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Deformations of Lie algebras using σ-derivations ☆

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Abstract

In this article we develop an approach to deformations of the Witt and Virasoro algebras based on σ-derivations. We show that σ-twisted Jacobi type identity holds for generators of such deformations. For the σ-twisted generalization of Lie algebras modeled by this construction, we develop a theory of central extensions. We show that our approach can be used to construct new deformations of Lie algebras and their central extensions, which in particular include naturally the q-deformations of the Witt and Virasoro algebras associated to q-difference operators, providing also corresponding q-deformed Jacobi identities.

Keywords: Lie algebras; Deformation theory; σ-Derivations; Extensions; Jacobi type identities; q-Witt algebras; q-Virasoro algebras

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1. Introduction

Multiplicative deformations–discretizations of derivatives have many applications in models of quantum phenomena, as well as in analysis of complex systems and processes exhibiting complete or partial scaling invariance. The key algebraic property which is shared by these differential and difference type operators and making them so useful is that they satisfy some versions of the Leibniz rule explaining how to calculate the operator on products given its action on each factor. It is desirable therefore to have a single unifying differentiation theory, which would be concerned with operators of a certain general class, satisfying generalized Leibniz rule and containing as examples the classical differentiation and other well-known derivations and differences.

The infinite-dimensional Lie algebra of complex polynomial vector fields on the unit circle, the Witt algebra, is an important example in the classical differential and integral calculus, relating it to topology and geometry, and at the same time responsible for many of its key algebraic properties. The universal enveloping algebra of the Witt algebra is isomorphic to an associative algebra with an infinite number of generators $\{d_j\colon\ j \in \mathbb{Z}\}$ and defining relations

$$[d_n, d_m] = d_n d_m - d_n d_m = (n - m)d_{n+m} \quad \text{for } n, m \in \mathbb{Z}. \tag{1}$$

The Witt algebra can also be defined as the complex Lie algebra of derivations on the algebra of Laurent polynomials $\mathbb{C}[t, t^{-1}]$ in one variable, that is, the Lie algebra of linear operators D on $\mathbb{C}[t, t^{-1}]$ satisfying the ordinary Leibniz rule $D(ab) = D(a)b + aD(b)$, with the commutator taken as the Lie algebra product. This definition will be most important in this article, as it will be taken as a starting point for generalization of the Witt algebra, incorporating operators obeying a generalized Leibniz rule twisted by an endomorphism (Definition 10).

Important examples of such twisted derivation-type operators, extensively investigated in physics and engineering and lying at the foundations of q-analysis, are the Jackson q-derivative

$$D_q(f)(t) = \frac{f(qt) - f(t)}{qt - t} \quad \text{and} \quad M_t D_q(f)(t) = \frac{f(qt) - f(t)}{q - 1}$$

acting on $\mathbb{C}[t, t^{-1}]$ or various function spaces. It satisfies a σ_q-twisted (q-deformed) Leibniz rule $D(fg) = D(f)g + \sigma_q(f)D(g)$ for the re-scaling automorphism $\sigma_q(f)(t) = f(qt)$. In this special case our general construction yields a natural q-deformation of the Witt algebra which becomes the usual Witt algebra defined by (1) when $q = 1$ (Theorem 27). This deformation is closely related to the q-deformations of the Witt algebra introduced and studied in [1,6,7,11,13,14,22,37–39]. However, our defining commutation relations in this case look somewhat different, as we obtained them, not from some conditions aiming to resolve specifically the case of q-deformations, but rather by choosing $\mathbb{C}[t, t^{-1}]$ as an example of the underlying coefficient algebra and specifying σ to be the automorphism σ_q in our general construction for σ-derivations. By simply choosing a different coefficient algebra or basic σ-derivation one can construct many other analogues and deformations of the Witt algebra. We demonstrate this by examples, constructing a class of deformations of the Witt algebra parameterized by integers defining arbitrary endomorphisms of $\mathbb{C}[t, t^{-1}]$ (Theorem 31). Also, we construct a multi-dimensional analogue of the Witt algebra by taking the underlying algebra to be the Laurent polynomials in several variables $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_n^{\pm 1}]$ and choosing σ to be an endomorphism mapping $z_1, \ldots, z_n$ to monomials (Theorem 37). The important feature of our approach is that, as in the non-deformed case, the deformations and analogues of the Witt algebra obtained by various choices of the underlying coefficient algebra, endomorphism σ and of the basic σ-derivation, are precisely the natural algebraic structures for the differential and integral type calculi and geometry based on the corresponding classes of generalized derivation and difference type operators.

The non-deformed Witt algebra has a unique, up to multiplication by a scalar, one-dimensional central Lie algebra extension, the Virasoro Lie algebra. Its universal enveloping algebra, also usually called the Virasoro algebra, is the algebra with infinite set of generators $\{d_j \colon j \in \mathbb{Z}\} \cup \{\mathbf{c}\}$ and defining relations

$$\begin{gathered}[d_j, d_k] = d_j d_k - d_k d_j = (j-k)d_{j+k} + \delta_{j+k,0}\frac{1}{12}(j+1)j(j-1)\mathbf{c}, \\ [\mathbf{c}, d_k] = \mathbf{c}d_k - d_k\mathbf{c} = 0, \quad \text{for } j, k \in \mathbb{Z}. \end{gathered} \tag{2}$$

We develop in this article a framework for the construction of central extensions of deformed Witt algebras built on σ-derivations. To this end we show first that our generalization of the Witt algebra to general σ-derivations satisfies skew-symmetry and a generalized (twisted) Jacobi identity (Theorem 5). The generalized Jacobi identity (22) has six terms, three of them twisted from inside and the other three twisted on the outside. This defines a class of non-associative algebras with multiplication satisfying skew-symmetry and such generalized Jacobi identities, and containing Lie algebras as the untwisted case. Sometimes the twisting can be put on the inside of all terms of the generalized Jacobi identity in the same way, and the terms can be coupled to yield the generalized Jacobi identity with three terms. For example, this is the case for the q-deformation of the Witt algebra in Theorem 27. Armed with this observation we define the corresponding class of non-associative algebras, calling this class hom-Lie algebras (Definition 14, Section 2.3), since it is associated with a twisting homomorphism. When the twisting homomorphism is the identity map, the generalized Jacobi identity becomes twice the usual Jacobi identity for Lie algebras, making Lie algebras into an example of hom-Lie algebras. In Section 2.4, for the class of hom-Lie algebras, we develop the central extension theory, providing homological type conditions useful for showing existence of central extensions and for their construction. Here, we required that the central extension of a hom-Lie algebra is also a hom-Lie algebra. In particular, the standard theory of central extensions of Lie algebras becomes a natural special case of the theory for hom-Lie algebras when no non-identity twisting is present. In particular, this implies that in the specific examples of deformation families of Witt and Virasoro type algebras constructed within this framework, the corresponding non-deformed Witt and Virasoro type Lie algebras are included as the algebras corresponding to those specific values of deformation parameters which remove the non-trivial twisting. In Section 4, we demonstrate the use of the central extension theory for hom-Lie algebras by applying it to the construction of a central hom-Lie algebra extension of the q-deformed Witt algebra from Theorem 27, which is a q-deformation of Virasoro Lie algebra. For $q = 1$ one indeed recovers the usual Virasoro Lie algebra as is expected from our general approach.

It should be mentioned that the use of q-deformed Jacobi identities for constructing q-deformations of the Witt and Virasoro algebras has been considered in physical and mathematical literature [1,6,7,12,26,27,37–41]. In particular, in [1] the two identities, skew-symmetry and a twisted from inside three-term Jacobi identity, almost as the one for hom-Lie algebras, have been clearly stated as a definition of a class of non-associative algebras, and then used as the conditions required to be satisfied by the central extension of a q-deformation of the Witt algebra from [14]. This results in a q-deformation of the Virasoro Lie algebra somehow related to that in the example we described in Section 4. Whether a particular deformation of the Witt or Virasoro algebra obtained by various constructions satisfy some kinds of Jacobi type identities is considered to be an important problem. The generalized twisted 6-term Jacobi identity obtained in our construction, gives automatically by specialization the deformed Jacobi identities satisfied by the corresponding particular deformations of the Witt and Virasoro algebras. There are also works employing usual and super Jacobi identities as conditions on central extensions and their deformations (for example [2,16,23,45]). Putting these works within context of our approach would be of interest.

We would also like to note that in the works [5,28,42,43], in the case of usual derivations on Laurent polynomials, it has been specifically noted that a Lie bracket can be defined by expressions somewhat resembling a special case of (20). We also would like to mention that q-deformations of the Witt and Virasoro algebras were considered indirectly as an algebra of pseudo q-difference operators based on the q-derivative on Laurent polynomials in [21, 22,24]. We believe that it should be possible, and would be of direct interest, to extend the results of these works to our general context of σ-derivations. For the reader's convenience, we have also included in the bibliography some works we know of [3,4,8–10,15,17–20, 25,29–36,44] concerned with other specific examples of deformations of Witt algebras and their applications that we believe could be considered in our framework, leaving the possibility of this as an open question for the moment.

We also feel that the further development should include using our construction for building more examples of deformed or twisted Witt and Virasoro type algebras based on differential and difference type operators on function spaces studied extensively in analysis and in numerical mathematics, and on functions on algebraic varieties important in algebraic geometry and its applications. It could be of interest to extend our constructions and examples over fields of finite characteristic, or various number fields. Development of the representation theory for the parametric families of Witt and Virasoro type algebras arising within our method, and understanding to which extent the representations of non-deformed Witt and Virasoro algebras appear as limit points will be important for applications in physics.

2. Some general considerations

2.1. Generalized derivations on commutative algebras and on UFDs

We begin with some definitions. Throughout this section, $\mathcal{A}$ is an associative $\mathbb{C}$-algebra, and σ and τ denote two different algebra endomorphisms on $\mathcal{A}$.

Definition 1. A (σ,τ)-derivation D on $\mathcal{A}$ is a $\mathbb{C}$-linear map satisfying

$$D(ab) = D(a)\tau(b) + \sigma(a)D(b),$$

where $a, b \in \mathcal{A}$. The set of all (σ,τ)-derivations on $\mathcal{A}$ is denoted by $\mathfrak{D}_{(\sigma,\tau)}(\mathcal{A})$.

Definition 2. A σ-derivation on $\mathcal{A}$ is a (σ,id)-derivation, i.e., a $\mathbb{C}$-linear map D satisfying

$$D(ab) = D(a)b + \sigma(a)D(b),$$

for $a, b \in \mathcal{A}$. We denote the set of all σ-derivations by $\mathfrak{D}_\sigma(\mathcal{A})$.

From now on, when speaking of unique factorization domains (UFD), we shall always mean a commutative associative algebra over $\mathbb{C}$ with unity 1 and with no zero-divisors, such that any element can be written in a unique way (up to a multiple of an invertible element) as a product of irreducible elements, i.e., elements which cannot be written as a product of two non-invertible elements. Examples of unique factorization domains include $\mathbb{C}[x_1,\dots,x_n]$, and the algebra $\mathbb{C}[t,t^{-1}]$ of Laurent polynomials.

When $\sigma(x)a = a\sigma(x)$ (or $\tau(x)a = a\tau(x)$) for all $x, a \in \mathcal{A}$ and in particular when $\mathcal{A}$ is commutative, $\mathfrak{D}_{(\sigma,\tau)}(\mathcal{A})$ carries a natural left (or right) $\mathcal{A}$-module structure by $(a, D) \mapsto$

$a \cdot D : x \mapsto aD(x)$. If $a, b \in \mathcal{A}$ we shall write $a \mid b$ if there is an element $c \in \mathcal{A}$ such that $ac = b$. If $S \subseteq \mathcal{A}$ is a subset of $\mathcal{A}$, a greatest common divisor, $\gcd(S)$, of S is defined as an element of $\mathcal{A}$ satisfying

$$\gcd(S) \mid a \quad \text{for } a \in S, \tag{3}$$

and

$$b \mid a \quad \text{for } a \in S \quad \Longrightarrow \quad b \mid \gcd(S). \tag{4}$$

It follows directly from the definition that

$$S \subseteq T \subseteq \mathcal{A} \quad \Longrightarrow \quad \gcd(T) \mid \gcd(S) \tag{5}$$

whenever $\gcd(S)$ and $\gcd(T)$ exist. If $\mathcal{A}$ is a unique factorization domain one can show that a $\gcd(S)$ exists for any nonempty subset S of $\mathcal{A}$ and that this element is unique up to a multiple of an invertible element in $\mathcal{A}$. Thus we are allowed to speak of *the* gcd.

Lemma 3. *Let $\mathcal{A}$ be a commutative algebra. Let σ and τ be two algebra endomorphisms on $\mathcal{A}$, and let D be a (σ, τ)-derivation on $\mathcal{A}$. Then*

$$D(x)\bigl(\tau(y) - \sigma(y)\bigr) = 0$$

for all $x \in \ker(\tau - \sigma)$ and $y \in \mathcal{A}$. Moreover, if $\mathcal{A}$ has no zero-divisors and $\sigma \neq \tau$, then

$$\ker(\tau - \sigma) \subseteq \ker D. \tag{6}$$

Proof. Let $y \in \mathcal{A}$ and let $x \in \ker(\tau - \sigma)$. Then

$$\begin{aligned} 0 &= D(xy - yx) = D(x)\tau(y) + \sigma(x)D(y) - D(y)\tau(x) - \sigma(y)D(x) \\ &= D(x)\bigl(\tau(y) - \sigma(y)\bigr) - D(y)\bigl(\tau(x) - \sigma(x)\bigr) = D(x)\bigl(\tau(y) - \sigma(y)\bigr). \end{aligned}$$

Furthermore, if $\mathcal{A}$ has no zero-divisors and if there is a $y \in \mathcal{A}$ such that $\tau(y) \neq \sigma(y)$ then $D(x) = 0$. □

Theorem 4. *Let σ and τ be different algebra endomorphisms on a unique factorization domain $\mathcal{A}$. Then $\mathcal{D}_{(\sigma,\tau)}(\mathcal{A})$ is free of rank one as an $\mathcal{A}$-module with generator*

$$\Delta := \frac{(\tau - \sigma)}{g} : \quad x \mapsto \frac{(\tau - \sigma)(x)}{g}, \tag{7}$$

where $g = \gcd((\tau - \sigma)(\mathcal{A}))$.

Proof. We note first that $(\tau - \sigma)/g$ is a (σ, τ)-derivation on $\mathcal{A}$:

$$\begin{aligned}\frac{(\tau - \sigma)(xy)}{g} &= \frac{\tau(x)\tau(y) - \sigma(x)\sigma(y)}{g} \\ &= \frac{(\tau(x) - \sigma(x))\tau(y) + \sigma(x)(\tau(y) - \sigma(y))}{g} \\ &= \frac{(\tau - \sigma)(x)}{g} \cdot \tau(y) + \sigma(x) \cdot \frac{(\tau - \sigma)(y)}{g},\end{aligned}$$

for $x, y \in \mathcal{A}$. Next we show that $(\tau - \sigma)/g$ generates a free $\mathcal{A}$-module of rank one. So suppose that

$$x \cdot \frac{\tau - \sigma}{g} = 0, \tag{8}$$

for some $x \in \mathcal{A}$. Since $\tau \neq \sigma$, there is an $y \in \mathcal{A}$ such that $(\tau - \sigma)(y) \neq 0$. Application of both sides in (8) to this y yields

$$x \cdot \frac{(\tau - \sigma)(y)}{g} = 0.$$

Since $\mathcal{A}$ has no zero-divisors, it then follows that $x = 0$. Thus

$$\mathcal{A} \cdot \frac{\tau - \sigma}{g}$$

is a free $\mathcal{A}$-module of rank one.

It remains to show that $\mathcal{D}_{(\sigma,\tau)}(\mathcal{A}) \subseteq \mathcal{A} \cdot \frac{\tau - \sigma}{g}$. Let D be a (σ, τ)-derivation on $\mathcal{A}$. We want to find $a_D \in \mathcal{A}$ such that

$$D(x) = a_D \cdot \frac{(\tau - \sigma)(x)}{g} \tag{9}$$

for $x \in \mathcal{A}$. We will define

$$a_D = \frac{D(x) \cdot g}{(\tau - \sigma)(x)} \tag{10}$$

for some x such that $(\tau - \sigma)(x) \neq 0$. For this to be possible, we must show two things. First of all, that

$$(\tau - \sigma)(x) \mid D(x) \cdot g \quad \text{for any } x \text{ with } (\tau - \sigma)(x) \neq 0 \tag{11}$$

and secondly, that

$$\frac{D(x) \cdot g}{(\tau - \sigma)(x)} = \frac{D(y) \cdot g}{(\tau - \sigma)(y)} \quad \text{for any } x, y \text{ with } (\tau - \sigma)(x) \neq 0 \neq (\tau - \sigma)(y). \tag{12}$$

Suppose for a moment that (11) and (12) were true. Then it is clear that if we define a_D by (10), the formula (9) holds for any $x \in \mathcal{A}$ satisfying $(\tau - \sigma)(x) \neq 0$. But (9) also holds when $x \in \mathcal{A}$ is such that $(\tau - \sigma)(x) = 0$, because then $D(x) = 0$ also, by Lemma 3.

We first prove (11). Let $x, y \in \mathcal{A}$ be such that $(\tau - \sigma)(x) \neq 0 \neq (\tau - \sigma)(y)$. Then we have

$$\begin{aligned} 0 &= D(xy - yx) = D(x)\tau(y) + \sigma(x)D(y) - D(y)\tau(x) - \sigma(y)D(x) \\ &= D(x)\big(\tau(y) - \sigma(y)\big) - D(y)\big(\tau(x) - \sigma(x)\big), \end{aligned}$$

so that

$$D(x)\big(\tau(y) - \sigma(y)\big) = D(y)\big(\tau(x) - \sigma(x)\big). \tag{13}$$

Now define a function $h : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by setting

$$h(z, w) = \gcd\big(\tau(z) - \sigma(z), \tau(w) - \sigma(w)\big) \quad \text{for } z, w \in \mathcal{A}.$$

By the choice of x and y, we have $h(x, y) \neq 0$. Divide both sides of (13) by $h(x, y)$:

$$D(x)\frac{\tau(y) - \sigma(y)}{h(x, y)} = D(y)\frac{\tau(x) - \sigma(x)}{h(x, y)}. \tag{14}$$

It is true that

$$\gcd\left(\frac{\tau(y) - \sigma(y)}{h(x, y)}, \frac{\tau(x) - \sigma(x)}{h(x, y)}\right) = 1.$$

Therefore, using that $\mathcal{A}$ is a UFD, we deduce from (14) that

$$\frac{\tau(x) - \sigma(x)}{h(x, y)} \,\Big|\, D(x),$$

that is, that,

$$(\tau - \sigma)(x) \mid D(x) \cdot h(x, y) \tag{15}$$

for any $x, y \in \mathcal{A}$ with $(\tau - \sigma)(x) \neq 0 \neq (\tau - \sigma)(y)$. Let $S = \mathcal{A} \setminus \ker(\tau - \sigma)$. Then from (15) and property (4) of the gcd we get

$$(\tau - \sigma)(x) \mid D(x) \cdot \gcd\big(h(x, S)\big) \tag{16}$$

for all $x \in \mathcal{A}$ with $(\tau - \sigma)(x) \neq 0$. But

$$\begin{aligned}\gcd\bigl(h(x,S)\bigr) &= \gcd\bigl(\bigl\{\gcd\bigl((\tau-\sigma)(x),(\tau-\sigma)(s)\bigr)\colon s\in S\bigr\}\bigr)\\ &= \gcd\bigl((\tau-\sigma)(S)\cup\bigl\{(\tau-\sigma)(x)\bigr\}\bigr)\\ &= \gcd\bigl((\tau-\sigma)(\mathcal{A})\cup\bigl\{(\tau-\sigma)(x)\bigr\}\bigr)\\ &= g.\end{aligned}$$

Thus (16) is equivalent to (11) which was to be proved.

Finally, we prove (12). Let $x, y \in \mathcal{A}$ be such that $(\tau-\sigma)(x) \neq 0 \neq (\tau-\sigma)(y)$. Then

$$\begin{aligned}0 &= D(xy-yx) = D(x)\tau(y)+\sigma(x)D(y)-D(y)\tau(x)-\sigma(y)D(x)\\ &= D(x)\bigl(\tau(y)-\sigma(y)\bigr)-D(y)\bigl(\tau(x)-\sigma(x)\bigr),\end{aligned}$$

which, after multiplication by g and division by $(\tau-\sigma)(x)\cdot(\tau-\sigma)(y)$ proves (12). This completes the proof of the existence of a_D, and hence the proof of the theorem. $\square$

2.2. *A bracket on σ-derivations*

The Witt algebra is isomorphic to the Lie algebra $\mathcal{D}(\mathbb{C}[t,t^{-1}])$ of all derivations of the commutative unital algebra of all complex Laurent polynomials:

$$\mathbb{C}\bigl[t,t^{-1}\bigr] = \left\{\sum_{k\in\mathbb{Z}} a_k t^k\colon a_k\in\mathbb{C},\ \text{only finitely many non-zero}\right\}.$$

In this section we will use this fact as a starting point for a generalization of the Witt algebra to an algebra consisting of σ-derivations.

We let $\mathcal{A}$ be a commutative associative algebra over $\mathbb{C}$ with unity 1, as in the example $\mathcal{A}=\mathbb{C}[t,t^{-1}]$ from the previous paragraph. When we speak of homomorphisms (endomorphisms) in the sequel we will always mean *algebra* homomorphisms (endomorphisms), except where otherwise indicated. If $\sigma:\mathcal{A}\to\mathcal{A}$ is a homomorphism of algebras, we denote, as before, the $\mathcal{A}$-module of all σ-derivations on $\mathcal{A}$ by $\mathcal{D}_\sigma(\mathcal{A})$. For clarity we will denote the module multiplication by $\cdot$ and the algebra multiplication in $\mathcal{A}$ by juxtaposition. The *annihilator* $\operatorname{Ann}(D)$ of an element $D\in\mathcal{D}_\sigma(\mathcal{A})$ is the set of all $a\in\mathcal{A}$ such that $a\cdot D=0$. It is easy to see that $\operatorname{Ann}(D)$ is an ideal in $\mathcal{A}$ for any $D\in\mathcal{D}_\sigma(\mathcal{A})$.

We now fix a homomorphism $\sigma:\mathcal{A}\to\mathcal{A}$, an element $\Delta\in\mathcal{D}_\sigma(\mathcal{A})$, and an element $\delta\in\mathcal{A}$, and we assume that these objects satisfy the following two conditions:

$$\sigma\bigl(\operatorname{Ann}(\Delta)\bigr)\subseteq\operatorname{Ann}(\Delta), \tag{17}$$

$$\Delta\bigl(\sigma(a)\bigr)=\delta\sigma\bigl(\Delta(a)\bigr),\quad \text{for } a\in\mathcal{A}. \tag{18}$$

Let

$$\mathcal{A}\cdot\Delta=\{a\cdot\Delta\colon a\in\mathcal{A}\}$$

denote the cyclic $\mathcal{A}$-submodule of $\mathfrak{D}_\sigma(\mathcal{A})$ generated by Δ. We have the following theorem, which introduces a $\mathbb{C}$-algebra structure on $\mathcal{A}\cdot\Delta$.

Theorem 5. *If* (17) *holds then the map*

$$\langle\cdot,\cdot\rangle_\sigma : \mathcal{A}\cdot\Delta\times\mathcal{A}\cdot\Delta\to\mathcal{A}\cdot\Delta$$

defined by setting

$$\langle a\cdot\Delta, b\cdot\Delta\rangle_\sigma = \big(\sigma(a)\cdot\Delta\big)\circ(b\cdot\Delta)-\big(\sigma(b)\cdot\Delta\big)\circ(a\cdot\Delta),\quad \textit{for } a,b\in\mathcal{A}, \tag{19}$$

where $\circ$ denotes composition of functions, is a well-defined $\mathbb{C}$-algebra product on the $\mathbb{C}$-linear space $\mathcal{A}\cdot\Delta$, satisfying the following identities for $a,b,c\in\mathcal{A}$:

$$\langle a\cdot\Delta, b\cdot\Delta\rangle_\sigma = \big(\sigma(a)\Delta(b)-\sigma(b)\Delta(a)\big)\cdot\Delta, \tag{20}$$

$$\langle a\cdot\Delta, b\cdot\Delta\rangle_\sigma = -\langle b\cdot\Delta, a\cdot\Delta\rangle_\sigma. \tag{21}$$

In addition, if (18) *holds, then we have the Jacobi-like identity*:

$$\begin{aligned}
&\big\langle\sigma(a)\cdot\Delta,\langle b\cdot\Delta,c\cdot\Delta\rangle_\sigma\big\rangle_\sigma+\delta\cdot\big\langle a\cdot\Delta,\langle b\cdot\Delta,c\cdot\Delta\rangle_\sigma\big\rangle_\sigma\\
&\quad+\big\langle\sigma(b)\cdot\Delta,\langle c\cdot\Delta,a\cdot\Delta\rangle_\sigma\big\rangle_\sigma+\delta\cdot\big\langle b\cdot\Delta,\langle c\cdot\Delta,a\cdot\Delta\rangle_\sigma\big\rangle_\sigma\\
&\quad+\big\langle\sigma(c)\cdot\Delta,\langle a\cdot\Delta,b\cdot\Delta\rangle_\sigma\big\rangle_\sigma+\delta\cdot\big\langle c\cdot\Delta,\langle a\cdot\Delta,b\cdot\Delta\rangle_\sigma\big\rangle_\sigma=0.
\end{aligned} \tag{22}$$

Remark 6. An important thing to notice is that the bracket $\langle\cdot,\cdot\rangle_\sigma$ defined in the theorem depends on the generator Δ of the cyclic submodule $\mathcal{A}\cdot\Delta$ of $\mathfrak{D}_\sigma(\mathcal{A})$ in an essential way. This reveals that one should in fact write $\langle\cdot,\cdot\rangle_{\sigma,\Delta}$ to explicitly indicate which Δ is chosen. Suppose, however, we choose another generator Δ' of $\mathcal{A}\cdot\Delta$. Then $\Delta'=u\Delta$ for an element $u\in\mathcal{A}$ (not necessarily a unit). Take elements $a\cdot\Delta', b\cdot\Delta'\in\mathcal{A}\cdot\Delta$. Then the following calculation shows how two different brackets relate when changing the generator (we use the commutativity of $\mathcal{A}$ freely):

$$\begin{aligned}
&\sigma(u)\langle a\cdot\Delta', b\cdot\Delta'\rangle_{\sigma,\Delta'}\quad \text{[the definition of the bracket]}\\
&\quad=\big(\sigma(a)u\sigma(u)\cdot\Delta\big)\circ(bu\Delta)-\big(\sigma(b)u\sigma(u)\cdot\Delta\big)\circ(au\cdot\Delta)\\
&\quad=u\cdot\Big(\big(\sigma(au)\cdot\Delta\big)\circ(bu\cdot\Delta)-\big(\sigma(bu)\cdot\Delta\big)\circ(au\cdot\Delta)\Big)\\
&\quad=u\cdot\langle au\cdot\Delta, bu\cdot\Delta\rangle_{\sigma,\Delta}=u\cdot\langle a\cdot\Delta', b\cdot\Delta'\rangle_{\sigma,\Delta}
\end{aligned}$$

so the “base change”-relation is

$$\sigma(u)\cdot\langle a\cdot\Delta', b\cdot\Delta'\rangle_{\sigma,\Delta'}=u\cdot\langle a\cdot\Delta', b\cdot\Delta'\rangle_{\sigma,\Delta}.$$

For the most part of this paper, we have a fixed generator and so we suppress the dependence on the generator from the bracket notation and simply write $\langle\cdot,\cdot\rangle_\sigma$. On the other hand, if $\mathcal{A}$ has no zero-divisors, we shall see later in Proposition 9 that the dependence of the generator Δ is not essential.

Remark 7. The identity (20) is just a formula expressing the product defined in (19) as an element of $\mathcal{A} \cdot \Delta$. Identities (21) and (22) are more essential, expressing, respectively, skew-symmetry and a generalized ((σ, δ)-twisted) Jacobi identity for the product defined by (19).

Before coming to the proof of the theorem we introduce a convenient notation. If $f : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A} \cdot \Delta$ is a function, we will write

$$\circlearrowleft_{a,b,c} f(a, b, c)$$

for the cyclic sum

$$f(a, b, c) + f(b, c, a) + f(c, a, b).$$

We note the following properties of the cyclic sum:

$$\circlearrowleft_{a,b,c} \big(x \cdot f(a, b, c) + y \cdot g(a, b, c)\big) = x \cdot \circlearrowleft_{a,b,c} f(a, b, c) + y \cdot \circlearrowleft_{a,b,c} g(a, b, c),$$
$$\circlearrowleft_{a,b,c} f(a, b, c) = \circlearrowleft_{a,b,c} f(b, c, a) = \circlearrowleft_{a,b,c} f(c, a, b),$$

where $f, g : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A} \cdot \Delta$ are two functions, and $x, y \in \mathcal{A}$. Combining these two identities we obtain

$$\begin{aligned} \circlearrowleft_{a,b,c} \big(f(a, b, c) + g(a, b, c)\big) &= \circlearrowleft_{a,b,c} \big(f(a, b, c) + g(b, c, a)\big) \\ &= \circlearrowleft_{a,b,c} \big(f(a, b, c) + g(c, a, b)\big). \end{aligned} \tag{23}$$

With this notation, (22) can be written

$$\circlearrowleft_{a,b,c} \big\{\big\langle\sigma(a) \cdot \Delta, \langle b \cdot \Delta, c \cdot \Delta\rangle_\sigma\big\rangle_\sigma + \delta \cdot \big\langle a \cdot \Delta, \langle b \cdot \Delta, c \cdot \Delta\rangle_\sigma\big\rangle_\sigma\big\} = 0. \tag{24}$$

We now turn to the proof of Theorem 5.

Proof. We must first show that $\langle \cdot, \cdot \rangle_\sigma$ is a well-defined function. That is, if $a_1 \cdot \Delta = a_2 \cdot \Delta$, then

$$\langle a_1 \cdot \Delta, b \cdot \Delta\rangle_\sigma = \langle a_2 \cdot \Delta, b \cdot \Delta\rangle_\sigma \tag{25}$$

and

$$\langle b \cdot \Delta, a_1 \cdot \Delta \rangle_\sigma = \langle b \cdot \Delta, a_2 \cdot \Delta \rangle_\sigma, \tag{26}$$

for $b \in \mathcal{A}$. Now $a_1 \cdot \Delta = a_2 \cdot \Delta$ is equivalent to $a_1 - a_2 \in \operatorname{Ann}(\Delta)$. Therefore, using the assumption (17), we also have $\sigma(a_1 - a_2) \in \operatorname{Ann}(\Delta)$. Hence

$$\begin{aligned}
&\langle a_1 \cdot \Delta, b \cdot \Delta \rangle_\sigma - \langle a_2 \cdot \Delta, b \cdot \Delta \rangle_\sigma \\
&\quad = \big(\sigma(a_1) \cdot \Delta\big) \circ (b \cdot \Delta) - \big(\sigma(b) \cdot \Delta\big) \circ (a_1 \cdot \Delta) - \big(\sigma(a_2) \cdot \Delta\big) \circ (b \cdot \Delta) \\
&\qquad + \big(\sigma(b) \cdot \Delta\big) \circ (a_2 \cdot \Delta) \\
&\quad = \big(\sigma(a_1 - a_2) \cdot \Delta\big) \circ (b \cdot \Delta) - \big(\sigma(b) \cdot \Delta\big) \circ \big((a_1 - a_2) \cdot \Delta\big) \\
&\quad = 0,
\end{aligned}$$

which shows (25). The proof of (26) is analogous.

Next we prove (20), which also shows that $\mathcal{A} \cdot \Delta$ is closed under $\langle \cdot, \cdot \rangle_\sigma$. Let $a, b, c \in \mathcal{A}$ be arbitrary. Then, since Δ is a σ-derivation on $\mathcal{A}$ we have

$$\begin{aligned}
&\langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma(c) \\
&\quad = \big(\sigma(a) \cdot \Delta\big)\big((b \cdot \Delta)(c)\big) - \big(\sigma(b) \cdot \Delta\big)\big((a \cdot \Delta)(c)\big) \\
&\quad = \sigma(a)\Delta\big(b\Delta(c)\big) - \sigma(b)\Delta\big(a\Delta(c)\big) \\
&\quad = \sigma(a)\big(\Delta(b)\Delta(c) + \sigma(b)\Delta\big(\Delta(c)\big)\big) - \sigma(b)\big(\Delta(a)\Delta(c) + \sigma(a)\Delta\big(\Delta(c)\big)\big) \\
&\quad = \big(\sigma(a)\Delta(b) - \sigma(b)\Delta(a)\big)\Delta(c) + \big(\sigma(a)\sigma(b) - \sigma(b)\sigma(a)\big)\Delta\big(\Delta(c)\big).
\end{aligned}$$

Since $\mathcal{A}$ is commutative, the last term is zero. Thus (20) is true. The skew-symmetry identity (21) is clear from the definition (19). Using the linearity of σ and Δ, and the definition of $\langle \cdot, \cdot \rangle_\sigma$, or the formula (20), it is also easy to see that $\langle \cdot, \cdot \rangle_\sigma$ is bilinear.

It remains to prove (22). Using (20) and that Δ is a σ-derivation on $\mathcal{A}$ we get

$$\begin{aligned}
&\big\langle \sigma(a) \cdot \Delta, \langle b \cdot \Delta, c \cdot \Delta \rangle_\sigma \big\rangle_\sigma \\
&\quad = \big\langle \sigma(a) \cdot \Delta, \big(\sigma(b)\Delta(c) - \sigma(c)\Delta(b)\big) \cdot \Delta \big\rangle_\sigma \\
&\quad = \big\{\sigma^2(a)\Delta\big(\sigma(b)\Delta(c) - \sigma(c)\Delta(b)\big) - \sigma\big(\sigma(b)\Delta(c) - \sigma(c)\Delta(b)\big)\Delta\big(\sigma(a)\big)\big\} \cdot \Delta \\
&\quad = \big\{\sigma^2(a)\big(\Delta\big(\sigma(b)\big)\Delta(c) + \sigma^2(b)\Delta^2(c) - \Delta\big(\sigma(c)\big)\Delta(b) - \sigma^2(c)\Delta^2(b)\big) \\
&\qquad - \big(\sigma^2(b)\sigma\big(\Delta(c)\big) - \sigma^2(c)\sigma\big(\Delta(b)\big)\big)\Delta\big(\sigma(a)\big)\big\} \cdot \Delta \\
&\quad = \sigma^2(a)\Delta\big(\sigma(b)\big)\Delta(c) \cdot \Delta + \sigma^2(a)\sigma^2(b)\Delta^2(c) \cdot \Delta - \sigma^2(a)\Delta\big(\sigma(c)\big)\Delta(b) \cdot \Delta \\
&\qquad - \sigma^2(a)\sigma^2(c)\Delta^2(b) \cdot \Delta - \sigma^2(b)\sigma\big(\Delta(c)\big)\Delta\big(\sigma(a)\big) \cdot \Delta \\
&\qquad + \sigma^2(c)\sigma\big(\Delta(b)\big)\Delta\big(\sigma(a)\big) \cdot \Delta,
\end{aligned} \tag{27}$$

where $\sigma^2 = \sigma \circ \sigma$ and $\Delta^2 = \Delta \circ \Delta$. Applying cyclic summation to the second and fourth term in (27) we get

$$\begin{aligned}
&\circlearrowleft_{a,b,c} \left\{\sigma^2(a)\sigma^2(b)\Delta^2(c)\cdot\Delta - \sigma^2(a)\sigma^2(c)\Delta^2(b)\cdot\Delta\right\} \\
&\quad = \circlearrowleft_{a,b,c} \left\{\sigma^2(a)\sigma^2(b)\Delta^2(c)\cdot\Delta - \sigma^2(b)\sigma^2(a)\Delta^2(c)\cdot\Delta\right\} = 0,
\end{aligned}$$

using (23) and that $\mathcal{A}$ is commutative. Similarly, if we apply cyclic summation to the fifth and sixth term in (27) and use the relation (18) we obtain

$$\begin{aligned}
&\circlearrowleft_{a,b,c} \left\{-\sigma^2(b)\sigma\big(\Delta(c)\big)\Delta\big(\sigma(a)\big)\cdot\Delta + \sigma^2(c)\sigma\big(\Delta(b)\big)\Delta\big(\sigma(a)\big)\cdot\Delta\right\} \\
&\quad = \circlearrowleft_{a,b,c} \left\{-\sigma^2(b)\sigma\big(\Delta(c)\big)\delta\sigma\big(\Delta(a)\big)\cdot\Delta + \sigma^2(c)\sigma\big(\Delta(b)\big)\delta\sigma\big(\Delta(a)\big)\cdot\Delta\right\} \\
&\quad = \delta\cdot\circlearrowleft_{a,b,c} \left\{-\sigma^2(b)\sigma\big(\Delta(c)\big)\sigma\big(\Delta(a)\big)\cdot\Delta + \sigma^2(b)\sigma\big(\Delta(a)\big)\sigma\big(\Delta(c)\big)\cdot\Delta\right\} \\
&\quad = 0,
\end{aligned}$$

where we again used (23) and the commutativity of $\mathcal{A}$. Consequently, the only terms in the right-hand side of (27) which do not vanish when we take cyclic summation are the first and the third. In other words,

$$\begin{aligned}
&\circlearrowleft_{a,b,c} \big\langle\sigma(a)\cdot\Delta, \langle b\cdot\Delta, c\cdot\Delta\rangle_\sigma\big\rangle_\sigma \\
&\quad = \circlearrowleft_{a,b,c} \left\{\sigma^2(a)\Delta\big(\sigma(b)\big)\Delta(c)\cdot\Delta - \sigma^2(a)\Delta\big(\sigma(c)\big)\Delta(b)\cdot\Delta\right\}.
\end{aligned} \tag{28}$$

We now consider the other term in (24). First note that from (20) we have

$$\langle b\cdot\Delta, c\cdot\Delta\rangle_\sigma = \big(\Delta(c)\sigma(b) - \Delta(b)\sigma(c)\big)\cdot\Delta$$

since $\mathcal{A}$ is commutative. Using first this and then (20) we get

$$\begin{aligned}
&\delta\cdot\big\langle a\cdot\Delta, \langle b\cdot\Delta, c\cdot\Delta\rangle_\sigma\big\rangle_\sigma \\
&\quad = \delta\cdot\big\langle a\cdot\Delta, \big(\Delta(c)\sigma(b) - \Delta(b)\sigma(c)\big)\cdot\Delta\big\rangle_\sigma \\
&\quad = \delta\Big(\sigma(a)\Delta\big(\Delta(c)\sigma(b) - \Delta(b)\sigma(c)\big) - \sigma\big(\Delta(c)\sigma(b) - \Delta(b)\sigma(c)\big)\Delta(a)\Big)\cdot\Delta \\
&\quad = \delta\Big\{\sigma(a)\Big(\Delta^2(c)\sigma(b) + \sigma\big(\Delta(c)\big)\Delta\big(\sigma(b)\big) - \Delta^2(b)\sigma(c) - \sigma\big(\Delta(b)\big)\Delta\big(\sigma(c)\big)\Big) \\
&\qquad - \Big(\sigma\big(\Delta(c)\big)\sigma^2(b) - \sigma\big(\Delta(b)\big)\sigma^2(c)\Big)\Delta(a)\Big\}\cdot\Delta \\
&\quad = \delta\sigma(a)\Delta^2(c)\sigma(b)\cdot\Delta + \delta\sigma(a)\sigma\big(\Delta(c)\big)\Delta\big(\sigma(b)\big)\cdot\Delta - \delta\sigma(a)\Delta^2(b)\sigma(c)\cdot\Delta \\
&\qquad - \delta\sigma(a)\sigma\big(\Delta(b)\big)\Delta\big(\sigma(c)\big)\cdot\Delta - \delta\sigma\big(\Delta(c)\big)\sigma^2(b)\Delta(a)\cdot\Delta \\
&\qquad + \delta\sigma\big(\Delta(b)\big)\sigma^2(c)\Delta(a)\cdot\Delta.
\end{aligned}$$

Using (18), this is equal to

$$\begin{aligned}
&\delta\sigma(a)\Delta^2(c)\sigma(b)\cdot\Delta+\sigma(a)\Delta\big(\sigma(c)\big)\Delta\big(\sigma(b)\big)\cdot\Delta-\delta\sigma(a)\Delta^2(b)\sigma(c)\cdot\Delta\\
&\quad-\sigma(a)\Delta\big(\sigma(b)\big)\Delta\big(\sigma(c)\big)\cdot\Delta-\Delta\big(\sigma(c)\big)\sigma^2(b)\Delta(a)\cdot\Delta+\Delta\big(\sigma(b)\big)\sigma^2(c)\Delta(a)\cdot\Delta\\
&\;=\delta\sigma(a)\Delta^2(c)\sigma(b)\cdot\Delta-\delta\sigma(a)\Delta^2(b)\sigma(c)\cdot\Delta-\Delta\big(\sigma(c)\big)\sigma^2(b)\Delta(a)\cdot\Delta\\
&\quad+\Delta\big(\sigma(b)\big)\sigma^2(c)\Delta(a)\cdot\Delta.
\end{aligned}$$

The first two terms of this last expression vanish after a cyclic summation and using (23), and so we get

$$\begin{aligned}
&\circlearrowleft_{a,b,c}\delta\cdot\big\langle a\cdot\Delta,\langle b\cdot\Delta,c\cdot\Delta\rangle_\sigma\big\rangle_\sigma\\
&\quad=\circlearrowleft_{a,b,c}\big\{-\Delta\big(\sigma(c)\big)\sigma^2(b)\Delta(a)\cdot\Delta+\Delta\big(\sigma(b)\big)\sigma^2(c)\Delta(a)\cdot\Delta\big\}.
\end{aligned}\tag{29}$$

Finally, combining this with (28) we deduce

$$\begin{aligned}
&\circlearrowleft_{a,b,c}\big\{\big\langle\sigma(a)\cdot\Delta,\langle b\cdot\Delta,c\cdot\Delta\rangle_\sigma\big\rangle_\sigma+\delta\big\langle a\cdot\Delta,\langle b\cdot\Delta,c\cdot\Delta\rangle_\sigma\big\rangle_\sigma\big\}\\
&\quad=\circlearrowleft_{a,b,c}\big\langle\sigma(a)\cdot\Delta,\langle b\cdot\Delta,c\cdot\Delta\rangle_\sigma\big\rangle_\sigma+\circlearrowleft_{a,b,c}\delta\big\langle a\cdot\Delta,\langle b\cdot\Delta,c\cdot\Delta\rangle_\sigma\big\rangle_\sigma\\
&\quad=\circlearrowleft_{a,b,c}\big\{\sigma^2(a)\Delta\big(\sigma(b)\big)\Delta(c)\cdot\Delta-\sigma^2(a)\Delta\big(\sigma(c)\big)\Delta(b)\cdot\Delta\big\}\\
&\qquad+\circlearrowleft_{a,b,c}\big\{-\Delta\big(\sigma(c)\big)\sigma^2(b)\Delta(a)\cdot\Delta+\Delta\big(\sigma(b)\big)\sigma^2(c)\Delta(a)\cdot\Delta\big\}\\
&\quad=\circlearrowleft_{a,b,c}\big\{\sigma^2(a)\Delta\big(\sigma(b)\big)\Delta(c)\cdot\Delta-\sigma^2(a)\Delta\big(\sigma(c)\big)\Delta(b)\cdot\Delta\big\}\\
&\qquad+\circlearrowleft_{a,b,c}\big\{-\Delta\big(\sigma(b)\big)\sigma^2(a)\Delta(c)\cdot\Delta+\Delta\big(\sigma(c)\big)\sigma^2(a)\Delta(b)\cdot\Delta\big\}\\
&\quad=0,
\end{aligned}$$

as was to be shown. □

Remark 8. If $\mathcal{A}$ is not assumed to be commutative, the construction still works if one imposes on Δ the additional condition that

$$[a,b]\Delta(c)=0\quad\text{for all } a,b,c\in\mathcal{A}.$$

Then the mapping $x\cdot\Delta: b\mapsto x\Delta(b)$ is again a σ-derivation for all $x\in\mathcal{A}$. As before $\mathcal{A}\cdot\Delta$ is a left $\mathcal{A}$-module. Then Theorem 5 remains valid with the same proof. We only need to note that, although $\mathcal{A}$ is not commutative we have $[a,b]\cdot\Delta=0$ which is to say that

$$ab\cdot\Delta=ba\cdot\Delta.$$

Proposition 9. *If $\mathcal{A}$ is a commutative $\mathbb{C}$-algebra without zero-divisors, and if $0 \neq \Delta \in \mathcal{D}_\sigma(\mathcal{A})$ and $0 \neq \Delta' \in \mathcal{D}_\sigma(\mathcal{A})$ generate the same cyclic $\mathcal{A}$-submodule $\mathcal{M}$ of $\mathcal{D}_\sigma(\mathcal{A})$, where $\sigma : \mathcal{A} \to \mathcal{A}$ is an algebra endomorphism, then there is a unit $u \in \mathcal{A}$ such that*

$$\langle x, y\rangle_{\sigma,\Delta} = u \cdot \langle x, y\rangle_{\sigma,\Delta'}. \tag{30}$$

Furthermore, if $u \in \mathbb{C}$ then

$$\big(\mathcal{M}, \langle\cdot,\cdot\rangle_{\sigma,\Delta}\big) \cong \big(\mathcal{M}, \langle\cdot,\cdot\rangle_{\sigma,\Delta'}\big).$$

Proof. That Δ and Δ' generates the same cyclic submodule implies that there are u_1, u_2 such that $\Delta = u_1\Delta'$ and $\Delta' = u_2\Delta$. This means that $u_1u_2\Delta = u_1\Delta' = \Delta$ or equivalently $(u_1u_2 - 1)\Delta = 0$. Choose $a \in \mathcal{A}$ such that $\Delta(a) \neq 0$. Then $(u_1u_2 - 1)\Delta(a) = 0$ implies that $u_1u_2 - 1 = 0$ and so u_1 and u_2 are both units. We now use Remark 6 to get

$$\sigma(u_2) \cdot \langle x, y\rangle_{\sigma,\Delta'} = u_2 \cdot \langle x, y\rangle_{\sigma,\Delta}.$$

Then $u = \sigma(u_2)/u_2$ satisfies (30). Now, if $u \in \mathbb{C}$ define $\varphi : (\mathcal{M}, \langle\cdot,\cdot\rangle_{\sigma,\Delta}) \to (\mathcal{M}, \langle\cdot,\cdot\rangle_{\sigma,\Delta'})$ by $\varphi(x) = ux$. Then

$$\varphi\big(\langle x, y\rangle_{\sigma,\Delta}\big) = u\langle x, y\rangle_{\sigma,\Delta} = u^2\langle x, y\rangle_{\sigma,\Delta'} = \langle ux, uy\rangle_{\sigma,\Delta'} = \big\langle\varphi(x), \varphi(y)\big\rangle_{\sigma,\Delta'}. \qquad \square$$

Definition 10. Let $\mathcal{A}$ be commutative and associative algebra, $\sigma : \mathcal{A} \to \mathcal{A}$ an algebra endomorphism and Δ a σ-derivation on $\mathcal{A}$. Then, a $(\mathcal{A}, \sigma, \Delta)$-*Witt algebra* (or a *generalized Witt algebra*) is the non-associative algebra $(\mathcal{A} \cdot \Delta, \langle\cdot,\cdot\rangle_{\sigma,\Delta})$ with the product defined by

$$\langle a \cdot \Delta, b \cdot \Delta\rangle_{\sigma,\Delta} = \big(\sigma(a)\Delta(b) - \sigma(b)\Delta(a)\big) \cdot \Delta.$$

Example 11. Take $\mathcal{A} = \mathbb{C}[t, t^{-1}]$, $\sigma = \mathrm{id}_{\mathcal{A}}$, the identity operator on $\mathcal{A}$, $\Delta = \frac{d}{dt}$, and $\delta = 1$. In this case one can show that $\mathcal{A} \cdot \Delta$ is equal to the whole $\mathcal{D}_\sigma(\mathcal{A})$. The conditions (17) and (18) are trivial to check. The definition (19) coincides with the usual Lie bracket of derivations, and (22) reduces to twice the usual Jacobi identity. Hence we recover the ordinary Witt algebra.

Example 12. Let $\mathcal{A}$ be a unique factorization domain, and let $\sigma : \mathcal{A} \to \mathcal{A}$ be a homomorphism, different from the identity. Then by Theorem 4,

$$\mathcal{D}_\sigma(\mathcal{A}) = \mathcal{A} \cdot \Delta,$$

where $\Delta = \frac{\mathrm{id}-\sigma}{g}$ and $g = \gcd((\mathrm{id} - \sigma)(\mathcal{A}))$. Furthermore, let $y \in \mathcal{A}$ and set

$$x = \Delta(y) = \frac{\mathrm{id} - \sigma}{g}(y) = \frac{y - \sigma(y)}{g}.$$

Then we have

$$\sigma(g)\sigma(x) = \sigma(gx) = \sigma(y) - \sigma^2(y) = (\mathrm{id} - \sigma)\big(\sigma(y)\big). \tag{31}$$

From the definition of g we know that it divides $(\mathrm{id} - \sigma)(g) = g - \sigma(g)$. Thus g also divides $\sigma(g)$. When we divide (31) by g and substitute the expression for x we obtain

$$\frac{\sigma(g)}{g}\sigma\left(\frac{\mathrm{id} - \sigma}{g}(y)\right) = \frac{\mathrm{id} - \sigma}{g}\big(\sigma(y)\big),$$

or, with our notation $\Delta = \frac{\mathrm{id}-\sigma}{g}$,

$$\frac{\sigma(g)}{g}\sigma\big(\Delta(y)\big) = \Delta\big(\sigma(y)\big).$$

This shows that (18) holds with

$$\delta = \sigma(g)/g. \tag{32}$$

Since $\mathcal{A}$ has no zero-divisors and $\sigma \neq \mathrm{id}$, it follows that $\mathrm{Ann}(\Delta) = 0$, and so (17) is clearly true. Hence we can use Theorem 5 to define an algebra structure on $\mathcal{D}_\sigma(\mathcal{A}) = \mathcal{A} \cdot \Delta$ which satisfies (21) and (22) with $\delta = \sigma(g)/g$. Since the choice of greatest common divisor is ambiguous (we can choose any associated element, that is, the greatest common divisor is only unique up to a multiple by an invertible element) this δ can be replaced by any $\delta' = u \cdot \delta$ where u is a unit (that is, an invertible element). To see this, note that if g' is another greatest common divisor related to g by $g' = u \cdot g$, then

$$\delta' = \frac{\sigma(ug)}{ug} = \frac{\sigma(u)\sigma(g)}{ug} = \frac{\sigma(u)}{u}\delta$$

and $\sigma(u)/u$ is clearly a unit since u is a unit. Therefore (18) becomes,

$$\Delta' = \frac{\Delta}{u} = \frac{\mathrm{id} - \sigma}{gu} \quad \Longrightarrow \quad \Delta'\big(\sigma(a)\big) = \frac{\sigma(u)}{u}\delta\sigma\big(\Delta'(a)\big).$$

Remark 13. If we choose a multiple $\Delta' = f \cdot \Delta$ of the generator $\Delta = \frac{\mathrm{id}-\sigma}{g}$ of $\mathcal{D}_\sigma(\mathcal{A})$, it will generate a proper $\mathcal{A}$-submodule $\mathcal{A} \cdot \Delta'$ of $\mathcal{D}_\sigma(\mathcal{A})$, unless f is a unit. To see this, suppose on the contrary that $\mathcal{A} \cdot \Delta' = \mathcal{D}_\sigma(\mathcal{A})$. Then there is some $g \in \mathcal{A}$ such that $g \cdot \Delta' = \Delta$. Since $\sigma \neq \mathrm{id}$ there is some $x \in \mathcal{A}$ such that $\sigma(x) \neq x$. Then

$$\Delta(x) = g \cdot \Delta'(x) = gf \cdot \Delta(x).$$

Since $\Delta(x) \neq 0$ and $\mathcal{A}$ has no zero-divisors, we must have $gf = 1$.

2.3. hom-Lie algebras

Let us now make the following definition.

Definition 14. A *hom-Lie algebra* (L, ς) is a non-associative algebra L together with an algebra homomorphism $\varsigma : L \to L$, such that

$$\langle x, y\rangle_L = -\langle y, x\rangle_L,$$
$$\big\langle (\mathrm{id}+\varsigma)(x), \langle y, z\rangle_L\big\rangle_L + \big\langle (\mathrm{id}+\varsigma)(y), \langle z, x\rangle_L\big\rangle_L + \big\langle (\mathrm{id}+\varsigma)(z), \langle x, y\rangle_L\big\rangle_L = 0,$$

for all $x, y, z \in L$, where $\langle\cdot,\cdot\rangle_L$ denotes the product in L.

Example 15. Taking $\varsigma = \mathrm{id}$ in the above definition gives us the definition of a Lie algebra. Hence hom-Lie algebras include Lie algebras as a subclass, thereby motivating the name 'hom-Lie algebras' as a deformation of Lie algebras twisted by a homomorphism.

Example 16. Letting $\mathfrak{a}$ be any vector space (finite- or infinite-dimensional) we put

$$\langle x, y\rangle_{\mathfrak{a}} = 0$$

for any $x, y \in \mathfrak{a}$. Then $(\mathfrak{a}, \varsigma_{\mathfrak{a}})$ is obviously a hom-Lie algebra for any linear map $\varsigma_{\mathfrak{a}}$ since the above conditions are trivially satisfied. As in the Lie case, we call these algebras *abelian* or *commutative* hom-Lie algebras.

Example 17. Suppose $\mathcal{A}$ is a commutative associative algebra, $\sigma : \mathcal{A} \to \mathcal{A}$ a homomorphism, $\Delta \in \mathfrak{D}_\sigma(\mathcal{A})$ and $\delta \in \mathcal{A}$ satisfy (17)–(18). Then since $\sigma(\mathrm{Ann}(\Delta)) \subseteq \mathrm{Ann}(\Delta)$, the map σ induces a map

$$\bar\sigma : \mathcal{A}\cdot\Delta \to \mathcal{A}\cdot\Delta, \quad \bar\sigma : a\cdot\Delta \mapsto \sigma(a)\cdot\Delta.$$

This map has the following property:

$$\begin{aligned}
\big\langle \bar\sigma(a\cdot\Delta), \bar\sigma(b\cdot\Delta)\big\rangle_\sigma &= \big\langle \sigma(a)\cdot\Delta, \sigma(b)\cdot\Delta\big\rangle_\sigma \\
&= \big(\sigma^2(a)\Delta\big(\sigma(b)\big) - \sigma^2(b)\Delta\big(\sigma(a)\big)\big)\cdot\Delta \\
&= \big(\sigma^2(a)\delta\sigma\big(\Delta(b)\big) - \sigma^2(b)\delta\sigma\big(\Delta(a)\big)\big)\cdot\Delta \\
&= \delta\sigma\big(\sigma(a)\Delta(b) - \sigma(b)\Delta(a)\big)\cdot\Delta \\
&= \delta\cdot\bar\sigma\big(\langle a\cdot\Delta, b\cdot\Delta\rangle_\sigma\big).
\end{aligned}$$

We suppose now that $\delta \in \mathbb{C}\setminus\{0\}$. Dividing both sides of the above calculation by δ^2 and using bilinearity of the product, we see that Theorem 5 makes $\mathcal{A}\cdot\Delta$ with the product $\langle\cdot,\cdot\rangle_\sigma$ into a hom-Lie algebra with $(1/\delta)\bar\sigma$ as its homomorphism ς.

By a *homomorphism of hom-Lie algebras* $\varphi : (L_1, \varsigma_1) \to (L_2, \varsigma_2)$ we mean an algebra homomorphism from L_1 to L_2 such that $\varphi \circ \varsigma_1 = \varsigma_2 \circ \varphi$, or, in other words, such that the diagram

$$\begin{tikzcd} L_1 \arrow[r, "\varphi"] \arrow[d, "\varsigma_1"] & L_2 \arrow[d, "\varsigma_2"] \\ L_1 \arrow[r, "\varphi"] & L_2 \end{tikzcd}$$

commutes. We now have the following proposition.

Proposition 18. *Let (L, ς) be a hom-Lie algebra, and let N be any non-associative algebra. Let*

$$\varphi : L \to N$$

be an algebra homomorphism. Then the following two conditions are equivalent:

(1) *There exists a linear subspace $U \subseteq N$ containing $\varphi(L)$ and a linear map*

$$k : U \to N$$

such that

$$\varphi \circ \varsigma = k \circ \varphi. \tag{33}$$

(2) $\ker \varphi \subseteq \ker(\varphi \circ \varsigma)$.

Moreover, if these conditions are satisfied, then

(i) *k is uniquely determined on $\varphi(L)$ by φ and ς,*
(ii) *$k|_{\varphi(L)}$ is a homomorphism,*
(iii) *$(\varphi(L), k|_{\varphi(L)})$ is a hom-Lie algebra, and*
(iv) *φ is a homomorphism of hom-Lie algebras.*

Remark 19. It is easy to check that condition (2) can equivalently be written

$$\varsigma(\ker \varphi) \subseteq \ker \varphi.$$

Proof. Assume that condition (1) holds, and let $x \in \ker \varphi$. Then

$$\varphi\big(\varsigma(x)\big) = k\big(\varphi(x)\big) = k(0) = 0,$$

so that $x \in \ker(\varphi \circ \varsigma)$. Thus (2) holds. Conversely, assume (2) is true. Take $U = \varphi(L)$ and define $k : \varphi(L) \to N$ by $k(\varphi(x)) = \varphi(\varsigma(x))$. This is well defined, since if $\varphi(x) = \varphi(y)$ we have

$$x - y \in \ker \varphi \subseteq \ker(\varphi \circ \varsigma)$$

by assumption. Hence $\varphi(\varsigma(x)) = \varphi(\varsigma(y))$ so k is well defined. Equation (33) holds by definition of k.

Assume now that the conditions (1) and (2) hold. To prove (i), assume that we have two linear maps $k_1 : U_1 \to N$ and $k_2 : U_2 \to N$ where U_i are subspaces of N with $\varphi(L) \subseteq U_i$. Suppose they both satisfy (33). Then

$$(k_1 - k_2)\big(\varphi(x)\big) = \varphi\big(\varsigma(x)\big) - \varphi\big(\varsigma(x)\big) = 0$$

for any $x \in L$. This shows that k_1 and k_2 coincide on $\varphi(L)$. For (ii) we use again the identity (33), and that φ is a homomorphism (we denote the product in N by $\{\cdot,\cdot\}$ to indicate its non-associativity):

$$\begin{aligned} k\big(\big\{\varphi(x), \varphi(y)\big\}\big) &= k\big(\varphi\big(\langle x, y\rangle_L\big)\big) = \varphi\big(\varsigma\big(\langle x, y\rangle_L\big)\big) = \varphi\big(\big\langle \varsigma(x), \varsigma(y)\big\rangle_L\big) \\ &= \big\{\varphi\big(\varsigma(x)\big), \varphi\big(\varsigma(y)\big)\big\} = \big\{k\big(\varphi(x)\big), k\big(\varphi(y)\big)\big\}, \end{aligned}$$

for $x, y \in L$.

Using (33) and that (L, ς) is a hom-Lie algebra we get

$$\big\{\varphi(x), \varphi(y)\big\} = \varphi\big(\langle x, y\rangle_L\big) = \varphi\big(-\langle y, x\rangle_L\big) = -\big\{\varphi(y), \varphi(x)\big\}$$

for $x, y \in L$ and

$$\begin{aligned} \circlearrowleft_{x,y,z} \big\{(\mathrm{id} + k)\big(\varphi(x)\big), \big\{\varphi(y), \varphi(z)\big\}\big\} &= \circlearrowleft_{x,y,z} \big\{\varphi(x) + k\big(\varphi(x)\big), \varphi\big(\langle y, z\rangle_L\big)\big\} \\ &= \circlearrowleft_{x,y,z} \big\{\varphi(x) + \varphi\big(\varsigma(x)\big), \varphi\big(\langle y, z\rangle_L\big)\big\} \\ &= \varphi\Big(\circlearrowleft_{x,y,z} \big\langle x + \varsigma(x), \langle y, z\rangle_L\big\rangle_L\Big) = 0 \end{aligned}$$

for $x, y, z \in L$. This shows (iii), and then (iv) is true since φ is a homomorphism satisfying Eq. (33). □

2.4. Extensions of hom-Lie algebras

In this section we will concentrate our efforts on developing the general theory of *central* extensions for hom-Lie algebras, and providing general (co-)homological type conditions for existence of central extensions useful for their construction.

If U and V are vector spaces, let $\mathrm{Alt}^2(U, V)$ denote the space of skew-symmetric forms (alternating mappings)

$$U \times U \to V.$$

Exactly as in the Lie algebra case we define an extension of hom-Lie algebras with the aid of exact sequences.

Definition 20. An extension of a hom-Lie algebra (L, ς) by an abelian hom-Lie algebra $(\mathfrak{a}, \varsigma_{\mathfrak{a}})$ is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a} \arrow[r, "\iota"] \arrow[d, "\varsigma_{\mathfrak{a}}"'] & \hat{L} \arrow[r, "\mathrm{pr}"] \arrow[d, "\hat{\varsigma}"'] & L \arrow[r] \arrow[d, "\varsigma"'] & 0 \\ 0 \arrow[r] & \mathfrak{a} \arrow[r, "\iota"] & \hat{L} \arrow[r, "\mathrm{pr}"] & L \arrow[r] & 0, \end{tikzcd} \tag{34}$$

where $(\hat{L}, \hat{\varsigma})$ is a hom-Lie algebra. We say that the extension is *central* if

$$\iota(\mathfrak{a}) \subseteq \mathrm{Z}(\hat{L}) = \big\{x \in \hat{L}\colon \langle x, \hat{L}\rangle_{\hat{L}} = 0\big\}.$$

The question now arises: what are the conditions for being able to construct a central extension $\hat{L}$ of L? We will now derive a necessary condition for this. The sequence above splits (as vector spaces) just as in the Lie algebra case, meaning that there is a (linear) *section* $s : L \to \hat{L}$, i.e., a linear map such that $\mathrm{pr} \circ s = \mathrm{id}_L$. To construct a hom-Lie algebra extension we must do two things:

- define the hom-Lie algebra homomorphism $\hat{\varsigma}$, and
- construct the bracket $\langle\cdot,\cdot\rangle_{\hat{L}}$ with the desired properties.

Note first of all that

$$\mathrm{pr} \circ \hat{\varsigma}(x) = \varsigma \circ \mathrm{pr}(x) \quad \text{for } x \in \hat{L}$$

since pr is a hom-Lie algebra homomorphism. This means that

$$\mathrm{pr}\big(\hat{\varsigma}(x) - s \circ \varsigma \circ \mathrm{pr}(x)\big) = 0$$

and this leads to, by the exactness,

$$\hat{\varsigma}(x) = s \circ \varsigma \circ \mathrm{pr}(x) + \iota \circ f_s(x), \tag{35}$$

where $f_s : \hat{L} \to \mathfrak{a}$ is a function dependent on s. Note that combining (35) with the commutativity of the left square in (34) we get for $a \in \mathfrak{a}$ that

$$\iota \circ \varsigma_{\mathfrak{a}}(a) = \hat{\varsigma} \circ \iota(a) = s \circ \varsigma \circ \mathrm{pr} \circ \iota(a) + \iota \circ f_s \circ \iota(a) = \iota \circ f_s \circ \iota(a)$$

and hence since ι is injective,

$$\varsigma_{\mathfrak{a}}(a) = f_s \circ \iota(a). \tag{36}$$

Also

$$\mathrm{pr}\big(\big\langle s(x), s(y)\big\rangle_{\hat{L}} - s\langle x, y\rangle_L\big) = 0,$$

hence

$$\big\langle s(x), s(y)\big\rangle_{\hat{L}} = s\langle x, y\rangle_L + \iota \circ g_s(x, y) \tag{37}$$

for some $g_s \in \mathrm{Alt}^2(L, \mathfrak{a})$, a "2-cocycle." This means that we have a "lift" of the bracket in L to the bracket in $\hat{L}$ for elements x, y in L defined by the "2-cocycle" and the section s.

Using (35), (37) and the linearity of the product we get (we temporarily suppress the indices L and $\hat{L}$ in the brackets and the s in g_s), for $a, b, c \in L$,

$$\begin{aligned}
&\big\langle(\mathrm{id} + \hat{\varsigma})\big(s(a)\big), \big\langle s(b), s(c)\big\rangle\big\rangle \\
&\quad = \big\langle(\mathrm{id} + \hat{\varsigma})\big(s(a)\big), s\langle b, c\rangle + \iota \circ g(b, c)\big\rangle \\
&\quad = \big\langle s(a) + \hat{\varsigma}\big(s(a)\big), s\langle b, c\rangle + \iota \circ g(b, c)\big\rangle \\
&\quad = \big\langle s(a), s\langle b, c\rangle\big\rangle + \big\langle s(a), \iota \circ g(b, c)\big\rangle + \big\langle \hat{\varsigma}\big(s(a)\big), s\langle b, c\rangle\big\rangle + \big\langle \hat{\varsigma}\big(s(a)\big), \iota \circ g(b, c)\big\rangle \\
&\quad = s\big\langle a, \langle b, c\rangle\big\rangle + \iota \circ g\big(a, \langle b, c\rangle\big) + \big\langle s(a), \iota \circ g(b, c)\big\rangle \\
&\qquad + \big\langle s \circ \varsigma \circ \mathrm{pr}\big(s(a)\big) + f_s\big(s(a)\big), s\langle b, c\rangle\big\rangle + \big\langle s \circ \varsigma \circ \mathrm{pr}\big(s(a)\big) + f_s\big(s(a)\big), \iota \circ g(b, c)\big\rangle \\
&\quad = s\big\langle a, \langle b, c\rangle\big\rangle + \iota \circ g\big(a, \langle b, c\rangle\big) + \big\langle s(a), \iota \circ g(b, c)\big\rangle + \big\langle s \circ \varsigma(a) + f_s\big(s(a)\big), s\langle b, c\rangle\big\rangle \\
&\qquad + \big\langle s \circ \varsigma(a) + f_s\big(s(a)\big), \iota \circ g(b, c)\big\rangle \\
&\quad = s\big\langle a, \langle b, c\rangle\big\rangle + \iota \circ g\big(a, \langle b, c\rangle\big) + \big\langle s(a), \iota \circ g(b, c)\big\rangle + \big\langle s \circ \varsigma(a), s\langle b, c\rangle\big\rangle \\
&\qquad + \big\langle f_s\big(s(a)\big), s\langle b, c\rangle\big\rangle + \big\langle s \circ \varsigma(a), \iota \circ g(b, c)\big\rangle + \big\langle f_s\big(s(a)\big), \iota \circ g(b, c)\big\rangle \\
&\quad = s\big\langle a, \langle b, c\rangle\big\rangle + \iota \circ g\big(a, \langle b, c\rangle\big) + \big\langle s(a), \iota \circ g(b, c)\big\rangle + s\big\langle \varsigma(a), \langle b, c\rangle\big\rangle \\
&\qquad + \iota \circ g\big(\varsigma(a), \langle b, c\rangle\big) + \big\langle f_s\big(s(a)\big), s\langle b, c\rangle\big\rangle + \big\langle s \circ \varsigma(a), \iota \circ g(b, c)\big\rangle \\
&\qquad + \big\langle f_s\big(s(a)\big), \iota \circ g(b, c)\big\rangle \\
&\quad = s\big\langle(\mathrm{id} + \varsigma)(a), \langle b, c\rangle\big\rangle + \iota \circ g\big((\mathrm{id} + \varsigma)(a), \langle b, c\rangle\big),
\end{aligned}$$

where, in last step, we have used that the extension is central. Summing up cyclically we get

$$\circlearrowleft_{a,b,c} g_s\big((\mathrm{id} + \varsigma)(a), \langle b, c\rangle_L\big) = 0 \tag{38}$$

since (L, ς) and $(\hat{L}, \hat{\varsigma})$ are hom-Lie algebras.

Picking another section $\tilde{s}$, we have $\tilde{s}(x) - s(x) = (\tilde{s} - s)(x) \in \ker \mathrm{pr} = \iota(\mathfrak{a})$. Since the extension is central,

$$\begin{aligned} 0 &= \big\langle \tilde{s}(x), \tilde{s}(y)\big\rangle_{\hat{L}} - \big\langle s(x), s(y)\big\rangle_{\hat{L}} \\ &= \tilde{s}\langle x, y\rangle_L + \iota \circ g_{\tilde{s}}(x, y) - s\langle x, y\rangle_L - \iota \circ g_s(x, y) \\ &= (\tilde{s} - s)\big(\langle x, y\rangle_L\big) + \iota \circ g_{\tilde{s}}(x, y) - \iota \circ g_s(x, y). \end{aligned} \tag{39}$$

This shows that the condition (38) is independent of the section s. We have almost proved the following theorem.

Theorem 21. *Suppose (L, ς) and $(\mathfrak{a}, \varsigma_{\mathfrak{a}})$ are hom-Lie algebras with $\mathfrak{a}$ abelian. If there exists a central extension $(\hat{L}, \hat{\varsigma})$ of (L, ς) by $(\mathfrak{a}, \varsigma_{\mathfrak{a}})$ then for every section $s : L \to \hat{L}$ there is a $g_s \in \mathrm{Alt}^2(L, \mathfrak{a})$ and a linear map $f_s : \hat{L} \to \mathfrak{a}$ such that*

$$f_s \circ \iota = \varsigma_{\mathfrak{a}}, \tag{40}$$

$$g_s\big(\varsigma(x), \varsigma(y)\big) = f_s\big(\big\langle s(x), s(y)\big\rangle_{\hat{L}}\big) \tag{41}$$

and

$$\circlearrowleft_{x,y,z} g_s\big((\mathrm{id} + \varsigma)(x), \langle y, z\rangle_L\big) = 0 \tag{42}$$

for all $x, y, z \in L$. Moreover, Eq. (42) *is independent of the choice of section s.*

Proof. It only remains to verify Eq. (41). We use that $\hat{\varsigma}$ is a homomorphism. On the one hand, using (35) and (37) we have for $x, y \in L$ that

$$\begin{aligned} \hat{\varsigma}\big(\big\langle s(x), s(y)\big\rangle_{\hat{L}}\big) &= \hat{\varsigma}\big(s\big(\langle x, y\rangle_L\big) + \iota \circ g_s(x, y)\big) \\ &= s \circ \varsigma \circ \mathrm{pr} \circ s\big(\langle x, y\rangle_L\big) + \iota \circ f_s \circ s\big(\langle x, y\rangle_L\big) + s \circ \varsigma \circ \mathrm{pr} \circ \iota \circ g_s(x, y) \\ &\quad + \iota \circ f_s \circ \iota \circ g_s(x, y) \\ &= s \circ \varsigma\big(\langle x, y\rangle_L\big) + \iota \circ f_s\big(\big\langle s(x), s(y)\big\rangle_{\hat{L}}\big). \end{aligned}$$

On the other hand,

$$\begin{aligned} \big\langle \hat{\varsigma} \circ s(x), \hat{\varsigma} \circ s(y)\big\rangle_{\hat{L}} &= \big\langle s \circ \varsigma \circ \mathrm{pr} \circ s(x) + \iota \circ f_s \circ s(x), s \circ \varsigma \circ \mathrm{pr} \circ s(y) + \iota \circ f_s \circ s(y)\big\rangle_{\hat{L}} \\ &= \big\langle s \circ \varsigma(x), s \circ \varsigma(y)\big\rangle_{\hat{L}} = s\big(\big\langle \varsigma(x), \varsigma(y)\big\rangle_L\big) + \iota \circ g_s\big(\varsigma(x), \varsigma(y)\big) \\ &= s \circ \varsigma\big(\langle x, y\rangle_L\big) + \iota \circ g_s\big(\varsigma(x), \varsigma(y)\big). \end{aligned}$$

Since ι is injective, (41) follows. $\square$

We now make the following definition:

Definition 22. A central hom-Lie algebra extension $(\hat{L}, \hat{\varsigma})$ of (L, ς) by $(\mathfrak{a}, \varsigma_{\mathfrak{a}})$ is called *trivial* if there exists a linear section $s : L \to \hat{L}$ such that

$$g_s(x, y) = 0$$

for all $x, y \in L$.

Remark 23. Note that by using (39) one can show that the above definition is equivalent to the statement: "A central extension of hom-Lie algebras is trivial if and only if for any section $s : L \to \hat{L}$ there is a linear map $s_1 : L \to \hat{L}$ such that $(s + s_1)$ is a section and

$$\iota \circ g_s(x, y) = s_1\big(\langle x, y\rangle_L\big)$$

for all $x, y \in L$." Indeed, take a section s. Since the extension is trivial there is a section $\tilde{s}$ such that $g_{\tilde{s}}(x, y) = 0$ for all $x, y \in L$. Inserting this into (39) gives (using that ι is one-to-one)

$$\iota \circ g_s(x, y) = \iota \circ g_{\tilde{s}}(x, y) + (\tilde{s} - s)\langle x, y\rangle_L = (\tilde{s} - s)\langle x, y\rangle_L$$

and putting $s_1 = \tilde{s} - s$ gives necessity. On the other hand taking $\tilde{s} = s + s_1$ in (39) gives us sufficiency.

Theorem 24. *Suppose (L, ς) and $(\mathfrak{a}, \varsigma_{\mathfrak{a}})$ are hom-Lie algebras with $\mathfrak{a}$ abelian. Then for every $g \in \mathrm{Alt}^2(L, \mathfrak{a})$ and every linear map $f : L \oplus \mathfrak{a} \to \mathfrak{a}$ such that*

$$f(0, a) = \varsigma_{\mathfrak{a}}(a) \quad \textit{for } a \in \mathfrak{a}, \tag{43}$$

$$g\big(\varsigma(x), \varsigma(y)\big) = f\big(\langle x, y\rangle_L, g(x, y)\big) \tag{44}$$

and

$$\circlearrowleft_{x,y,z} g\big((\mathrm{id} + \varsigma)(x), \langle y, z\rangle_L\big) = 0, \tag{45}$$

for $x, y, z \in L$, there exists a hom-Lie algebra $(\hat{L}, \hat{\varsigma})$ which is a central extension of (L, ς) by $(\mathfrak{a}, \varsigma_{\mathfrak{a}})$.

Proof. As a vector space we set $\hat{L} = L \oplus \mathfrak{a}$. Define the product $\langle \cdot, \cdot \rangle_{\hat{L}}$ in $\hat{L}$ by setting

$$\big\langle (x, a), (y, b)\big\rangle_{\hat{L}} = \big(\langle x, y\rangle_L, g(x, y)\big) \quad \text{for } (x, a), (y, b) \in \hat{L} \tag{46}$$

and define $\hat{\varsigma} : \hat{L} \to \hat{L}$ by

$$\hat{\varsigma}(x, a) = \big(\varsigma(x), f(x, a)\big) \quad \text{for } (x, a) \in \hat{L}.$$

We claim that the linear map $\hat{\varsigma}$ is a homomorphism. Indeed,

$$\hat{\varsigma}\big(\big\langle (x,a),(y,b)\big\rangle_{\hat{L}}\big) = \hat{\varsigma}\big(\langle x,y\rangle_L, g(x,y)\big) = \big(\varsigma\big(\langle x,y\rangle_L\big), f\big(\langle x,y\rangle_L, g(x,y)\big)\big)$$

and

$$\big\langle \hat{\varsigma}(x,a), \hat{\varsigma}(y,b)\big\rangle_{\hat{L}} = \big\langle\big(\varsigma(x), f(x,a)\big), \big(\varsigma(y), f(y,b)\big)\big\rangle = \big(\big\langle\varsigma(x),\varsigma(y)\big\rangle, g\big(\varsigma(x),\varsigma(y)\big)\big).$$

These two expressions are equal because ς is a homomorphism and (44) holds. Next we prove that $(\hat{L}, \hat{\varsigma})$ is a hom-Lie algebra. Skew-symmetry of $\langle\cdot,\cdot\rangle_{\hat{L}}$ is immediate since g is alternating. The generalized Jacobi identity can be verified as follows:

$$\begin{aligned}
&\circlearrowleft_{(x,a),(y,b),(z,c)} \big\langle(\mathrm{id}+\hat{\varsigma})(x,a), \big\langle(y,b),(z,c)\big\rangle_{\hat{L}}\big\rangle_{\hat{L}} \\
&\quad= \circlearrowleft_{(x,a),(y,b),(z,c)} \big\langle\big(x+\varsigma(x), a+f(x,a)\big), \big(\langle y,z\rangle_L, g(y,z)\big)\big\rangle_{\hat{L}} \\
&\quad= \circlearrowleft_{(x,a),(y,b),(z,c)} \big(\big\langle x+\varsigma(x), \langle y,z\rangle_L\big\rangle_L, g\big(x+\varsigma(x), \langle y,z\rangle_L\big)\big) = 0,
\end{aligned}$$

where we used (45) and that (L,ς) is a hom-Lie algebra.

Next we define pr and ι to be the natural projection and inclusion, respectively:

$$\begin{aligned}
\mathrm{pr}: \hat{L} &\to L, \qquad \mathrm{pr}(x,a) = x; \\
\iota: \mathfrak{a} &\to \hat{L}, \qquad \iota(a) = (0,a).
\end{aligned}$$

That the diagram (34) has exact rows is now obvious. Next we show that the linear maps pr and ι are homomorphisms.

$$\mathrm{pr}\big(\big\langle(x,a),(y,b)\big\rangle_{\hat{L}}\big) = \mathrm{pr}\big(\langle x,y\rangle_L, g(x,y)\big) = \langle x,y\rangle_L = \big\langle \mathrm{pr}(x,a), \mathrm{pr}(y,b)\big\rangle_L,$$
$$\big\langle\iota(a),\iota(b)\big\rangle_{\hat{L}} = \big\langle(0,a),(0,b)\big\rangle_{\hat{L}} = (0,0) = \iota(0) = \iota\big(\langle a,b\rangle_{\mathfrak{a}}\big)$$

since $\mathfrak{a}$ was abelian. This shows that pr and ι are homomorphisms. In fact they are also hom-Lie algebra homomorphisms, because

$$\mathrm{pr}\circ\hat{\varsigma}(x,a) = \mathrm{pr}\big(\varsigma(x), f(x,a)\big) = \varsigma(x) = \varsigma\circ\mathrm{pr}(x,a)$$

and

$$\hat{\varsigma}\circ\iota(a) = \hat{\varsigma}(0,a) = \big(\varsigma(0), f(0,a)\big) = \big(0, \varsigma_{\mathfrak{a}}(a)\big) = \iota\circ\varsigma_{\mathfrak{a}}(a),$$

where we used (43). This proves that $(\hat{L}, \hat{\varsigma})$ is an extension of (L,ς) by $(\mathfrak{a}, \varsigma_{\mathfrak{a}})$. Finally, that the extension is central is clear from the definition of ι and (46). $\square$

3. Examples

3.1. A q-deformed Witt algebra

Let $\mathcal{A}$ be the complex algebra of Laurent polynomials in one variable t, i.e.,

$$\mathcal{A} = \mathbb{C}\big[t, t^{-1}\big] \cong \mathbb{C}[x, y]/(xy - 1).$$

Fix $q \in \mathbb{C} \setminus \{0, 1\}$, and let σ be the unique endomorphism on $\mathcal{A}$ determined by

$$\sigma(t) = qt.$$

Explicitly, we have

$$\sigma\big(f(t)\big) = f(qt), \quad \text{for } f(t) \in \mathcal{A}.$$

The set $\mathfrak{D}_\sigma(\mathcal{A})$ of all σ-derivations on $\mathcal{A}$ is a free $\mathcal{A}$-module of rank one, and the mapping

$$D : \mathcal{A} \to \mathcal{A},$$

defined by

$$D\big(f(t)\big) = t\frac{\sigma(f(t)) - f(t)}{\sigma(t) - t} = \frac{f(qt) - f(t)}{q - 1} \quad \text{for } f(t) \in \mathcal{A}, \tag{47}$$

is a generator.

To see that D indeed generates $\mathfrak{D}_\sigma(\mathcal{A})$, note that, since $\mathcal{A}$ is a UFD, a generator of $\mathfrak{D}_\sigma(\mathcal{A})$ is on the form

$$\frac{\mathrm{id} - \sigma}{\gcd((\mathrm{id} - \sigma)(\mathcal{A}))},$$

by Theorem 4. Now, a greatest common divisor on $(\mathrm{id} - \sigma)(\mathcal{A})$ is any element of $\mathcal{A}$ on the form ct^k, where $c \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{Z}$. This is because $\gcd((\mathrm{id} - \sigma)(\mathcal{A}))$ divides any element of $(\mathrm{id} - \sigma)(\mathcal{A})$ so in particular it divides $(\mathrm{id} - \sigma)(t) = -(q - 1)t$ which is a unit (when $q \neq 1$). This means that $q - 1$ is a $\gcd((\mathrm{id} - \sigma)(\mathcal{A}))$. Therefore,

$$D\big(f(t)\big) = \frac{f(qt) - f(t)}{q - 1} = -\frac{\mathrm{id} - \sigma}{q - 1}\big(f(t)\big)$$

and so $D = -\frac{\mathrm{id}-\sigma}{q-1}$ is a generator for $\mathfrak{D}_\sigma(\mathcal{A})$.

Remark 25. Note that $t^{-1}D = D_q$, the *Jackson q-derivative*.

Since D is a polynomial (over $\mathbb{C}$) in σ, D and σ commute. Let $\langle\cdot,\cdot\rangle_\sigma$ denote the product on $\mathcal{D}_\sigma(\mathcal{A})$ defined by

$$\langle f\cdot D, g\cdot D\rangle_\sigma = \bigl(\sigma(f)\cdot D\bigr)\circ(g\cdot D) - \bigl(\sigma(g)\cdot D\bigr)\circ(f\cdot D) \tag{48}$$

for $f, g\in\mathcal{A}$ coming from (19). It satisfies the following identities:

$$\langle f\cdot D, g\cdot D\rangle_\sigma = \bigl(\sigma(f)D(g) - \sigma(g)D(f)\bigr)\cdot D, \tag{49}$$

$$\langle f\cdot D, g\cdot D\rangle_\sigma = -\langle g\cdot D, f\cdot D\rangle_\sigma, \tag{50}$$

and

$$\begin{aligned}&\bigl\langle\bigl(\sigma(f)+f\bigr)\cdot D, \langle g\cdot D, h\cdot D\rangle_\sigma\bigr\rangle_\sigma + \bigl\langle\bigl(\sigma(g)+g\bigr)\cdot D, \langle h\cdot D, f\cdot D\rangle_\sigma\bigr\rangle_\sigma\\ &\quad+\bigl\langle\bigl(\sigma(h)+h\bigr)\cdot D, \langle f\cdot D, g\cdot D\rangle_\sigma\bigr\rangle_\sigma = 0,\end{aligned} \tag{51}$$

for all $f, g, h\in\mathcal{A}$. The identities (50) and (51) show that $\mathcal{D}_\sigma(\mathcal{A})$ is a hom-Lie algebra with

$$\varsigma : \mathcal{D}_\sigma(\mathcal{A})\to\mathcal{D}_\sigma(\mathcal{A})\quad \varsigma : f\cdot D\mapsto\sigma(f)\cdot D$$

as its homomorphism. As a $\mathbb{C}$-linear space, $\mathcal{D}_\sigma(\mathcal{A})$ has a basis $\{d_n\colon n\in\mathbb{Z}\}$, where

$$d_n = -t^n\cdot D. \tag{52}$$

Note that

$$\sigma\bigl(-t^n\bigr) = -q^n t^n, \tag{53}$$

which imply

$$\varsigma(d_n) = q^n d_n. \tag{54}$$

Note further that

$$D\bigl(-t^n\bigr) = \frac{-q^n t^n + t^n}{q-1} = -\{n\}_q t^n, \tag{55}$$

where $\{n\}_q$ for $n\in\mathbb{Z}$ denotes the q-*number*

$$\{n\}_q = \frac{q^n-1}{q-1}.$$

Using (49) with $f(t) = -t^n$ and $g(t) = -t^l$ we obtain the following important commutation relation:

$$\begin{aligned}\langle d_n, d_l\rangle_\sigma &= \big(\big(-q^n t^n\big)\cdot\big(-\{l\}_q t^l\big) - \big(-q^l t^l\big)\cdot\big(-\{n\}_q t^n\big)\big)\cdot D \\ &= \left(q^l \frac{q^n-1}{q-1} - q^n \frac{q^l-1}{q-1}\right)\cdot\big(-t^{n+l}\big)\cdot D \\ &= \frac{q^n-q^l}{q-1} d_{n+l} = \big(\{n\}_q - \{l\}_q\big) d_{n+l}, \end{aligned} \tag{56}$$

for $n, l \in \mathbb{Z}$, where the bracket is defined on generators by (19) as

$$\langle d_n, d_l\rangle_\sigma = q^n d_n d_l - q^l d_l d_n.$$

This means, in particular, that $\mathcal{D}_\sigma(\mathcal{A})$ admits a $\mathbb{Z}$-grading as an algebra:

$$\mathcal{D}_\sigma(\mathcal{A}) = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}\cdot d_i.$$

Remark 26. If $q = 1$ we simply define D to be $t\cdot\partial$ where $\partial = \frac{d}{dt}$, the ordinary differential operator. Then ∂ generates $\mathcal{D}_{\mathrm{id}}(\mathcal{A})$ even though Theorem 4 cannot be used. The relation

$$\langle d_n, d_l\rangle_\sigma = \big(\{n\}_q - \{l\}_q\big) d_{n+l}$$

then becomes the standard commutation relation in the Witt algebra:

$$\langle \partial_n, \partial_l\rangle = (n-l)\partial_{n+l},$$

where $\partial_n = -t^n\cdot D$.

It follows from (50) that

$$\langle d_n, d_l\rangle_\sigma = -\langle d_l, d_n\rangle_\sigma, \tag{57}$$

and substituting $f(t) = -t^n$, $g(t) = -t^l$ and $h(t) = -t^m$ into (51) we obtain the following q-deformation of the Jacobi identity:

$$\begin{aligned}&\big(q^n+1\big)\big\langle d_n, \langle d_l, d_m\rangle_\sigma\big\rangle_\sigma + \big(q^l+1\big)\big\langle d_l, \langle d_m, d_n\rangle_\sigma\big\rangle_\sigma + \big(q^m+1\big)\big\langle d_m, \langle d_n, d_l\rangle_\sigma\big\rangle_\sigma \\ &\quad = 0,\end{aligned} \tag{58}$$

for all $n, l, m \in \mathbb{Z}$. Hence,

Theorem 27. *Let $\mathcal{A} = \mathbb{C}[t, t^{-1}]$. Then the $\mathbb{C}$-linear space*

$$\mathcal{D}_\sigma(\mathcal{A}) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\cdot d_n,$$

where $D = -\frac{\mathrm{id}-\sigma}{q-1}$, $d_n = -t^n D$ *and* $\sigma(t) = qt$, *can be equipped with the bracket multiplication*

$$\langle\cdot,\cdot\rangle_\sigma : \mathcal{D}_\sigma(\mathcal{A}) \times \mathcal{D}_\sigma(\mathcal{A}) \to \mathcal{D}_\sigma(\mathcal{A})$$

defined on generators by (19) *as*

$$\langle d_n, d_m\rangle_\sigma = q^n d_n d_m - q^m d_m d_n$$

with commutation relations

$$\langle d_n, d_m\rangle_\sigma = \big(\{n\}_q - \{m\}_q\big) d_{n+m}.$$

This bracket satisfies skew-symmetry

$$\langle d_n, d_m\rangle_\sigma = -\langle d_m, d_n\rangle_\sigma$$

and a σ*-deformed Jacobi-identity*

$$\big(q^n + 1\big)\big\langle d_n, \langle d_l, d_m\rangle_\sigma\big\rangle_\sigma + \big(q^l + 1\big)\big\langle d_l, \langle d_m, d_n\rangle_\sigma\big\rangle_\sigma + \big(q^m + 1\big)\big\langle d_m, \langle d_n, d_l\rangle_\sigma\big\rangle_\sigma = 0.$$

Remark 28. The associative algebra with an infinite number of (abstract) generators $\{d_j\colon\ j \in \mathbb{Z}\}$ and defining relations

$$q^n d_n d_m - q^m d_m d_n = \big(\{n\}_q - \{m\}_q\big) d_{n+m}, \quad n, m \in \mathbb{Z},$$

is a well-defined associative algebra, since our construction, summarized in Theorem 27, yields at the same time its operator representation. Naturally, an outcome of our approach is that this parametric family of algebras is a deformation of the Witt algebra defined by relations (1) in the sense that (1) is obtained when $q = 1$.

3.2. Non-linearly deformed Witt algebras

With the aid of Theorems 4 and 5 we will now construct a non-linear deformation of the derivations of $\mathcal{A} = \mathbb{C}[t, t^{-1}]$, the algebra of Laurent polynomials. Take any $p(t) \in \mathcal{A}$ and assume that $\sigma(t) = p(t)$. In addition, we assume $\sigma(1) = 1$, since if this is not the case, we would have had $\sigma(1) = 0$ because $\mathcal{A}$ has no zero-divisors and so $\sigma(1) = 0$ would imply $\sigma = 0$ identically. This leads us to

$$1 = \sigma(1) = \sigma\big(t \cdot t^{-1}\big) = \sigma(t)\sigma\big(t^{-1}\big) \quad \Longrightarrow \quad \sigma\big(t^{-1}\big) = \sigma(t)^{-1},$$

implying two things:

(1) $\sigma(t)$ must be a unit, and
(2) $\sigma(t^{-1})$ is completely determined by $\sigma(t)$ as its inverse in $\mathcal{A}$.

Hence, since $\sigma(t)$ is a unit, $\sigma(t) = p(t) = qt^s$, for some $q \in \mathbb{C} \setminus \{0\}$ and $s \in \mathbb{Z}$. We will, however, continue writing $p(t)$ instead of qt^s except in the explicit calculations.

It suffices to compute a greatest common divisor of $(\mathrm{id} - \sigma)(\mathcal{A})$ on the generator t since $\sigma(t^{-1})$ is determined by $\sigma(t)$. Furthermore, any gcd is only determined up to a multiple of a unit. This gives us that

$$g = \eta^{-1} t^{k-1} (\mathrm{id} - \sigma)(t) = \eta^{-1} t^{k-1} \big(t - p(t)\big) = \eta^{-1} t^{k-1} \big(t - qt^s\big)$$

is a perfectly general gcd and so Theorem 4 tells us that

$$D = \frac{\mathrm{id} - \sigma}{\eta^{-1} t^{k-1}(t - p(t))} = \eta t^{-k+1} \frac{\mathrm{id} - \sigma}{t - qt^s} = \eta t^{-k} \frac{\mathrm{id} - \sigma}{1 - qt^{s-1}}$$

is a generator for $\mathcal{D}_\sigma(A)$.

Two direct consequences of this is that, firstly, if $r \in \mathbb{Z}_{\geqslant 0}$ then

$$\begin{aligned} D\big(t^r\big) &= \eta \cdot t^{-k+r} \frac{1 - q^r t^{r(s-1)}}{1 - qt^{s-1}} = \eta \sum_{l=0}^{r-1} q^l t^{(s-1)l + r - k} \\ &= \eta t^{-k} \sum_{l=0}^{r-1} p(t)^l t^{r-l} = \eta t^{-k} \sum_{l=0}^{r-1} p(t)^{r-1-l} t^{l+1} \end{aligned}$$

and secondly, if $r \in \mathbb{Z}_{<0}$ then

$$\begin{aligned} D\big(t^r\big) &= \eta \cdot t^{-k+r} \frac{1 - q^r t^{r(s-1)}}{1 - qt^{s-1}} = -\eta \cdot t^{-k+r} q^r t^{r(s-1)} \frac{1 - q^{-r} t^{-r(s-1)}}{1 - qt^{s-1}} \\ &= -\eta \cdot t^{-k+r} q^r t^{r(s-1)} \sum_{l=0}^{-r-1} q^l t^{l(s-1)} = -\eta \sum_{l=0}^{-r-1} q^{r+l} t^{(r+l)(s-1) - k + r}. \end{aligned}$$

The σ-derivations on $\mathbb{C}[t, t^{-1}]$ are of the form $f(t) \cdot D$ for $f \in \mathbb{C}[t, t^{-1}]$ and so, given that $t^{\mathbb{Z}}$ is a linear basis of $\mathbb{C}[t, t^{-1}]$ (over $\mathbb{C}$), $-t^{\mathbb{Z}} \cdot D$ is a linear basis (over $\mathbb{C}$ again) for $\mathcal{D}_\sigma(\mathbb{C}[t, t^{-1}])$. We now introduce a bracket on $\mathcal{D}_\sigma(\mathbb{C}[t, t^{-1}])$ in accordance with Theorem 5 as we did in the previous section. Once again,

$$\big\langle -t^n \cdot D, -t^m \cdot D \big\rangle_\sigma = \big(\sigma\big(-t^n\big) D\big(-t^m\big) - \sigma\big(-t^m\big) D\big(-t^n\big)\big) D.$$

To continue we consider three cases (1) $n, m > 0$, (2) $n > 0$, $m < 0$, and (3) $n, m < 0$.

Case 1. Assume $n, m > 0$. Thus the coefficient in the bracket is

$$\begin{aligned}
&\sigma\big(t^n\big)D\big(t^m\big) - \sigma\big(t^m\big)D\big(t^n\big) \\
&\quad = p(t)^n \cdot \eta t^{-k} \sum_{l=0}^{m-1} p(t)^{m-1-l} t^{l+1} - p(t)^m \cdot \eta t^{-k} \sum_{l=0}^{n-1} p(t)^{n-1-l} t^{l+1} \\
&\quad = \eta\left(\sum_{l=0}^{m-1} p(t)^{n+m-1-l} t^{l-k+1} - \sum_{l=0}^{n-1} p(t)^{n+m-1-l} t^{l-k+1}\right).
\end{aligned}$$

To re-write this we use the "sign function"

$$\operatorname{sign}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

So,

$$\begin{aligned}
&\eta\left(\sum_{l=0}^{m-1} p(t)^{n+m-1-l} t^{l-k+1} - \sum_{l=0}^{n-1} p(t)^{n+m-1-l} t^{l-k+1}\right) \\
&\quad = \eta \operatorname{sign}(m-n) \sum_{l=\min(n,m)}^{\max(n,m)-1} p(t)^{n+m-1-l} t^{l-k+1} \\
&\quad = \eta \operatorname{sign}(m-n) \sum_{l=\min(n,m)}^{\max(n,m)-1} q^{n+m-1-l} t^{(n+m-1)s-(s-1)l-(k-1)}
\end{aligned}$$

giving that (for $n, m \in \mathbb{Z}_{\geqslant 0}$)

$$\langle d_n, d_m\rangle_\sigma = \eta \operatorname{sign}(n-m) \sum_{l=\min(n,m)}^{\max(n,m)-1} q^{n+m-1-l} d_{(n+m-1)s-(s-1)l-(k-1)}. \tag{59}$$

Remark 29. Note that if we take $k = 0$, $s = 1$ and $\eta = 1$, the right-hand sum in (59) contains only the generator d_{n+m} multiplied by the coefficient

$$\begin{aligned}
&\operatorname{sign}(n-m) q^{n+m-1} \sum_{l=\min(n,m)}^{\max(n,m)-1} \big(q^{-1}\big)^l \\
&\quad = \operatorname{sign}(n-m) q^{n+m-1} \frac{(q^{-1})^{\max(n,m)} - (q^{-1})^{\min(n,m)}}{q^{-1} - 1} \\
&\quad = \operatorname{sign}(n-m) \frac{q^{\max(n,m)} - q^{\min(n,m)}}{q - 1}
\end{aligned}$$

$$\begin{aligned}&= \operatorname{sign}(n-m)\bigl(\{\max(n,m)\}_q - \{\min(n,m)\}_q\bigr)\\&= \{n\}_q - \{m\}_q.\end{aligned}$$

This means that commutation relation (59) reduces to relation (56) for the q-deformation of the Witt algebra described in Section 2.1.

Case 2. Now, suppose $n > 0$ and $m < 0$. We then get the bracket coefficient

$$\begin{aligned}&\sigma(t^n)D(t^m) - \sigma(t^m)D(t^n)\\&= -\eta\cdot q^n t^{ns}\sum_{l_1=0}^{-m-1} q^{m+l_1}t^{(m+l_1)(s-1)-k+m} - \eta\cdot q^m t^{ms}\sum_{l_2=0}^{n-1} q^{l_2}t^{(s-1)l_2+n-k}\\&= -\eta\left(\sum_{l_1=0}^{-m-1} q^{n+m+l_1}t^{(m+l_1)(s-1)-k+m+ns} + \sum_{l_2=0}^{n-1} q^{m+l_2}t^{(s-1)l_2+n-k+ms}\right).\end{aligned}$$

We now show that there is no overlap between these two sums. Knowing that $0 \leqslant l_2 \leqslant n-1$ we consider the difference in exponents of t:

$$\begin{aligned}&(m+l_1)(s-1) - k + m + ns - (s-1)l_2 - n + k - ms\\&\quad= (s-1)(l_1-l_2) + m(s-1) + (1-s)m + (s-1)n\\&\quad= (s-1)(l_1-l_2+n)\end{aligned}$$

and this is zero (for $s \neq 1$) when $n = l_2 - l_1$. But, $n = l_2 - l_1 \leqslant n-1-0 = n-1$ which is a contradiction and hence we cannot have any overlap. Hence, we see that the bracket becomes

$$\langle d_n, d_m\rangle_\sigma = \eta\left(\sum_{l=0}^{-m-1} q^{n+m+l}d_{(m+l)(s-1)+ns+m-k} + \sum_{l=0}^{n-1} q^{m+l}d_{(s-1)l+n+ms-k}\right). \tag{60}$$

Case 2′. By interchanging the role of n and m so that $m > 0$ and $n < 0$ we get instead

$$\begin{aligned}&\sigma(t^n)D(t^m) - \sigma(t^m)D(t^n)\\&\quad= \eta\left(\sum_{l_1=0}^{m-1} q^{n+l_1}t^{(s-1)l_1+m+ns-k} + \sum_{l_2=0}^{-n-1} q^{m+n+l_2}t^{(n+l_2)(s-1)+n+ms-k}\right),\end{aligned}$$

so the coefficient for the bracket becomes

$$\langle d_n, d_m\rangle_\sigma = -\eta\left(\sum_{l=0}^{m-1} q^{n+l}d_{(s-1)l+m+ns-k} + \sum_{l=0}^{-n-1} q^{m+n+l}d_{(n+l)(s-1)+n+ms-k}\right). \tag{61}$$

Remark 30. If we put $k = 0$, $s = 1$ and $\eta = 1$ in Case 2 and 2′, we once again get the single generator d_{n+m} multiplied with

$$q^{n+m}\sum_{l=0}^{-m-1} q^l + q^m \sum_{l=0}^{n-1} q^l = q^{m+n}\frac{1-q^{-m}}{1-q} + q^m \frac{1-q^n}{1-q}$$
$$= -q^n \frac{1-q^m}{1-q} + q^m \frac{1-q^n}{1-q} = \frac{q^m - q^n}{1-q}$$
$$= \{n\}_q - \{m\}_q,$$

just as we would expect from the case in the previous section.

Case 3. Both $n, m < 0$. This leads to

$$\begin{aligned}
&\sigma\big(t^n\big)D\big(t^m\big) - \sigma\big(t^m\big)D\big(t^n\big)\\
&\quad = -\eta\cdot q^n t^{ns}\sum_{l_1=0}^{-m-1} q^{m+l_1} t^{(m+l_1)(s-1)-k+m} + +\eta\cdot q^m t^{ms}\sum_{l_2=0}^{-n-1} q^{n+l_2} t^{(n+l_2)(s-1)-k+n}\\
&\quad = -\eta\left(\sum_{l=0}^{-m-1} q^{n+m+l} t^{(m+n)s+(s-1)l-k} - \sum_{l=0}^{-n-1} q^{n+m+l} t^{(n+m)s+(s-1)l-k}\right).
\end{aligned}$$

This leads to a bracket coefficient resembling that of Case 1, namely

$$\langle d_n, d_m\rangle_\sigma = \eta\, \mathrm{sign}(n-m) \sum_{l=\min(-n,-m)}^{\max(-n,-m)-1} q^{n+m+l} d_{(m+n)s+(s-1)l-k}. \tag{62}$$

We can now, from (32), calculate δ to get

$$\delta = \frac{\sigma(g)}{g} = \frac{\eta^{-1}t^{sk}(1-q^s t^{s(s-1)})}{\eta^{-1}t^k(1-qt^{s-1})} = q^k t^{k(s-1)} \sum_{r=0}^{s-1}\big(qt^{s-1}\big)^r.$$

This means, by the definition of δ, that D and σ span a "quantum plane"-like commutation relation

$$D\circ\sigma = q^k t^{k(s-1)} \sum_{r=0}^{s-1}\big(qt^{s-1}\big)^r \cdot \sigma\circ D.$$

To get a hom-Lie algebra it is enough for δ to be a (non-zero) complex number and this can be achieved only when $s = 1$, that is, when the deformation is linear (i.e., when σ homogeneous of degree zero).

Theorem 5 now tells us what a generalized Jacobi identity looks like

$$\circlearrowleft_{n,m,l}\left(q^n\big\langle d_{ns},\langle d_m,d_l\rangle_\sigma\big\rangle_\sigma+q^k t^{k(s-1)}\sum_{r=0}^{s-1}\big(qt^{s-1}\big)^r\big\langle d_n\langle d_m,d_l\rangle_\sigma\big\rangle_\sigma\right)=0.$$

The hom-Lie algebra Jacobi identity ($s=1$) becomes

$$\circlearrowleft_{n,m,l}\big(q^n+q^k\big)\big\langle d_n,\langle d_m,d_l\rangle_\sigma\big\rangle_\sigma=0.$$

We summarize our findings in a theorem.

Theorem 31. *Let $\mathcal{A}=\mathbb{C}[t,t^{-1}]$. Then the $\mathbb{C}$-linear space*

$$\mathcal{D}_\sigma(\mathcal{A})=\bigoplus_{n\in\mathbb{Z}}\mathbb{C}\cdot d_n,$$

where

$$D=\eta t^{-k+1}\frac{\mathrm{id}-\sigma}{t-qt^s},$$

$d_n=-t^nD$ and $\sigma(t)=qt^s$, can be equipped with the bracket product

$$\langle\cdot,\cdot\rangle_\sigma:\mathcal{D}_\sigma(\mathcal{A})\times\mathcal{D}_\sigma(\mathcal{A})\to\mathcal{D}_\sigma(\mathcal{A})$$

defined on generators by (19) *as*

$$\langle d_n,d_m\rangle_\sigma=q^nd_{ns}d_m-q^md_{ms}d_n$$

and satisfying defining commutation relations

$$\langle d_n,d_m\rangle_\sigma=\eta\,\mathrm{sign}(n-m)\sum_{l=\min(n,m)}^{\max(n,m)-1}q^{n+m-1-l}d_{s(n+m-1)-(k-1)-l(s-1)}$$

for $n,m\geqslant 0$;

$$\langle d_n,d_m\rangle_\sigma=\eta\left(\sum_{l=0}^{-m-1}q^{n+m+l}d_{(m+l)(s-1)+ns+m-k}+\sum_{l=0}^{n-1}q^{m+l}d_{(s-1)l+n+ms-k}\right)$$

for $n\geqslant 0,\ m<0$;

$$\langle d_n,d_m\rangle_\sigma=-\eta\left(\sum_{l=0}^{m-1}q^{n+l}d_{(s-1)l+m+ns-k}+\sum_{l=0}^{-n-1}q^{m+n+l}d_{(n+l)(s-1)+n+ms-k}\right)$$

for $m\geqslant 0,\ n<0$;

$$\langle d_n,d_m\rangle_\sigma=\eta\,\mathrm{sign}(n-m)\sum_{l=\min(-n,-m)}^{\max(-n,-m)-1}q^{n+m+l}d_{(m+n)s+(s-1)l-k}\quad\textit{for } n,m<0.$$

Furthermore, this bracket satisfies skew-symmetry

$$\langle d_n, d_m\rangle_\sigma = -\langle d_m, d_n\rangle_\sigma,$$

and a σ-deformed Jacobi identity

$$\circlearrowleft_{n,m,l}\left(q^n\big\langle d_{ns}, \langle d_m, d_l\rangle_\sigma\big\rangle_\sigma + q^k t^{k(s-1)}\sum_{r=0}^{s-1}\big(qt^{s-1}\big)^r\big\langle d_n\langle d_m, d_l\rangle_\sigma\big\rangle_\sigma\right) = 0.$$

Remark 32. The associative algebra with an infinite number of (abstract) generators $\{d_j\colon\ j\in\mathbb{Z}\}$ and defining relations

$$q^n d_{ns}d_m - q^m d_{ms}d_n = \begin{cases} \text{Eq. (59)} & \text{for } n, m \geqslant 0,\\ \text{Eq. (60)} & \text{for } n \geqslant 0,\ m < 0,\\ \text{Eq. (61)} & \text{for } m \geqslant 0,\ n < 0,\\ \text{Eq. (62)} & \text{for } n, m < 0 \end{cases}$$

is a well-defined associative algebra, since our construction yields at the same time its operator representation. Naturally, an outcome of our approach is that this parametric family of algebras is a deformation of the Witt algebra defined by relations (1) in the sense that (1) is obtained when $q = 1$ and $k = 0$, $s = 1$, $\eta = 1$.

3.2.1. A submodule of $\mathcal{D}_\sigma(\mathbb{C}[t, t^{-1}])$

We let, as before, $\mathcal{A} = \mathbb{C}[t, t^{-1}]$, the algebra of Laurent polynomials, and σ be some non-zero endomorphism such that $\sigma(t) = p(t)$. In the previous section we showed that any greatest common divisor of $(\mathrm{id} - \sigma)(\mathcal{A})$ has the form $\eta^{-1}t^{k-1}(t - p(t)) = \eta^{-1}t^k(1 - qt^{s-1})$ for $k \in \mathbb{Z}$ and non-zero $\eta \in \mathbb{C}$. As described in Remark 13, this means that

$$D = \frac{\mathrm{id} - \sigma}{\eta^{-1}\cdot t^k}$$

generates a proper cyclic $\mathcal{A}$-submodule $\mathcal{M}$ of $\mathcal{D}_\sigma(\mathcal{A})$, unless $p(t) = \beta t$ for some $\beta \in \mathbb{C}$.

As above we calculate δ using (32) and we find

$$\delta = q^k t^{(s-1)k}$$

which means that D and σ satisfy the following relation

$$D\circ\sigma = q^k t^{(s-1)k}\sigma\circ D.$$

We set

$$\tilde{d}_n = -t^n D.$$

Before we calculate the bracket, we note that $D(t) = \eta t^{-k+1}(1 - qt^{s-1})$, and

$$\eta\frac{\mathrm{id}-\sigma}{t^k}\left(t^r\right) = \eta t^{r-k}\left(1 - q^r t^{(s-1)r}\right).$$

The coefficient of D in the bracket $\langle \tilde{d}_n, \tilde{d}_m\rangle_\sigma$ then becomes

$$\begin{aligned}\sigma\left(t^n\right)Dt^m - \sigma\left(t^m\right)Dt^n &= \eta q^n t^{ns+m-k}\left(1 - q^m t^{(s-1)m}\right) - \eta q^m t^{ms+n-k}\left(1 - q^n t^{(s-1)n}\right)\\ &= \eta q^n t^{ns+m-k} - \eta q^m t^{ms+n-k}\end{aligned}$$

which means that

$$\langle \tilde{d}_n, \tilde{d}_m\rangle_\sigma = \eta q^m \tilde{d}_{ms+n-k} - \eta q^n \tilde{d}_{ns+m-k},$$

where

$$\langle \tilde{d}_n, \tilde{d}_m\rangle_\sigma = q^n \tilde{d}_{ns}\tilde{d}_m - q^m \tilde{d}_{ms}\tilde{d}_n$$

by (19). Putting $a = -t^n$, $b = -t^m$ and $c = -t^l$ in (22) with $\delta = q^k t^{(s-1)k}$ and $\Delta = D$, we get

$$\begin{aligned}&\circlearrowleft_{n,m,l}\left(\left\langle -q^n t^{ns}D, \left\langle -t^m D, -t^l D\right\rangle_\sigma\right\rangle_\sigma + q^k t^{(s-1)k}\left\langle -t^n D, \left\langle -t^m D, -t^l D\right\rangle_\sigma\right\rangle_\sigma\right)\\ &\quad = \circlearrowleft_{n,m,l}\left(q^n\left\langle \tilde{d}_{ns}, \langle \tilde{d}_m, \tilde{d}_l\rangle_\sigma\right\rangle_\sigma + q^k t^{(s-1)k}\left\langle \tilde{d}_n, \langle \tilde{d}_m, \tilde{d}_l\rangle_\sigma\right\rangle_\sigma\right) = 0.\end{aligned}$$

We summarize the obtained results in the following theorem.

Theorem 33. *The $\mathbb{C}$-linear space*

$$\mathcal{M} = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}\cdot\tilde{d}_i \quad \textit{with } \tilde{d}_i = -t^i D$$

allows a structure as an algebra with bracket defined on generators (*by* (19)) *as*

$$\langle \tilde{d}_n, \tilde{d}_m\rangle_\sigma = q^n \tilde{d}_{ns}\tilde{d}_m - q^m \tilde{d}_{ms}\tilde{d}_n$$

and satisfying relations

$$\langle \tilde{d}_n, \tilde{d}_m\rangle_\sigma = \eta q^m \tilde{d}_{ms+n-k} - \eta q^n \tilde{d}_{ns+m-k},$$

with $s \in \mathbb{Z}$ and $\eta \in \mathbb{C}$. The σ-deformed Jacobi identity becomes

$$\circlearrowleft_{n,m,l}\left(q^n\left\langle \tilde{d}_{ns}, \langle \tilde{d}_m, \tilde{d}_l\rangle_\sigma\right\rangle_\sigma + q^k t^{(s-1)k}\left\langle \tilde{d}_n, \langle \tilde{d}_m, \tilde{d}_l\rangle_\sigma\right\rangle_\sigma\right) = 0.$$

Remark 34. The only possible way to obtain a $\mathbb{Z}$-grading on $\mathcal{M}$ with the bracket $\langle\cdot,\cdot\rangle_\sigma$ is when $k=0$ and $s=1$ in the above theorem.

Remark 35. By performing a change of basis and considering instead $\tilde{d}_n = -t^{n+k}D$ in the definition of $\tilde{d}_n$ we can evade the use of k altogether. Hence we see that the k-shifted grading is something resulting from a choice of basis for $\mathcal{M}$.

Remark 36. The associative algebra with an infinite number of (abstract) generators $\{d_j\colon\ j \in \mathbb{Z}\}$ and defining relations

$$q^n d_{ns} d_m - q^m d_{ms} d_n = \eta q^m d_{ms+n-k} - \eta q^n d_{ns+m-k}, \quad n, m \in \mathbb{Z},$$

is a well-defined associative algebra, since our construction, summarized in Theorem 33, yields at the same time its operator representation. It is interesting that, when $q=1$, $k=0$ and $s=1$, we get a commutative algebra with countable number of generators instead of the Witt algebra.

3.2.2. Generalization to several variables

We let the boldface font denote an $\mathbb{Z}$-vector, e.g.,

$$\mathbf{k} = (k_1, k_2, \ldots, k_n), \quad k_i \in \mathbb{Z}.$$

Consider the algebra of Laurent polynomials in $z_1, z_2, \ldots, z_n$

$$\mathcal{A} = \mathbb{C}\big[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_n^{\pm 1}\big] \cong \frac{\mathbb{C}[z_1, \ldots, z_n, u_1, \ldots, u_n]}{(z_1 u_1 - 1, \ldots, z_n u_n - 1)}$$

and let

$$\sigma(z_1) = q_{z_1} z_1^{S_{1,1}} \cdots z_n^{S_{1,n}}, \quad \ldots, \quad \sigma(z_n) = q_{z_n} z_1^{S_{n,1}} \cdots z_n^{S_{n,n}}.$$

Notice that σ is determined by an integral matrix

$$S = [S_{i,j}]$$

and the complex numbers q_{z_k}. A common divisor on $(\mathrm{id} - \sigma)(\mathcal{A})$ is

$$g = Q^{-1} z_1^{G_1} \cdots z_n^{G_n}$$

and so the element

$$D = Q \frac{\mathrm{id} - \sigma}{z_1^{G_1} \cdots z_n^{G_n}}$$

generates an $\mathcal{A}$-submodule $\mathcal{M}'$ of $\mathcal{D}_\sigma(\mathcal{A})$. Using these D and σ we calculate δ to be (by formula (18))

$$\begin{aligned}\delta &= q_{z_1}^{G_1}\cdots q_{z_n}^{G_n} z_1^{(S_{1,1}-1)G_1+S_{2,1}G_2+\cdots+S_{n,1}G_n}\cdots z_n^{S_{1,n}G_1+S_{2,n}G_2+\cdots+(S_{n,n}-1)G_n}\\ &= q_{z_1}^{G_1}\cdots q_{z_n}^{G_n} z_1^{\delta_1}\cdots z_n^{\delta_n},\end{aligned}$$

where

$$\delta_k = S_{1,k}G_1 + S_{2,k}G_2 + \cdots + (S_{k,k}-1)G_k + \cdots + S_{n,k}G_n.$$

We also introduce the notation $\alpha_r(\mathbf{l}) = \sum_{i=1}^n S_{i,r}l_i$. Now,

$$\begin{aligned}\sigma(z_1)^{l_1}\cdots\sigma(z_n)^{l_n} &= q_{z_1}^{l_1}\cdots q_{z_n}^{l_n} z_1^{S_{1,1}l_1+\cdots+S_{n,1}l_n}\cdots z_n^{S_{1,n}l_1+\cdots+S_{n,n}l_n}\\ &= q_{z_1}^{l_1}\cdots q_{z_n}^{l_n} z_1^{\alpha_1(\mathbf{l})}\cdots z_n^{\alpha_n(\mathbf{l})}\end{aligned}$$

and so,

$$\begin{aligned}D\big(z_1^{l_1}\cdots z_n^{l_n}\big) &= Q\frac{\mathrm{id}-\sigma}{z_1^{G_1}\cdots z_n^{G_n}}\big(z_1^{l_1}\cdots z_n^{l_n}\big)\\ &= Q\frac{z_1^{l_1}\cdots z_n^{l_n} - \sigma(z_1)^{l_1}\cdots\sigma(z_n)^{l_n}}{z_1^{G_1}\cdots z_n^{G_n}}\\ &= Q\frac{z_1^{l_1}\cdots z_n^{l_n} - q_{z_1}^{l_1}\cdots q_{z_n}^{l_n} z_1^{\alpha_1(\mathbf{l})}\cdots z_n^{\alpha_n(\mathbf{l})}}{z_1^{G_1}\cdots z_n^{G_n}}\\ &= -Qz_1^{l_1-G_1}\cdots z_n^{l_n-G_n}\big(q_{z_1}^{l_1}\cdots q_{z_n}^{l_n} z_1^{\alpha_1(\mathbf{l})-l_1}\cdots z_n^{\alpha_n(\mathbf{l})-l_n} - 1\big).\end{aligned}$$

We now put $d_{\mathbf{l}} = d_{l_1,\dots,l_n} = -z_1^{l_1}\cdots z_n^{l_n}D$ and calculate the coefficient of the bracket $\langle d_{\mathbf{k}}, d_{\mathbf{l}}\rangle_\sigma$ with the aid of Theorem 5 as before:

$$\begin{aligned}&\sigma\big(z_1^{k_1}\cdots z_n^{k_n}\big)D\big(z_1^{l_1}\cdots z_n^{l_n}\big) - \sigma\big(z_1^{l_1}\cdots z_n^{l_n}\big)D\big(z_1^{k_1}\cdots z_n^{k_n}\big)\\ &\quad= -Qq_{z_1}^{k_1}\cdots q_{z_n}^{k_n} z_1^{\alpha_1(\mathbf{k})}\cdots z_n^{\alpha_n(\mathbf{k})} z_1^{l_1-G_1}\cdots z_n^{l_n-G_n}\big(q_{z_1}^{l_1}\cdots q_{z_n}^{l_n} z_1^{\alpha_1(\mathbf{l})-l_1}\cdots z_n^{\alpha_n(\mathbf{l})-l_n} - 1\big)\\ &\qquad+ Qq_{z_1}^{l_1}\cdots q_{z_n}^{l_n} z_1^{\alpha_1(\mathbf{l})}\cdots z_n^{\alpha_n(\mathbf{l})} z_1^{k_1-G_1}\cdots z_n^{k_n-G_n}\big(q_{z_1}^{k_1}\cdots q_{z_n}^{k_n} z_1^{\alpha_1(\mathbf{k})-k_1}\cdots z_n^{\alpha_n(\mathbf{k})-k_n} - 1\big)\\ &\quad= -Qq_{z_1}^{k_1+l_1}\cdots q_{z_n}^{k_n+l_n} z_1^{\alpha_1(\mathbf{k})+\alpha_1(\mathbf{l})-G_1}\cdots z_n^{\alpha_n(\mathbf{k})+\alpha_n(\mathbf{l})-G_n}\\ &\qquad+ Qq_{z_1}^{k_1}\cdots q_{z_n}^{k_n} z_1^{\alpha_1(\mathbf{k})+l_1-G_1}\cdots z_n^{\alpha_n(\mathbf{k})+l_n-G_n}\\ &\qquad+ Qq_{z_1}^{k_1+l_1}\cdots q_{z_n}^{k_n+l_n} z_1^{\alpha_1(\mathbf{k})+\alpha_1(\mathbf{l})-G_1}\cdots z_n^{\alpha_n(\mathbf{k})+\alpha_n(\mathbf{l})-G_n}\\ &\qquad- Qq_{z_1}^{l_1}\cdots q_{z_n}^{l_n} z_1^{\alpha_1(\mathbf{l})+k_1-G_1}\cdots z_n^{\alpha_n(\mathbf{l})+k_n-G_n}\\ &\quad= Qq_{z_1}^{k_1}\cdots q_{z_n}^{k_n} z_1^{\alpha_1(\mathbf{k})+l_1-G_1}\cdots z_n^{\alpha_n(\mathbf{k})+l_n-G_n} - Qq_{z_1}^{l_1}\cdots q_{z_n}^{l_n} z_1^{\alpha_1(\mathbf{l})+k_1-G_1}\cdots z_n^{\alpha_n(\mathbf{l})+k_n-G_n}.\end{aligned}$$

This gives us

$$\begin{aligned}\langle d_{\mathbf{k}}, d_{\mathbf{l}}\rangle_\sigma = {} & Q q_{z_1}^{l_1}\cdots q_{z_n}^{l_n} d_{\alpha_1(\mathbf{l})+k_1-G_1,\dots,\alpha_n(\mathbf{l})+k_n-G_n} \\ & - Q q_{z_1}^{k_1}\cdots q_{z_n}^{k_n} d_{\alpha_1(\mathbf{k})+l_1-G_1,\dots,\alpha_n(\mathbf{k})+l_n-G_n}.\end{aligned}$$

Using the δ we calculated before we can deduce a deformed Jacobi identity as

$$\circlearrowleft_{k,l,h} \left(q_{z_1}^{k_1}\cdots q_{z_n}^{k_n}\left\langle d_{\alpha_1(\mathbf{k}),\dots,\alpha_n(\mathbf{k})}, \langle d_{\mathbf{l}}, d_{\mathbf{h}}\rangle_\sigma\right\rangle_\sigma + q_{z_1}^{G_1}\cdots q_{z_n}^{G_n} z_1^{\delta_1}\cdots z_n^{\delta_n}\left\langle d_{\mathbf{k}}, \langle d_{\mathbf{l}}, d_{\mathbf{h}}\rangle_\sigma\right\rangle_\sigma\right) = 0$$

from which we see that we get a hom-Lie algebra if all δ_k are zero, that is, if

$$\begin{bmatrix} S_{1,1}-1 & S_{1,2} & \cdots & S_{1,n} \\ S_{2,1} & S_{2,2}-1 & \cdots & S_{2,n} \\ \vdots & \cdots & & \\ S_{n,1} & & \cdots & S_{n,n}-1 \end{bmatrix}^T \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{bmatrix} = 0$$

which means that

$$\ker \begin{bmatrix} S_{1,1}-1 & S_{1,2} & \cdots & S_{1,n} \\ S_{2,1} & S_{2,2}-1 & \cdots & S_{2,n} \\ \vdots & \cdots & & \\ S_{n,1} & & \cdots & S_{n,n}-1 \end{bmatrix}^T \cap \mathbb{Z}^n \neq \emptyset.$$

In this case we get the deformed Jacobi identity

$$\circlearrowleft_{k,l,h} \left(q_{z_1}^{k_1}\cdots q_{z_n}^{k_n}\left\langle d_{\alpha_1(\mathbf{k}),\dots,\alpha_n(\mathbf{k})}, \langle d_{\mathbf{l}}, d_{\mathbf{h}}\rangle_\sigma\right\rangle_\sigma + q_{z_1}^{G_1}\cdots q_{z_n}^{G_n}\left\langle d_{\mathbf{k}}, \langle d_{\mathbf{l}}, d_{\mathbf{h}}\rangle_\sigma\right\rangle_\sigma\right) = 0.$$

We summarize the obtained results in the following theorem.

Theorem 37. *The* $\mathbb{C}$*-linear space*

$$\mathcal{M}' = \bigoplus_{\mathbf{l}\in\mathbb{Z}^n} \mathbb{C}\cdot d_{\mathbf{l}}$$

spanned by $d_{\mathbf{l}} = -z_1^{l_1}\cdots z_n^{l_n} D$, *where* D *is given by*

$$D = Q\frac{\mathrm{id}-\sigma}{z_1^{G_1}\cdots z_n^{G_n}},$$

can be endowed with a bracket defined on generators (*by* (19)) *as*

$$\langle d_{\mathbf{k}}, d_{\mathbf{l}}\rangle_\sigma = q_{z_1}^{k_1}\cdots q_{z_n}^{k_n} d_{\alpha_1(\mathbf{k}),\dots,\alpha_n(\mathbf{k})} d_{\mathbf{l}} - q_{z_1}^{l_1}\cdots q_{z_n}^{l_n} d_{\alpha_1(\mathbf{l}),\dots,\alpha_n(\mathbf{l})} d_{\mathbf{k}}$$

and satisfying relations

$$\begin{aligned}\langle d_{\mathbf{k}}, d_{\mathbf{l}}\rangle_\sigma = {} & Qq_{z_1}^{l_1}\cdots q_{z_n}^{l_n} d_{\alpha_1(\mathbf{l})+k_1-G_1,\ldots,\alpha_n(\mathbf{l})+k_n-G_n} \\ & - Qq_{z_1}^{k_1}\cdots q_{z_n}^{k_n} d_{\alpha_1(\mathbf{k})+l_1-G_1,\ldots,\alpha_n(\mathbf{k})+l_n-G_n}.\end{aligned}$$

The bracket satisfies the σ-deformed Jacobi identity

$$\circlearrowleft_{k,l,h} \Big(q_{z_1}^{k_1}\cdots q_{z_n}^{k_n} \big\langle d_{\alpha_1(\mathbf{k}),\ldots,\alpha_n(\mathbf{k})}, \langle d_{\mathbf{l}}, d_{\mathbf{h}}\rangle_\sigma\big\rangle_\sigma + q_{z_1}^{G_1}\cdots q_{z_n}^{G_n} z_1^{\delta_1}\cdots z_n^{\delta_n} \big\langle d_{\mathbf{k}}, \langle d_{\mathbf{l}}, d_{\mathbf{h}}\rangle_\sigma\big\rangle_\sigma \Big) = 0.$$

Furthermore, a hom-Lie algebra is obtained if the eigenvalue problem

$$S\cdot G = G$$

(*where "$\cdot$" means product of matrices*) *has a solution $G\in\mathbb{Z}^n$. We then get the Jacobi-like identity*

$$\circlearrowleft_{k,l,h} \Big(q_{z_1}^{k_1}\cdots q_{z_n}^{k_n} \big\langle d_{\alpha_1(\mathbf{k}),\ldots,\alpha_n(\mathbf{k})}, \langle d_{\mathbf{l}}, d_{\mathbf{h}}\rangle_\sigma\big\rangle_\sigma + q_{z_1}^{G_1}\cdots q_{z_n}^{G_n} \big\langle d_{\mathbf{k}}, \langle d_{\mathbf{l}}, d_{\mathbf{h}}\rangle_\sigma\big\rangle_\sigma \Big) = 0.$$

4. A deformation of the Virasoro algebra

In this section we will prove existence and uniqueness (up to isomorphism of hom-Lie algebras) of a one-dimensional central extension of the hom-Lie algebra $(\mathcal{D}_\sigma(\mathbb{C}[t,t^{-1}]),\varsigma)$ constructed in Section 3.1, in the case when q is not a root of unity. The obtained hom-Lie algebra is a q-deformation of the Virasoro algebra.

4.1. Uniqueness of the extension

Let $\mathcal{A}=\mathbb{C}[t,t^{-1}]$, σ be the algebra endomorphism on $\mathcal{A}$ satisfying $\sigma(t)=qt$, where $0,1\neq q\in\mathbb{C}$ is not a root of unity, and set $L=\mathcal{D}_\sigma(\mathcal{A})$. Then L can be given the structure of a hom-Lie algebra (L,ς) as described in Section 3.1.

Let

$$0 \longrightarrow (\mathbb{C},\mathrm{id}_{\mathbb{C}}) \xrightarrow{\iota} (\hat{L},\hat{\varsigma}) \xrightarrow{\mathrm{pr}} (L,\varsigma) \longrightarrow 0$$

be a short exact sequence of hom-Lie algebras and hom-Lie algebra homomorphisms. In other words, let $(\hat{L},\hat{\varsigma})$ be a one-dimensional central extension of (L,ς) by $(\mathbb{C},\mathrm{id}_{\mathbb{C}})$. We also set $\mathbf{c}=\iota(1)$.

Choose a linear section $s:L\to\hat{L}$ and let $g_s\in\mathrm{Alt}^2(L,\mathbb{C})$ be the corresponding "2-cocycle" so that (37) is satisfied for $x,y\in L$. Let $\{d_n\}$ denote the basis (52) of L. Define a linear map $s':L\to\hat{L}$ by

$$s'(d_n)=\begin{cases} s(d_n) & \text{if } n=0,\\ s(d_n)-\frac{1}{\{n\}_q}\iota\circ g_s(d_0,d_n)\mathbf{c} & \text{if } n\neq 0.\end{cases}$$

Then s' is also a section. Using the calculation (39) and the commutation relation (56) we get

$$\iota \circ g_{s'}(d_m, d_n) = \iota \circ g_s(d_m, d_n) + (s - s')\big(\langle d_m, d_n\rangle_L\big) = \begin{cases} \iota \circ g_s(d_m, d_n) & \text{if } m+n=0, \\ \iota \circ g_s(d_m, d_n) + \frac{\{m\}_q - \{n\}_q}{\{m+n\}_q} \iota \circ g_s(d_0, d_{m+n}) & \text{if } m+n \neq 0. \end{cases}$$

In particular we have $g_{s'}(d_0, d_n) = 0$ for any $n \in \mathbb{Z}$. According to the calculations in Section 2.4, the "2-cocycle" $g_{s'}$ must satisfy (38) for any $a, b, c \in L$. Thus we have,

$$\circlearrowleft_{k,l,m} g_{s'}\big((\mathrm{id} + \varsigma)(d_k), \langle d_l, d_m\rangle_L\big) = 0$$

for $k, l, m \in \mathbb{Z}$. Substituting the definition (54) of ς and using the commutation relation (56) again we get

$$\circlearrowleft_{k,l,m} \big(1 + q^k\big)\big(\{l\}_q - \{m\}_q\big) a(k, l+m) = 0, \tag{63}$$

where we for simplicity have put $a(m, n) = g_{s'}(d_m, d_n)$ for $m, n \in \mathbb{Z}$. Using (63) with $k = 0$ and that $a(0, n) = 0$ for any $n \in \mathbb{Z}$ we obtain

$$\big(1 + q^l\big)\{m\}_q a(l, m) + \big(1 + q^m\big)\big(-\{l\}_q\big) a(m, l) = 0,$$

or, since a is alternating,

$$\left(\big(1 + q^l\big)\frac{q^m - 1}{q - 1} + \big(1 + q^m\big)\frac{q^l - 1}{q - 1}\right) a(l, m) = 0,$$

which simplifies to

$$2\frac{q^{l+m} - 1}{q - 1} a(l, m) = 0.$$

This shows that $a(l, m) = 0$ unless $l + m = 0$. Setting $b(m) = a(m, -m)$ we have so far

$$\big\langle s'(d_m), s'(d_n)\big\rangle_{\hat{L}} = \big(\{m\}_q - \{n\}_q\big) s'(d_{m+n}) + \delta_{m+n,0} b(m)\mathbf{c}.$$

Using (63) with $k = -n - 1$, $l = n$, $m = 1$ we get

$$\big(1 + q^{-n-1}\big)\frac{q^n - q}{q - 1} a(-n-1, n+1) + \big(1 + q^n\big)\frac{q - q^{-n-1}}{q - 1} a(n, -n) + (1 + q)\frac{q^{-n-1} - q^n}{q - 1} a(1, -1) = 0,$$

or, after multiplication by q^{n+1},

$$q\big(1+q^{n+1}\big)\{n-1\}_q b(n+1) = \big(1+q^n\big)\{n+2\}_q b(n) - (1+q)\{2n+1\}_q b(1). \quad (64)$$

This is a second order linear recurrence equation in b.

Lemma 38. *The functions $b_1, b_2 : \mathbb{Z} \to \mathbb{C}$ defined by*

$$b_1(m) = \frac{q^{-m}}{1+q^m}\{m-1\}_q\{m\}_q\{m+1\}_q,$$
$$b_2(m) = q^{-m}\{2m\}_q$$

are two linear independent solutions of (64).

Proof. Substituting b_1 for b in (64) the left-hand side equals

$$q\{n-1\}_q q^{-n-1}\{n\}_q\{n+1\}_q\{n+2\}_q,$$

while the right-hand side becomes

$$\{n+2\}_q q^{-n}\{n-1\}_q\{n\}_q\{n+1\}_q - 0.$$

These expressions are equal. To prove that b_2 is also a solution requires some calculations:

$$\begin{aligned}
&q\{n-1\}_q\big(1+q^{-1}\big)\{2n+2\}_q - \{n+2\}_q\big(1+q^{-n}\big)\{2n\}_q + \{2n+1\}_q\big(1+q^{-1}\big)\{2\}_q \\
&\quad = \big(\{n\}_q - q^{n-1}\big)\big(q+q^{-n}\big)\big(\{2n\}_q + q^{2n} + q^{2n+1}\big) \\
&\qquad - \big(\{n\}_q + q^n(q+1)\big)\big(1+q^{-n}\big)\{2n\}_q + \big(\{2n\}_q + q^{2n}\big)\big(1+q^{-1}\big)(1+q) \\
&\quad = \{2n\}_q\big(\{n\}_q\big(q+q^{-n}\big) - q^n - q^{-1} - \{n\}_q\big(1+q^{-n}\big) - (q+1)q^n - 1 - q + 2 \\
&\qquad + q + q^{-1}\big) + (q+1)q^{2n}\big(q\{n\}_q + q^{-n}\{n\}_q - q^n - q^{-1} + 1 + q^{-1}\big) \\
&\quad = \{2n\}_q\big(q^n - 1 - q^n - q^n(q+1) - 1 + 2\big) + q^{2n}\frac{q+1}{q-1}\big(-q^n + q^n\big) \\
&\quad = \frac{q^{2n}-1}{q-1}(q+1)\big(-q^n\big) + q^{2n}\frac{q+1}{q-1}\big(-q^n + q^n\big) \\
&\quad = \frac{q+1}{q-1}\big(q^n - q^{2n-n}\big) = 0.
\end{aligned}$$

It remains to show that b_1 and b_2 are linear independent. If

$$\lambda b_1 + \mu b_2 = 0,$$

then evaluation at $m = 1$ gives $\mu q^{-1}(1+q) = 0$ so $\mu = 0$. Since b_1 is non-zero, we must have $\lambda = 0$ also. □

Thus we have

$$b(m) = \alpha b_1(m) + \beta b_2(m)$$

for some $\alpha, \beta \in \mathbb{C}$. In terms of $g_{s'}$ this means that

$$g_{s'}(d_m, d_n) = \delta_{m+n,0}\big(\alpha b_1(m) + \beta b_2(m)\big).$$

Define now yet another section $s'' : L \to \hat{L}$ by

$$s''(d_m) = s'(d_m) + \delta_{m,0}\beta \mathbf{c}.$$

Then

$$\begin{aligned}
\iota \circ g_{s''}(d_m, d_n) &= \iota \circ g_{s'}(d_m, d_n) + (s' - s'')\big(\langle d_m, d_n\rangle_L\big) \\
&= \delta_{m+n,0}\big(\alpha b_1(m) + \beta b_2(m)\big)\mathbf{c} - \big(\{m\}_q - \{n\}_q\big)\delta_{m+n,0}\beta\mathbf{c} \\
&= \delta_{m+n,0}\big(\alpha b_1(m) + \beta b_2(m) - \beta q^{-m}\{2m\}_q\big)\mathbf{c} \\
&= \delta_{m+n,0}\alpha b_1(m)\mathbf{c},
\end{aligned}$$

where we used that $\{m\}_q - \{-m\}_q = q^{-m}\{2m\}_q$. If $\alpha = 0$ we have a trivial extension. Otherwise we set $\mathbf{c}' = 6\alpha\mathbf{c}$.

It remains to determine the homomorphism $\hat{\varsigma}$. Using (35) we have for $x \in \hat{L}$,

$$\hat{\varsigma}(x) = s'' \circ \varsigma \circ \mathrm{pr}(x) + \iota \circ f_{s''}(x)$$

for some linear function $f_{s''} : \hat{L} \to \mathbb{C}$. To determine $f_{s''}$, first use (36):

$$f_{s''}(\mathbf{c}') = f_{s''}\big(\iota(6\alpha)\big) = \mathrm{id}_{\mathbb{C}}(6\alpha) = 6\alpha.$$

Hence

$$\hat{\varsigma}(\mathbf{c}') = \mathbf{c}'.$$

Next, we use (41) in Theorem 21 to get

$$f_{s''}\big(\big\langle s''(d_m), s''(d_n)\big\rangle_{\hat{L}}\big) = g_{s''}\big(\varsigma(d_m), \varsigma(d_n)\big).$$

By (37) we see that

$$\begin{aligned}
\big\langle s''(d_m), s''(d_n)\big\rangle_{\hat{L}} &= s''\big(\langle d_m, d_n\rangle_L\big) + \iota \circ g_{s''}(d_m, d_n) \\
&= \big(\{m\}_q - \{n\}_q\big)s''(d_{m+n}) + \iota \circ g_{s''}(d_m, d_n)
\end{aligned}$$

and so

$$\big(\{m\}_q - \{n\}_q\big) f_{s''}\big(s''(d_{m+n})\big) + f_{s''}\big(\iota \circ g_{s''}(d_m, d_n)\big) = q^{m+n} g_{s''}(d_m, d_n)$$

which is equivalent to

$$\big(\{m\}_q - \{n\}_q\big) f_{s''}\big(s''(d_{m+n})\big) = \big(q^{m+n} - 1\big) g_{s''}(d_m, d_n)$$

for all integers m, n. But the right-hand side is identically zero ($g_{s''}$ being a multiple of $\delta_{m+n,0}$). Hence, taking $m \neq 0$ and $n = 0$ we get $f_{s''}(s''(d_m)) = 0$ for all non-zero m. But if we take $m = 1$ and $n = -1$ we also get $f_{s''}(s''(d_0)) = 0$ because $\{1\}_q - \{-1\}_q = 1 + q^{-1} \neq 0$ since q is not a root of unity.

Putting $\hat{L} \ni L_n := s''(d_n)$ we have proved the following theorem.

Theorem 39. *Every nontrivial one-dimensional central extension of the hom-Lie algebra* $(\mathcal{D}_\sigma(\mathcal{A}), \varsigma)$*, where* $\mathcal{A} = \mathbb{C}[t, t^{-1}]$*, is isomorphic to the hom-Lie algebra* $\mathrm{Vir}_q = (\hat{L}, \hat{\varsigma})$*, where* $\hat{L}$ *is the non-associative algebra with basis* $\{L_n \colon n \in \mathbb{Z}\} \cup \{\mathbf{c}\}$ *and relations*

$$\langle \mathbf{c}, \hat{L} \rangle_{\hat{L}} = 0,$$
$$\langle L_m, L_n \rangle_{\hat{L}} = \big(\{m\}_q - \{n\}_q\big) L_{m+n} + \delta_{m+n,0} \frac{q^{-m}}{6(1+q^m)} \{m-1\}_q \{m\}_q \{m+1\}_q \mathbf{c},$$

and $\hat{\varsigma} : \hat{L} \to \hat{L}$ *is the endomorphism of* $\hat{L}$ *defined by*

$$\hat{\varsigma}(L_n) = q^n L_n, \qquad \hat{\varsigma}(\mathbf{c}) = \mathbf{c}.$$

4.2. *Existence of a non-trivial extension*

We now proceed to prove the following result. Let $\mathcal{A} = \mathbb{C}[t, t^{-1}]$.

Theorem 40. *There exists a non-trivial central extension of* $(\mathcal{D}_\sigma(\mathcal{A}), \varsigma)$ *by* $(\mathbb{C}, \mathrm{id}_{\mathbb{C}})$.

Proof. We set $L := \mathcal{D}_\sigma(\mathcal{A})$ for brevity and define $g : L \times L \to \mathbb{C}$ by setting

$$g(d_m, d_n) := \delta_{m+n,0} \frac{q^{-m}}{1+q^m} \{m-1\}_q \{m\}_q \{m+1\}_q, \quad \text{for } m, n \in \mathbb{Z},$$

and extending using the bilinearity. We also define a linear map $f : L \oplus \mathbb{C} \to \mathbb{C}$ by

$$f(x, a) = a \quad \text{for } x \in L,\ a \in \mathbb{C}.$$

Our goal is to use Theorem 24 which means that we have to verify that g and f satisfy the necessary conditions. First of all, using that $\{-n\}_q = -q^{-n}\{n\}_q$, we note

$$\begin{aligned}
g(d_m, d_n) &= \delta_{m+n,0}\frac{q^{-m}}{1+q^m}\{m-1\}_q\{m\}_q\{m+1\}_q \\
&= \delta_{n+m,0}\frac{q^{n}}{1+q^{-n}}\{-n-1\}_q\{-n\}_q\{-n+1\}_q \\
&= -\delta_{n+m,0}\frac{q^{-n}}{1+q^n}\{n-1\}_q\{n\}_q\{n+1\}_q = -g(d_n, d_m).
\end{aligned}$$

This shows that g is alternating. That (43) holds is immediate. To check (44) let $m, n \in \mathbb{Z}$. Then

$$\begin{aligned}
g\big(\varsigma(d_m), \varsigma(d_n)\big) = g\big(q^m d_m, q^n d_n\big) &= q^{m+n} g(d_m, d_n) \\
&= q^{m+n}\delta_{m+n,0}\frac{q^{-m}}{1+q^m}\{m-1\}_q\{m\}_q\{m+1\}_q \\
&= g(d_m, d_n) = f\big(\langle d_m, d_n\rangle_L, g(d_m, d_n)\big).
\end{aligned}$$

It remains to verify (45). By trilinearity it is enough to assume that $(x, y, z) = (d_k, d_l, d_m)$ for some $k, l, m \in \mathbb{Z}$. Moreover, if $k+m+l \neq 0$ then (45) holds trivially due to the Kronecker delta in the definition of g. Thus we can assume $k+m+l=0$. We then have

$$\begin{aligned}
&\circlearrowleft_{k,l,m} g\big((\mathrm{id}+\varsigma)d_k, \langle d_l, d_m\rangle_L\big) \\
&\quad = \circlearrowleft_{k,l,m} \big(1+q^k\big)\big(\{l\}_q - \{m\}_q\big)g(d_k, d_{l+m}) \\
&\quad = \big(\{l\}_q - \{-k-l\}_q\big)q^{-k}\{k-1\}_q\{k\}_q\{k+1\}_q \\
&\qquad + \big(\{-k-l\}_q - \{k\}_q\big)q^{-l}\{l-1\}_q\{l\}_q\{l+1\}_q \\
&\qquad + \big(\{k\}_q - \{l\}_q\big)q^{k+l}\{-k-l-1\}_q\{-k-l\}_q\{-k-l+1\}_q \\
&\quad = -\{-k-l\}_q\big(q^{-k}\{k-1\}_q\{k\}_q\{k+1\}_q - q^{-l}\{l-1\}_q\{l\}_q\{l+1\}_q \\
&\qquad - \big(\{k\}_q - \{l\}_q\big)q^{k+l}\{-k-l-1\}_q\{-k-l+1\}_q\big) \\
&\qquad + q^{-k-l}\{k\}_q\{l\}_q\big(q^l\{k-1\}_q\{k+1\}_q - q^k\{l-1\}_q\{l+1\}_q\big).
\end{aligned}$$

The second factor in the first term equals

$$\begin{aligned}
&q^{-k}\{k-1\}_q\{k\}_q\{k+1\}_q - q^{-l}\{l-1\}_q\{l\}_q\{l+1\}_q \\
&\qquad - \big(\{k\}_q - \{l\}_q\big)q^{k+l}\{-k-l-1\}_q\{-k-l+1\}_q \\
&\quad = q^{-k}\big(\{k\}_q - q^{k-1}\big)\{k\}_q\big(\{k\}_q + q^k\big) - q^{-l}\big(\{k\}_q - q^{l-1}\big)\{l\}_q\big(\{l\}_q + q^l\big) \\
&\qquad - \big(\{k\}_q - \{l\}_q\big)q^{k+l}q^{-2k-2l}\big(\{k+l\}_q + q^{k+l}\big)\big(\{k+l\}_q - q^{k+l-1}\big) \\
&\quad = q^{-k}\{k\}_q^3 + \big(1-q^{-1}\big)\{k\}_q^2 - q^{k-1}\{k\}_q - q^{-l}\{l\}_q^3 - \big(1-q^{-1}\big)\{l\}_q^2 + q^{l-1}\{l\}_q
\end{aligned}$$

$$\begin{aligned}
&- \big(\{k\}_q - \{l\}_q\big)\big(q^{-k-l}\big(\{k\}_q + q^k\{l\}_q\big)\big(\{l\}_q + q^l\{k\}_q\big) \\
&+ \big(1-q^{-1}\big)\{k+l\}_q - q^{k+l-1}\big) \\
={}& \big(1-q^{-1}\big)\{k\}_q^2 - q^{k-1}\{k\}_q - \big(1-q^{-1}\big)\{l\}_q^2 + q^{l-1}\{l\}_q \\
&- q^{-k-l}\{k\}_q\{l\}_q\big(\big(1+q^{k+l}\big)\big(\{k\}_q - \{l\}_q\big) + q^k\{l\}_q - q^l\{k\}_q\big) \\
&- \big(\{k\}_q - \{l\}_q\big)\big(\big(1-q^{-1}\big)\{k+l\}_q - q^{k+l-1}\big) \\
={}& \{k\}_q\big(\big(1-q^{-1}\big)\big(\{k\}_q - \{k+l\}_q\big) - q^{k-1} + q^{k+l-1}\big) \\
&- \{l\}_q\big(\big(1-q^{-1}\big)\big(\{l\}_q - \{k+l\}_q\big) - q^{l-1} + q^{k+l-1}\big) \\
&- q^{-k-l}\{k\}_q\{l\}_q\big(\big(1+q^{k+l}\big)\big(\{k\}_q - \{l\}_q\big) + q^k\{l\}_q - q^l\{k\}_q\big).
\end{aligned}$$

The first of the three terms in the last equality above is equal to

$$\begin{aligned}
&\big(1-q^{-1}\big)\big(\{k\}_q - \{k+l\}_q\big) - q^{k-1} + q^{k+l-1} \\
&\quad = q^{-1}(q-1)\frac{q^k - 1 - q^{k+l} + 1}{q-1} - q^{k-1} + q^{k+l-1} = 0.
\end{aligned}$$

Similarly the second term vanishes. Using that $\{a+b\}_q = \{a\}_q + q^a\{b\}_q$ we see that the whole expression is equal to

$$\begin{aligned}
&q^{-k-l}\{k\}_q\{l\}_q\big(q^l\{k-1\}_q\{k+1\}_q - q^k\{l-1\}_q\{l+1\}_q \\
&\quad + \{-k-l\}_q\big(\big(1+q^{k+l}\big)\big(\{k\}_q - \{l\}_q\big) + q^k\{l\}_q - q^l\{k\}_q\big)\big) \\
&= q^{-k-l}\{k\}_q\{l\}_q\big(q^l\big(\{k\}_q^2 + q^k\big(1-q^{-1}\big)\{k\}_q - q^{2k-1}\big) \\
&\quad - q^k\big(\{l\}_q^2 + q^l\big(1-q^{-1}\big)\{l\}_q - q^{2l-1}\big) - \{k+l\}_q\big(\{k\}_q - \{l\}_q\big)\big) \\
&= q^{-k-l}\{k\}_q\{l\}_q\big(q^l\{k\}_q^2 - q^{k+l+1} - q^k\{l\}_q^2 + q^{k+l+1} - \{k+l\}_q\big(\{k\}_q - \{l\}_q\big)\big) \\
&= q^{-k-l}\{k\}_q\{l\}_q\{k+l\}_q\big(q^l\{k\}_q - q^k\{l\}_q - \{k\}_q + \{l\}_q\big) \\
&= \{k\}_q\{l\}_q\{k+l\}_q\big(\{l+k\}_q - \{k+l\}_q\big) = 0.
\end{aligned}$$

Thus (45) holds for all $x, y, z \in L$. Hence, by Theorem 24, there exists a central extension of (L, ς) by $(\mathfrak{a}, \varsigma_{\mathfrak{a}})$.

Suppose this extension is trivial. Then by Remark 23 there is a linear map s_1 such that

$$g(d_m, d_n) = s_1\big(\langle d_m, d_n\rangle_L\big),$$

or

$$\delta_{m+n,0}\frac{q^{-m}}{1+q^m}\{m-1\}_q\{m\}_q\{m+1\}_q = s_1\big(d_{m+n}\big(\{m\}_q - \{n\}_q\big)\big),$$

for $m, n \in \mathbb{Z}$. Taking $m = 1$ and $n = -1$ gives $s_1(d_0) = 0$. On the other hand, setting $m = 2$ and $n = -2$ yields $s_1(d_0) \neq 0$. This contradiction shows that the extension is non-trivial. $\square$

Remark 41. The coefficient in the central extension part in Theorem 39 is $1/6 \cdot g(d_m, d_n)$, where $g(d_m, d_n)$ is from the above theorem. This factor $1/6$ is easily obtained by rescaling **c**. The reason for this factor in Theorem 39 is that for the classical undeformed Virasoro algebra one usually rescales by a factor $1/12$ in the central extension term. Now by taking $q = 1$ in Theorem 39, we thus get the classical undeformed Virasoro algebra including the usually chosen scaling factor $1/12$.

Remark 42. If τ is an automorphism of $\mathcal{A}$ and Δ is a (σ, τ)-derivation on A we can still define a product on $\mathcal{A} \cdot \Delta$ by

$$\begin{aligned}\langle a \cdot \Delta, b \cdot \Delta\rangle_{\sigma,\tau,\Delta} &= \big(\sigma(a) \cdot \Delta\big) \circ (b \cdot \Delta) - \big(\sigma(b) \cdot \Delta\big) \circ (a \cdot \Delta)\\ &= \big(\sigma(a)\Delta(b) - \sigma(b)\Delta(a)\big) \cdot \Delta.\end{aligned}$$

For example, if we take $\mathcal{A} = \mathbb{C}[t, t^{-1}]$, $\sigma(t) = qt$ and $\tau = \sigma^{-1}$, and the symmetric q-difference operator

$$\Delta = \frac{\tau - \sigma}{q^{-1} - q} : f(t) \mapsto \frac{f(q^{-1}t) - f(qt)}{q^{-1} - q},$$

then our bracket will be

$$\langle d_n, d_m\rangle = q^n d_n d_m - q^m d_m d_n = \frac{q^{n-m} - q^{m-n}}{q - q^{-1}} d_{n+m} = [n - m]_q d_{n+m}, \tag{65}$$

where $d_n = -t^n \cdot \Delta$ and $[k]_q = (q^k - q^{-k})/(q - q^{-1})$ is the symmetric q-number. This is easily calculated by direct substitution. The right-hand side of this commutation relation (65) coincides with the right-hand side of the defining relations (1) in the q-deformation of Witt algebra considered in [1,14], but the left-hand side of the bracket turns out to be slightly different. When $q \to 1$ the defining relations for the classical Witt algebra are recovered in both cases.

References

[1] N. Aizawa, H.-T. Sato, q-deformation of the Virasoro algebra with central extension, Phys. Lett. B 256 (2) (1999) 185–190.

[2] J. Avan, L. Frappat, M. Rossi, P. Sorba, Central extensions of classical and quantum q-Virasoro algebras, Phys. Lett. A 251 (1) (1999) 13–24.

[3] A.A. Belov, K.D. Chaltikian, q-deformation of Virasoro algebra and lattice conformal theories, Modern Phys. Lett. A 8 (13) (1993) 1233–1242.

[4] A.A. Belov, K.D. Chaltikian, q-Deformation of Virasoro algebra and lattice conformal theories, in: Differ. Geom. Gruppy Li i Mekh. 15-2, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 235 (1996) 217–227, 306; translation in J. Math. Sci. (New York) 94 (4) (1999) 1581–1588.

[5] S. Bloch, Zeta values and differential operators on the circle, J. Algebra 182 (1996) 476–500.

[6] M. Chaichian, A.P. Isaev, J. Lukierski, Z. Popowicz, P. Prešnajder, q-Deformations of Virasoro algebra and conformal dimensions, Phys. Lett. B 262 (1) (1991) 32–38.

[7] M. Chaichian, P. Kulish, J. Lukierski, q-Deformed Jacobi identity, q-oscillators and q-deformed infinite-dimensional algebras, Phys. Lett. B 237 (3–4) (1990) 401–406.

[8] M. Chaichian, Z. Popowicz, P. Prešnajder, q-Virasoro algebra and its relation to the q-deformed KdV system, Phys. Lett. B 249 (1) (1990) 63–65.

[9] M. Chaichian, P. Prešnajder, Sugawara construction and the q-deformation of Virasoro algebra, in: Quantum Groups and Related Topics, Wrocław, 1991, in: Math. Phys. Stud., vol. 13, Kluwer Academic, Dordrecht, 1992, pp. 3–12.

[10] M. Chaichian, P. Prešnajder, On the q-Sugawara construction for the Virasoro (super)algebra, in: Quantum Symmetries, Clausthal, 1991, World Scientific, River Edge, NJ, 1993, pp. 352–365.

[11] M. Chaichian, P. Prešnajder, q-Virasoro algebra, q-conformal dimensions and free q-superstring, Nuclear Phys. B 482 (1–2) (1996) 466–478.

[12] R. Chakrabarti, R. Jagannathan, A (p,q)-deformed Virasoro algebra, J. Phys. A 25 (9) (1992) 2607–2614.

[13] W.-S. Chung, Two parameter deformation of Virasoro algebra, J. Math. Phys. 35 (5) (1994) 2490–2496.

[14] T.L. Curtright, C.K. Zachos, Deforming maps for quantum algebras, Phys. Lett. B 243 (3) (1990) 237–244.

[15] C. Devchand, M.V. Saveliev, Comultiplication for quantum deformations of the centreless Virasoro algebra in the continuum formulation, Phys. Lett. B 258 (3–4) (1991) 364–368.

[16] D.B. Fairlie, J. Nuyts, C.K. Zachos, A presentation for the Virasoro and super-Virasoro algebras, Comm. Math. Phys. 117 (1988) 595–614.

[17] J. Hartwig, D. Larsson, S.D. Silvestrov, Deformations of Lie algebras using σ-derivations, Preprints in Mathematical Sciences 2003:32, LUTFMA-5036-2003, Centre for Mathematical Sciences, Department of Mathematics, Lund Institute of Technology, Lund University, 2003; preprint, math.QA/0408064.

[18] L. Hellström, S.D. Silvestrov, Commuting Elements in q-Deformed Heisenberg Algebras, World Scientific, ISBN: 981-02-4403-7, 2000, 256 pp.

[19] N. Hu, Quantum group structure of the q-deformed Virasoro algebra, Lett. Math. Phys. 44 (2) (1998) 99–103.

[20] A. Jellal, H.-T. Sato, FFZ realization of the deformed super Virasoro algebra—Chaichian–Prešnajder type, Phys. Lett. B 483 (2000) 451–455.

[21] V. Kac, A. Radul, Quasifinite highest weight modules over the Lie algebra of differential operators on the circle, Comm. Math. Phys. 157 (1993) 429–457.

[22] C. Kassel, Cyclic homology of differential operators, the Virasoro algebra and a q-analogue, Comm. Math. Phys. 146 (1992) 343–351.

[23] R. Kemmoku, H.-T. Sato, Deformed fields and Moyal construction of deformed super Virasoro algebra, Nucl. Phys. B 595 (2001) 689–709.

[24] B. Khesin, V. Lyubashenko, C. Roger, Extensions and contractions of Lie algebra of q-pseudodifferential symbols on the circle, J. Func. Anal. 143 (1997) 55–97.

[25] E. Kirkman, C. Procesi, L. Small, A q-analogue for Virasoro algebra, Comm. Algebra 22 (1994) 3755–3774.

[26] D. Larsson, S.D. Silvestrov, Quasi-Lie algebras, in: Non-Commutative Geometry and Representation Theory in Mathematical Physics, Karlstad, July 2004, Contemp. Math., AMS, in press.

[27] D. Larsson, S.D. Silvestrov, Quasi-hom-Lie algebra, central extensions and 2-cocycle-like identities, J. Algebra 288 (2005) 321–344.

[28] W.-L. Li, 2-Cocycles on the algebra of differential operators, J. Algebra 122 (1989) 64–80.

[29] K.Q. Liu, Some results on q-deformations of the Virasoro algebra, in: Proc. Conf. on Quantum Topology, Manhattan, KS, 1993, World Scientific, River Edge, NJ, 1994, pp. 259–268.

[30] K.Q. Liu, Indecomposable representations of the q-deformed Virasoro algebra, Math. Z. 217 (1) (1994) 15–35.

[31] K.Q. Liu, A class of Harish-Chandra modules for the q-deformed Virasoro algebra, J. Algebra 171 (2) (1995) 606–630.

[32] V. Mazorchuck, On simple modules over q-analog for the Virasoro algebra, Hadronic J. 21 (5) (1998) 541–550.
[33] N. Mebarki, H. Aissaoui, A. Boudine, A. Maasmi, The q-deformed Virasoro algebra, Czechoslovak J. Phys. 47 (8) (1997) 755–759.
[34] F.J. Narganes-Quijano, Cyclic representations of a q-deformation of the Virasoro algebra, J. Phys. A 24 (3) (1991) 593–601.
[35] J.M. Osborn, D.S. Passman, Derivations of skew-polynomial rings, J. Algebra 176 (1995) 417–448.
[36] J.M. Osborn, K. Zhao, A characterization of the Block Lie algebra and its q-forms in characteristic 0, J. Algebra 207 (2) (1998) 367–408.
[37] A.P. Polychronakos, Consistency conditions and representations of a q-deformed Virasoro algebra, Phys. Lett. B 256 (1) (1991) 35–40.
[38] H.-T. Sato, Realizations of q-deformed Virasoro algebra, Progr. Theoret. Phys. 89 (2) (1993) 531–544.
[39] H.-T. Sato, q-Virasoro operators from an analogue of the Noether currents, Z. Phys. C 70 (2) (1996) 349–355.
[40] H.-T. Sato, OPE formulae for deformed super-Virasoro algebras, Nucl. Phys. B 471 (1996) 553–569.
[41] H.-T. Sato, Deformation of super Virasoro algebra in non-commutative quantum superspace, Phys. Lett. B 415 (1997) 170–174.
[42] Y. Su, 2-Cocycles on the Lie algebras of generalized differential operators, Comm. Algebra 30 (2) (2002) 763–782.
[43] Y. Su, Classification of quasifinite modules over the Lie algebras of Weyl type, Adv. Math. 174 (2003) 57–68.
[44] K. Zhao, The q-Virasoro-like algebra, J. Algebra 188 (1997) 506–512.
[45] C.-Z. Zha, W.-Z. Zhao, The q-deformation of super high-order Virasoro algebra, J. Math. Phys. 36 (2) (1995) 967–979.