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ELSEVIER Topology and its Applications 61 (1995) 241–256

TOPOLOGY AND ITS APPLICATIONS

Spaces with unique Hausdorff extensions

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Received 3 February 1993; revised 7 February 1994

Abstract

H-closed extensions of Hausdorff spaces have been studied extensively as a generalization of compactifications of Tychonoff spaces. The collection of H-closed extensions of a space is known to have an upper semilattice structure. Little work has been done to characterize spaces whose collections of H-closed extensions have specified upper semilattice structures. In 1970 J.R. Porter found necessary and sufficient conditions on a space so that it would have exactly one one-point H-closed extension. He asked for a characterization of those spaces which have exactly one H-closed extension. This is the same as having exactly one Hausdorff extension. In this paper we answer Porter's question and give an example of such a space. Topological sums of this space give spaces which have two, five, or in general, $p(n)$ many H-closed extensions where $p(n)$ is the number of ways a set of size n can be partitioned. This space is also an example of a space with exactly one free prime open filter which gives an answer to a question asked by J. Pelant, P. Simon, and J. Vaughan. As a preliminary for obtaining the above results, we find necessary and sufficient conditions on a space so that the S- and θ-equivalence relations defined by J.R. Porter and C. Votaw are equivalent.

Keywords: Extension; Hausdorff; H-closed extension; H-bounded set; Open filter; Prime open filter; Lattice; Semilattice

AMS (MOS) Subj. Class.: Primary 54D25, 54D35; secondary 54A20

1. Introduction

This paper will consider only Hausdorff spaces. Recall that an extension of a space X is any space containing X as a dense subspace. Recall also that a space is H-closed if it is closed in every Hausdorff space in which it is embedded. Thus only

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SSDI 0166-8641(94)00031-W

non-H-closed spaces can have extensions. A useful characterization of the H-closed property is that every open cover of an H-closed space has a finite subcollection which is dense. Thus the H-closed property is a generalization of compactness. H-closed extensions are important in the study of extensions of Hausdorff spaces in the same way that compactifications are important in the study of extensions of Tychonoff spaces. Since compactifications are H-closed extensions, H-closed extension theory contains the theory of compactifications and in general is much richer. Further, of the extensions which can be defined on a Hausdorff space, the H-closed extensions are maximal extensions in the sense that if one adds points to an H-closed extension of a space (i.e., embeds the extension into a larger space) either the original space is no longer dense or the new space is not Hausdorff.

Although H-closed extensions of Hausdorff spaces have been studied extensively, to date there has been very little work on either the H-closed extensions of a specific space or on spaces which have a prescribed H-closed extension structure. One particular line of inquiry began with the work of Obreanu who looked at the existence of one-point H-closed extensions. In [14] he showed that a non-H-closed space has a one-point H-closed extension if and only if it is locally H-closed (every point has a neighborhood which is H-closed). Unlike one-point compactifications which must be unique, a locally H-closed space may have more than one H-closed extension. In 1970 Porter [16] found that a non-H-closed space has exactly one one-point extension if and only if every closed nowhere dense set is a subset of an H-closed set. Porter asked for necessary and sufficient conditions on a space so that it has exactly one H-closed extension, which is the same as having exactly one Hausdorff extension. This question was brought up again by Girou in [6]. In Section 5 it will be shown that a non-H-closed space has exactly one H-closed extension if and only if it is almost H-closed and every closed nowhere dense set is the subset of an H-closed set. In Section 6, an example of such a space will be constructed. Several applications of this space will be considered. It is an example of a non-H-closed space with exactly one free prime open filter answering a question of Pelant, Simon, and Vaughan [15].

While the result mentioned above seems to finish the series of questions concerning one-point H-closed extensions, it actually opens a new line of inquiry. To state that a space has exactly one H-closed extension specifies the collection of H-closed extensions completely, namely it is a one-point set. As far as the author is aware, this is the only known situation in which the structure of the H-closed extensions of a particular space has been characterized. This also is the first time that, given a specified structure for a semilattice, a characterization has been found for those spaces whose semilattice of H-closed extensions has that structure.

In the process of answering Porter's question, a general theorem on the structure of the set of H-closed extensions of a space is proven. The collection of H-closed extensions of a space X will be denoted by $H(X)$. The set $H(X)$ is given a partial ordering as follows: $Y \geqslant Z$ if and only if there is a continuous function f from Y onto Z such that $f|_X = \mathrm{id}_X$. It is well known that with this ordering the

H-closed extensions of a space form a complete upper semilattice. Two extensions Y and Z will be considered to be equivalent if $Y \geqslant Z$ and $Z \geqslant Y$. This equivalence will be denoted by $Y \equiv_X Z$. The symbol $O_Y^p = \{U \cap X$: U is an open neighborhood of p in $Y\}$ will denote the open neighborhood trace filter on X of a point p in an extension Y of X. In general, points in the remainder will be identified with their open neighborhood trace filters. An excellent reference on these topics is the book by Porter and Woods [21].

There are two common equivalence relations due to Porter and Votaw [19] on $H(X)$ (or more generally on $E(X)$, the collection of all Hausdorff extensions of X) which will be under consideration. The first is θ-equivalence. Recall that a function f from X to Y is θ-continuous at x if for every neighborhood V of $f(x)$ there is a neighborhood U of x such that $f(\mathrm{cl}_X U) \subseteq \mathrm{cl}_Y V$. Let Y and Z be extensions of X. We say that Y is θ-equivalent to Z (and write $Y \equiv_\theta Z$) if and only if there is a θ-homeomorphism between Y and Z fixing X. A more useful characterization will be given later. The second equivalence relation is S-equivalence. We say that Y is S-equivalent to Z (and write $Y \equiv_S Z$) if and only if they have the same neighborhood trace filters, that is $\{O_Y^p\colon p \in Y\} = \{O_Z^p\colon p \in Z\}$. In general the S-equivalence classes refine the θ-equivalence classes. We will examine conditions under which S- and θ-equivalence are the same. Our theorem is that the S- and θ-equivalence relations are the same if and only if every closed nowhere dense set is H-bounded. H-bounded sets are examined in Section 2.

The notion of S-equivalence suggests that there can be multiple topologies on an extension of a space X which yield the same neighborhood filter trace on X. Two topologies of importance are the simple and strict extension topologies. The simple extension topology on an extension Y of X has a base consisting of sets of the form $U \cup \{p\}$ where $U \in O^p$. The strict extension topology on Y has a base consisting of sets of the form $o(U) = \{p \in Y\colon U \in O^p\}$ where U is open in X.

Two named extensions will be of interest to us. The Katětov extension κX consists of the points $X \cup \{\mathscr{U}\colon \mathscr{U}$ is a free open ultrafilter$\}$ with the simple extension topology. The Katětov extension is the largest H-closed extension in the order mentioned above. Thus for every H-closed extension Y of X, there will be a continuous surjection f_Y from κX to Y which keeps X fixed. This map will be called the Katětov map. The Fomin extension σX has the same point set as κX, but with the strict extension topology. It will be important to note that κX is the largest H-closed extension of X in its S-equivalence class with the ordering noted above. The Fomin extension is the smallest H-closed extension in the S-equivalence class that contains κX. Thus κX and σX represent the maximum and minimum elements in the topmost S-equivalence class. For more background on these two extensions see [21].

2. H-bounded sets

The notion of an H-bounded set was introduced by Lambrinos [11] and studied further in [13]. H-bounded sets are related to H-closed sets and H-sets which have been more throughly examined in the literature, see for example [10,17,21–24]. Our interest will be exclusively with H-closed sets and H-bounded sets; we mention H-sets just for the sake of comparison. In this section these three properties will be defined and briefly compared. The properties of H-bounded sets that will be needed are presented.

There are several ways one might choose to define an H-closed subset. Three are listed below, and it is interesting that they define distinct properties.

Definition 2.1. Let $A \subseteq X$.

(1) The set A is *H-closed* if and only if every open cover of A by open sets in A has a finite subcollection whose closures in A cover A. Thus an H-closed subset is an H-closed subspace.

(2) [22] The set A is an *H-set* if and only if every open cover of A by open sets in X has a finite subcollection whose closures in X cover A.

(3) [11] The set A is an *H-bounded* subset if and only if every open cover of X has a finite subcollection whose closures in X cover A.

We will state below a few basic properties of H-bounded sets and compare them with H-sets and H-closed sets. For a detailed study of their properties see [11] or [13].

Let us begin by noting that for closed subsets of a regular space, all these properties are equivalent to compactness.

Proposition 2.2. *In a regular space X a closed set is H-bounded if and only if it is an H-set if and only if it is H-closed if and only if it is compact.*

The next example is a (necessarily nonregular) space for which these three properties are distinct.

Example 2.3. Let $X = \{(1/n, 1/m): n \text{ and } m \in \mathbb{N}\} \cup \{(1/n, 0): n \in \mathbb{N}\} \cup \{p\}$. Points in the plane have the topology inherited as a subspace of $\mathbb{R}^2$. A basic neighborhood of p has the form $\{p\} \cup \{(1/n, 1/m): n \geqslant N, m > 0\}$ for some positive integer N.

This space is H-closed but not compact. The points on the x-axis together with p is an H-set which is not H-closed, and the x-axis alone is an H-bounded set which is not an H-set.

From Definition 2.1 and Example 2.3, it is evident that a compact set is an H-set, and an H-set is an H-bounded set, but none of these implications is reversible.

The next proposition is obvious from Definition 2.1. An immediate consequence of it is that an H-bounded set need not be closed (H-sets and H-closed sets are closed), although we will be interested in only closed H-bounded sets.

Proposition 2.4. *Let $A \subseteq X$. If A is a subset of an H-set, then A is H-bounded.*

It is not true that all H-bounded sets are of this form, but there is essentially only one known example which is not of this form, see [13]. In what follows, we will frequently be interested in locally H-closed spaces which have the property that all H-bounded sets are of the type described in Proposition 2.4.

Proposition 2.5. *If X is a locally H-closed space, then every H-bounded subset of X is contained in an H-closed set.*

This section will conclude with an especially useful characterization of H-bounded sets involving open filters. Note that our use of the term "open filter" will mean a filter on the lattice of open sets. For a filter to *meet* a set means that the intersection of the set with each element of the filter is nonempty. The *adherence* of an open filter $\mathscr{F}$, denoted by $\mathrm{ad}(\mathscr{F})$, is the set $\bigcap\{\mathrm{cl}(U)\colon U \in \mathscr{F}\}$. If $\mathrm{ad}(\mathscr{F}) = \emptyset$, then $\mathscr{F}$ is said to be *free*, otherwise $\mathscr{F}$ is said to be *fixed*.

Proposition 2.6. *Let A be a subset of X.*

(1) *The set A is an H-set if and only if every open filter on X meeting A has an adherent point in A.*

(2) *The set A is an H-bounded set if and only if every open filter on X meeting A has an adherent point. (That is, no free filters meet A.)*

3. A tool: Prime open filters

When studying extensions, the open neighborhood trace filters of points in the remainder of the extension are free open filters. In this section we look at open filters which satisfy an additional property and are called prime.

Definition 3.1. An open filter on a space is *prime* if whenever the union of two open sets is in the filter, one of the open sets is in the filter.

Open ultrafilters are well-known examples of prime open filters. We will be interested in the existence of free, nonmaximal, prime open filters.

A useful technique is borrowed from several fields including commutative ring theory [9] and lattice theory [7]. An ideal in a commutative ring which is maximal with respect to the exclusion of a multiplicative set is prime. Translating into the language of open filters gives the following proposition:

Proposition 3.2. *An open filter on a space X which is maximal with respect to the exclusion of some collection of open sets of X which is closed under finite unions is prime.*

It is well known that each filter is the intersection of the ultrafilters containing it. It is not generally true that each open filter is the intersection of the open ultrafilters containing it, but we do have the following corollary to Proposition 3.2:

Corollary 3.3 [2]. *Each open filter on a space is the intersection of the prime open filters on that space that contain it.*

Working with dense open sets allows us to strengthen Proposition 3.2 and provides a characterization of prime open filters involving dense open sets.

Proposition 3.4. *An open filter on a space X is prime if and only if it is maximal with respect to the exclusion of some collection of dense open sets of X which is closed under finite unions.*

Proof. Suppose that $\mathscr{F}$ is a prime open filter on X. Let $\mathscr{D}(\mathscr{F})=\{D\colon D$ is dense open and $D\notin\mathscr{F}\}$. Since $\mathscr{F}$ is prime, $\mathscr{D}(\mathscr{F})$ is closed under finite unions. To show that $\mathscr{F}$ is maximal with respect to exclusion of $\mathscr{D}(\mathscr{F})$, suppose that $\mathscr{G}$ excludes $\mathscr{D}(\mathscr{F})$ and $\mathscr{G}$ contains $\mathscr{F}$. If $\mathscr{D}(\mathscr{G})=\{D\colon D$ is dense open and $D\notin\mathscr{G}\}$, it follows easily that $\mathscr{D}(\mathscr{F})=\mathscr{D}(\mathscr{G})$. Suppose $U\in\mathscr{G}\setminus\mathscr{F}$. Then $U\cup\operatorname{int}(X\setminus U)\in\mathscr{G}$, but since $\mathscr{F}$ is prime $U\cup\operatorname{int}(X\setminus U)\notin\mathscr{F}$. Thus $\mathscr{D}(\mathscr{F})\neq\mathscr{D}(\mathscr{G})$ which is a contradiction.

Conversely, suppose that $\mathscr{F}$ is maximal with respect to the exclusion of some collection of dense open sets $\mathscr{D}$ which is closed under finite unions. Suppose that for open sets U and V, $U\cup V\in\mathscr{F}$, but $U\notin\mathscr{F}$ and $V\notin\mathscr{F}$. The filter $\langle\mathscr{F},U\rangle$ on X generated by $\mathscr{F}$ and U must then meet $\mathscr{D}$. Similarly for $\langle\mathscr{F},V\rangle$. Hence there exists D_1 and D_2 in $\mathscr{D}$ such that $D_1=(W_1\cup U)\cap F_1$ and $D_2=(W_2\cup V)\cap F_2$ where the W_i are dense open and the F_i are in $\mathscr{F}$. Now, $D_1\cup D_2=((W_1\cup U)\cap F_1)\cup((W_2\cup V)\cap F_2)$ and this is equal to $(W_1\cup W_2\cup U\cup V)\cap((W_1\cup U)\cup F_2)\cap(F_1\cup(W_2\cup V))\cap(F_1\cup F_2)$. Each of the four terms in this intersection is an element of $\mathscr{F}$, so $D_1\cup D_2\in\mathscr{F}$. This contradicts the assumption that $\mathscr{D}$ is closed under finite unions. $\square$

The next proposition is a useful tool.

Proposition 3.5. *An open filter on a space is an open ultrafilter if and only if it is maximal with respect to containing every dense open set.*

Proof. A dense open set meets every element of an open ultrafilter and thus must be an element of it. The maximality follows from being an ultrafilter. Conversely, suppose an open filter $\mathscr{F}$ is maximal with respect to containing all of the dense open sets. Then it is maximal with respect to excluding the empty set of dense open sets and is, therefore, prime. Let U be any open set. The set $U\cup(X\setminus\operatorname{cl}U)$ is a dense open set and hence in $\mathscr{F}$. Thus either U or $X\setminus\operatorname{cl}U$ is in $\mathscr{F}$, which implies that $\mathscr{F}$ is an open ultrafilter. $\square$

Definition 3.6. An open filter on a space X which contains every dense open set of X is called *saturated* or *balanced.*

The next two corollaries are immediate from Proposition 3.5.

Corollary 3.7. *An open filter on a space is saturated if and only if it is either an open ultrafilter or an intersection of open ultrafilters.*

Corollary 3.8. *An open filter on a space is an ultrafilter if and only if it is prime and saturated.*

It is well known (see [7]) that if $\mathscr{B}$ is a Boolean algebra then a $\mathscr{B}$-filter is prime if and only if it is maximal. This is not the case for open filters.

As mentioned in the introduction to this section, open ultrafilters are easily seen to be prime. The next proposition is a corollary to Proposition 3.5 and gives conditions for the existence of nonmaximal prime open filters.

Proposition 3.9. *The following are equivalent for a space X:*

(1) *There is a nonmaximal prime open filter on X;*
(2) *X has a proper (not the whole space) dense open set;*
(3) *X has a nonempty nowhere dense set;*
(4) *X is not a discrete space.*

Proof. The equivalence of (1) and (2) follows from Propositions 3.4 and 3.5. □

When looking at extensions, it will be important for us to know if nonmaximal free prime open filters exist.

Proposition 3.10. *A space has a nonmaximal free prime open filter if and only if there exists a closed nowhere dense set which is not H-bounded.*

Proof. By Proposition 2.6, a closed nowhere dense set is H-bounded if and only if its complement is in every free open filter. If every closed nowhere dense set is H-bounded, then every free open filter is saturated. Thus all free prime open filters are saturated, and hence they are open ultrafilters by Corollary 3.8. Conversely, if the space has a non-H-bounded, closed nowhere dense set A, then there is a free open filter $\mathscr{F}$ which does not contain the complement of A. By Zorn's lemma, there is a filter maximal with respect to containing $\mathscr{F}$ and excluding the complement of A. Since it contains $\mathscr{F}$, this filter is free and by Proposition 3.4, it is prime. Thus there is a free prime open filter which is not saturated, and, hence, not an open ultrafilter. □

For more information on prime open filters see [3].

4. When S-equivalence and θ-equivalence are equivalent

In the introduction, it is mentioned that the S-equivalence relation always refines the θ-equivalence relation. This section characterizes those spaces for which the S- and θ-equivalence relations are the same. This result is interesting by itself and will also be central to characterizing spaces with exactly one H-closed extension in the next section.

Recall our convention that points in the remainder of a space are identified with their neighborhood trace filters, which are free open filters. Thus the remainder of an extension of a space X will be considered as a subset of $FREE(X)$, the collection of all free open filters on X. Since we are interested only in Hausdorff extensions, not every subset of $FREE(X)$ can be the remainder of some extension. The following definition is needed:

Definition 4.1 [18]. Two open filters, $\mathscr{F}$ and $\mathscr{G}$, on a space X, will be said to be *Hausdorff separated* if there is an $F \in \mathscr{F}$ and a $G \in \mathscr{G}$ such that $F \cap G = \emptyset$. A collection of open filters on X will be said to be a Hausdorff separated subset of $FREE(X)$ if every pair of open filters is Hausdorff separated.

Proposition 4.2. *Let X be a space. If $\mathscr{A}$ is a Hausdorff separated collection of free open filters on X, then $\mathscr{A}$ is the remainder of some Hausdorff extension of X.*

Proof. Let $Y = X \cup \mathscr{A}$, and give Y the simple extension topology. This is a standard construction of an extension and the fact that the added points are Hausdorff separated, open filters implies that the extension is Hausdorff. □

For a space X, we will consider $FREE(X)$ to be a poset by defining $\leqslant$ to be reverse inclusion, that is, $\mathscr{F} \leqslant \mathscr{G}$ if and only if $\mathscr{F} \supseteq \mathscr{G}$. To simplify discussion, a Hausdorff separated subset of $FREE(X)$ can be considered to be an antichain in $(FREE(X), \leqslant)$.

Proposition 4.3. *Let X be a space. Two open filters on X are Hausdorff separated if and only if there is no other open filter on X that contains them both.*

Proposition 4.4. *Let X be a space and $\mathscr{A}$ a subset of $FREE(X)$. The set $\mathscr{A}$ is Hausdorff separated if and only if $\mathscr{A}$ is an antichain in $FREE(X)$.*

Thus every antichain corresponds to an extension of X. The next problem is to determine which antichains correspond to H-closed extensions.

Proposition 4.5 [18]. *The remainder of an H-closed extension of X is a maximal antichain in $FREE(X)$. Conversely every maximal antichain in $FREE(X)$ determines an H-closed extension, unique up to S-equivalence.*

Proof. Let Y be an H-closed extension of X. Since Y is Hausdorff, $Y\backslash X$ is an antichain in *FREE*(X). Suppose $Y\backslash X$ were not maximal. Then there is an open filter $\mathscr{F}$ on X such that $\mathscr{F}$ is not compatible with any element of $Y\backslash X$. Thus for each $p \in Y\backslash X$, there exists a $U \in p$ and $V \in \mathscr{F}$ such that $U \cap V = \emptyset$. Since $o(U) \cap o(V) = \emptyset$, $\{o(U)\colon U \in \mathscr{F}\}$ does not adhere to any point in $Y\backslash X$. Since $\mathscr{F}$ is free, $\{o(U)\colon U \in \mathscr{F}\}$ does not adhere to any point in X. Therefore $\{o(U)\colon U \in \mathscr{F}\}$ is a free open filter on Y, contradicting the assumption that Y is H-closed. Conversely, given a maximal antichain, it can be attached to X and given any topology between the strict and simple extension topologies. (All of the extensions constructed in this way are S-equivalent to one another.) Let Y denote this new extension. Suppose that $\mathscr{F}$ is a free open filter on Y. Then $\mathscr{G} = \{F \cap X\colon F \in \mathscr{F}\}$ is a free open filter on X. Since $Y\backslash X$ is maximal, there is a $p \in Y\backslash X$ such that p meets $\mathscr{G}$ and hence $\mathscr{F}$. Thus every neighborhood of p meets every element of $\mathscr{F}$. It follows that $p \in \operatorname{ad}(\mathscr{F}) = \emptyset$, which is a contradiction. Thus Y is an H-closed extension. □

Note that the maximal antichain corresponding to the S-equivalence class containing σX and κX is the set of all free open ultrafilters on X which will be denoted by *MAXFREE*(X) or by $\sigma X\backslash X$ if we are considering *MAXFREE*(X) as a subspace of σX.

Next let us turn our attention to θ-equivalence. See [18] for more information. Each open filter on a space has a set of open ultrafilters which contain it. If $\mathscr{F}$ is a free open filter on X, let $P(\mathscr{F})$ denote the set of all open ultrafilters which contain $\mathscr{F}$. If $\mathscr{A}$ is a maximal antichain in *FREE*(X), then $\{P(\mathscr{F})\colon \mathscr{F} \in \mathscr{A}\}$ is a partition of *MAXFREE*(X). The next proposition examines these partition elements.

Proposition 4.6 [18]. *If $\mathscr{F}$ is a free open filter on X, then $P(\mathscr{F})$ is a compact subset of $\sigma X\backslash X$. Conversely, if A is a compact subset of $\sigma X\backslash X$, then $\bigcap A$ is a free open filter and $P(\bigcap A) = A$. (Note that $\bigcap A$ is the intersection of open filters and is, therefore, an open filter.)*

Observe that if Y is an extension of X, $p \in Y\backslash X$, and f_Y is the Katětov map, then $P(O^p) = f_Y^{\leftarrow}(p)$.

A consequence of Proposition 4.6 is the following very useful result.

Proposition 4.7 [18]. *Each H-closed extension determines a partition of $\sigma X\backslash X$ into compact sets (in the topology of $\sigma X\backslash X$). Conversely each partition of $\sigma X\backslash X$ into compact sets determines an H-closed extension which is unique up to θ-equivalence.*

Proposition 4.7 implies that every partition of $\sigma X\backslash X$ into compact sets determines a θ-equivalence class and conversely every θ-equivalence class determines a partition of $\sigma X\backslash X$ into compact sets.

The proof of Proposition 4.7 is sketched below for the ideas in it will be useful to us later.

Proof. It was shown above that an H-closed extension of X determines a maximal antichain in $FREE(X)$, which induces a partition of $\sigma X \backslash X$ into sets, which Proposition 4.6 shows are compact. Conversely, the collection of the intersections of the compact partition elements forms a maximal antichain in $FREE(X)$, which can be attached to X to yield an H-closed extension. If Y and Z are θ-equivalent, then there is a θ-homeomorphism between them which keeps X fixed. Thus the partitions of $\kappa X \backslash X$ (and hence $\sigma X \backslash X$) induced by the inverse images of points by the Katětov maps, $\{f_Y^{\leftarrow}(p)\colon p \in Y \backslash X\}$ and $\{f_Z^{\leftarrow}(p)\colon p \in Z \backslash X\}$, are equivalent. If Z is in a different θ-equivalence class then the partitions obtained by taking inverse images of points under the Katětov maps will be different. □

Two S-equivalence classes refine the same θ-equivalence class if and only if they induce the same partition of $\sigma X \backslash X$. It is natural to ask under what conditions a partition of $\sigma X \backslash X$ can be induced by two distinct antichains in $FREE(X)$. One antichain can always be formed by intersecting the partition elements. Its elements (open filters) are intersections of open ultrafilters, and hence saturated. Not all open filters are the intersections of open ultrafilters, but all open filters are the intersection of the prime open filters containing them. A free open filter is the intersection of all free prime open filters containing it. In particular, distinct maximal antichains in $FREE(X)$ induce distinct partitions of $\sigma X \backslash X$ if and only if there are no nonmaximal free prime open filters on X. From this discussion we conclude:

Theorem 4.8. *The following are equivalent for a space X:*
(1) *The S- and θ-equivalence classes are the same;*
(2) *there are no nonmaximal free prime open filters;*
(3) *every closed nowhere dense set is H-bounded.*

A reasonable question is whether the property that all θ-equivalence classes have no strictly finer S-refinement is equivalent to the property that there exists a θ-equivalence class which has no strictly finer S-refinement. The answer depends on which θ-equivalence class is examined. By Theorem 5.5 below, we see that if the θ-equivalence class containing the one-point H-closed extensions (the bottom-most class) consists of only one point and hence has no S-refinement, then all the θ-equivalence classes have no strictly finer S-refinement. On the other hand by results in [13], the real line has the property that the θ-equivalence class containing the Katětov extension (the top-most class) has no strictly finer S-refinement, but the θ-equivalence class containing the one-point H-closed extensions contains infinitely many S-equivalence classes.

5. A characterization of those X for which $|H(X)|=1$

Theorem 4.8 will be useful in answering Porter's question about the existence of spaces with exactly one H-closed extension. Two preliminaries are needed.

Proposition 5.1 [5]. *The extensions σX and κX are equivalent if and only if $|\kappa X \backslash X| < \omega$.*

Definition 5.2. A space X is *almost H-closed* if and only if $|\kappa X \backslash X| \leqslant 1$ (i.e., if and only if X has at most one free open ultrafilter).

The notion of almost H-closed is analogous to almost compactness. More information on both almost H-closed and almost compactness can be found in [21]. Almost H-closed spaces have the following well-known internal characterization.

Proposition 5.3. *A space X is almost H-closed if and only if for each pair of disjoint open sets, U and V, either $\mathrm{cl}_X U$ or $\mathrm{cl}_X V$ is H-closed.*

Theorem 5.4. *For a non-H-closed space X the following are equivalent:*

(1) $|H(X)|=1$;

(2) *X is almost H-closed and every closed nowhere dense set is H-bounded;*

(3) *X is almost H-closed and every closed nowhere dense set is a subset of an H-closed set;*

(4) *X has a unique free prime open filter;*

(5) *X has a unique free open filter;*

(6) *X has a unique extension.*

Proof. (1) *implies* (2). The equality $|H(X)|=1$ trivially implies that the θ- and S-equivalence relations are the same, so by Theorem 4.8 every closed nowhere dense set is H-bounded. Also $|H(X)|=1$ implies $\sigma X \equiv_X \kappa X$; hence by Proposition 5.1, $|\sigma X \backslash X| < \omega$. By Proposition 4.7, every partition of $\sigma X \backslash X$ into compact sets yields at least one distinct H-closed extension. Since $\sigma X \backslash X$ is finite, every partition of it is a partition into compact sets. Therefore $|\kappa X \backslash X| \leqslant 1$ so X is almost H-closed.

(2) *implies* (3). This follows from Proposition 2.5 and the fact that an almost H-closed space is locally H-closed.

(3) *implies* (4). By Definition 5.2, X has a unique free open ultrafilter $\mathscr{U}$. Let $\mathscr{F}$ be a free prime open filter. Since every closed nowhere dense set is a subset of an H-closed set, Proposition 2.5 implies that every closed nowhere dense set is H-bounded. Hence $\mathscr{F}$ is an open ultrafilter by Proposition 3.10. Hence $\mathscr{F}=\mathscr{U}$ and (4) follows.

(4) *implies* (5). Since every free open filter is the intersection of the free prime

open filters containing it, there can be only one free open filter if there is only one free prime open filter.

(5) *implies* (6). Since X is not H-closed, it has at least one extension, namely κX. Suppose Y and Z are both Hausdorff extensions of X. Since there is only one free open filter on X, both Y and Z must be one-point extensions, and both must be in the same S-equivalence class. It is easy to verify that for a finite point extension, the simple and strict extension topologies are the same. Thus the S-equivalence class of a finite point extension consists of one element. Therefore $Y \equiv_X Z$.

(6) *implies* (1). If X is non-H-closed then it has at least one H-closed extension, namely the Katětov extension. If X has exactly one extension, then the extension must be H-closed. Conversely if it has more than one H-closed extension then trivially it must have more than one extension. □

We conclude this section by comparing Theorem 5.4 with Porter's theorem characterizing spaces with exactly one one-point H-closed extension.

Theorem 5.5 [16]. *A non-H-closed, locally H-closed space has exactly one one-point H-closed extension if and only if every closed nowhere dense set is the subset of an H-closed set.*

Clearly Theorem 5.5 is a restatement of the equivalence of (1) and (3) in Theorem 5.4.

6. Examples and applications

The goal of this section is to exhibit a space with exactly one H-closed extension. Essentially the same construction will be used to construct two spaces: one with a dense set of isolated points and one with no isolated points. We will conclude by looking at several applications of these spaces.

One example is [20, Example 8] and the other is essentially the same construction with a different base space. The construction involves a technique originally due to Herrlich [8] and is presented here in the form discussed in [21, [7S]]. This technique can be used for any positive integer n, but here only the case $n = 2$ will be considered.

Definition 6.1. Let D be a dense subset of a space X. Define $X(D^2) = (D \times \{0, 1\}) \cup (X \setminus D)$. A set $U \subseteq X(D^2)$ is open in $X(D^2)$ if $U \cap (D \times \{i\})$ is open in $D \times \{i\}$ (with the product topology) for $i \in \{0, 1\}$ and $x \in U \cap (X \setminus D)$ implies that $x \in ((V \cap D) \times \{0, 1\}) \cup (V \cap D) \subseteq U$ for some open set V in X. Essentially we have taken a dense set, doubled it, and made each copy open. Neighborhoods of points in $X \setminus D$ reach to both copies of the dense set.

Proposition 6.2 (see [21, [7S]]). *Let D be a dense subset of a space X.*

(1) *If X is H-closed, so is $X(D^2)$.*

(2) *If X is semiregular, then $X(D^2)$ is semiregular also.*

(3) *Thus if X is minimal Hausdorff, so is $X(D^2)$.*

The following is an example of a space with exactly one H-closed extension.

Example 6.3. Let $p \in \kappa\omega \setminus \omega$ and $D = \omega \cup \{p\}$. Define $X = \kappa\omega(D^2) \setminus \{(p, 0)\}$.

Proposition 6.4. *The space $X = \kappa\omega(D^2) \setminus \{(p, 0)\}$ has exactly one free open filter. By Proposition* 5.4, *X has exactly one H-closed extension.*

Proof. Since $\kappa\omega(D^2) \setminus (\omega \times \{0\})$ is homeomorphic to $\kappa\omega$, it is H-closed. Thus the complement of $\omega \times \{0\}$ is H-bounded, which implies that $\omega \times \{0\}$ is an element of every free open filter on X. Since $\omega \times \{0\}$ consists only of isolated points, every free open filter on X is saturated. By Corollary 3.8 every prime open filter is an open ultrafilter on X.

We claim that there is exactly one free open ultrafilter on X. Let $\mathscr{F}$ be any free open ultrafilter on X and $q = \{F \cap (\omega \times \{0\})\colon F \in \mathscr{F}\}$. The set q is an ultrafilter on $\omega \times \{0\}$ and can be thought of as a point in $\kappa\omega(D^2) \setminus (\omega \times \{0\})$. Now if $\mathscr{U}$ is the open filter on X generated by q, then it follows easily that $\mathscr{U} = \mathscr{F}$. As a point of $\kappa\omega(D^2) \setminus (\omega \times \{0\})$, q must be the point p for otherwise $q \in \mathrm{ad}(\mathscr{U}) = \mathrm{ad}(\mathscr{F})$, contradicting the assumption that $\mathscr{F}$ is free. We have shown then that the trace onto $\omega \times \{0\}$ of every free open ultrafilter on X is p, and the open filter generated on X by p is the open ultrafilter we started with. Therefore there can be only one free open ultrafilter on X.

It has been shown that X has exactly one free prime open filter. Since every free open filter is the intersection of free prime open filters, this is the only free open filter. $\square$

It is instructive to compare the space of Example 6.3 and Proposition 6.4 with the similar, but simpler space $\kappa\omega \setminus \{p\}$ where $p \in \kappa\omega \setminus \omega$. The space $\kappa\omega \setminus \{p\}$ is almost H-closed, and consequently has a unique free open ultrafilter. The set $(\kappa\omega \setminus \{p\}) \setminus \omega$, however, is a non-H-bounded, closed nowhere dense set so there exists a nonmaximal free prime open filter on $\kappa\omega \setminus \{p\}$. Since attaching the free open ultrafilter to $\kappa\omega \setminus \{p\}$ gives an extension distinct from the extension obtained by attaching a nonmaximal free open filter, $\kappa\omega \setminus \{p\}$ does not have a unique extension.

The next example was the first known since it had been used by Porter and Woods in another context. It is essentially the same as the previous example except that $\mathbb{Q}$ is used instead of ω. It is interesting because it does not have any isolated points and because it makes use of remote points.

Example 6.5 [20]. Let p be a remote point of $\beta\mathbb{Q}$. Then $D = \mathbb{Q} \cup \{p\}$ is dense in $\beta\mathbb{Q}$. Let $p_0 = (p, 0)$. Let $X = \beta\mathbb{Q}(D^2) \setminus \{p_0\}$.

Proposition 6.6. *The space $X = \beta\mathbb{Q}(D^2) \setminus \{p_0\}$ has exactly one free open filter or equivalently exactly one H-closed extension.*

Proof. Let $\mathscr{F}$ be a free open filter on X. $\mathscr{F}$ is an open filter base on $\beta\mathbb{Q}(D^2)$. Let $\mathscr{G}$ be the open filter on $\beta\mathbb{Q}(D^2)$ generated by $\mathscr{F}$. $\beta\mathbb{Q}(D^2)$ is H-closed so $\mathscr{G}$ adheres. $\mathscr{F}$ is free so $\mathscr{G}$ adheres uniquely at p_0. The space $\beta\mathbb{Q}(D^2)$ is minimal Hausdorff so $\mathscr{G}$ converges. This means that $\mathscr{G} \supseteq \mathscr{N}_{p_0}$, which implies that $\mathscr{F}$ contains $O^{p_0} = \{U \cap X\colon U \in \mathscr{N}_{p_0}\}$. We claim that O^p_X is an open ultrafilter on X. If so, then every free open filter on X contains the open ultrafilter O^{p_0}. Thus X has exactly one free open filter which (necessarily) is an open ultrafilter. This implies that X is almost H-closed. This also implies that if A is a closed nowhere dense set then every free open filter misses it and hence A is H-bounded. These are the conditions which imply that X has exactly one H-closed extension.

We finish then by proving the claim. Carlson has shown [1] that the Stone–Čech compactification of a normal space has an open neighborhood filter trace consisting of prime open filters. Since $\mathbb{Q}$ is normal, $\beta\mathbb{Q}$ is prime. Thus $O^p_{\mathbb{Q}} = \{U \cap \mathbb{Q}\colon U \in \mathscr{N}_p\}$ is prime. If U is open in X then $U \cup (X \setminus \mathrm{cl}(U))$ is dense in X. The set $(U \cap \mathbb{Q}) \cup ((X \setminus \mathrm{cl}(U)) \cap \mathbb{Q})$ is dense in $\mathbb{Q}$. Since p is a remote point (saturated trace filter), the set $(U \cap \mathbb{Q}) \cup ((X \setminus \mathrm{cl}(U)) \cap \mathbb{Q})$ is in O^p. The trace filter $O^p_{\mathbb{Q}}$ is prime so either $U \cap \mathbb{Q}$ or $(X \setminus \mathrm{cl}(U)) \cap \mathbb{Q}$ is in $O^p_{\mathbb{Q}}$. But $O^p_{\mathbb{Q}} \subseteq O^p_X$ so either U or $X \setminus \mathrm{cl}\, U$ is in $O^{p_0}_X$. Therefore $O^{p_0}_X$ is an open ultrafilter. $\square$

There are several applications of these examples. In [15] Pelant, Simon, and Vaughan compute the number of free prime closed filters on spaces with different separation axioms. They show that every noncompact completely regular space has at least $\aleph_2$ free prime closed filters, every noncompact Hausdorff space has at least $\aleph_1$ free prime closed filters, but there exists a T_1 noncompact space with exactly one free prime closed filter. They also show that if a space is regular the number of free prime closed filters is the same as the number of free prime open filters. They ask for the smallest number of free prime open filters possible on a noncompact space. The answer is of course zero, and any H-closed, noncompact space (there are many known spaces of this type) is an example. What they probably intended to ask is: What is the smallest number of free prime open filters possible on a non-H-closed space? The above examples give spaces with exactly one free prime open filter. Thus for free prime open filters we have the following proposition:

Proposition 6.7. (1) *A non-H-closed, completely regular space has at least $\aleph_2$ many free open filters.*

(2) *A non-H-closed, regular space has at least $\aleph_1$ many free prime open filters.*

(3) *There exists a Hausdorff space with exactly one free prime open filter.*

It is interesting to note that $\beta\mathbb{Q}\setminus\{p\}$ has at least $\aleph_2$ many free prime open filters, but $\beta\mathbb{Q}(D^2)\setminus\{p\}$ has only one free prime open filter if p is a remote point.

Another application of the above examples is to generate spaces with finite H-closed extensions. The topological sum of two spaces with exactly one H-closed extension each has exactly two H-closed extensions; a two-point and a one-point. Similarly a space with exactly five H-closed extensions can be constructed by taking the topological sum of three spaces each with exactly one extension. It is clear that this method cannot be used to construct a space with n extensions for every positive integer n. It is possible, however, to construct spaces with exactly $p(n)$ H-closed extensions where $p(n)$ is the number of ways a set of size n can be partitioned, which is known as the Bell number. For more information on Bell numbers see the books by Chandler [4] or Lovász [12].

We conclude with some questions.

Question 6.8. Is there a space with exactly three H-closed extensions? Four? Other non-Bell numbers?

Question 6.9. Is there a space with two one-point H-closed extensions and no other H-closed extensions?

Question 6.10. Find necessary and sufficient conditions on a space X such that X has finitely many extensions.

Instead of asking for the existence of a space for a particular lattice structure, we may ask which lattice structures are possible candidates for an H-closed extension lattice.

Question 6.11. Characterize all complete upper semilattices, $\mathscr{L}$, for which there is a space X such that $\mathscr{L} = H(X)$.

I would like to thank Jack Porter for his advice, support, and encouragement on this project.

References

[1] J.W. Carlson, Prime extensions and nearness structures, Comment. Math. Univ. Carolin. 21 (1980) 663–677.
[2] J.W. Carlson, Prime Wallman compactification, Topology Proc. 12 (1987) 217–238.
[3] J.W. Carlson, Balanced filters, Topology Proc. 14 (1989) 25–40.
[4] R. Chandler, Hausdorff Compactifications (Marcel Dekker, New York, 1976).
[5] J. Flachsmeyer, Zur theorie der H-abgeschlossen Erweiterungen, Math. Z. 94 (1966) 349–381.
[6] M. Girou, Locally H-closed spaces, Ph.D. Thesis, University of Missouri, Columbia, MO (1989).
[7] G. Grätzer, Lattice theory: first concepts and distributive lattices (Freeman, San Francisco, CA, 1971).

[8] H. Herrlich, Nicht alle T_2-minimalen Raume sind von 2. Kategorie, Math. Z. 91 (1966) 185.
[9] I. Kaplansky, Commutative Rings (Allyn & Bacon, Newton, MA, 1970).
[10] R.L. Krystock, On H-sets and open filter adherences, Canad. Math. Bull. 31 (1988) 37–44.
[11] P.Th. Lambrinos, Some weaker forms of topological boundedness, Ann. Soc. Sci. Bruxelles Sér. I 90 (1976) 109–124.
[12] L. Lovász, Combinatorial Problems and Exercises (North-Holland, Amsterdam, 1979).
[13] D. Mooney, H-bounded sets, Topology Proc., to appear.
[14] F. Obreanu, Spatti local absolut inchise, An. Acad. Rep. Pop. Romîne Secţ. Ştiinţ. Mat. Fiz. Chem. Ser. A 3 (1950) 375–394.
[15] J. Pelant, P. Simon and J.E. Vaughan, The smallest number of free prime closed filters, Fund. Math. 131 (1988) 215–221.
[16] J.R. Porter, On locally H-closed spaces, Proc. London Math. Soc. (3) 20 (1970) 193–204.
[17] J. Porter and J. Thomas, On H-closed and minimal Hausdorff spaces, Trans. Amer. Math. Soc. 138 (1969) 159–170.
[18] J.R. Porter and C. Votaw, H-closed extensions I, Gen. Topology Appl. 3 (1973) 211–224.
[19] J.R. Porter and C. Votaw, H-closed extensions II, Trans. Amer. Math. Soc. 202 (1975) 193–208.
[20] J.R. Porter and R.G. Woods, Accumulation points of nowhere dense sets in H-closed spaces, Proc. Amer. Math. Soc. 93 (1985) 539–542.
[21] J.R. Porter and R.G. Woods, Extensions and Absolutes of Hausdorff Spaces (Springer, Berlin, 1988).
[22] N.V. Velichko, H-closed topological spaces, Amer. Math. Soc. Transl. 78 (1967) 103–118.
[23] J. Vermeer, Expansions of H-closed spaces, Ph.D. Thesis, Vrije Universiteit, Amsterdam (1983).
[24] J. Vermeer, Closed subspaces of H-closed spaces, Pacific J. Math. 118 (1985) 229–247.