1 \neq \hat{K}$, and $K_2$ is an immediate extension of $K$.

Suppose further that $K$ contains a primitive $p$th root of unity or char $K = p$. It is well known (cf. [L, Chap. VIII, Theorems 10 and 11]) that then there are elements $a_i \in K^*$ and $\xi_i \in K$, such that $K_i = K(\xi_i)$ and $\xi_i^p - c.\xi_i - a_i = 0$: $i = 1, 2$, where $c = 0$ in case of char $K \neq p$ and $c = 1$, otherwise. Lemma 2.3, formula (1.3), and the noted properties of $D'$ imply that $D_1 \otimes_K D_2$ is a division algebra in each of the following special cases:

Case 1. If the co-classes of $v(a_1)$, $v(b_1)$, $v(a_2)$, and $v(b_2)$ generate a subgroup of $v(K)/p.v(K)$ of order $p^4$. 


Case 2. If the subgroup of $v(K)/p.v(K)$ generated by the co-classes of $v(a_1), v(b_1), \text{ and } v(b_2)$ is of order $p^2$ and one of the following two statements is true:

(i) $v(K_2) = v(K)$;

(ii) $\text{char } \hat{K} = p$ and the quotient group $v(K_2)/v(K)$ is cyclic of order $p^2$.

Case 3. If $\text{char } \hat{K} = p$, $v(b_1) = v(b_2) = 0$, and if the extension $K_3/K$ and the residue classes $b_1, b_2 \in \hat{K}$ satisfy the following conditions:

(i) The quotient group $v(K_3)/v(K)$ is cyclic of order $p^2$ or $\hat{K_3}/\hat{K}$ is a purely inseparable simple extension of degree $p^2$;

(ii) The degree of the field extension $\hat{K}_3(b_1, b_2)/\hat{K}_3$ is equal to $p^2$.

Case 4. If $\text{char } \hat{K} = p$, $v(b_1) \neq p.v(K_3)$, $v(b_2) = 0$, the residue class $b_2 \in \hat{K}$ is out of $\hat{K}_3$ and the extension $K_3/K$ satisfies condition (i) of Case 3.

The four cases are considered without much difficulty because of the fact that the residue division ring $D'$ of $(D', v')$ is a field. Moreover, $D' = \hat{K}$ in Cases 1 and 2(ii), and $D'/\hat{K}$ is a purely inseparable extension in Cases 3 and 4. Hence, $D'$ has no $\hat{K}$-automorphism other than the identity except, possibly, in Case 2(i). The following lemma contains sufficient conditions for $K$ to admit field extension $K_1$, $K_2$, and $K_3$ as in Case 3.

Lemma 2.4. Assume that $(K, v)$ is a Henselian valued field with a residue field $\hat{K}$ of characteristic $p > 0$ and that $K$ contains a primitive $p$th root of unity unless $\text{char } K = p$. Assume also that $K$ has some of the following properties:

(i) $\text{char } K = 0$ and $v(p) \neq p.v(K)$;

(ii) $\text{char } K = 0$, $v(p) \in p.v(K)$ and $K$ contains an element $\pi$, such that $v(p) > v(\pi) > 0$ and $v(\pi) \notin p.v(K)$;

(iii) $\hat{K}$ is not a perfect field and either $\text{char } K = p$ or $\text{char } K = 0$ and $v(p) \in p.v(K)$;

(iv) $\text{char } K = p$, $v(K) \neq p.v(K)$ and $\hat{K}$ has more than $p$ elements.

Then the algebraic closure $\hat{K}$ of $K$ contains as subfields some cyclic extensions $K_1$ and $K_2$ of $K$ of degree $p$, such that the compositum $K_3 = K_1.K_2$ satisfies condition (i) of Case 3.

Proof. The fields $K_1$ and $K_2$ can be obtained by adjoining to $K$ some roots $\xi_1$ and $\xi_2$ in $\hat{K}$ of the equations $x^p - c.x - a_i = 0$: $i = 1, 2$, for certain elements $a_1, a_2 \in K$, $c$ being equal to 0 or 1 depending on whether
char $K = 0$ or $p$, respectively. The elements $a_1$ and $a_2$ can be determined as follows:

(A) If char $K = 0$, then $a_2 = a_1 + 1$, where

(a) $a_1 = p$ in case of $v(p) \neq p.v(K)$;
(b) $a_1 = \pi$, if $p$ and $\pi$ are as in Lemma 2.4(ii);
(c) $v(a_1) = 0$ and the residue class $\hat{a}_1$ is not a $p$th power in $\hat{K}$, provided that $\hat{K}$ is not a perfect field and $v(p) \notin p.v(K)$.

(B) If char $K = p$, then $a_2 = \lambda^p.a_1$, for some $\lambda \in K$ with $v(\lambda) = 0$ and a residue class $\hat{\lambda}$ out of the prime subfield of $\hat{K}$, and $a_1$ satisfying one of the following two conditions:

(a) $v(a_1) < 0$ and $v(a_1) \notin p.v(K)$, if $v(K) \neq p.v(K)$;
(b) if the field $\hat{K}$ is not perfect, then $a_1 = b^{-p}.a_1'$, for some elements $b, a_1' \in K^*$, such that $v(a_1') = 0 < v(b)$ and the residue class $\hat{a}_1'$ is not a $p$th power in $\hat{K}$.

Denote by $\xi_3$ the element $\xi_1 + 1 - \xi_2$ when char $K = 0$ and put $\xi_3 = \lambda.\xi_1 - \xi_2$ in case that char $K = p$.

Observe that if $a_1$ and $a_2$ are fixed as required by (A) or (B), then the equations $x^p - c.x - a_i = 0: i = 1, 2, 11$ have no root in $K$. In view of the elementary properties of the valuation $v$, this is easily verified by straightforward calculations. Furthermore, it follows from Kummer's theory and its analogue based on the Artin–Schreier theorem (cf. [L, Chap. VIII, Theorems 11, 13, 14, and 15; CF, Chap. II, Lemma 2.3)] that the fields $K_1 = K(\xi_1)$ and $K_2 = K(\xi_2)$ are distinct cyclic extensions of $K$ of degree $p^2$; hence, their compositum $K_3$ is a noncyclic abelian extension of degree $p^2$. Let $v_3$ be the valuation of $K_3$ extending $v$, and let $\bar{\xi}$ be the norm of $\xi_3$ from $K_2$ over $K_1$. A standard computation shows that $\bar{\xi}_3 = (\xi_1 + 1)^p - \xi_1^p$ if char $K = 0$, and $\bar{\xi}_3 = (\lambda^p - 1).\xi_1$ if char $K = p$. This implies that if $a_1$ and $a_2$ are determined as in (A)(a), (A)(b), or (B)(a), then $v_3(\bar{\xi}_3) \notin v(K)$. Therefore, the equality $v_3(\bar{\xi}_3) = v_3(\bar{\xi}_3)$ and formula (1.3) indicate that under any of these conditions, the quotient group $v(K_3)/v(K)$ is cyclic of order $p^2$ and is generated by the co-class of $v_3(\xi_3)$. Denote by $\xi'_3$ the element $b.\bar{\xi}_3$ if $b, a_1$ and $a_2$ are determined as in (B)(b), and put $\xi'_3 = \beta.\xi_3$, for some element $\beta \in K$ such that $v(\beta^p) = v(p^{-1}),$ if $a_1$ and $a_2$ satisfy the condition (A)(c). Arguing as above, we obtain in each of these cases that $v_3(\bar{\xi}'_3) = 0$, the residue class $\bar{\xi}'_3$ is a primitive element for $K_3$ over $K$, and the field extension $K_3/K$ is purely inseparable of degree $p^2$. Lemma 2.4 is proved.

Suppose that $(K, v)$ is a Henselian valued stable field. It follows from Lemma 2.4 and the discussion preceding it that $K, v, \tau(p)$, and $K$ satisfy the conditions of Theorem 2.1(i). A nother result of this discussion is that if
$	au(p) \geq 2$, then the maximal $p$-extension $\hat{K}(p)$ of $\hat{K}$ contains as a subfield at most one cyclic extension of $\hat{K}$ of degree $p$. In view of Galois theory and [Se, Chap. I, Proposition 25], this means that $G(\hat{K}(p)/\hat{K})$ is a pro-$p$-group of rank $\leq 1$. Applying now [Ch4, Remark 3.5(ii) and Lemma 3.8], one concludes that if $K(p) \neq K$, then $G(\hat{K}(p)/K) \cong \mathbb{Z}_p$ unless $p = 2$ and $\hat{K}$ is formally real. Thus Theorem 2.1(iii) reduces to a consequence of Theorem 2.1(iv).

We turn to the proof of Theorem 2.1(iv). Let $\hat{K}$ be a formally real field and $v(K) \neq 2.v(K)$. It is clear from Proposition 1.1 and [Ch4, Lemma 3.8] that $\hat{K}$ is Pythagorean and $\hat{K}(2) = \hat{K}(\sqrt{-1})$. Taking also into consideration that $K$ is formally real, one obtains that the equation $x^2 + y^2 = -1$ has no solution over $K$ and $\hat{K}$. In view of [P, Sect. 15.1, Lemma], this indicates that $A_{-1}(-1, -1; K)$ is an inertial central division algebra over $K$. Note also that if $\tau(2) \geq 2$, and $a$ and $b$ are elements of $K$, such that the subgroup of $v(K) / 2.v(K)$ generated by the co-classes of $v(a)$ and $v(b)$ is of order 4, then $A_{-1}(a, b; K)$ is a totally ramified central division $K$-algebra. It is now easy to see from (1.3) and Lemma 2.3 that $A_{-1}(-1, -1; K) \otimes_K A_{-1}(a, b; K)$ is a division algebra, in contradiction with the stability of its centre $K$. As $v(K) \neq 2.v(K)$, the obtained contradiction shows that $\tau(2) = 1$, which completes the proof of Theorem 2.1(iv).

It remains for us to prove Theorem 2.1(ii). Assume that $K$ contains a primitive $p$th root of unity and $\tau(p) = 3$. Lemma 2.4 and Case 2(i) of the discussion preceding it, combined with formula (1.3), imply that then $K$ and $\hat{K}$ satisfy the following conditions:

(2.1) $v(K') \neq v(K)$, for every cyclic field extension $K'/K$ of degree $p$; in particular, $\hat{K}$ has no cyclic extensions of degree $p$. Moreover, if $\text{char } \hat{K} = p$, then $\hat{K}$ is a perfect field and $t.v(p) < v(\pi)$, for all integers $t$ and every $\pi \in K$ with $v(\pi) > 0$ and $v(\pi) \neq p.v(K)$.

It is clear from (2.1) as well as from the last assertion of Theorem 2.1(i) that $\hat{K}(p) = \hat{K}$. Let $c_1$, $c_2$, and $c_3$ be elements of $K$, such that the co-classes of $v(c_1)$, $v(c_2)$, and $v(c_3)$ generate the quotient group $v(K) / p.v(K)$. As $K$ contains a primitive $p$th root of unity, its maximal $p$-extension $K(p)$ contains $p^n$th roots of $c_1$, $c_2$, and $c_3$, for all $n \in \mathbb{N}$. Denote by $K_{p}(p)$ the extension of $K$ generated by the set of these roots. We prove the remaining part of Theorem 2.1(ii) by establishing successively the following properties of $K$ and $K_{p}(p)$:

(2.2) $K$ contains a primitive $p^n$th root of unity for every $n \in \mathbb{N}$, $K_{p}(p)/K$ is an abelian extension with a Galois group isomorphic to $\mathbb{Z}_p^n$, and $K_{p}(p) = K(p)$.
Suppose first that \( p \neq \text{char } \hat{K} \). Since \( \hat{K}(p) = \hat{K} \), one obtains in this case from the Henselian property of \((K, v)\) and the availability of a primitive \( p \)-th root of unity in \( K \) that \( \hat{K} \) and \( K \) contain primitive \( p \)-th roots of unity, for every \( n \in \mathbb{N} \). Hence, \( K_0(p)/K \) is an abelian extension. More precisely, it follows from Kummer’s theory and the assumption on \( c_1 \), \( c_2 \), and \( c_3 \) that the Galois group \( G(K_0(p)/K) \) has a divisible character group and that a finite group \( G \) can be realized as a quotient group of \( G(K_0(p)/K) \) by an appropriate open subgroup \( U_G \), if and only if \( G \) is an abelian \( p \)-group of rank \( \leq 3 \). As the class of these abelian groups is closed under the formation of subgroups and quotient groups, it is easy to see from Galois theory that the considered properties of \( G(K_0(p)/K) \) are preserved by \( G(K_0(p)/K') \), for every finite extension \( K' \) of \( K \) in \( K_0(p) \). In view of [F, Theorem 23.1] and Pontrjagin’s duality [Po, Chap. 6; F, Theorem 23.1], this indicates that the character group of \( G(K_0(p)/K) \) can be presented as a direct sum of three isomorphic copies of the quasi-cyclic group \( \mathbb{Z}(p^\infty) \) and that there exists a topological group isomorphism \( G(K_0(p)/K') \cong \mathbb{Z}_p^3 \). Taking once again into account that \( \hat{K}(p) = \hat{K} \) and observing that the residue field \( K' \) of \( K \) with respect to the valuation \( v' \) extending \( v \) is a \( p \)-extension of \( \hat{K} \), one obtains that \( \hat{K}' = \hat{K} \). Since \( K \) contains a primitive \( p \)-th root of unity and \( p \neq \text{char } \hat{K} \), this implies that \( \hat{K}' = \hat{K}' \). Hence, by (1.2) and the Henselian property of \( v' \), the quotient groups \( K'^*/K'^p \) and \( v'(K')/p.v'(K') \) are isomorphic. By Remark 2.2, these abelian \( p \)-groups are of rank \( \tau(p) = 3 \), so one sees from Kummer’s theory and the established properties of \( G(K_0(p)/K') \) that every extension of \( K' \) in \( K(p) \) of degree \( p \) is a subfield of \( K_0(p) \). It is now easily proved by induction on the degree over \( K \) that every finite extension of \( K \) in \( K(p) \) is contained in \( K_0(p) \). This shows that \( K_0(p) = K(p) \) and completes the proof of (2.2) in the case of \( p \neq \text{char } \hat{K} \).

Assume now that \( \text{char } \hat{K} = p \). It suffices to show that there is a Henselian valuation of \( K \) with a residue field of zero characteristic and a value group satisfying the condition of Theorem 2.1(ii) with respect to \( p \). Since the set \( I = \{ \mu \in K : v(\mu) > t.v(p) \} \) for all integers \( t \) is a prime ideal of the valuation ring \( O_K \) of \((K, v)\), one can attach to it a valuation \( w \) of \( K \) in the same way as in [R, Sect. 1, (A.7)]. The value group \( w(K) \) is the quotient group of \( v(K) \) by the subgroup \( (h, -h \in v(K) : h = v(\beta_h)) \), for some element \( \beta_h \in (O_K \setminus I) \). The ordering \( \leq \) on \( w(K) \) is canonically induced by the ordering \( \leq \) on \( v(K) \) so that \( \omega(g_1) \leq \omega(g_2) \) whenever \( g_1, g_2 \in v(K) \) and \( g_1 \leq g_2 \), where \( \omega \) is the natural group homomorphism of \( v(K) \) on \( w(K) \). By definition, the valuation \( w \) acts on the elements of \( K^* \) as the composition \( \omega \circ v \), which indicates that it induces the trivial valuation on the prime subfield of \( K \), or what amounts to the same, that the residue field of \((K, w)\) is of characteristic zero. By Corollary 3 of [R,
Sect. 2], \((K, w)\) is a Henselian valued field; also, it follows from (2.1) that the quotient group \(w(K)/p, w(K)\) is of rank 3, and that it is generated by the co-classes of \(w(c_i)\); \(i = 1, 2, 3\). As the established properties of \(K\) and \(w\) imply (2.2), Theorem 2.1 is proved.

**Corollary 2.5.** With the assumptions being as in Theorem 2.1(ii), then \(K(p)/K\) is a totally ramified extension.

*Proof.* Let \(c_1, c_2, \) and \(c_3\) be elements of \(K\) determined as in the proof of Theorem 2.1(iii). It follows from (2.2) that the field \(K(p)\) can be presented as a union \(\bigcup_{m=1}^{\infty} K'_m\), where \(K'_m\) is an extension of \(K\) generated by the \(p^m\)th roots of \(c_1, c_2, \) and \(c_3\) in \(K(p)\), for every \(m \in N\). It becomes clear from (1.3) that \(K'_m/K\) are totally ramified extensions, for all \(m \in N\). This proves Corollary 2.5, since every finite extension \(K'\) of \(K\) in \(K(p)\) is a subfield of some \(K'_m\) (depending on \(K'\)) and \(e(K'_m/K) = e(K'_m/K')e(K'/K)\).

**Corollary 2.6.** Let \((K, v)\) be a Henselian valued stable closed field and \(p\) a prime number. Then \(\tau(p), \hat{K}, \) and each Sylow pro-\(p\)-subgroup \(\hat{G}_p\) of the absolute Galois group \(G_{\hat{K}}\) satisfy the following conditions:

(i) If \(\tau(p) = 1\) and \(\hat{G}_p \neq \{1\}\), then the open subgroups of \(\hat{G}_p\) are \(p\)-groups of Demushkin type, unless \(p = 2\) and \(\hat{K}\) is formally real;

(ii) \(\tau(p) \leq 3\) and the equality \(\tau(p) = 3\) is possible only in the case of \(v\)’s, \(\hat{K} = p\) and \(\hat{G}_p = \{1\}\). Moreover, if \(\tau(p) = 2\) and \(\hat{G}_p \neq \{1\}\), then \(\hat{G}_p\) is isomorphic to \(Z_p^\infty\);

(iii) \(\hat{K}\) is an almost perfect field whenever \(\tau(p) = \hat{K} = p\) and \(\tau(p) \geq 1\); furthermore, if \(\hat{K} = p\) and \(\tau(p) \geq 2\), then \(\hat{K}\) is perfect.

*Proof.* (i) and (ii) Let \(K_p\) be an extension of \(K\) in \(K_{sep}\) corresponding by Galois theory to some Sylow pro-\(p\)-subgroup of \(G_K\). Then \(K_p\) contains a primitive \(p\)th root of unity unless \(\hat{K} = p\), and by [Ch.4, Lemma 4.3], \(K_p\) is a stable closed field. It follows from the general properties of inertial extensions of \(K\) in \(K_{sep}\) and of the Sylow subgroups of profinite groups (cf. [J W, p. 135; Se, Chap. I, 1.4]) that \(G_p\) is isomorphic to the absolute Galois group of the residue field \(\hat{K}_p\) of the valuation \(v_p\) of \(K_p\) extending \(v\). Since the degrees of the finite extensions of \(K\) in \(K_p\) are not divisible by \(p\), Remark 2.2 indicates that the \(p\)-group \(v_p(K_p)/pv_p(K_p)\) is of rank \(\tau(p)\). It is now clear from Proposition 1.1 and the proof of Corollary 2.6 in [Ch.4] that \(K_p\) is a quasi-local field, provided that \(\tau(p) > 0\). These observations show that Corollary 2.6(i)–(ii) can be obtained as a consequence of Proposition 1.2(i) and Theorem 2.1, applied to \(K_p\) and \((K_p, v_p)\), respectively.

(iii) If \(\hat{K} = p\), our conclusion is contained in Theorem 2.1(i) and (ii), so it suffices to consider the special case of \(\hat{K} = 0\) and \(\hat{K} = p\).
Since $[\hat{L}:\hat{L}^p] = [\hat{K}:\hat{K}^p]$, for every finite extension $\hat{L}/\hat{K}$, it is easily obtained from (1.3) and Remark 2.2 that one can also assume for the proof that $K$ contains a primitive $p$th root of unity. In this way, Corollary 2.6(iii) reduces to a special case of Theorem 2.1(i) and (ii).

**Corollary 2.7.** Let $(K,v)$ be a Henselian valued stable closed field with a totally indivisible value group, and let $D$ be a central division algebra over $K$. Then every extension $\hat{L}$ of $K$ of degree dividing $\text{ind}(D)$ embeds in $D$ as a $K$-subalgebra.

**Proof.** By Corollary 2.6 of [Ch$_{4}$], and Theorem 2.1(i), $\hat{K}$ is a quasi-local almost perfect field. This, of course, also applies to the separable closure $\hat{L}_0$ of $\hat{K}$ in $\hat{L}$. Note further that $\hat{L}/\hat{L}_0$ is a purely inseparable extension whenever $\text{char} \hat{K} \neq 0$ (cf. [L$_1$, Chap. VII, Proposition 10]). These observations show that our assertion can be deduced from [Ch$_{4}$, Corollary 4.7 and Lemma 4.4], and the $\hat{L}$-isomorphism $D \otimes_{\hat{K}} \hat{L} \cong \left( D \otimes_{\hat{K}} \hat{L}_0 \right) \otimes_{\hat{K}} \hat{L}$.

3. **A SUFFICIENT CONDITION FOR STABILITY OF HENSELIAN VALUED FIELDS**

The main result of this section shows that a Henselian valued field $(K,v)$ is stable whenever $\text{char} K = 0$ and the properties of $\hat{K}$ and $v(K)$ are admissible by Proposition 1.1 and Theorem 2.1. On the other hand, it is not difficult to see that Proposition 1.1 and Theorem 2.1 do not provide generally valid sufficient conditions for stability of $K$. For example, statements [TY, Theorem 4.1] (with its proof), (1.3), and [P, Sect. 13.4, Corollary] imply, for each prime $q$, the existence of a Henselian valued field $(L_q,w_q)$ of characteristic $q$ with an algebraically closed residue field and a divisible value group, admitting (immediate) central division $L_q$-algebras of exponent $q$ and index $q^n$, for every $n \in \mathbb{N}$. This gives a good idea of the possible difficulties in analyzing the relations between Schur indices and exponents of central division $K$-algebras of $q$-primary dimensions when $\text{char} K = q$. In this section we prove that if the properties of $\hat{K}$ and $v(K)$ are admissible by Proposition 1.1 and Theorem 2.1, then only these relations can be an obstruction to the stability of $K$. We also show that this obstruction vanishes in several interesting special cases.

**Theorem 3.1.** Assume that $(K,v)$ is a Henselian valued field with a residue field $\hat{K}$ and invariants $\tau(p)$ of the value group $v(K)$ satisfying the stability conditions established by Proposition 1.1 and Theorem 2.1, for every prime number $p$. Then:

(a) $K$ is a stable field if and only if either $\text{char} \hat{K} = 0$ or $\text{char} \hat{K} = q > 0$ and $\text{exp}(\Delta_q) = \text{ind}(\Delta_q)$, for every central division $K$-algebra $\Delta_q$ of $q$-power dimension;
(b) $K$ is stable in each of the following special cases:

(i) $K$ is an almost perfect field and $\text{char } K = \text{char } \hat{K}$;

(ii) $\hat{K}$ is a perfect field of characteristic $q > 0$, $\tau(q) \leq 1$, and all central division $K$-algebras are defectless;

(iii) $\text{char } \hat{K} = q \neq \text{char } K$, $\tau(q) = 3$, and $K$ contains a primitive $q$th root of unity.

Proof. Theorem 3.1(a) is equivalent to the statement that $\exp(\Delta) = \text{ind}(\Delta) = p^k$, for every central division $K$-algebra $\Delta$ of dimension not divisible by $\text{char } \hat{K}$. It suffices to prove this statement in the special case of $\text{ind}(\Delta) = p^k$, where $p$ is a prime number not equal to $\text{char } K$ and $k \in \mathbb{N}$. Our proof is based on the following well-known result (implied by [JW, Decomposition Lemmas 5.14 and 6.2, Theorem 5.15, and Corollary 1.5]):

(3.1) $\Delta$ is similar to the $K$-algebra $S \otimes_K V \otimes_K T$, for some central division $K$-algebras $S$, $V$, and $T$ of $p$-power dimensions, such that $S$ is inertial and $T$ is totally ramified over $K$, $V$ possesses a maximal subfield $V_1$ inertial over $K$, and another maximal subfield $V_2$ totally ramified over $K$.

Moreover, if $D$ is the underlying division $K$-algebra of $S \otimes_K V$, then $\nu(V_2) = \nu(D)$ and $V_1$ is $K$-isomorphic to the centre $Z(D)$ of $D$.

We first prove that $V$ is a cyclic $K$-algebra, $\text{ind}(V) = \exp(V)$ and $\text{ind}(D) = \exp(D)$. It follows from ([JW, Theorem 2.8(b)]) and the properties of $S$ that the residue division ring $\hat{S}$ is a central algebra over $\hat{K}$; since $\hat{K}$ is assumed to be stable, the quoted theorem also shows that $\text{ind}(S) = \text{ind}(\hat{S}) = \exp(S) = \exp(S)$. As (1.3) implies that $T = V = K$ and $\Delta \cong D \cong S$ in the special case of $\nu(K) = p, \nu(K)$, one can assume further that $\nu(K) \neq p, \nu(K)$. In view of the Henselian property of $(K, \nu)$, formula (1.3), and [JW, Proposition 1.7], $Z(D)/K$ and $V_1/K$ are abelian $p$-extensions with Galois groups isomorphic to the quotient group $\nu(D)/\nu(K) = \nu(V)/\nu(K)$. Clearly, $\nu(D)/\nu(K)$ is a cyclic group, if $\tau(p) \leq 1$; so is the Galois group $G(Z(D)/K)$ in case of $\tau(p) \geq 2$, by stability condition (c) of Theorem 2.1. Hence, $Z(D)/K$ and $V_1/K$ are cyclic field extensions of degree $\text{ind}(V) = p^k$, so it follows from (3.1.2) and ([P, Sect. 15.1, Proposition 1]; [JW, Proposition 1.7]), that $V$ can be presented as a cyclic $K$-algebra $(V_1, \sigma_1, \nu)$, where $\sigma_1$ is a $K$-automorphism of the field $V_1$, of order $p^{k_1}$, and $\nu$ is some element of $K^*$, for which $\nu \in p, \nu(K)$. As the group $\nu(V)/\nu(K)$ is cyclic of order $p^{k_1}$, one obtains from ([P, Sect. 14.4, Proposition b(ii); PY, (3.19)], that $\exp(V) = p^{k_1} = \text{ind}(V)$. Put $\text{ind}(S) = \text{ind}(\hat{S}) = p^{k_2}$, $k = \min(k_1, k_2)$, and $k' = \max(k_1, k_2)$. By Galois theory, $Z(D)$ possesses a subfield $\hat{K}'$, which is a cyclic extension of $\hat{K}$ of degree $p^k$; the field $\hat{K}'$ can
be embedded as a $\hat{K}$-subalgebra in $\hat{S}$ since $\nu(K) \neq p, \nu(K)$ and central division $K$-algebras of $p$-primary dimensions satisfy the stability condition established by Proposition 1.1. As the algebras $(\hat{S} \otimes_{K} \hat{K'}) \otimes_{K'} Z(\hat{D})$ and $S \otimes_{K} Z(D)$ are isomorphic over $Z(D)$ (cf. [P, Sect. 9.4, Corollary a]), one sees from Wedderburn’s structure theorem and [Ch1, Lemma 3.5] that the underlying division algebra of $\hat{S} \otimes_{K} Z(\hat{D})$ is of index $p^{k_2-k}$ over its centre $Z(\hat{D})$. Taking also into consideration that $\nu(D)/\nu(K)$ is a cyclic group of order $p^{k_1}$, one obtains from [JW, Theorem 5.15], that $\text{ind}(D) = \exp(D) = p^{k}$. This proves that $\text{ind}(\Delta) = \exp(\Delta)$ in the case of $T = K$, since $\Delta$ is then $K$-isomorphic to $D$. Assuming further that $T \neq K$ and $\exp(T) = p^{t}$, we show that $K, S, T,$ and $V$ have the following properties:

(3.2) (i) $K$ contains a primitive root of unity $\epsilon$ of degree $p'$, $2 \leq \tau(p) \leq 3$ and either $\hat{K}(p) = \hat{K}$ or $\tau(p) = 2$ and $\hat{K}(p)/\hat{K}$ is a $\mathbb{Z}_p$-extension;

(ii) $S = K$, $T$ is a symbol $K$-algebra, and $\text{ind}(T) = p'$. Moreover, if $T = A_4(\lambda, \mu; K)$ for some $\lambda, \mu \in K^*$, then the group $\nu(T)/\nu(K)$ is generated by the co-classes of $p^{-1}.\nu(\lambda)$ and $p^{-1}.\nu(\mu)$;

(iii) $V$ is isomorphic to the cyclic $K$-algebra $(V_1, \sigma_1, \nu_1)$, for some element $\nu_1 \in K$ which is a $p'$th power in $T$ but is not a $p$th power in $K$.

It follows from [JW, Theorem 1.10] and the assumptions on $T$ that $\tau(p) \geq 2$ and $K$ contains a primitive $p'$th root of unity $\epsilon$. Since $K$ satisfies the stability conditions established by Theorem 2.1, this proves (3.2)(i). Applying (3.2)(ii) and Galois cohomology (cf. [Se, Chap. 1, Sect. 4]), one obtains that $H^2(G(\hat{K}(p)/K), F_{p}) = \{0\}$. By the Merkurjev–Suslin theorem (cf. [M-S, (11.5) and Sect. 9] or [S, Sects. 18, 19, and 21]), this means that $Br(\hat{K}) = \{0\}$; hence, $S = K$ and $S = K$. As $\tau(p) \leq 3$, it is not difficult to see from (1.3) that $\nu(T)/\nu(K)$ is an abelian $p$-group of rank $\leq 3$. More precisely, Theorem 1.10 of [JW], this observation, and the fact that $T$ is totally ramified over $K$ indicate that $\nu(T)/\nu(K)$ is isomorphic to the direct sum of two cyclic groups of order $\text{ind}(T)$. It is therefore clear from Proposition 3.14 of [PY], with its proof, that $T$ has the properties required by (3.2)(ii). Statement (3.2)(iii) is now obvious for $V = K$, so we prove it assuming that $V \neq K$ and $T = A_4(\lambda, \mu; K)$, for some $\lambda, \mu \in K^*$. We have already seen that $V$ can be identified with the cyclic $K$-algebra $(V_1, \sigma_1, \nu)$, for some $\nu \in K^*$ of value $\nu(\nu) \notin p, \nu(K)$. The assumption that $V \neq K$ shows that $V_1 \neq K$ and $\hat{K}(p) \neq K$. In view of (3.1) and (3.2)(i–ii), this means that $\tau(p) = 2$ and the co-classes of $\nu(\lambda)$ and $\nu(\mu)$ generate the group $\nu(K)/p, \nu(K)$. Therefore, $\nu$ can be presented as a product $\nu = \lambda^i.\mu^j.\nu'^{i+j}$, for suitably chosen elements $\nu \in K^*$ and $\nu' \in K^*$ with $\nu(\nu') = 0$, and some integers $i, j \geq 0$ with a greatest common divisor relatively prime to $p$. It is clear from the definition of $V$ that it is
$K$-isomorphic to $(V'_1, \sigma_1, \lambda', \mu', \nu')$, i.e., $\nu$ can be taken so that $\overline{v} = 1$. As $V'_1/K$ is an inertial cyclic $p$-extension and $Br(K)_p = \{0\}$, one obtains from [JW, Theorem 5.6] that the $K$-algebra $(V'_1, \sigma_1, u)$ is similar to $K$, whenever $u \in K$ and $v(u) = 0$. By the lemma in [P, Sect. 15.1], this is equivalent to the assertion that the norm group $N(V'_1/K)$ contains all elements of $K$ of value zero. This, applied to the elements $\nu'$ and $-\nu'$, also implies that $V$ is isomorphic to the $K$-algebras $(V'_1, \sigma_1, \lambda', \mu')$ and $(V'_1, \sigma_1, -\lambda', \mu')$. At the same time, it follows from the definition of $T$ that at least one of the elements $\lambda', \mu'$ and $-\lambda', \mu'$ is a $p'$th power in $T$. Note also that $\lambda', \mu'$ and $-\lambda', \mu'$ are not $p$th powers in $K$, since $v(\lambda', \mu') = v(-\lambda', \mu')$ is not an element of $p.v(K)$. Thus (3.2) is proved.

We are now ready to show that $\text{ind}(\Delta) = \exp(\Delta)$ in the case of $T \neq K$. Our argument is based on the fact that $S = K$, i.e., $\Delta$ is similar to $V \otimes_K T$. Let $\overline{K}$ be an algebraic closure of $K$ and $\nu_1$ an element of $K$ fixed as in (3.2)(iii). Put $m = \min\{\ell, k_1\}$, $m' = \max\{\ell, k_1\}$, and denote by $K_1$ the extension of $K$ in $\overline{K}$ generated by some $p^m$th root of $\nu_1$. Clearly, $[K_1:K] = p^m$ and $K_1$ is isomorphic to some $K$-subalgebras of $V$ and $T$. This, combined with [Ch., Lemma 3.5] and the general properties of tensor products (cf. [P, Sect. 9.3, Corollary b, and Sect. 9.4, Corollary a]), proves the existence of the following sequences of isomorphisms of $K_1$-algebras,

$$(V \otimes_K T) \otimes_{K_1} K_1 \equiv (M_{a}(\Delta)) \otimes_{K_1} K_1 \equiv M_{a}(M_{b}(\Delta_{1})) \equiv M_{a b}(\Delta_{1})$$

and

$$(V \otimes_K T) \otimes_{K_1} K_1 \equiv (V \otimes_{K_1} K_1) \otimes_{K_1} (T \otimes_{K_1} K_1) \equiv M_{p^m_{\infty}}(V'_1) \otimes_{K_1} M_{p^m}(T_1) \equiv M_{p^m_{\infty}}(M_{a}(\Delta_{1})) \equiv M_{p^m_{\infty a}}(\Delta_{1})$$

for some central division $K_1$-algebras $\Delta_{1}$, $V'_1$, and $T_1$, and some positive integers $a$, $b$, and $a'$, such that $a+p^m_{\infty} = \exp(V)$ and $b+p^m_{\infty} = \exp(T) = p^1$; in view of the above and [P, Sect. 14.4, Proposition b(ii)], this shows that $\text{ind}(\Delta) = p^m$.

Assume now that $\ell = k_1 = m$. In this case, $V'_1$ is a Kummer extension of $K$ generated by a $p'$th root of some element $\rho \in K$. As $V'_1$ is inertial over $K$, $\rho$ can be chosen so that $v(\rho) = 0$. Since the $K$-algebras $(V'_1, \sigma_1, \nu_1)$ and $(V'_1, \sigma_1', \nu_1')$ are isomorphic, for every integer $n$ not divisible by $p$ (cf. [P, Sect. 15.1, Corollary a]), one can assume without loss of generality that $\sigma_1(\rho) = \epsilon^{-1}. \rho$. In other words, $V$ can be identified with the symbol $K$-algebras $A_{e^{-1}}(\rho, \nu_1; K)$ and $A_e(\nu_1; \rho^{-1}; K)$. Moreover, statements
(3.2(iii) and [P, Sect. 15.1, Propositions a, b, and Lemma] imply that $T$ can be identified as a $K$-algebra with $A_1(v_2, \mu_2; K)$, and $\Delta$ with the underlying division algebra of $A_1(v_1, \rho^{-1}\mu_2; K)$, for some $\mu_2 \in K^*$. The valuation of the field $K$ extending $v$ (and also denoted by $v$) is unique, so we have $v(\theta) = v(\sigma(\theta))$, for every $K$-automorphism $\sigma$ of $K$ and all $\theta \in K^*$. It follows from (1.3) and (3.2(iii)) that the group $v(K)/v(K)$ is cyclic of order $p' - 1$ and is generated by the co-class of $v(\tau_1)$, $\tau_1$ being a $p'$th root of $v_1$ in $K$. Therefore, each element $\gamma \in N(K_1/K)$ can be presented as a product $\gamma = v_1(\gamma_1^{q^j} \gamma')$, for some number $j \in \{0, 1, \ldots, p' - 1\}$ and some elements $\gamma_1 \in K^*$ and $\gamma' \in K^*$ with $v(\gamma') = 0$. Applying now the second assertion of (3.2(ii)) to $v_1$ and $\mu_2$, one obtains that $\mu_1^{q^j} \not\in N(K_1/K)$ and $(\rho^{-1} \mu_2)^{p' - 1} \not\in N(K_1/K)$. Hence, by [P, Sect. 15.1, Corollary 5], $A_1(v_1, \rho^{-1}\mu_2; K)$ is a division $K$-algebra (isomorphic to $\Delta$) of exponent and index $p'$. This completes the proof of Theorem 3.1(a).

The proof of Theorem 3.1(b) is much easier. Theorem 3.1(a) shows that $\Delta$ is a central division $K$-algebra and $\text{ind}(\Delta) = \text{exp}(\Delta)$, provided that $\Delta$ is a central division $K$-algebra and $\text{ind}(\Delta)$ is not divisible by char $K$. This, combined with [A, Chap. VII, Theorem 22; Ch, Lemma 4.4], proves Theorem 3.1(b)(ii). It follows from the proof of (2.2) that if char $K = 0$, char $K = q > 0$ and $t \cdot v(q) < v(\pi)$ for all integers $t$ and every $\pi \in K^*$, such that $v(\pi) > 0$ and $v(\pi) \not\equiv q \cdot v(K)$, then $v$ induces on $K$ a Henselian valuation $w$ with a residue field $k$ of zero characteristic and a value group $w(K)$, for which $w(K)/q \cdot w(K)$ is a $q$-group of rank $\tau(q)$. Using the second part of the proof of (2.2), one also obtains that if $K$ and $v(K)$ satisfy the stability conditions of Theorem 2.1(ii) with respect to $q$, then the same applies to $K$, $k$, and $w(K)$. Thus Theorem 3.1(b)(ii) reduces to a special case of Theorem 3.1(a).

Suppose now that $K$ is a perfect field, char $K = q > 0$, $\tau(q) \leq 1$, and $\Delta_q$ is a defectless central division algebra over $K$ of $q$-power dimension. The condition $\tau(q) \leq 1$ and (1.3) imply that the quotient group $v(\Delta_q)/v(K)$ is cyclic and $\Delta_q$ possesses a subfield $T_q$ containing $K$ and such that $v(T_q) = v(\Delta_q)$; since $[T_q: K]\text{ind}(\Delta_q)$ (by [P, Sect. 13.1, Corollary 1]), this shows that the ramification index $e(\Delta_q/K)$ is bounded by $\text{ind}(\Delta_q)$. As $K$ is perfect, it is clear from [A, Chap. VII, Theorem 22] and (1.3) that the residue division $K$-algebra $\Delta_q$ is a field; moreover, by [JW, Theorem 2.9], the inertial lift $\Delta_q$ of $\Delta_q$ can be embedded as a $K$-subalgebra of $\Delta_q$, so it follows from the Henselian property of $(K, v)$ and [JW, Proposition 1.7], that $\Delta_q/K$ and $\Delta_q/K$ are cyclic field extensions of degree dividing $e(\Delta_q/K)$. It can now easily be seen from the assumption that $\Delta_q$ is defectless over $K$ that $[\Delta_q: K] = [\Delta_q: K] = e(\Delta_q/K) = [T_q: K] = \text{ind}(\Delta_q)$, and $\Delta_q$ is a tame division $K$-algebra in the sense of [JW, Sect. 6; PY, Sect. 3]. Therefore, the equality $\text{ind}(\Delta_q) = \text{exp}(\Delta_q)$ is obtained in a straightforward way from [PY, (3.19)], or [JW, Corollary 6.10]. Theorem 3.1 is proved.
**Corollary 3.2.** Let $K$ be a perfect field with a Henselian valuation $v$, such that the value group $v(K)$ is divisible and $\text{char } K = \text{char } \hat{K}$. Then $K$ is stable if and only if so is $\hat{K}$.

**Proof.** Corollary 3.2 follows from [JW, Theorem 2.8(b)], since the degrees of central division $K$-algebras are not divisible by $\text{char } K$ and therefore, Decomposition Lemmas 6.8 and 5.14 of [JW] imply that these algebras are inertial over $K$.

**Corollary 3.3.** Assume that $(K, v)$ is a Henselian valued field satisfying the following conditions:

(i) $\hat{K}$ is perfect and every central division $K$-algebra is defectless;

(ii) the quotient groups $v(K)/p.v(K)$ are of order $p$, for all prime numbers $p$.

Then $K$ is a stable field if and only if $\hat{K}$ is stable and each cyclic extension $L$ of $K$ is embeddable as a $K$-subalgebra in every central division $\hat{K}$-algebra of Schur index divisible by $[L:K]$.

**Proof.** This is an immediate consequence of Theorem 3.1 and [Ch$_4$, Corollary 2.5].

**Proposition 3.4.** Let $(K, v)$ be a Henselian valued stable field satisfying some of the following two conditions:

(i) $K$ is an almost perfect field and $\text{char } K = \text{char } \hat{K}$;

(ii) $\hat{K}$ is a perfect field, $\text{char } \hat{K} = q > 0$, $\tau(q) \leq 1$, and every finite separable extension of $K$ is defectless.

Then every totally ramified algebraic extension of $K$ is a stable field.

To prove Proposition 3.4(ii), we need the following lemma.

**Lemma 3.5.** Let $(K, v)$ be a Henselian valued field with a residue field $\hat{K}$ of characteristic $q > 0$ and a value group for which $\tau(q) \leq 1$. Then the following conditions are equivalent:

(i) Every finite separable extension of $K$ is defectless;

(ii) Every finite dimensional division $K$-algebra is defectless over its centre.

**Proof.** The implication (ii) $\rightarrow$ (i) follows from [TY, Proposition 3.1; M, Sect. 4, Theorem 2], so we prove that (i) $\rightarrow$ (ii). In view of the general structure of central simple algebras (cf. [P, Sect. 14.4]), and of statements (1.3) and [JW, Corollary 1.5], it suffices to establish the defectlessness of central division algebras of $q$-primary dimensions over an arbitrary finite extension $L$ of $K$. Lemma 3.4 of [TY] and Proposition 14 of [L$_1$, Chap.
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XII indicate that $L$ satisfies condition (i) of the lemma with respect to its valuation $v_L$ extending $v$; hence, by Theorem 3.1 of [TY], central division $L$-algebras of index $q$ are defectless. Taking also into account that the group $v_L(L)/q.v_L(L)$ is of order $q^{\pi(n)}$ (see Remark 2.2) and arguing as in the proof of Theorem 3.1(b)(ii), one obtains that each algebra of this kind contains as a maximal subfield an inertial extension of $L$. This, combined with Theorem 2 of [M, Sect. 4], the general properties of tensor products, and the basic results about inertial extensions of $L$ in $L_{sep}$ (cf. [P, Sect. 9.4, Corollary 2.2] and arguing as in the proof of Theorem 3.1 b ii , one obtains that each algebra of this kind contains as a maximal subfield an inertial extension of $L$. This, combined with Theorem 2 of [M, Sect. 4], the general properties of tensor products, and the basic results about inertial extensions of $L$ in $L_{sep}$ (cf. [P, Sect. 9.4, Corollary 2.2]), implies that every central division $L$-algebra of exponent $q$ has a splitting field inertial over $L$. Suppose now that $D$ is a central division $L$-algebra of exponent $q^n$, for some $n \in \mathbb{N}$, and denote by $\Delta$ the $q^{n-1}$th tensor power of $D$ over $L$. Since $\Delta$ is a central simple $L$-algebra of exponent $q$, $D \otimes L M$ is a central simple $M$-algebra of exponent dividing $q^{n-1}$ whenever $M$ is a splitting field of $\Delta$. Proceeding by induction on $n$, one obtains now without difficulty that $D$ is split over $L$ by some inertial extension of $L$. Therefore, by Lemma 5.1 of [JW], $D$ is defectless over $L$, so Lemma 3.5 is proved.

Proof of Proposition 3.4. Let $L$ be a totally ramified algebraic extension of $K$ and $\tau(L)$ the value group of the valuation $v_L$ on $L$ extending $v$. Arguing as in the proof of Lemma 4.3 of [Ch], one obtains that it suffices to establish the stability of $L$ on the additional assumption that $L/K$ is a finite extension. In this case, by Remark 2.2, the $p$-groups $v(L)/v(K)$ and $v(L)/v(K)$ are isomorphic for each prime $p$. As $K$ is stable and $K$ is the residue field of $(L,v_L)$, this means that $L$ and $v_L$ satisfy the necessary conditions for stability given by Theorem 2.1. Note also that $K$ is an almost perfect field if and only if $L$ is one. These observations, combined with Theorem 3.1 and Lemma 3.5, prove our assertion.

4. CHARACTERIZATIONS AND EXAMPLES OF HENSELIAN VALUED STABLE FIELDS

First we show that the spectrum of invariants $\tau(p)$ of Henselian valued stable fields with a given quasi-local perfect residue field $F_0$ is fully determined by the restrictions established by Theorem 2.1, when $p$ runs through the set of all prime numbers not equal to char $F_0$. In view of Corollary 2.6, this gives a possibility of constructing a series of Henselian valued stable fields which are not stable closed (see also Remark 4.2(ii)).

Proposition 4.1. Let $F_0$ be a quasi-local perfect field of characteristic $q \geq 0$ and let $\{n(p): p \in P_0\}$ be a sequence of integers $\geq 0$, indexed by a set of prime numbers $P_0$ containing $q$, if $q \neq 0$, and including the union $P(F_0) = P_1(F_0) \cup P_2(F_0)$, where $P_1(F_0)$ is the set of all prime numbers $p'$, for which
$F_0$ contains a primitive $p'$th root of unity, and $P_2(F_0)$ consists of these primes $p'$, for which the Galois group of the maximal $p'$-extension $F_0(p')$ of $F_0$ is not pro-cyclic. Assume that the numbers $n(p)$ satisfy the following conditions: (i) $n(p) \leq 1$, for all $p \in P_2(F_0)$, also for $p = q$ in case of $q \neq 0$, and for $p = 2$ in case that $F$ is formally real; (ii) $n(p) \leq 3 - r(p)$, for every $p \in (P(F_0) \setminus P_2(F_0))$, $r(p)$ being equal to 1, if $F_0(p) \neq F_0$, and to 0, otherwise.

Then there is a Henselian valued stable field $(K, v)$ of characteristic $q$ with a residue field $F_0$ and a value group $v(K)$ whose invariants $\tau(p)$ are equal to $n(p)$, for all $p \in P_0$, and to infinity, for all primes $p \not= P_0$.

**Proof.** We show that a field $K$ with the required properties can be found among the algebraic extensions of the union $F = \bigcup_{n=1}^{n} F_n^n$, where \( \{F_n = F_{n-1}(z_n): n \in N\} \) is an inductively defined tower of Laurent formal power series fields. Let $Z_n$ be an inversely lexicographically ordered countable direct sum of isomorphic copies of the additive group $Z$ of integer numbers, and let $v_n$ be the standard $Z^n$-valued valuations of the fields $F_n$ acting trivially on $F_0$. Since $v_n$ is Henselian and is extended to $F_{n+1}$ by $v_{n+1}$, for each $n \in N$, the sequence \( \{v_n: n \in N\} \) defines in an obvious manner a Henselian valuation $v_F$ of $F$ with a residue field $F_0$ and a value group isomorphic to $Z_n$. We denote by $v$ the unique valuation extending $v_F$ of each intermediate field of the extension $\overline{F}/F$, $\overline{F}$ being an algebraic closure of $F$, and we associate with every $p \in P_0$ a set of elements $\Sigma_p = \{\eta_{(p)}(m) \in \overline{F}: m \in N, j(p) \in N, j(p) > n(p)\}$, such that $\eta_{(p)}(1)^p = x_{j(p)}$ and $\eta_{(p)}(m)^p = \eta_{(p)}(m - 1)$, for every pair of integers $j(p) > n(p)$, $m \geq 2$. It is easy to see that the extension $K_p$ of $F$ generated by $\Sigma_p$ is totally ramified and that each finite extension of $F$ in $K_p$ is of $p$-power degree, for every $p \in P_0$. Moreover, the $p$-group $v(K_p)/p.v(K_p)$ is of rank $n(p)$ while the ranks of the $p'$-groups $v(K_p)/p'.v(K_p)$ are infinite for all prime numbers $p' \not= p$. Therefore, the compositum $K$ of the fields $K_p$, for all $p \in P_0$, is also totally ramified over $F$, $F_0$ is the residue field of $(K, v)$, the ranks of $v(K)/p.v(K)$ are equal to $n(p)$ when $p$ runs over the elements of $P_0$, and to infinity for every prime $p$ out of $P_0$. As the field $F_0$ is algebraically closed in $F$, this indicates that it is algebraically closed in $K$ as well; in particular, all roots of unity in $K$ are elements of $F_0$. In view of the assumptions in Proposition 4.1 and the established properties of $v(K)$, this implies that the triple $(K, v(K), \hat{K} = F_0)$ satisfies the conditions of Proposition 1.1 and Theorem 2.1 with respect to every prime number $p$, so it follows from Theorem 3.1 that $K$ is a stable field. Proposition 4.1 is proved.

**Remark 4.2.** (i) With the assumptions and notations being as in Proposition 4.1, assume also that $P_0$ is the set of all prime numbers and $n(p) \leq n$, for some $n \in N$. It is clear from the proof of Proposition 4.1
that the field $K$ can be obtained as a totally ramified algebraic extension of the iterated formal power series field $F_n$ in $n$ variables over $F$.

(ii) Let $(n(p): p \in P)$ be a sequence of elements of $\mathbb{N} \cup \{0, \infty\}$, indexed by the set $P$ of all prime numbers. Theorem 2.1 and Proposition 4.1 indicate that if $F_0$ is a field of characteristic zero not containing a primitive $p$th root of unity, for any $p \in (P \setminus \{2\})$, and such that $G_{F_0}$ is a metabelian group of cohomological dimension one (cf. [Ch$_2$, Example 1.3; Ch$_3$, Sect. 2]), then there exists a Henselian valued stable field with a residue field $F_0$ and invariants $\tau(p) = n(p)$, for all $p \in P$, if and only if $n(2) \leq 2$. Similarly, if $F_0$ is a finite field with $q^n$ elements and $n(q) \leq 1$, where $q = \text{char } F_0$, then we have $n(p) = \tau(p)$, for some Henselian valued stable field with a residue field $F_0$, if and only if $n(p) \leq 2$, for every $p \in P$ dividing $q^n - 1$.

In the rest of this section, we characterize the stable closed fields in some of the most interesting classes of Henselian valued fields. We consider almost exclusively nonreal fields, since the case of formally real fields with 2-indivisible value groups has been an object of a detailed research in Section 3 of [Ch$_2$].

**Proposition 4.3.** Let $(K, v)$ be a Henselian valued field satisfying at least one of the conditions (i) and (ii) of Proposition 3.4. Then $K$ is stable closed if and only if $\hat{K}$, $G_{\hat{K}}$, and $v(K)$ have the following properties:

(i) $\hat{K}$ is stable closed;

(ii) $\tau(p) \leq 3$, for every prime number $p$;

(iii) If $\tau(p) \geq 1$ and $\text{cd}_p(G_{\hat{K}}) > 0$, for some prime $p$, then $\tau(p) \leq 2$ and each open subgroup $P$ of a Sylow pro-$p$-subgroup of $G_{\hat{K}}$ is of Demushkin type; moreover, $P$ is isomorphic to $\mathbb{Z}$ in case $\tau(p) = 2$.

**Proof.** We show that $K$ is a stable closed field, provided that $\hat{K}$, $G_{\hat{K}}$, and $v(K)$ have properties (i), (ii), and (iii); the inverse implication follows from Proposition 1.1, Corollary 2.6, and the fact that $\mathbb{Z}_p$ is continuously isomorphic to its open subgroups. It is not difficult to see from [Ch$_4$, Lemma 4.5], formula (1.3), and the proof of Corollary 2.6 that one may consider only the special case in which $G_K$ is a pro-$p$-group, for some prime $p$. If $p = \text{char } K$, our assertion can be deduced from Theorem 3.1, Lemma 3.5, and Remark 2.2. Suppose that $p \neq \text{char } K$. Since $G_{\hat{K}}$ is a pro-$p$-group, we obtain from (iii) and [Ch$_4$, Lemma 3.11], that $\hat{K}$ is a quasi-local field, provided that $\tau(p) \geq 1$. It is therefore clear from (1.3), Remark 2.2, and our assumptions on $\hat{K}$, $G_{\hat{K}}$, and $v(K)$ that the properties of the value groups and residue fields of valued finite extensions of $(K, v)$ are admissible by Proposition 1.1 and Theorem 2.1. This, combined with Theorem 3.1, implies that $K$ is a stable closed field when $p \neq \text{char } K$, which completes our proof.
COROLLARY 4.4. Let $K = \hat{K}((X_1))(X_2) \ldots ((X_n))$ be an iterated formal Laurent power series field in $n \geq 3$ indeterminates over a field $\hat{K}$. Then the following statements are true:

(i) $K$ is a stable field if and only if $n = 3$, char $\hat{K} = 0$, and $\hat{K}$ does not admit proper abelian extensions;

(ii) $K$ is stable closed if and only if $n = 3$, char $\hat{K} = 0$, and $\hat{K}$ is algebraically closed.

Proof. We consider $K$ with respect to its standard $\mathbb{Z}^n$-valued valuation. This valuation is Henselian with a residue field $\hat{K}$ and invariants $\tau(p) = n$, for every prime $p$. Hence, by Theorem 2.1(i), applied to the case of $p = 2$, $K$ is not a stable field unless $n = 3$ and char $\hat{K} = 0$. Our assertions concerning the case of $n = 3$ follow from Theorem 2.1(ii), Proposition 4.3, and the elementary properties of cyclotomic extensions and Euler's $\varphi$-function (cf. [L1, Chap. VIII, Sect. 3; IR, Chap. 4, Sect. 1]).

COROLLARY 4.5. Let $K = \hat{K}((X))(Y)$ be an iterated formal power series field in two indeterminates over a field $\hat{K}$, and let $P(K)$ be the set of all prime numbers for which $\hat{K}(p) \neq \hat{K}$. Then the following is true:

(i) $K$ is a stable field if and only if $\hat{K}$ is perfect and stable, central division $\hat{K}$-algebras of index $p$ are cyclic, and the Galois group $G(\hat{K}(p)/\hat{K})$ is isomorphic to $\mathbb{Z}_p$, for every $p \in P(K)$;

(ii) $K$ is stable closed if and only if $\hat{K}$ is a perfect field with a metabelian absolute Galois group of cohomological dimension $\leq 1$.

Proof. If char $\hat{K} = 0$, our assertion can be proved by applying Theorem 3.1 and Proposition 4.3 to the standard $\mathbb{Z}^2$-valued valuation of $K$. Similarly, the necessity of the conditions on $\hat{K}$ in statements (i) and (ii) follows from Proposition 1.1, Theorem 2.1, and Corollary 2.6. Conversely, the sufficiency of the condition on $\hat{K}$ in (i) can be obtained as a consequence of Theorem 3.1 and the following statement:

Let $\hat{K}$ be an iterated formal power series field in 2 indeterminates over a perfect field $K$ of characteristic $q > 0$, such that either $\hat{K}(q) = \hat{K}$ or $G(\hat{K}(q)/\hat{K})$ is isomorphic to $\mathbb{Z}_q$. Then $\text{ind}(\Delta) = \text{exp}(\Delta)$, for every central division $K$-algebra $\Delta$ of $q$-primary dimension.

Let $\Delta$ be a central division algebra of exponent $q^n$ over a field $K$ satisfying the conditions of (4.1). For the case of $n = 1$, Tignol has established (cf. [AJ]) that $\Delta$ is a cyclic $K$-algebra of index $q$, so one may assume further that $n \geq 2$. Tignol's result, applied to the underlying division algebra of the $q^{n-1}$th tensor power of $\Delta$ over $K$, shows that there is an extension $K'$ of $K$ in $K(q)$ of degree $q$, such that $\Delta \otimes_K K'$ is a central simple $K'$-algebra of exponent $q^{n-1}$. It is not difficult to see from
the general properties of complete discrete valued fields (cf. [E, (5.7), (5.9), and (18.8)]) that \(K_1\) is isomorphic to an iterated formal power series field in 2 indeterminates over an extension \(\hat{K}_1\) of \(\hat{K}\) of degree 1 or \(q\). Since \(\hat{K}'(q) = \hat{K}(q)\) and \(G(\hat{K}(q)/\hat{K}')\) is isomorphic to \(\mathbb{Z}_q\) whenever \(\hat{K}' \neq \hat{K}\) (cf. [K, Chap. 9; Se, Chap. I, 4.5]), this enables one to prove inductively the existence of a subfield \(\Delta'\) of \(K(q)\) that is a splitting field of \(\Delta\), and an extension of \(\hat{K}\) of degree \(q^n\). Statement (4.1) follows from the obtained result and the basic theory of simple algebras (cf. [P, Sects. 13.4 and 14.4]), so Corollary 4.5(i) is proved. Observing finally that the Brauer groups of the finite extensions of \(\hat{K}\) are trivial in case \(\hat{K}\) is perfect with \(cd(\hat{G}_{\hat{K}}) \leq 1\) (cf. [Se, Chap. II, Proposition 6(b)], and also, that every finite extension \(\mathcal{L}\) of \(\hat{K}\) is isomorphic to an iterated formal power series field in 2 indeterminates over a finite extension \(\hat{L}\) of \(\hat{K}\), one concludes that Corollary 4.5(ii) can be deduced from Corollary 4.5(i), Galois theory, and [Ch₃, Lemma 1.2]. This completes our proof.

**Example.** Let \(\overline{Q}\) be an algebraic closure of the field \(Q\) of rational numbers, and let \(Q_0\) be the compositum of all normal finite extensions of \(Q\) in \(\overline{Q}\) with solvable Galois groups. It is well known (cf. [Se, Chap. II, 3.1]) that \(Q_0\) is not an algebraically closed field, and that it does not admit proper abelian extensions. Therefore, by Corollary 4.4, the power series field \(Q_0((X))(Y)((Z))\) is stable but not stable closed. The field \(Q_0\) is a normal extension of \(Q\), so it follows from [Ch₃, Lemmas 1.2 and 3.5] that every profinite abelian group \(\Gamma\) of cohomological dimension \(\leq 1\) is isomorphic to the Galois group \(G(Q_0/Q_\Gamma)\), for some subfield \(Q_\Gamma\) of \(Q_0\). It is clear from Galois theory and our assumptions that \(\Gamma\) is isomorphic to the topological group product \(\prod_p G(Q_\Gamma(p)/Q_\Gamma)\), taken over the set of prime numbers. In particular, \(G(Q_\Gamma(p)/Q_\Gamma)\) is isomorphic to the Sylow pro-\(p\)-subgroup of \(\Gamma\), for each prime \(p\); in view of [Ch₃, Lemma 1.2], this means that either \(Q_\Gamma(p)/Q_\Gamma\) is a \(Z_p\)-extension or \(Q_\Gamma(p) = Q_\Gamma\). Note also that each central division \(Q_\Gamma\)-algebra \(D\) is cyclic with \(\exp(D) = \text{ind}(D)\); since \(D\) is a locally finite dimensional algebra over \(Q\), this can be deduced from the stability of algebraic number fields, and the cyclicity of their central division algebras (cf. Lemma 4.3 of [Ch₃], with its proof). These observations, combined with Corollary 4.5, indicate that the field \(Q_\Gamma((X))(Y))\) is stable but not stable closed.

**Corollary 4.6.** A Henselian discrete valued field \((K, v)\) with a perfect residue field \(\hat{K}\) is stable closed if and only if \(\hat{K}\) is quasi-local.

**Proof.** Since finite separable extensions of \(K\) are defectless (cf. [TY, Propositions 2.2 and 3.1]) and \(\tau(p) = 1\), for every prime \(p\), this can be deduced from [Ch₂, Proposition 3.1], and Propositions 4.3(jj) and 1.2. Our
conclusion also follows from Corollary 3.3 and [Ch₄, Corollary 2.6; TY, Proposition 2.2].

**Corollary 4.7.** Let \( P \) be the set of prime numbers and \((n(p); p \in P)\) a sequence of nonnegative integer numbers. Then there exists a stable closed field \( K \), such that the \( p \)-component of \( Br(K) \) is isomorphic to a direct sum of a divisible group and a cyclic group of order \( p^{n(p)} \), for each \( p \in P \).

Corollary 4.7 shows that \( p^{n(p)}Br(K) \), for each \( p \in P \), this gives an affirmative answer to Question 2 of [FS].

**Proof.** It follows from [Ch₃, Proposition 3.4 and Corollary 2.5], the structure of Demushkin groups, and Pontrjagin's duality (cf. [Lab₂, p. 106; Po, Chap. 6]) that there is a quasi-local nonreal field \( F \) of zero characteristic, for which the \( p \)-component of the character group \( X(G_p) \) is isomorphic to the direct sum of the quasi-cyclic group \( Z(p^n) \) and a cyclic group of order \( p^{n(p)} \), for each \( p \in P \). Let \( K \) be a formal power series field in one indeterminate over \( F \). The natural discrete valuation of \( K \) is Henselian with a residue field \( F \), so Corollary 4.6 shows that \( K \) is stable closed. In view of the Witt isomorphism \( Br(K) \cong Br(F) \oplus X(G_p) \) [Wi], it now remains to be seen that \( Br(F) \) is a divisible group. Since \( cd(G_p) \leq 2 \) [Ch₂, Proposition 4.1], this can be deduced from [Se, Chap. II, Proposition 4], so Corollary 4.7 is proved.

**Corollary 4.8.** Assume that \((K, \nu)\) is a Henselian valued field satisfying at least one of the conditions (i) and (ii) of Proposition 3.4. Assume also that \( \nu(K) \) is totally indivisible and \( G_K \) is of cohomological dimension \( \leq 2 \). Then \( K \) is stable closed.

**Proof.** The conditions on \( \nu(K) \) and \( G_K \) imply the inequalities \( 1 \leq \tau(p) \leq 2 \) and \( cd_p(G_K) \leq 2 - \tau(p) \), for every prime \( p \neq \text{char} \, K \) (cf. [Ch₂, Lemma 1.2]). The assumptions on \((K, \nu)\) and the fact that \( \nu(K) \) is totally indivisible indicate that \( K \) is a perfect field of characteristic \( q \geq 0 \), and also, that \( \tau(q) = 1 \) in case \( q > 0 \). It is clear from these results and Propositions 3 and 5 of [Se, Chap. II], that \( Br(K_i) = \{0\} \) for every finite extension \( K_i \) of \( K \). Our assertion follows now directly from Proposition 4.3.

**REFERENCES**


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