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Communication

On 4-chromatic edge-critical regular graphs of high connectivity

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Abstract

New examples of 4-chromatic edge-critical r-regular and r-connected graphs are presented for $r = 6, 8, 10$.

Keywords: Circulants; Edge-critical graph; Graph coloring

All graphs considered are simple (without loops and multiple edges), finite, and undirected. A k-chromatic graph G is called *edge-critical* (or simply k-critical) if it becomes $(k-1)$-chromatic after removing any edge. For $k \leqslant 3$ all such graphs have a simple structure, but for $k \geqslant 4$ they can be quite complicated. In 1989, Erdős [5] conjectured that 4-critical r-regular graphs exist for every $r \geqslant 3$. He also noted there that he did not know such graphs for $r \geqslant 6$. In 1960, Dirac [3,4] posed the conjecture that r-connected 4-critical graphs exist for every $r \geqslant 3$. Note that for $r = 3$ there exists the only such example—complete graph K_4, as follows from Brooks's theorem [1]. Graphs satisfying the conjectures of Erdős and Dirac will be called *Erdős's* and *Dirac's graphs*, respectively.

Now, we list some known constructions of 4-critical 4-regular graphs. The first examples (an infinite family) of such graphs were constructed by Gallai [7]. Other graphs, including planar examples, were reported by Koester [12,13]. There are also constructions of Evstigneev and Melnikov [6], and Youngs [17]. None of these constructions

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yields vertex-transitive graphs. Note that there exists a vertex-transitive 4-regular 4-critical graph. It is so-called circulant $C(13;1,5)$ (see the definition below). This graph was mentioned in the papers of Jensen and Royle [10], Chao [2], Göbel and Neutel [8]. Nonvertex-transitive 4-critical 5-regular graphs were constructed by Jensen [9]. Additional information on this and related problems can be found in [11].

A graph $G(V,E)=C(n;a_0,\ a_1,\dots,a_k)$ is called a *circulant* if $V(G)=\{0,\ 1,\dots,n-1\}$ and $E(G)=\{(i,j)\colon\ |i-j|\in\{a_0,\ a_1,\dots,a_k\}\ (\mathrm{mod}\ n)\}$, where $1\leqslant a_0<a_1<\cdots<a_k\leqslant n/2$. If $a_k<n/2$ then G is a $(2k+2)$-regular graph; otherwise, G is $(2k+1)$-regular. It is clear that circulants are vertex-transitive graphs. If $(n,a_i)=1$ then the edges of distance a_i induce a Hamiltonian cycle. We call such cycle *a_i-cycle*. Throughout this note we have $a_0=1$ and 1-cycle is called the *main* cycle.

The first example of 6-regular Erdős's graph has been found by Pyatkin in 2001 [15]. It is the circulant $C(157;1,8,14)$. There was also posed the conjecture that the circulant $C(1669;1,8,14,326)$ is a 4-critical (8-regular) graph. This conjecture has been recently disproved by Dobrynin and Melnikov, who have showed that this circulant is 3-chromatic.

In this note, we report new 4-critical r-regular and r-connected graphs for $r\in\{6,8,10\}$. The main result is the following

Theorem 1. *The following graphs are vertex-transitive 4-critical r-connected r-regular graphs, i.e. they are Erdős's and Dirac's graphs*

(a) *for* $r=6$:

$C(97;1,\mathbf{23},\mathbf{44})$.

(b) *for* $r=8$:

$C(289;1,38,110,134)$, $C(\mathbf{337};1,35,89,\mathbf{167})$, $C(391;1,50,\mathbf{173},\mathbf{176})$,
$C(403;1,\mathbf{137},\mathbf{140},164)$, $C(433;1,\mathbf{65},\mathbf{116},\mathbf{179})$, $C(469;1,101,\mathbf{170},\mathbf{173})$,
$C(469;1,\mathbf{104},170,\mathbf{206})$, $C(541;1,\mathbf{14},\mathbf{26},167)$, $C(541;1,\mathbf{83},\mathbf{164},191)$,
$C(589;1,\mathbf{47},\mathbf{92},215)$, $C(589;1,\mathbf{47},\mathbf{92},101)$, $C(\mathbf{691};1,\mathbf{110},\mathbf{245},\mathbf{338})$.

(c) *for* $r=10$:

$C(1063;1,89,\mathbf{236},368,\mathbf{470})$, $C(1369;1,\mathbf{98},\mathbf{194},230,425)$,
$C(1843;1,\mathbf{233},377,\mathbf{464},623)$, $C(1891;1,\mathbf{65},\mathbf{68},863,902)$.

Now, we present a sketch of the proof of Theorem 1. Different sets of parameters may define the same circulants. If $(n,a_i)=1$ then the a_i-cycle can be considered as the main cycle for any i. Then we obtain another representation of the same circulant with respective parameters. For example, Pyatkin's circulant has three different forms:

$$C(157;1,8,14)\cong C(157;59,1,41)\cong C(157;56,23,1).$$

Two last forms will be called the *inversions* of the first form. Sometimes inversions could have the same parameters as the initial form, as happens, for instance, for $C(13;1,5)$. Let $a_0=1$ and $A=\{a_1,\dots,a_k\}=A^e\cup A^o$ where A^e and A^o consist of even and odd a_i, respectively. Suppose that $(n, a_i) = 1$ for every i. Then for fixed $a \in A$ and any $b \in (A \cup \{a_0\}) \backslash \{a\}$ define the function

$$r_{n,a}(b) = \min\{r > 0 \mid ra \equiv \pm b \pmod n\}.$$

This function determines the parameters of the inversion when the a-cycle is considered as the main cycle.

A circulant $C(n; 1,\ a_1, \ldots, a_k)$ is a *normal* circulant if $n \equiv 1 \pmod 6$, $(n, a_i) = 1$, $a_i \equiv 2 \pmod 3$ for every $i \in \{1, \ldots, k\}$ and $r_{n,a}(b) \equiv 2 \pmod 3$ for every $a \in A$ and any $b \in (A \cup \{a_0\}) \backslash \{a\}$.

The idea of studying normal circulants first appeared in [15], where the following lemma was proved.

Lemma 2. *If G is normal circulant then for every* $e \in E(G)$ *we have* $\chi(G \backslash e) = 3$.

Therefore, if a normal circulant is 4-chromatic, then it is edge-critical. In order to prove that a circulant is 4-chromatic, we need first to deduce some properties of 3-chromatic circulants. Each 3-coloring of a circulant can be presented as a cyclic word of period n over the alphabet $\{1, 2, 3\}$, for instance

$$\ldots 2, \mathit{3}, 1, 3, \mathit{1}, 2, 1, 2, 1, \mathit{2}, 3, 2, 3, 2, 3, 2, \mathit{3}, 1, 3, \mathit{1}, 2, 1, \mathit{2}, \mathit{3}, \mathit{1}, 2, \ldots \quad . \qquad (*)$$

We call a 3-coloring f *periodic,* if $f_i \neq f_{i+1}$ for every i and every maximum subword induced by any two colors has an even length. The marginal symbols of such a subword will be called *outer* symbols and all other subword's symbols are *inner*. Note that the symbol f_i is outer if and only if $f_{i-1} \neq f_{i+1}$. For example, all outer symbols of $(*)$ are marked by italic font. Denote the coordinates of the outer symbols lying in the interval $[1, n]$ by $(c_1, \ldots, c_s)$. It is easy to observe that in any periodic 3-coloring the outer symbols induce a word $\ldots 123123123 \ldots$. Moreover, the subword induced by the inner symbols lying between c_i and c_{i+1} has an even length $2l_i$ (maybe, $l_i = 0$). If all $l_i \in \{0, 1\}$ then the 3-coloring is *strongly periodic*. We call a 3-chromatic circulant G *periodic* (*strongly periodic*) if every its proper 3-coloring is periodic (strongly periodic). Since n is odd, $s \equiv 3 \pmod 6$ and the number of vertices in any normal periodic circulant can be presented as follows:

$$n = 6t + 3 + 2 \sum_{i=1}^{6t+3} l_i.$$

The next lemma presents some sufficient conditions for periodic circulants.

Lemma 3. *Let* $G = C(n; 1, a_1, \ldots, a_k)$ *be a 3-chromatic circulant. Then*

(1) *If there are some* p *and* q *such that* $a_p = a_q + 3$, *then G is strongly periodic.*
(2) *If there are some* p *and* q *(maybe,* $p = q$*) such that* $a_p + a_q = n \pm 3$, *then G is strongly periodic.*

(3) *If there are some* p, q *and* r (*maybe*, $p=q$) *such that* $a_p+a_q-2=a_r$, *then* G *is periodic.*
(4) *If there are some* p, q *and* r (*maybe*, *some of them are the same*) *such that* $a_p+a_q+a_r=n+2$, *then* G *is periodic.*

Note, that all circulants in Theorem 1 except $C(289;1,38,110,134)$ satisfy at least one of the conditions (1)–(4) of Lemma 3. The corresponding parameters are marked by bold font.

Let $l_{i,j}=l_i+l_{i+1}+\cdots+l_j$ for $i\leqslant j$. For every integer $m\geqslant 0$ and $a\in A$ define the predicate $I_a(m)$ by the following rule: $I_a(m)$ is true if and only if

$$(a\in A^o \text{ and } \forall i,\ l_{i,6m+i}\geqslant (a-6m-3)/2+1) \quad \text{or}$$

$$(a\in A^e \text{ and } \forall i,\ l_{i,6m+i+3}\geqslant (a-6m-6)/2+1).$$

Analogously, for every integer $m\geqslant 0$ and $a\in A$ introduce the predicate $J_a(m)$: it is true if and only if

$$(a\in A^o \text{ and } \forall j,\ l_{j,6m+j+4}\leqslant (a-6m-3)/2-1) \quad \text{or}$$

$$(a\in A^e \text{ and } \forall j,\ l_{j,6m+j+7}\leqslant (a-6m-6)/2-1).$$

Then any periodic circulant must satisfy the following

Lemma 4. *Let* $G=C(n;1,a_1,\ldots,a_k)$ *be a* 3-*chromatic periodic normal circulant. Then every its proper* 3-*coloring must satisfy the following conditions*

(1) *For every* $a\in A^e$, $l_{i,i+1}\leqslant a/2-1$.
(2) *For every integer* m *and* $a\in A$, $I_a(m)$ *or* $J_a(m)$ *holds.*

Note, that the first condition of Lemma 4 can be interpreted as $J_a(-1)$. We can also observe that $I_a((a-5)/6)$ for odd a and $I_a((a-8)/6)$ for even a are true. Our main tool is the following

Lemma 5. *Let* $G=C(n;1,a_1,\ldots,a_k)$ *be a* 3-*chromatic periodic normal circulant. Then there exists a nonnegative integer* t *such that*

(1) *For every* $a\in A^o$ *there exists a nonnegative integer* $m_a\leqslant (a-5)/6$ *such that* $n\geqslant 6at+3a-6m_an\geqslant -n$.
(2) *For every* $a\in A^e$ *there exists a nonnegative integer* $m_a\leqslant (a-8)/6$ *such that* $4n\geqslant 6at+3a-6m_an\geqslant 2n$.

For strongly periodic normal circulants the lower bounds $m_a\geqslant\lceil (a-3)/18\rceil$ for odd a and $m_a\geqslant\lceil (a-12)/18\rceil$ for even a hold.

Suppose that a circulant $G=C(n;1,a_1,\ldots,a_k)$ satisfies the conditions of Lemmas 2 and 3. Then for every $a\in A$ there is a finite number of suitable m_a from Lemma 5. We can check them all and, using the inequalities of Lemma 5, find the set T_a of all available t. If the intersection $\bigcup_{a\in A} T_a$ is empty, then G is an Erdős's graph. This proof is applicable for all graphs from Theorem 1 except the circulant $C(289;1,38,110,134)$. The last graph does not satisfy Lemma 3; nevertheless, it is also an Erdős's graph. This was showed by computer calculations.

Note, that all circulants from Theorem 1 and Pyatkin's graph are also Dirac's graphs. Since all considered circulants are vertex-transitive and do not contain triangles, this is a simple consequence of the following result of Mader [14] and Watkins [16] that if G is a connected vertex-transitive graph without K_4, then the vertex connectivity of G is equal to its maximum degree.

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