Europ. J. Combinatorics (2002) **23**, 121–129
doi:10.1006/eujc.2001.0511
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A Four-Class Subscheme of the Association Scheme Coming from the Action of $PGL(2, 4^f)$

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Let Ω denote the set of two-element subsets of the projective line $PG(1, q)$, where $q = 2^e$, $e \geq 1$. The character table of the association scheme $\mathfrak{X}(PGL(2, q), \Omega)$ is calculated. Then by using this character table, we prove that the conjectured subscheme of de Caen and van Dam is indeed an association scheme.

1. INTRODUCTION

Let G be a finite group acting transitively on a finite set X. Then G acts naturally on $X \times X$ and the orbits of G acting on $X \times X$ form the relations of an association scheme (these orbits are called the *orbitals*). We denote this association scheme by $\mathfrak{X}(G, X)$. In [3], de Caen and van Dam studied the association scheme coming from the action of the projective general linear group $PGL(2, q)$ on the set of two-element subsets of the projective line $PG(1, q)$. (This action is naturally induced by the action of $PGL(2, q)$ on $PG(1, q)$.) Then they conjectured the existence of an interesting four-class subscheme in the case $q = 4^f$, $f \geq 2$. Theoretically, if we can compute the character table of the association scheme, then by applying a lemma of Bannai [1, Lemma 1], we would be able to determine all subschemes. Fortunately, a method for computing this character table is given in Kwok [5], where he computed the character table of the association scheme $\mathfrak{X}(O(3, q), O(3, q)/O^+(2, q))$ for odd q (this information was communicated to the author by Professor Bannai in his lecture at Kyushu University). In this paper we calculate the character table explicitly and then, using this table, we prove that the conjectured subscheme of de Caen and van Dam is indeed an association scheme.

Throughout this paper, we always assume $q = 2^e$, $e \geq 1$. For each nonzero vector $\binom{x}{y}$ in $\mathbb{F}_q^2$, we denote the projective point in $PG(1, q)$ which contains $\binom{x}{y}$ by $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. Let Ω be the set of two-element subsets of $PG(1, q)$. Notice that since q is even we can uniquely take an element of the special linear group $SL(2, q)$ as a representative of each element in $PGL(2, q) = GL(2, q)/Z(GL(2, q))$, where $Z(GL(2, q))$ is the center of $GL(2, q)$. Therefore in this case $SL(2, q)$ is isomorphic to $PGL(2, q)$ and we may consider the action of $SL(2, q)$ on Ω instead of $PGL(2, q)$. In what follows we let $G = SL(2, q)$ for brevity. Let H be the stabilizer of $\left\{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right\} \in \Omega$ in G, then $\mathfrak{X}(G, \Omega)$ is exactly the same as $\mathfrak{X}(G, G/H)$, and this association scheme is found to be symmetric (see Section 2.2).

Now, let $P = (p_i(j))$ be the character table of $\mathfrak{X}(G, G/H)$, where the (j, i)-entry of P is $p_i(j)$. Then each entry of P is given by

$$\begin{aligned} p_i(j) &= \frac{1}{|H|} \sum_{x \in Hg_iH} \chi_j(x) \\ &= \frac{1}{|H|} \sum_k |Hg_iH \cap C_k| \chi_j(c_k) \qquad (1) \end{aligned}$$

where $1_H^G = \chi_0 + \chi_1 + \cdots + \chi_d$ is the decomposition of the permutation character into irreducible characters, $\{Hg_iH \mid 0 \leq i \leq d\}$ is the set of double cosets of H in G, $\{C_k\}$ is the

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set of conjugacy classes of G and c_k is a class representative of C_k (cf. Bannai-Ito [2, p. 174 Corollary 11.7]).

REMARK. $\mathfrak{X}(G, G/H)$ is commutative if and only if 1_H^G is multiplicity free. Hence χ_0, $\chi_1, \dots, \chi_d$ are distinct irreducible characters.

REMARK. There is a natural one-to-one correspondence between the relations of $\mathfrak{X}(G, G/H)$ and the set of double cosets of H in G. Namely the orbital which contains $(g_1H, g_2H) \in G/H \times G/H$ corresponds to $Hg_1^{-1}g_2H$.

Thus in the same way as Kwok [5], we determine

(i) the set of conjugacy classes and the group character table of G,
(ii) the set of double cosets of H in G,
(iii) the order of the intersection of each conjugacy class and each double coset,
(iv) the decomposition of the permutation character 1_H^G into irreducible characters.

2. COMPUTATIONS

2.1. Conjugacy classes and group character table of $G = SL(2, q)$. These facts are found in Steinberg [6]. Let ρ and σ be primitive elements of $\mathbb{F}_q$ and $\mathbb{F}_{q^2}$ respectively, where $\rho = \sigma^{q+1}$. Then we have the conjugacy classes of G as follows:

$$I = \left\{\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right\} : \text{size } 1$$
$$U = \left\{g \in G \;\middle|\; g \sim \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\right\} : \text{size } q^2 - 1$$
$$T_k = \left\{g \in G \;\middle|\; g \sim \begin{pmatrix} \rho^k & \\ & \rho^{-k} \end{pmatrix}\right\} \ (k = 1, 2, \dots, \tfrac{q}{2} - 1) \quad : \text{size } q(q+1)$$
$$S_l = \left\{g \in G \;\middle|\; g \sim \begin{pmatrix} \sigma^{(q-1)l} & \\ & \sigma^{-(q-1)l} \end{pmatrix}\right\} \ (l = 1, 2, \dots, \tfrac{q}{2}) \quad : \text{size } q(q-1).$$

The character table of G is given as follows.

	χ_1	χ_q	$\chi_{q+1}^{(n)}$ $n = 1, 2, \dots, \frac{q}{2} - 1$	$\chi_{q-1}^{(m)}$ $m = 1, 2, \dots, \frac{q}{2}$
I	1	q	$q+1$	$q-1$
U	1	0	1	-1
T_k $k = 1, 2, \dots, \frac{q}{2} - 1$	1	1	$\varepsilon^{nk} + \varepsilon^{-nk}$	0
S_l $l = 1, 2, \dots, \frac{q}{2}$	1	-1	0	$-(\tau^{ml} + \tau^{-ml})$

where ε and τ are primitive $(q-1)$th and $(q+1)$th roots of unity in the complex number field, respectively.

2.2. Double cosets of H in G. Clearly the stabilizer of $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ in G is given by

$$H = \left\{ \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix}, \begin{pmatrix} & z \\ z^{-1} & \end{pmatrix} \,\middle|\, z \in \mathbb{F}_q^* \right\}.$$

The double cosets of H in G are in one-to-one correspondence with the orbits of H acting on Ω, that is, for each H-orbit Λ in Ω, the corresponding double coset consists of the elements in G which map $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ into Λ.

Orbits	Corresponding double cosets	Double coset size
$\Lambda_0 = \left\{ \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} \right\}$	$H_0 = H$	$2(q-1)$
$\Lambda'_0 = \left\{ \begin{matrix} \left\{\begin{bmatrix}1\\z\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}, \\ \left\{\begin{bmatrix}1\\z\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\} \end{matrix} \,\middle|\, z \in \mathbb{F}_q^* \right\}$	$H'_0 = \left\{ \begin{matrix} \begin{pmatrix} \alpha & \\ \beta & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} & \alpha \\ \alpha^{-1} & \beta \end{pmatrix}, \\ \begin{pmatrix} \beta & \alpha \\ \alpha^{-1} & \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ & \alpha^{-1} \end{pmatrix} \end{matrix} \,\middle|\, \alpha, \beta \in \mathbb{F}_q^* \right\}$	$4(q-1)^2$
$\Lambda_t = \left\{ \left\{\begin{bmatrix}1\\\rho^a\end{bmatrix}, \begin{bmatrix}1\\\rho^b\end{bmatrix}\right\} \,\middle|\, \begin{matrix} a-b \equiv \pm t \\ (\text{mod } q-1) \end{matrix} \right\}$ $(t = 1, 2, \ldots, \frac{q}{2} - 1)$	$H_t = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \,\middle|\, \begin{matrix} \alpha, \beta, \gamma, \delta \in \mathbb{F}_q^*, \\ \alpha^{-1}\delta^{-1}\beta\gamma = \rho^{\pm t} \end{matrix} \right\}$	$2(q-1)^2$

Obviously $g^{-1} \in HgH$ for all $g \in G$. This implies that $\mathfrak{X}(G, G/H)$ is a symmetric association scheme.

2.3. Order of the intersection of each conjugacy class and each double coset. In order to describe these numbers, we define

$$\begin{aligned} \varphi_k &= \rho^k + \rho^{-k} \qquad (k = 1, 2, \ldots, \tfrac{q}{2} - 1), \\ \varphi'_l &= \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\sigma^{l(q-1)}) = \sigma^{l(q-1)} + \sigma^{-l(q-1)} \qquad (l = 1, 2, \ldots, \tfrac{q}{2}), \\ \psi_t &= (1 + \rho^t)^{-1} \qquad (t = 1, 2, \ldots, q-2), \end{aligned}$$

where $\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}$ is the trace map from $\mathbb{F}_{q^2}$ onto $\mathbb{F}_q$. The elements $\psi_1, \psi_2, \ldots, \psi_{q-2}$ are distinct, so that we have

$$\mathbb{F}_q \backslash \{0, 1\} = \{\psi_1, \psi_2, \ldots, \psi_{q-2}\}.$$

Also it follows from the definition of ψ_t that

$$\begin{aligned} \psi_t + \psi_{-t} &= (1+\rho^t)^{-1}(1+\rho^{-t})^{-1}\{(1+\rho^t) + (1+\rho^{-t})\} \\ &= (1+\rho^t)^{-1}(1+\rho^{-t})^{-1}(1+\rho^t)(1+\rho^{-t}) \\ &= 1 \end{aligned} \tag{2}$$

for $t = 1, 2, \ldots, q-2$. Notice that the characteristic polynomial of an element in T_k (respectively S_l) is given by $x^2 + \varphi_k x + 1$ ($x^2 + \varphi'_l x + 1$). Thus for each element $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in G, g belongs to T_k (S_l) if and only if $\alpha + \delta = \varphi_k$ ($\alpha + \delta = \varphi'_l$). This also implies that φ_k ($k = 1, 2, \ldots, \frac{q}{2} - 1$), φ'_l ($l = 1, 2, \ldots, \frac{q}{2}$) are distinct elements in $\mathbb{F}_q^*$, that is,

$$\mathbb{F}_q^* = \{\varphi_1, \varphi_2, \ldots, \varphi_{\frac{q}{2}-1}, \varphi'_1, \varphi'_2, \ldots, \varphi'_{\frac{q}{2}}\}.$$

For the proof of the following lemma, we refer to Hirschfeld [4, p. 3].

LEMMA 2.1. *Let a and b be elements of $\mathbb{F}_q^*$. Then the equation $x^2 + ax + b = 0$ has two (distinct) solutions in $\mathbb{F}_q$ if and only if $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(a^{-2}b) = 0$.*

As an immediate consequence of Lemma 2.1, we obtain

$$\{\varphi_1, \varphi_2, \dots, \varphi_{\frac{q}{2}-1}\} = \{z \in \mathbb{F}_q^* \mid \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(z^{-1}) = 0\}, \tag{3}$$

and

$$\{\varphi'_1, \varphi'_2, \dots, \varphi'_{\frac{q}{2}}\} = \{z \in \mathbb{F}_q^* \mid \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(z^{-1}) = 1\}. \tag{4}$$

Now we can find the order of the intersection of each conjugacy class of G and each double coset of H in G as follows.

For example, suppose that the conjugacy class is S_l and the double coset is H_t, $t \neq 0$. If an element $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$ belongs to $S_l \cap H_t$, then g must satisfy the following equations:

$$\alpha\delta + \beta\gamma = 1,$$
$$\alpha + \delta = \varphi'_l,$$

and in addition, either one of the following two equations:

$$\beta\gamma = \rho^t \alpha\delta, \tag{5a}$$
$$\beta\gamma = \rho^{-t} \alpha\delta. \tag{5b}$$

If g satisfies (5a), then α is a solution of the quadratic equation

$$x^2 + \varphi'_l x + \psi_t = 0,$$

and for each α there are exactly $q-1$ choices for the other entries β, γ and δ. By Lemma 2.1, the number of solutions of this quadratic equation is equal to

$$1 + (-1)^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\varphi'^{-2}_l \psi_t)},$$

that is, the number of elements $g \in S_l$ which satisfy (5a) is given by

$$(q-1)\{1 + (-1)^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\varphi'^{-2}_l \psi_t)}\}.$$

Similarly, the number of elements $g \in S_l$ which satisfy (5b) is given by

$$(q-1)\{1 + (-1)^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\varphi'^{-2}_l \psi_{-t})}\}.$$

Therefore we have

$$|S_l \cap H_t| = (q-1)\{2 + (-1)^{T(\varphi'_l, \psi_t)} + (-1)^{T(\varphi'_l, \psi_{-t})}\},$$

where

$$T(a, b) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(a^{-2}b) \tag{6}$$

for $a, b \in \mathbb{F}_q^*$. Furthermore it follows from (2) and (4) that

$$\begin{aligned} T(\varphi'_l, \psi_t) + T(\varphi'_l, \psi_{-t}) &= \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\varphi'^{-2}_l(\psi_t + \psi_{-t})) \\ &= \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\varphi'^{-2}_l) \\ &= 1, \end{aligned} \tag{7}$$

so that

$$|S_l \cap H_t| = 2(q-1).$$

After doing similar calculations, we obtain the following table.

	$H_0 = H$	H_0'	H_t $t = 1, 2, \dots, \frac{q}{2} - 1$
I	1	0	0
U	$q-1$	$2(q-1)$	$2(q-1)$
T_k $k = 1, 2, \dots, \frac{q}{2} - 1$	2	$6(q-1)$	$2(q-1)\left\{1 + (-1)^{T(\varphi_k, \psi_t)}\right\}$
S_l $l = 1, 2, \dots, \frac{q}{2}$	0	$2(q-1)$	$2(q-1)$

It follows from the above table that

$$|H_t| = 2(q-1)^2 = 2(q-1) + \sum_{k=1}^{\frac{q}{2}-1} 2(q-1)\{1 + (-1)^{T(\varphi_k, \psi_t)}\} + \frac{q}{2} \cdot 2(q-1)$$

for $t = 1, 2, \dots, \frac{q}{2} - 1$, from which we obtain

$$\sum_{k=1}^{\frac{q}{2}-1} (-1)^{T(\varphi_k, \psi_t)} = -1. \tag{8}$$

Similarly

$$\sum_{t=1}^{\frac{q}{2}-1} (-1)^{T(\varphi_k, \psi_t)} = -1 \tag{9}$$

for $k = 1, 2, \dots, \frac{q}{2} - 1$.

2.4. Decomposition of 1_H^G into irreducible characters. In order to find the irreducible characters which appear in the decomposition of 1_H^G, we need to compute the multiplicity $\langle 1_H^G, \chi \rangle_G$ of each irreducible character χ of G in 1_H^G. By Frobenius reciprocity we have

$$\begin{aligned} \langle 1_H^G, \chi \rangle_G &= \langle 1_H, \chi \downarrow_H \rangle_H \\ &= \frac{1}{|H|} \sum_k |H \cap C_k| \chi(c_k) \end{aligned} \tag{10}$$

where $\{C_k\}$ is the set of conjugacy classes of G, c_k is a class representative of C_k, and $\chi \downarrow_H$ is the restriction of χ to H. The numbers $|H \cap C_k|$ are given in the previous subsection, and we can easily verify that

$$1_H^G = \chi_1 + \chi_q + \chi_{q+1}^{(1)} + \dots + \chi_{q+1}^{(\frac{q}{2}-1)}. \tag{11}$$

3. Character Table of $\mathfrak{X}(G, \Omega)$

Applying the results obtained in the previous section to (1), we have the character table P of $\mathfrak{X}(G, \Omega)$.

	$H_0 = H$	H_0'	H_t $t = 1, 2, \ldots, \frac{q}{2} - 1$
χ_1	1	$2(q-1)$	$q-1$
χ_q	1	$q-3$	-2
$\chi_{q+1}^{(n)}$ $n = 1, 2, \ldots, \frac{q}{2} - 1$	1	-2	$\sum_{k=1}^{\frac{q}{2}-1} (-1)^{T(\varphi_k, \psi_t)} (\varepsilon^{nk} + \varepsilon^{-nk})$

4. Construction of a Four-class Subscheme of $\mathfrak{X}(G, \Omega)$

As mentioned before, in this section we verify that the putative subscheme conjectured in [3] is indeed a subscheme of $\mathfrak{X}(G, \Omega)$.

From now on, we assume $q = 4^f$, $f \geq 2$. Then, in our terminology the conjecture in [3] is translated into the following form.

THEOREM 4.1. *Partition the set of double cosets of H in G into five subsets as follows:*

$$\{H_0\},\ \{H_0'\},\ \{H_t \mid (2^f + 1)|t\},\ \{H_t \mid (2^f - 1)|t\},\ \{H_t \mid (2^f \pm 1) \nmid t\}.$$

Then this partition forms a subscheme of $\mathfrak{X}(G, \Omega)$.

PROOF OF THEOREM 4.1. We also partition the set of the irreducible characters which appear in the decomposition of 1_H^G into five subsets:

$$\{\chi_1\},\ \{\chi_q\},\ \{\chi_{q+1}^{(n)} \mid (2^f + 1)|n\},\ \{\chi_{q+1}^{(n)} \mid (2^f - 1)|n\},\ \{\chi_{q+1}^{(n)} \mid (2^f \pm 1) \nmid n\}.$$

Then, from these two partitions, we have $5 \times 5 = 25$ submatrices of the character table P of $\mathfrak{X}(G, \Omega)$. The key condition which we need to check is:

— *each submatrix of P defined above has constant row sum.*

The theorem is then an immediate consequence of a lemma in Bannai [1, Lemma 1].

From now on, in order to verify this condition we observe the values of $T(\varphi_k, \psi_t)$ in detail. For convenience in the discussion, we let

$$\boldsymbol{t_k} = \left(T(\varphi_k, \psi_1), \ldots, T(\varphi_k, \psi_{\frac{4^f}{2}-1})\right), \qquad k = 1, 2, \ldots, \frac{4^f}{2} - 1.$$

First of all, notice that

$$\{\varphi_k^{-2}\psi_t \mid t = 1, 2, \ldots, 4^f - 2\} = \mathbb{F}_{4^f} \setminus \{0, \varphi_k^{-2}\}.$$

There are $\frac{4^f}{2}$ elements α in $\mathbb{F}_{4^f}$ such that $\mathrm{Tr}_{\mathbb{F}_{4^f}/\mathbb{F}_2}(\alpha) = 0$, and in particular $\mathrm{Tr}_{\mathbb{F}_{4^f}/\mathbb{F}_2}(\varphi_k^{-2}) = 0$ by (3). Since in the same way as (7) we have

$$T(\varphi_k, \psi_t) = T(\varphi_k, \psi_{-t}), \tag{12}$$

hence we conclude that for each $k = 1, 2, \ldots, \frac{4^f}{2} - 1$, $\boldsymbol{t_k}$ has $(4^{f-1} - 1)$ zeros.

Next it follows from (3) and (4) that

$$\{\varphi_{u(2^f+1)} \mid u = 1, 2, \ldots, 2^{f-1} - 1\} = \{z \in \mathbb{F}_{2^f}^* \mid \operatorname{Tr}_{\mathbb{F}_{2^f}/\mathbb{F}_2}(z^{-1}) = 0\}, \tag{13}$$

and

$$\{\varphi_{v(2^f-1)} \mid v = 1, 2, \ldots, 2^{f-1}\} = \{z \in \mathbb{F}_{2^f}^* \mid \operatorname{Tr}_{\mathbb{F}_{2^f}/\mathbb{F}_2}(z^{-1}) = 1\}. \tag{14}$$

We check the values of $T(\varphi_k, \psi_t)$ in each of the following six cases.

(i) *Case* $(2^f + 1)|k,\ (2^f + 1)|t$: in this case, both φ_k and ψ_t belong to $\mathbb{F}_{2^f}$. Thus $T(\varphi_k, \psi_t) = \operatorname{Tr}_{\mathbb{F}_{4^f}/\mathbb{F}_2}(\varphi_k^{-2}\psi_t) = 0$.

(ii) *Case* $(2^f + 1)|k,\ (2^f - 1)|t$: let $k = u(2^f + 1)$ and let $t = w(2^f - 1)$. Then by (2) we have

$$(\psi_{w(2^f-1)})^{2^f} = (1 + \rho^{w(4^f-2^f)})^{-1} = \psi_{-w(2^f-1)} = \psi_{w(2^f-1)} + 1. \tag{15}$$

Hence it follows from (13) that

$$\begin{aligned} T(\varphi_k, \psi_t) &= \operatorname{Tr}_{\mathbb{F}_{2^f}/\mathbb{F}_2}\left(\varphi_k^{-2}(\psi_t + \psi_t^{2^f})\right) \\ &= \operatorname{Tr}_{\mathbb{F}_{2^f}/\mathbb{F}_2}(\varphi_{u(2^f+1)}^{-2}) \\ &= 0. \end{aligned}$$

(iii) *Case* $(2^f - 1)|k,\ (2^f + 1)|t : T(\varphi_k, \psi_t) = 0$.

(iv) *Case* $(2^f - 1)|k,\ (2^f - 1)|t : T(\varphi_k, \psi_t) = 1$.

(v) *Case* $(2^f \pm 1) \nmid k,\ (2^f + 1)|t$: let $t = w(2^f + 1)$. In this case $\psi_t \in \mathbb{F}_{2^f}$, $\varphi_k \notin \mathbb{F}_{2^f}$ and

$$T(\varphi_k, \psi_t) = \operatorname{Tr}_{\mathbb{F}_{2^f}/\mathbb{F}_2}\left((\varphi_k^{-2} + (\varphi_k^{-2})^{2^f})\psi_t\right).$$

Also we have

$$\left\{ (\varphi_k^{-2} + (\varphi_k^{-2})^{2^f})\psi_{w(2^f+1)} \;\middle|\; w = 1, 2, \ldots, 2^f - 2 \right\} = \mathbb{F}_{2^f} \backslash \left\{0,\ \varphi_k^{-2} + (\varphi_k^{-2})^{2^f}\right\}.$$

There are 2^{f-1} elements α in $\mathbb{F}_{2^f}$ such that $\operatorname{Tr}_{\mathbb{F}_{2^f}/\mathbb{F}_2}(\alpha) = 0$ and by (3) we have

$$\begin{aligned} \operatorname{Tr}_{\mathbb{F}_{2^f}/\mathbb{F}_2}\left(\varphi_k^{-2} + (\varphi_k^{-2})^{2^f}\right) &= \operatorname{Tr}_{\mathbb{F}_{4^f}/\mathbb{F}_2}(\varphi_k^{-2}) \\ &= 0. \end{aligned}$$

Therefore (12) implies that $\boldsymbol{t_k}$ has $(2^{f-2} - 1)$ zeros in the positions $(2^f + 1)|t$.

(vi) *Case* $(2^f \pm 1) \nmid k,\ (2^f - 1)|t$: let $t = w(2^f - 1)$. Then it follows from (15) that $\{\psi_{w(2^f-1)} \mid w = 1, 2, \ldots, 2^f\}$ is the set of roots of the equation

$$x^{2^f} + x + 1 = 0,$$

which implies that for each $w = 1, 2, \ldots, 2^f$ there is a unique element z in $\mathbb{F}_{2^f}$ such that

$$\psi_{w(2^f-1)} = \psi_{2^f-1} + z. \tag{16}$$

Then since

$$T(\varphi_k, \psi_{w(2^f-1)}) = T(\varphi_k, \psi_{2^f-1}) + \operatorname{Tr}_{\mathbb{F}_{2^f}/\mathbb{F}_2}\left((\varphi_k^{-2} + (\varphi_k^{-2})^{2^f})\, z\right),$$

there are 2^{f-1} elements in $\{\psi_{w(2^f-1)} \mid w = 1, 2, \ldots, 2^f\}$ with $T(\varphi_k, \psi_{w(2^f-1)}) = T(\varphi_k, \psi_{2^f-1})$, and $T(\varphi_k, \psi_{w(2^f-1)}) = T(\varphi_k, \psi_{2^f-1}) + 1$ for the other 2^{f-1} elements. Thus there are exactly 2^{f-1} elements in $\{\psi_{w(2^f-1)} \mid w = 1, 2, \ldots, 2^f\}$ with $T(\varphi_k, \psi_{w(2^f-1)}) = 0$, and it follows from (12) that $\boldsymbol{t_k}$ has 2^{f-2} zeros in the positions $(2^f - 1)|t$.

The above observations give us the following:

$$\sum_{\substack{1 \le t \le \frac{4^f}{2}-1 \\ (2^f+1)|t}} (-1)^{T(\varphi_k,\psi_t)} = \begin{cases} 2^{f-1} - 1 & \text{if } (2^f+1)|k \\ 2^{f-1} - 1 & \text{if } (2^f-1)|k \\ -1 & \text{otherwise,} \end{cases} \tag{17}$$

$$\sum_{\substack{1 \le t \le \frac{4^f}{2}-1 \\ (2^f-1)|t}} (-1)^{T(\varphi_k,\psi_t)} = \begin{cases} 2^{f-1} & \text{if } (2^f+1)|k \\ -2^{f-1} & \text{if } (2^f-1)|k \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Finally it follows from (9) that

$$\sum_{\substack{1 \le t \le \frac{4^f}{2}-1 \\ (2^f\pm1)\nmid t}} (-1)^{T(\varphi_k,\psi_t)} = \begin{cases} -2^{f} & \text{if } (2^f+1)|k \\ 0 & \text{if } (2^f-1)|k \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

With these preparations, the previous condition on constant row sums is verified by direct calculations as follows. Thus we have finished the proof of Theorem 4.1.

	H_0	H_0'	H_t, $(2^f+1)\|t$	H_t, $(2^f-1)\|t$	H_t, $(2^f\pm1)\nmid t$
χ_1	1	$2(4^f-1)$	$(2^{f-1}-1)(4^f-1)$	$2^{f-1}(4^f-1)$	$2^f(2^{f-1}-1)(4^f-1)$
χ_q	1	4^f-3	$2-2^f$	-2^f	$-2^f(2^f-2)$
$\chi_{q+1}^{(n)}$, $(2^f+1)\|n$	1	-2	$2^{f-1}(2^f-1)+1$	$-2^{f-1}(2^f+1)$	2^f
$\chi_{q+1}^{(n)}$, $(2^f-1)\|n$	1	-2	$(2^{f-1}-1)(2^f-1)$	$2^{f-1}(2^f-1)$	$-2^f(2^f-2)$
$\chi_{q+1}^{(n)}$, $(2^f\pm1)\nmid n$	1	-2	$1-2^f$	0	2^f

□

REMARK. The above table gives the character table of this four-class subscheme of $\mathfrak{X}(G, \Omega)$, which is exactly the same as that conjectured in [3].

ACKNOWLEDGEMENTS

The author would like to thank Professor Eiichi Bannai for his suggestion of this problem. I am also indebted to the referee, whose detailed report has improved the presentation of the paper.

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Received 10 March 2000 and accepted 28 March 2001

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