Global existence and blow-up for a degenerate reaction–diffusion system with nonlocal sources

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\section*{ABSTRACT}

This work investigates a degenerate reaction–diffusion system with homogeneous Dirichlet boundary data. Using the self-similar form of function and super-solution sub-solution techniques, together with results from interactions among the multi-nonlinearities in the system, described using ten exponents, the global existence and blow-up criteria for nonnegative solutions are determined.

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\section*{1. Introduction and description of the results}

In this work, we investigate the global existence and finite time blow-up of nonnegative solutions for the following degenerate reaction–diffusion system with nonlocal sources:

\begin{equation}
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u^m + \|u^{p_1} v^{q_1}\|^\alpha_{\Omega}, & (x, t) \in \Omega \times (0, T), \\
\frac{\partial v}{\partial t} &= \Delta v^n + \|u^{p_2} v^{q_2}\|^\beta_{\Omega}, & (x, t) \in \Omega \times (0, T), \\
\quad u(x, 0) &= u_0(x), & x \in \Omega, \\
\quad v(x, 0) &= v_0(x), & x \in \Omega, \\
\quad u(x, t) = v(x, t) = 0, & (x, t) \in \partial \Omega \times (0, T),
\end{aligned}
\end{equation}

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) \((N \geq 1)\) with smooth boundary \(\partial \Omega\), and constants \(m, n > 1, \alpha, \beta > 1, p, q > 0, p_1, q_1 \geq 0,\) and \(p_2 > 0\), where \(u_0(x)\) and \(v_0(x)\) are nonnegative bounded functions on \(\Omega\), and where \(\| \cdot \|_{\Omega} = \int_{\Omega} | \cdot |^\alpha d\Omega\).

System (1.1) models phenomena such as heat propagation in a two-component combustible mixture \([1]\), chemical processes \([2]\), the interaction of two non-self-limiting biological groups \([3]\), etc. And it is worth studying because of the applications to heat and mass transport processes. In addition, there exist interesting interactions among the multi-nonlinearities described by these exponents in the problem (1.1).

In the past two decades, many physical phenomena were formulated into nonlocal mathematical models (see \([4–12]\) and references therein) and studied by many authors. Here, we will recall some of those results concerning the first initial–boundary problem.

Deng in \([7]\), Song et al. in \([10]\) and Lei and Zheng in \([11]\) considered the following problem:

\begin{equation}
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u^m + u^p v^q, & v_t &= \Delta v^n + u^q v^\beta,
\end{aligned}
\end{equation}
by different methods. Some results which concern the global boundedness of and blow-up criteria for solutions are determined. In addition, Kong and Wang in [8] and Deng et al. in [12] studied the systems

\[ \begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + \int_{\Omega} u^m(x, t)v^n(x, t) \, dx, \\
\frac{\partial v}{\partial t} &= \Delta v + \int_{\Omega} v^p(x, t)v^q(x, t) \, dx.
\end{align*} \tag{1.3} \]

\[ \begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + a\|v\|_p^p, \\
\frac{\partial v}{\partial t} &= \Delta v^p + b\|u\|_q^q.
\end{align*} \tag{1.4} \]

respectively. Several interesting results concerning the blow-up and global existence are established.

In this work, by means of a detailed classification of exponents, we give several criteria for the global existence and blow-up of nonnegative solutions to problem (1.1). The main purpose is to extend Escobedo’s method from [4] and results from [12]. Not only do our results cover those of [8, 12], but also our methods are different from—and even easier than—the ones used by them.

Let \( D = (pq_1)(qp_2) - (m - pp_1)(n - qq_2) \), and let us state our main results.

**Theorem 1.** Suppose that one of the following conditions holds:

1. \( m > pp_1, n > qq_2 \) and \( D < 0 \);
2. \( m > pp_1, n > qq_2, D > 0 \), and the initial data \( u_0(x), v_0(x) \) are sufficiently small;
3. \( m > pp_1, n > qq_2, D = 0 \) and the domain \( (\Omega) \) is sufficiently small.

Then every nonnegative solution of system (1.1) exists globally.

**Theorem 2.** Suppose that one of the following conditions holds:

1. \( m > pp_1, n > qq_2, D > 0 \) and the initial data \( u_0(x), v_0(x) \) are sufficiently large;
2. \( m > pp_1, n > qq_2, D = 0 \), the domain contains a sufficiently large ball, and the initial data \( u_0(x), v_0(x) \) are sufficiently large.

Then the nonnegative solution of system (1.1) blows up in finite time.

**2. Proof of Theorem 1**

According to the comparison principle, we only need to construct bounded, positive super-solutions for any \( T > 0 \). Let \( \varphi(x) \) be the unique positive solution of the following linear elliptic problem:

\[-\Delta \varphi(x) = 1, \quad x \in \Omega; \quad \varphi(x) = 1, \quad x \in \partial \Omega.\]

Define \( C = \max_{x \in \Omega} \varphi(x) \); then \( 1 \leq \varphi(x) \leq C \). Now, we define the functions \( \tilde{u}, \tilde{v} \) as

\[ \begin{align*}
\tilde{u}(x, t) &= k_1 \varphi^{\frac{1}{m}}(x), \\
\tilde{v}(x, t) &= k_2 \varphi^{\frac{1}{n}}(x)
\end{align*} \tag{2.1} \]

with positive constants \( k_1, k_2 \) to be determined later. Clearly, for any \( T > 0 \), \((\tilde{u}, \tilde{v})\) is a bounded function and \( \tilde{u} \geq k_1 > 0, \quad \tilde{v} \geq k_2 > 0 \). Then, a series of computations yields

\[ \begin{align*}
\left\{ \begin{array}{ll}
\frac{\partial \tilde{u}}{\partial t} - \Delta \tilde{u}^m &= \frac{\partial \varphi^m}{\partial t} \leq k_1^m \|\varphi\|_m^m \leq k_1^{pp_1} k_2^{pq_1} C^{pp_1 \left( \frac{q_1}{m} + \frac{n}{m} \right) |\Omega|^{\frac{q_1}{m}}}, \\
\frac{\partial \tilde{v}}{\partial t} - \Delta \tilde{v}^n &= \frac{\partial \varphi^m}{\partial t} \leq k_1^n \|\varphi\|_n^n \leq k_1^{pp_1} k_2^{pq_1} C^{pp_1 \left( \frac{q_1}{m} + \frac{n}{m} \right) |\Omega|^{\frac{q_1}{m}}}. \end{array} \right. \tag{2.2} \]

Define

\[ C_1 = C^{\left( \frac{q_1}{m} + \frac{n}{m} \right) \frac{m}{m - pp_1}} |\Omega|^{\frac{q_1}{m - pp_1}}, \quad C_2 = C^{-\frac{1}{pq} \left( \frac{p_1}{q_1} + \frac{q_2}{q_2} \right) |\Omega|^{\frac{1}{pq}}} \]

(1) If \( m > pp_1, n > qq_2, D < 0 \), then \( D < 0 \) implies \( \frac{pq_1}{m - pp_1} < \frac{n - qq_2}{qq_2} \). So there exist two constants \( k_1, k_2 > 0 \) sufficiently large that \( k_1 \geq \|u_0\|_\infty, \quad k_2 \geq \|v_0\|_\infty \) and

\[ C_1 k_1^{pq_1 \frac{m}{m - pp_1}} \leq k_1 \leq C_2 k_2^{n - qq_2} \tag{2.3} \]

Now, it follows from (2.2)–(2.3) that \((\tilde{u}, \tilde{v})\) defined by (2.1) is a positive super-solution of (1.1).

(2) If \( m > pp_1, n > qq_2, D > 0 \), then \( D > 0 \) implies \( \frac{pq_1}{m - pp_1} > \frac{n - qq_2}{qq_2} \). So there exists a constant \( k_2 \in (0, 1) \) sufficiently small that \( C_1 k_1^{pq_1 \frac{m}{m - pp_1}} < C_2 k_2^{n - qq_2} \). Thus, we can choose \( k_1 > 0 \) such that

\[ C_1 k_1^{pq_1 \frac{m}{m - pp_1}} \leq k_1 \leq C_2 k_2^{n - qq_2} \tag{2.4} \]

Therefore, provided \( u_0(x), \) \( v_0(x) \) are sufficiently small and satisfy

\[ \begin{align*}
\tilde{u}(x, 0) &= k_1 \varphi^{\frac{1}{m}}(x) \geq u_0(x), \\
\tilde{v}(x, 0) &= k_2 \varphi^{\frac{1}{n}}(x) \geq v_0(x), \quad x \in \Omega,
\end{align*} \tag{2.5} \]

then from (2.4)–(2.5), we know that \((\tilde{u}, \tilde{v})\) defined by (2.1) is a positive super-solution of (1.1).
(3) If \( m > pp_1, n > qq_2 \) and \( D = 0 \), then \( D = 0 \) implies \( \frac{m}{m - pp_1} = \frac{n - qq_2}{qq_2} \). Without loss of generality, we may assume that \( \Omega \subset B \), where \( B \) is a sufficiently large ball. And we denote as \( \varphi_B(x) \) the unique positive solution of the following linear elliptic problem:

\[ -\Delta \varphi(x) = 1, \quad x \in B; \quad \varphi(x) = 1, \quad x \in \partial B. \]

Let \( C_0 = \max_{x \in B} \varphi_B(x) \); then \( C \leq C_0 \). Thus, as long as \( \Omega \) is sufficiently small, we have that

\[ |\Omega| < \min \left\{ \left( \frac{1}{C_0} \right)^{A_2/A_1}, \left( \frac{1}{C_0} \right)^{A_4/A_2} \right\}, \tag{2.6} \]

where

\[ A_1 = \frac{p}{\alpha(m - pp_1)} + \frac{1}{pp_1\beta}, \quad A_2 = \left( \frac{p_1}{m} + \frac{q_1}{n} \right) \frac{p}{m - pp_1} + \frac{1}{pp_1} \left( \frac{p_2}{m} + \frac{q_2}{n} \right), \]

\[ A_3 = \frac{q}{\beta(n - qq_2)} + \frac{1}{qq_2\alpha}, \quad A_4 = \left( \frac{p_2}{m} + \frac{q_2}{n} \right) \frac{q}{n - qq_2} + \frac{1}{qq_2} \left( \frac{p_1}{m} + \frac{q_1}{n} \right). \]

Furthermore, we choose \( k_1, k_2 \) large enough to satisfy (2.5) with \( \psi \) replaced by \( \varphi_B \). Therefore, it follows from (2.5) and (2.6) that \((\tilde{u}, \tilde{v})\) is a positive super-solution of (1.1).

Thus, according to the comparison principle, the proof of Theorem 1 is completed.

3. Proof of Theorem 2

Due to the requirements of the comparison principle we will construct blow-up sub-solutions in some subdomain of \( \Omega \) in which \( u, v > 0 \). We use an idea from Souplet [13] and apply it to degenerate parabolic equations. By translation, one may assume without loss of generality that \( 0 \in \Omega \).

(1) Let \( B_R = B(0, R) \subset \Omega \) be an open ball with radius \( R \) and \( \psi(x) \) a nontrivial nonnegative continuous function, vanishing on \( \partial \Omega \) and with \( \psi(0) > 0 \). Let the functions \( \tilde{u}, \tilde{v} \) be as follows, in self-similar forms:

\[ \tilde{u}(x, t) = \frac{1}{(T - t)^{l_1}} V \left[ \frac{|x|}{(T - t)^{\sigma}} \right], \quad \tilde{v}(x, t) = \frac{1}{(T - t)^{l_2}} V \left[ \frac{|x|}{(T - t)^{\sigma}} \right] \tag{3.1} \]

with

\[ V(r) = \frac{R^3}{6} - \frac{R^2}{2} + \frac{1}{3} r^3, \quad r = \frac{|x|}{(T - t)^\sigma}, \quad 0 \leq r \leq R; \quad V(r) = 0, \quad r \geq R; \]

where \( l_1, l_2, \sigma > 0 \) and \( 0 < T < 1 \) are to be determined later. Clearly, \( 0 \leq V(r) \leq \frac{R^3}{6} \) and \( V(r) \) is nonincreasing since \( V'(r) = r - R \leq 0 \). Note that, for \( T \) small enough,

\[ \text{Supp} \tilde{u}(\cdot, t) = \text{Supp} \tilde{v}(\cdot, t) = \overline{B}(0, R(T - t)^{\sigma}) \subset \overline{B}(0, RT^{\sigma}) \subset \overline{B}(0, R) \subset \Omega, \quad 0 \leq t < T. \tag{3.2} \]

Obviously, \((\tilde{u}, \tilde{v})\) becomes unbounded as \( t \to T^- \) at the point \( x = 0 \). Calculating directly, we obtain

\[ \tilde{u}_t - \Delta \tilde{u}^{m} \leq \frac{l_1 R^3}{(T - t)^{l_1 + 1}} + \frac{m V^{m-1}(r)}{(T - t)^m + (N + 1)r}, \quad (x, t) \in B_R \times (0, T), \tag{3.3} \]

\[ \tilde{v}_t - \Delta \tilde{v}^{n} \leq \frac{l_2 R^3}{(T - t)^{l_2 + 1}} + \frac{n V^{n-1}(r)}{(T - t)^n + (N + 1)r}, \quad (x, t) \in B_R \times (0, T), \tag{3.4} \]

and we notice that \( T < 1 \) is sufficiently small.

If \( 0 \leq r \leq r_0 = NR/(N + 1) \), we have \( V(r) \geq V(r_0) = \frac{R^3(3N + 1)}{6(N + 1)^2} \); then

\[ \|\tilde{u}^{p_1 \tilde{u}^{q_1}}\|_{L^\infty(B_R)} \geq \left( \int_{B_R} \tilde{u}_R^{p_1} \tilde{u}^{q_1} \right)^{p_1/p_{1+q_1}} \geq \frac{|B_R|^{p_1/\alpha}}{(T - t)^{p_1(l_1 + q_1)}} \left( \frac{R^3(3N + 1)}{6(N + 1)^3} \right)^{p_1(p_1 + q_1)}, \]

\[ \|\tilde{u}^{p_2 \tilde{u}^{q_2}}\|_{L^\infty(B_R)} \geq \left( \int_{B_R} \tilde{u}_R^{p_2} \tilde{u}^{q_2} \right)^{p_2/p_{2+q_2}} \geq \frac{|B_R|^{q_2/\beta}}{(T - t)^{p_2(l_2 + q_2)}} \left( \frac{R^3(3N + 1)}{6(N + 1)^3} \right)^{q_2(q_2 + q_2)}. \]
Hence,
\[
\tilde{u}_t - \Delta \tilde{u}^m - \|\tilde{u}^{p1}\tilde{v}^{q1}\|_\alpha^p \leq \frac{l_1R^3/6}{(T-t)^{l_1+1}} + \frac{mV^{m-1}(r)}{(T-t)^{m1+2\sigma}}(NR - (N + 1)r) \nonumber \\
- \frac{|B_r|^{p/\alpha}}{(T-t)^{p(p1\tilde{l}_1+q1\tilde{l}_2)}} \left( \frac{R^3(3N + 1)}{6(N + 1)^3} \right)^{p(p1+q1)}.
\] (3.5)
\[
\tilde{v}_t - \Delta \tilde{v}^n - \|\tilde{u}^{p2}\tilde{v}^{q2}\|_\beta^q \leq \frac{l_2R^3/6}{(T-t)^{l_2+1}} + \frac{nV^{n-1}(r)}{(T-t)^{n2+2\sigma}}(NR - (N + 1)r) \nonumber \\
- \frac{|B_r|^{q/\beta}}{(T-t)^{q(p2\tilde{l}_1+q2\tilde{l}_2)}} \left( \frac{R^3(3N + 1)}{6(N + 1)^3} \right)^{q(p2+q2)}.
\] (3.6)

Similarly, if \(NR/(N + 1) < r \leq R\), then
\[
\tilde{u}_t - \Delta \tilde{u}^m - \|\tilde{u}^{p1}\tilde{v}^{q1}\|_\alpha^p \leq \frac{l_1R^3/6}{(T-t)^{l_1+1}} + \frac{mV^{m-1}(r)}{(T-t)^{m1+2\sigma}}(NR - (N + 1)r),
\] (3.7)
\[
\tilde{v}_t - \Delta \tilde{v}^n - \|\tilde{u}^{p2}\tilde{v}^{q2}\|_\beta^q \leq \frac{l_2R^3/6}{(T-t)^{l_2+1}} + \frac{nV^{n-1}(r)}{(T-t)^{n2+2\sigma}}(NR - (N + 1)r).
\] (3.8)

If \(m > p1, n > qq2\) and \(D > 0\), then \(D > 0\) implies \(\frac{p1}{m-p1} \frac{qq2}{n-qq2} > 1\). Therefore, there exist two positive constants \(l_1, l_2\) large enough that
\[
p(p1l_1 + q1l_2) > ml_1, \quad q(p2l_1 + q2l_2) > nl_2 \quad \text{and} \quad (m - 1)l_1 > 1, \quad (n - 1)l_2 > 1.
\] (3.9)

Then, we can choose \(\sigma > 0\) sufficiently small that
\[
p(p1l_1 + q1l_2) > ml_1 + 2\sigma > ml_1 > l_1 + 1, \quad q(p2l_1 + q2l_2) > nl_2 + 2\sigma > nl_2 > l_2 + 1.
\]

Hence, for sufficiently small \(T > 0\), (3.5)-(3.6) or (3.7)-(3.8) imply that
\[
\tilde{u}_t - \Delta \tilde{u}^m - \|\tilde{u}^{p1}\tilde{v}^{q1}\|_\alpha^p \leq 0, \quad \tilde{v}_t - \Delta \tilde{v}^n - \|\tilde{u}^{p2}\tilde{v}^{q2}\|_\beta^q \leq 0, \quad (x, t) \in B_R \times (0, T).
\]

Since \(\psi(0) > 0\) and \(\psi(x)\) is continuous, there exist two positive constants \(\rho\) and \(\varepsilon\) such that \(\psi(x) \geq \varepsilon\) for all \(x \in B(0, \rho) \subseteq B(0, R)\). Choose \(T\) small enough to ensure that \(B(0, RT) \subseteq B(0, \rho)\); hence \(\tilde{u} \leq 0, \tilde{v} \leq 0\) on \(\partial \Omega \times (0, T)\), and from (3.2) it follows that \(\tilde{u}(x, 0) \leq K(\psi(x), \tilde{v}(x, 0) \leq K(\psi(x)\) for sufficiently large \(K\). By the comparison principle, we have \((\tilde{u}, \tilde{v}) \leq (u, v)\) provided that \(u_0(x) \geq K(\psi(x)\) and \(v_0(x) \geq K(\psi(x)\). It follows that \((u, v)\) blows up in finite time.

(2) Next, we consider the case \(m > pp1, n > qq2\) and \(D = 0\). Clearly, there exist two positive constants \(l_1, l_2\) such that
\[
ml_1 = p1l_2 + p2l_1, \quad nl_2 = q1l_1 + q2l_2, \quad \text{and} \quad (m - 1)l_1 > 1, \quad (n - 1)l_2 > 1.
\] (3.10)

Denote by \(\lambda_{B_R} > 0\) and \(\phi_k(r)\) the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem:
\[
-\phi''(r) - \frac{N-1}{r} \phi'(r) = \lambda \phi(r), \quad r \in (0, R) ; \quad \phi'(0) = 0, \quad \phi(R) = 0.
\]

It is well known that \(\phi_k(r)\) can be normalized as \(\phi_k(r) > 0\) in \(B_R\) and \(\phi_k(0) = \max_{B_R} \phi_k(r) = 1\). By the properties (let \(\tau = r/R\)) of the eigenvalues and eigenfunctions we see that \(\lambda_{B_R} = R^{-2}\lambda_{B_1}\), and \(\phi_k(r) = \phi_k(r/R) = \phi_k(\tau)\), where \(\lambda_{B_1}\) and \(\phi_k(\tau)\) are the first eigenvalue and the corresponding normalized eigenfunction of the eigenvalue problem in the unit ball \(B_1(0)\). Moreover,
\[
\max_{B_1} \phi_k(\tau) = \phi_k(0) = \phi_k(0) = \max_{B_R} \phi_k(r) = 1.
\]

Like for (3.1), we define the functions \(\tilde{u}(x, t), \tilde{v}(x, t)\) in the form
\[
\tilde{u}(x, t) = \frac{1}{(T-t)\tau} \phi_1^l(|x|), \quad \tilde{v}(x, t) = \frac{1}{(T-t)^2} \phi_2^l(|x|).
\] (3.11)

In the following, we will prove that \((\tilde{u}, \tilde{v})\) blows up in the ball \(B_R = B(0, R)\). Because of this, \((\tilde{u}, \tilde{v})\) blows up in the larger domain \(\Omega\). Calculating directly, we have
\[
\tilde{u}_t - \Delta \tilde{u}^m - \|\tilde{u}^{p1}\tilde{v}^{q1}\|_\alpha^p \leq \frac{\phi_1^l}{(T-t)^{l_1+1}} \left( l_1 - \frac{1}{(T-t)^{m1-l_1-1}} (c_1 - \lambda_{B_R} ml_1) \right),
\] (3.12)
\[
\tilde{v}_t - \Delta \tilde{v}^n - \|\tilde{u}^{p2}\tilde{v}^{q2}\|_\beta^q \leq \frac{\phi_2^l}{(T-t)^{l_2+1}} \left( l_2 - \frac{1}{(T-t)^{n2-l_2-1}} (c_2 - \lambda_{B_R} nl_2) \right),
\] (3.13)
where
\[ c_1 = \| \phi_{R_1}^{p_1 l_1 + q_1 l_2} \|_p \leq K_1 R^{\frac{N p_1}{\alpha}} \quad \text{and} \quad c_2 = \| \phi_{R_2}^{p_2 l_1 + q_2 l_2} \|_q \leq K_2 R^{\frac{N q_2}{\beta}} \]
and \( K_1, K_2 \) are constants independent of \( R \). Then, in view of the relation \( \lambda R = R^{-2} \lambda R \), we may assume that \( R \), that is, the ball \( B_R \), is sufficiently large that
\[ \lambda R_1 < \min \left\{ \frac{c_1}{m l_1}, \frac{c_2}{n l_2} \right\}. \quad (3.14) \]
Hence, for sufficiently small \( T > 0 \), (3.12) and (3.13) imply that
\[ \tilde{u}_t - \Delta \tilde{u}^{m} - \| \tilde{u}^{p_1} \tilde{v}^{q_1} \|_p \leq 0, \quad \tilde{v}_t - \Delta \tilde{v}^{n} - \| \tilde{u}^{p_2} \tilde{v}^{q_2} \|_q \leq 0. \]
Therefore, \((\tilde{u}, \tilde{v})\) is a positive sub-solution of (1.1) in the ball \( B_R \), which blows up in finite time provided the initial data are sufficiently large that
\[ \tilde{u}(x, 0) = T^{-l_1} \phi_{R_1}^{l_1} (|x|) \leq u_0(x), \quad \tilde{v}(x, 0) = T^{-l_2} \phi_{R_2}^{l_2} (|x|) \leq v_0(x), \quad x \in B_R. \]
Thus the proof of Theorem 2 is completed.

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