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The Asymptotic Number of Rooted 2-Connected Triangular Maps on a Surface

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In this paper, we continue the study of the asymptotic number of rooted maps on general surfaces initiated by Bender and Canfield. Let $\vec{\Delta}_g(n)$ (respectively, $\tilde{\Delta}_g(n)$) be the number of n-vertex rooted 2-connected triangular maps on the orientable (respectively, non-orientable) surface of type g. We shall prove that, as $n \to \infty$,

$$\vec{\Delta}_g(n) \sim \vec{t}_g (An)^{5(g-1)/2} (27/2)^n \quad \text{and} \quad \tilde{\Delta}_g(n) \sim \tilde{t}_g (An)^{5(g-1)/2} (27/2)^n,$$

where $A = 3^{6/5}/2^{7/5}$, $\vec{t}_g$ and $\tilde{t}_g$ are the constants defined in an earlier paper by the author (*J. Combin. Theory Ser. B* **52** (1991), 236–249).

1. INTRODUCTION

A (rooted) triangular map on a surface is a (rooted) map on the surface [2] such that each face has valency three; a (rooted) near-triangular map on a surface is a (rooted) map on the surface such that all faces except possibly the root face and some other distinguished faces have valency three. As in [2], we use $g = 1 - \chi/2$ to denote the type of a surface with Euler Characteristic χ.

Consider rooted loopless near-triangular maps which have some distinguished faces indexed by a finite set I. Let $\vec{\Delta}_g(x, y, \mathbf{z}_I)$ be the generating function for such maps on the orientable surface of type g, where x marks the number of non-root vertices, y marks the root face valency, and $\mathbf{z}_I = \{z_i : i \in I\}$ marks the valencies of the distinguished faces. We similarly define $\tilde{\Delta}_g(x, y, \mathbf{z}_I)$ for non-orientable surfaces and define $\Delta_g(x, y, \mathbf{z}_I) = \vec{\Delta}_g(x, y, \mathbf{z}_I) + \tilde{\Delta}_g(x, y, \mathbf{z}_I)$. For convenience, we shall simply use $\vec{\Delta}_g(x, y, I)$ to denote $\vec{\Delta}_g(x, y, \mathbf{z}_I)$, etc. throughout the rest of the paper. Let $[\cdot]$ be the usual coefficient operator. Define

$$\vec{\Delta}_{g,r}(x, I) = [y^r]\, \vec{\Delta}_g(x, y, I), \qquad \tilde{\Delta}_{g,r}(x, I) = [y^r]\, \tilde{\Delta}_g(x, y, I),$$

$$\vec{\Delta}_g(n) = [x^{n-1}]\, \vec{\Delta}_{g,3}(x, \varnothing), \quad \text{and} \quad \tilde{\Delta}_g(n) = [x^{n-1}]\, \tilde{\Delta}_{g,3}(x, \varnothing).$$

Then $\vec{\Delta}_g(n)$ (respectively, $\tilde{\Delta}_g(n)$) is the number of n-vertex rooted loopless

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triangular maps on the orientable (respectively, non-orientable) surface of type g.

Although it seems very difficult to obtain the exact expressions of generating functions of rooted non-planar maps with high connectivity (the only known results are for 2-connected maps and triangular maps on the projective plane [4, 5]), Bender and Wormald obtained the asymptotic number of rooted 2-connected maps on general surfaces [3]. They observed that the relation between rooted 2-connected planar maps and rooted planar maps also holds for non-planar ones except for a "negligible fraction." In this paper, we shall use the following proposition to get around the connectivity difficulty. (See [5] for a proof.)

PROPOSITION. *A triangular map is 2-connected if and only if it is loopless.*

We shall prove:

THEOREM 1. *For fixed g and $n \to \infty$,*

$$\vec{\Delta}_g(n) \sim \vec{t}_g (An)^{5(g-1)/2} (27/2)^n,$$
$$\tilde{\Delta}_g(n) \sim \tilde{t}_g (An)^{5(g-1)/2} (27/2)^n,$$

where $A = 3^{6/5}/2^{7/5}$ and $\vec{t}_g$ and $\tilde{t}_g$ are the constants defined in [6, *Theorem* 1].

The rest of the paper is organized as follows: In Section 2, we show that $\Delta_g(x, y, I)$ satisfy the typical recursion described in [2, 6] with some extra negligible terms; In Section 3, we show that these "extra negligible terms" are indeed negligible and thereby obtain the asymptotic expression for $\Delta_{g,2}(x, \emptyset)$; Section 4 gives similar asymptotic expression for $\vec{\Delta}_{g,2}(x, \emptyset)$ and uses the preceding results and the following lemma to complete the proof of Theorem 1.

LEMMA 1. *For $g > 0$,*

$$\tilde{\Delta}_{g,3}(x, \emptyset) = \tilde{\Delta}_{g,2}(x, \emptyset) \quad \text{and} \quad \vec{\Delta}_{g,3}(x, \emptyset) = \vec{\Delta}_{g,2}(x, \emptyset).$$

Proof. The proof is exactly the same as the proof of Lemma 2 in [6]. ∎

We will assume that the reader is familiar with [2, 6]. For those notations not defined here, we refer to [2].

2. FUNCTIONAL EQUATIONS FOR $\Delta_g(x, y, I)$

Let

$$F(x, y, I) = \sum a(j, k, f_I)\, x^j y^k \prod_{i \in I} z_i^{f_i} \text{ and } G(x, y, I) = \sum b(j, k, f_I)\, x^j y^k \prod_{i \in I} z_i^{f_i}$$

be formal power series with non-negative coefficients. We say $F(x, y, I) \leqslant G(x, y, I)$ if $a(j, k, f_I) \leqslant b(j, k, f_I)$ for all j, k, f_I. In this section, we prove:

THEOREM 2. *Let $w, w' \notin I$ be distinct integers and $(g, I) \neq (0, \emptyset)$. Then,*

$$\begin{aligned}
&\Delta_g(x, y, I)\\
&\quad = xy^2 \sum_{j=0/2}^{g} \sum_{S \subseteq I} \Delta_j(x, y, S)\, \Delta_{g-j}(x, y, I-S)\\
&\qquad + 2y^2 \left[y \frac{\partial}{\partial z_w} \Delta_{g-1}(x, y, I+\{w\}) - L_{T,g-1}(x, y, I+\{w\}) \right]_{z_w = y}\\
&\qquad + y^2 \left[\frac{\partial}{\partial y} (y\Delta_{g-1/2}(x, y, I)) - L_{P,g-1/2}(x, y, I) \right]\\
&\qquad + y^{-1} [\Delta_g(x, y, I) - L_g(x, y, I)]\\
&\qquad + \sum_{i \in I} y z_i \left[\frac{1}{z_i - y} (z_i \Delta_g(x, z_i, I-\{i\}) - y\Delta_g(x, y, I-\{i\}))\right.\\
&\qquad \left. - L_{D,g}(x, y, I-\{i\}) \right],
\end{aligned} \tag{1}$$

where

$$\begin{aligned}
0 &\leqslant L_{T,g-1}(x, y, I+\{w\})\\
&\leqslant \sum_{j=0/2}^{g-1} \sum_{S \subseteq I} \Delta_j(x, z_w, S)\, \Delta_{g-1-j}(x, y, I+\{w\}-S)\\
&\quad + 2x^{-1} z_w \frac{\partial}{\partial z_{w'}} \Delta_{g-2}(x, y, I+\{w\}+\{w'\})|_{z_{w'} = z_w}\\
&\quad + x^{-1} \frac{\partial}{\partial z_w} (z_w \Delta_{g-3/2}(x, y, I+\{w\})),
\end{aligned} \tag{2}$$

$$\begin{aligned}
0 &\leqslant L_{P,g-1/2}(x, y, I)\\
&\leqslant \sum_{j=0/2}^{g-1/2} \sum_{S \subseteq I} \Delta_j(x, y, S)\, \Delta_{g-1/2-j}(x, y, I-S)\\
&\quad + 2x^{-1} z_w \frac{\partial}{\partial z_w} \Delta_{g-3/2}(x, y, I+\{w\})|_{z_w = y}\\
&\quad + x^{-1} \frac{\partial}{\partial y} (y\Delta_{g-1}(x, y, I)),
\end{aligned} \tag{3}$$

$$0 \leqslant L_{D,g}(x, y, I-\{i\})$$

$$\leqslant \sum_{j=0/2}^{g} \sum_{S \subseteq I-\{i\}} \Delta_j(x, z_i, S)\, \Delta_{g-j}(x, y, I-\{i\}-S)$$

$$+2x^{-1} z_i \frac{\partial}{\partial z_i} \Delta_{g-1}(x, y, I)$$

$$+x^{-1} \frac{1}{z_i - y} \left(z_i \Delta_{g-1/2}(x, z_i, I-\{i\}) - y \Delta_{g-1/2}(x, y, I-\{i\})\right), \tag{4}$$

and

$$L_g(x, y, I) = \sum_{j=0/2}^{g} \sum_{S \subseteq I} y^2 \Delta_{j,2}(x, S)\, \Delta_{g-j}(x, y, I-S) + N_g(x, y, I), \tag{5}$$

with

$$0 \leqslant N_g(x, y, I) \leqslant 4x^{-1} \Delta_{g-1}(x, y, I+\{w\})|_{z_w=y} + x^{-1} \Delta_{g-1/2}(x, y, I), \tag{6}$$

and

$$\Delta_{-1/2}(x, y, I) \equiv \Delta_{-1}(x, y, I) \equiv 0.$$

Proof. The proof of (1) is very similar to that of [6, Theorem 2] except that there are some extra L-terms here. These L-terms arise from subcases A_1, B_1, and B_2 (cf. the proof of [6, Theorem 2]) when adding a new root edge creates a loop. We omit the proof of (2)–(6) because it is quite lengthy. Interested readers may refer to [7] for the details. ▮

3. Asymptotic Evaluation of $\Delta_{g,2}(x, \varnothing)$

This section is very similar to Section 4 of [6]. Multiplying (1) by y and collecting terms in $\Delta_g(x, y, I)$, we obtain

$$A(x, y)\, \Delta_g(x, y, I)$$

$$= -xy^3 \sum_{\substack{j=0/2 \\ (j,S) \neq (0,\varnothing),\,(g,I)}}^{g} \sum_{S \subseteq I} \Delta_j(x, y, S)\, \Delta_{g-j}(x, y, I-S)$$

$$-2y^4 \frac{\partial}{\partial z_w} \Delta_{g-1}(x, y, I+\{w\})|_{z_w=y} + 2y^3 L_{T,g-1}(x, y, I+\{w\})|_{z_w=y}$$

$$-y^3 \frac{\partial}{\partial y} (y \Delta_{g-1/2}(x, y, I)) + y^3 L_{P,g-1/2}(x, y, I)$$

$$
\begin{aligned}
&+ \sum_{\substack{j=0/2\\(j,S)\neq(0,\varnothing),(g,I)}}^{g} \sum_{S\subseteq I} y^2\Delta_{j,2}(x,S)\,\Delta_{g-j}(x,y,I-S)+y^2\Delta_{g,2}(x,I)\\
&\times\Delta_0(x,y,\varnothing)-N_g(x,y,I)\\
&-\sum_{i\in I} y^2 z_i\left[\frac{1}{z_i-y}\,(z_i\Delta_g(x,z_i,I-\{i\})-y\Delta_g(x,y,I-\{i\}))\right.\\
&\left.-L_{D,g}(x,y,I-\{i\})\right], \qquad (7)
\end{aligned}
$$

where

$$A(x,y)=2xy^3\Delta_0(x,y,\varnothing)+1-y-y^2\Delta_{0,2}(x,\varnothing).$$

For the remainder of the paper, let $f(x)=\sum_{k\geqslant 0} f_k x^k$ be the unique power series defined by

$$
\begin{aligned}
f&=\frac{1}{1-t},\\
x&=t(1-2t)^2.
\end{aligned}
$$

Using Lagrange's inversion formula, we have

$$f_k=\frac{1}{k}\sum_{l=0}^{k}\binom{2k+l-1}{l}(k-l)\,2^l\geqslant 0.$$

From [5], we know $A(x,f)=0$. Setting $y=f$ in (7), we have

$$
\begin{aligned}
&\Delta_0(x,f,\varnothing)\,\Delta_{g,2}(x,I)\\
&\quad= xf \sum_{\substack{j=0/2\\(j,S)\neq(0,\varnothing),(g,I)}}^{g} \sum_{S\subseteq I} \Delta_j(x,f,S)\,\Delta_{g-j}(x,f,I-S)\\
&\quad+2f^2\frac{\partial}{\partial z_w}\Delta_{g-1}(x,f,I+\{w\})|_{z_w=f}-2fL_{T,g-1}(x,f,I+\{w\})|_{z_w=f}\\
&\quad+f\frac{\partial}{\partial y}(y\Delta_{g-1/2}(x,y,I))|_{y=f}-fL_{P,g-1/2}(x,f,I)\\
&\quad- \sum_{\substack{j=0/2\\(j,S)\neq(0,\varnothing),(g,I)}}^{g} \sum_{S\subseteq I} \Delta_{j,2}(x,S)\,\Delta_{g-j}(x,f,I-S)+\frac{1}{f^2}N_g(x,f,I)\\
&\quad+\sum_{i\in I} z_i\left[\frac{1}{z_i-f}\,(z_i\Delta_g(x,z_i,I-\{i\})-f\Delta_g(x,f,I-\{i\}))\right.\\
&\quad\left.-L_{D,g}(x,f,I-\{i\})\right]. \qquad (8)
\end{aligned}
$$

Before proceeding, we need to introduce some notations. Let

$$g(x)=\sum_{n\geqslant 0} g_n x^n, \qquad h(x)=\sum_{n\geqslant 0} h_n x^n$$

be two formal power series. We use

$$g(x)=O(h(x)) \qquad \text{to mean } g_n=O(h_n),$$

$$g(x)=o(h(x)) \qquad \text{to mean } g_n=o(h_n).$$

From [2], we know that $g(x)\approx h(x)$ implies $g(x)=h(x)+o(h(x))$. Let $\boldsymbol{\alpha}=(\ldots\alpha_i\ldots)$ be a vector of non-negative integers such that $\alpha_i=0$ for $i\notin I$ and define

$$H_g^{(n)}(x, I, \boldsymbol{\alpha})=\frac{\partial^{n+|\boldsymbol{\alpha}|}}{\partial y^n \prod_{i\in I} \partial z_i^{\alpha_i}} H_g(x, y, I)|_{y=z_i=f}$$

for any function $H_g(x, y, I)$ (here, and in the following, $|\boldsymbol{\alpha}|$ denotes $\sum_i \alpha_i$.). Let $\mathbf{0}$ denote the zero vector and $\mathbf{e}_w$ denote the wth unit vector. Our goal in this section is to prove the following theorem and use it to estimate $\Delta_{g,2}(x, \varnothing)$.

THEOREM 3. *Let* $e=(10g+2n+5|I|+2|\boldsymbol{\alpha}|-3)/4$. *Then*

$$\Delta_{g,2}^{(0)}(x, I, \boldsymbol{\alpha})=O\left(\left(1-\frac{27}{2}x\right)^{-e+3/4+n/2}\right) \qquad \textit{for} \quad (g, I)\neq(0, \varnothing),$$

and there is a collection of constants $\Delta_g^{(n)}(I, \boldsymbol{\alpha})$ *such that*

$$\Delta_g^{(n)}(x, I, \boldsymbol{\alpha})=\Delta_g^{(n)}(I, \boldsymbol{\alpha})\left(1-\frac{27}{2}x\right)^{-e}+O\left(\left(1-\frac{27}{2}x\right)^{-e+1/4}\right)$$

for $(g, |I|, n)\neq(0, 0, 0)$ *and* $\Delta_g^{(n)}(I, \boldsymbol{\alpha})>0$ *for* $(g, |I|, n)\neq(0, 0, 1)$.

We will prove Theorem 3 by induction using the lexicographic ordering on $(g, |I|, n)$. The following lemma covers $(0, 0, n)$.

LEMMA 2. *For* $n>0$,

$$A^{(n)}=d_n\left(1-\frac{27}{2}x\right)^{-(2n-3)/4}+O\left(\left(1-\frac{27}{2}x\right)^{-(2n-3)/4+1/4}\right)$$

with

$$d_n=-\left(\frac{16}{27}\right)^{1/4}\binom{1/2}{n-1}\left(-\frac{25\sqrt{3}}{36}\right)^{n-1} n!,$$

and Theorem 3 *holds for* $\Delta_0^{(n)}(0, \varnothing)=(125/32)d_n$.

Proof. The proof is very similar to that of [6, Lemma 4.1]. ∎

In proving Theorem 3, the following lemmas will be used frequently without explicit reference.

LEMMA 3. *Let $F(x, y, I)$ and $G(x, y, I)$ be two formal power series with non-negative coefficients. If*

$$F(x, y, I) \leqslant G(x, y, I),$$

then

$$\frac{\partial^{n+|\alpha|}}{\partial y^n \prod_{i\in I} \partial z_i^{\alpha_i}} F(x, y, I) \leqslant \frac{\partial^{n+|\alpha|}}{\partial y^n \prod_{i\in I} \partial z_i^{\alpha_i}} G(x, y, I),$$

and

$$F^{(n)}(x, I, \boldsymbol{\alpha}) \leqslant G^{(n)}(x, I, \boldsymbol{\alpha}).$$

Proof. The proof is trivial once we recall that $f(x)$ has non-negative coefficients. ∎

LEMMA 4. *If $F(x, y)$ is analytic and non-zero at $(2/27, 6/5)$ and*

$$G^{(k)} = g_k\left(1 - \frac{27}{2}x\right)^{-\alpha_k} + o\left(\left(1 - \frac{27}{2}x\right)^{-\alpha_k}\right)$$

where $\alpha_k \notin \{0, -1, -2, ...\}$ is strictly increasing, then

$$(FG)^{(k)} = F(2/27, 6/5)\, G^{(k)} + o(G^{(k)}).$$

Proof. See the proof of [2, Lemma 1]. ∎

We now complete the proof of Theorem 3. Let $(g, I) \neq (0, \varnothing)$ and $n \geqslant 0$, assume that Theorem 3 is true for all indices before $(g, |I|, n)$. Then it follows from (2)–(6) and Lemma 3 that

$$L_{T, g-1}^{(n+1-k)}(x, I + \{w\}, \boldsymbol{\alpha} + k\mathbf{e}_w) = O\left(\left(1 - \frac{27}{2}x\right)^{-e+3/4}\right), \tag{9}$$

$$L_{P, g-1/2}^{(n+1)}(x, I, \boldsymbol{\alpha}) = O\left(\left(1 - \frac{27}{2}x\right)^{-e+3/4}\right), \tag{10}$$

$$L_{D, g}^{(n+1)}(x, I - \{i\}, \boldsymbol{\alpha}) = O\left(\left(1 - \frac{27}{2}x\right)^{-e+3/4}\right), \tag{11}$$

$$N_g^{(n+1)}(x, I, \boldsymbol{\alpha}) = O\left(\left(1 - \frac{27}{2}x\right)^{-e+3/4}\right), \tag{12}$$

where e is defined in Theorem 3. Applying

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\prod_{i\in I}\partial z_i^{\alpha_i}}$$

to both sides of (8), setting $z_i = f$, and using (9)–(12), we obtain

$$\Delta_{g,2}^{(0)}(x, I, \boldsymbol{\alpha}) = O\left(\left(1-\frac{27}{2}x\right)^{-e+3/4+n/2}\right). \tag{13}$$

Applying

$$\frac{\partial^{n+1+|\boldsymbol{\alpha}|}}{\partial y^{n+1}\prod_{i\in I}\partial z_i^{\alpha_i}}$$

to both sides of (7), setting $y = z_i = f$, and using (9)–(13), we obtain

$$\Delta_g^{(n)}(x, I, \boldsymbol{\alpha}) = \Delta_g^{(n)}(I, \boldsymbol{\alpha})\left(1-\frac{27}{2}x\right)^{-e} + O\left(\left(1-\frac{27}{2}x\right)^{-e+1/4}\right),$$

with $\Delta_g^{(n)} > 0$ given by the recursion

$$\begin{aligned}
&(n+1)\left(\frac{16}{27}\right)^{1/4}\Delta_g^{(n)}(I, \boldsymbol{\alpha})\\
&\quad= \sum_{k=0}^{n-1}\binom{n+1}{k} d_{n+1-k}\Delta_g^{(k)}(I, \boldsymbol{\alpha})\\
&\qquad+\frac{2}{27}\left(\frac{6}{5}\right)^3 \sum_{\substack{j=0/2\\ (j,S)\neq(0,\varnothing),(g,I)}}^{g}\ \sum_{S\subseteq I}\ \sum_{k=0}^{n+1}\binom{n+1}{k}\\
&\qquad\times\Delta_j^{(k)}(S, \boldsymbol{\alpha}|_S)\,\Delta_{g-j}^{(n+1-k)}(I-S, \boldsymbol{\alpha}|_{I-S})\\
&\qquad+2\left(\frac{6}{5}\right)^4\sum_{k=0}^{n+1}\binom{n+1}{k}\Delta_{g-1}^{(n+1-k)}(I+\{w\}, \boldsymbol{\alpha}+(k+1)\,\boldsymbol{l}w)\\
&\qquad+\left(\frac{6}{5}\right)^4\Delta_{g-1/2}^{(n+2)}(I, \boldsymbol{\alpha})\\
&\qquad+\left(\frac{6}{5}\right)^4\sum_{i\in I}\frac{(n+1)!\,\alpha_i!}{(n+\alpha_i+2)!}\Delta_g^{(n+\alpha_i+2)}(I-\{i\}, \boldsymbol{\alpha}|_{I-\{i\}}).
\end{aligned}\tag{14}$$

Thus Theorem 3 is proved.

Setting $I=\varnothing$ in (8) and using Theorem 3, we obtain

$$\begin{aligned}\Delta_{g,2}(x,\varnothing)=&\left[\frac{16}{225}\sum_{j=1/2}^{g-1/2}\Delta_j^{(0)}(\varnothing,\mathbf{0})\,\Delta_{g-j}^{(0)}(\varnothing,\mathbf{0})\right.\\&+\frac{288}{125}\Delta_{g-1}^{(0)}(\{w\},\mathbf{e}_w)\\&\left.+\frac{144}{125}\Delta_{g-1/2}^{(1)}(\varnothing,\mathbf{0})\right]\left(1-\frac{27}{2}x\right)^{-(10g-6)/4}\\&+o\left(\left(1-\frac{27}{2}x\right)^{-(10g-6)/4}\right).\end{aligned}\tag{15}$$

4. Proof of Theorem 1

Similar to Section 3, we have

THEOREM 4. *Let* $e=(10g+2n+5|I|+2|\boldsymbol{\alpha}|-3)/4$. *Then*

$$\vec{\Delta}_{g,2}^{(0)}(x,I,\boldsymbol{\alpha})=O\left(\left(1-\frac{27}{2}x\right)^{-e+3/4+n/2}\right)\qquad \textit{for}\quad (g,I)\neq(0,\varnothing),$$

and there is a collection of constants $\vec{\Delta}_g^{(n)}(I,\boldsymbol{\alpha})$ *such that*

$$\vec{\Delta}_g^{(n)}(x,I,\boldsymbol{\alpha})=\vec{\Delta}_g^{(n)}(I,\boldsymbol{\alpha})\left(1-\frac{27}{2}x\right)^{-e}+O\left(\left(1-\frac{27}{2}x\right)^{-e+1/4}\right)$$

for $(g,|I|,n)\neq(0,0,0)$ *and* $\vec{\Delta}_g^{(n)}(I,\boldsymbol{\alpha})>0$ *for* $(g,|I|,n)\neq(0,0,1)$.

The analogs of (14) and (15) turn out to be

$$\begin{aligned}&(n+1)\left(\frac{16}{27}\right)^{1/4}\vec{\Delta}_g^{(n)}(I,\boldsymbol{\alpha})\\&=\sum_{k=0}^{n-1}\binom{n+1}{k}d_{n+1-k}\vec{\Delta}_g^{(k)}(I,\boldsymbol{\alpha})\\&\quad+\frac{2}{27}\left(\frac{6}{5}\right)^3\sum_{\substack{j=0\\(j,S)\neq(0,\varnothing),(g,I)}}^{g}\sum_{S\subseteq I}\sum_{k=0}^{n+1}\binom{n+1}{k}\\&\quad\times\vec{\Delta}_j^{(k)}(S,\boldsymbol{\alpha}|_S)\,\vec{\Delta}_{g-j}^{(n+1-k)}(I-S,\boldsymbol{\alpha}|_{I-S})\\&\quad+\left(\frac{6}{5}\right)^4\sum_{k=0}^{n+1}\binom{n+1}{k}\vec{\Delta}_{g-1}^{(n+1-k)}(I+\{w\},\boldsymbol{\alpha}+(k+1)\,\mathbf{l}w)\\&\quad+\left(\frac{6}{5}\right)^4\sum_{i\in I}\frac{(n+1)!\,\alpha_i!}{(n+\alpha_i+2)!}\vec{\Delta}_g^{(n+\alpha_i+2)}(I-\{i\},\boldsymbol{\alpha}|_{I-\{i\}}),\end{aligned}\tag{16}$$

and

$$\vec{\Delta}_{g,2}(x, \varnothing) = \left[\frac{16}{225} \sum_{j=1}^{g-1} \vec{\Delta}_j^{(0)}(\varnothing, \mathbf{0})\, \vec{\Delta}_{g-j}^{(0)}(\varnothing, \mathbf{0}) + \frac{144}{125} \vec{\Delta}_{g-1}^{(0)}(\{w\}, lw)\right] \left(1 - \frac{27}{2} x\right)^{-(10g-6)/4} + o\left(\left(1 - \frac{27}{2} x\right)^{-(10g-6)/4}\right). \tag{17}$$

The following lemma together with (15) and (17) and [1, Theorem 4] complete the proof of Theorem 1 (cf. [6, Sect. 5] for details).

LEMMA 5. *Let* $T_g^{(n)}(I, \boldsymbol{\alpha})$, $\vec{T}_g^{(n)}(I, \boldsymbol{\alpha})$ *be the constants defined in* [5, *Theorem* 3] *and* $\Delta_g^{(n)}(I, \boldsymbol{\alpha})$, $\vec{\Delta}_g^{(n)}(I, \boldsymbol{\alpha})$ *be the constants defined in Theorem* 3 *and Theorem* 5. *Then for* $(g, |I|, n) \neq (0, 0, 0)$, *we have*

$$\vec{\Delta}_g^{(n)}(I, \boldsymbol{\alpha}) = \beta_0 \beta_2^{n+|\boldsymbol{\alpha}|} \beta_2^{|I|+2g} \vec{T}_g^{(n)}(I, \boldsymbol{\alpha}),$$

$$\Delta_g^{(n)}(I, \boldsymbol{\alpha}) = \beta_0 \beta_1^{n+|\boldsymbol{\alpha}|} \beta_2^{|I|+2g} T_g^{(n)}(I, \boldsymbol{\alpha}),$$

with

$$\beta_0 = \frac{15(3-\sqrt{3})}{2^{3/4}}, \qquad \beta_1 = \frac{25(3-\sqrt{3})^2}{2^{5/2} \times 3^{3/2}}, \qquad \beta_2 = \frac{3^{3/2}}{2^{7/4}}.$$

Proof. The proof is straightforward by comparing the recursions (14) and (16) with (4.6) and (4.7) of [6] (cf. the proof of [6, Lemma 5.1]). ∎

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