List edge and list total colorings of planar graphs without 4-cycles

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Abstract

Let \( G \) be a planar graph with maximum degree \( \Delta \) such that \( G \) has no cycle of length from 4 to \( k \), where \( k \geq 4 \). Then the list chromatic index \( \chi'_l(G) = \Delta \) and the list total chromatic number \( \chi''_l(G) = \Delta + 1 \) if \( (\Delta, k) \in \{(7, 4), (6, 5), (5, 8)\} \). Furthermore, \( \chi'_l(G) = \Delta \) if \( (\Delta, k) \in \{(4, 14)\} \).

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1. Introduction

We consider finite simple graphs. Any undefined notation follows that of Bondy and Murty [1]. We use \( V(G), E(G), \delta(G) \) and \( \Delta(G) \) to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph \( G \), respectively. Let \( d(v) \) denote the degree of vertex \( v \).

The mapping \( L \) is said to be a total assignment for the graph \( G \) if it assigns a list \( L(x) \) of possible colors to each element \( x \in V \cup E \). If \( G \) has a total coloring \( \phi \) such that \( \phi(x) \in L(x) \) for all \( x \in V \cup E \), and no two adjacent or incident elements receive the same color, then we say that \( G \) is total-L-colorable. Let \( f : V \cup E \rightarrow \mathbb{N} \) be a function into the positive integers. We say that \( G \) is total-\( f \)-choosable if it is total-L-colorable for every total assignment \( L \) satisfying \( |L(x)| = f(x) \) for all elements \( x \in V \cup E \). The list total chromatic number \( \chi''_l(G) \) of \( G \) is the smallest integer \( k \) such that \( G \) is totally \( f \)-choosable when \( f(x) = k \) for each \( x \in V \cup E \). The list chromatic number \( \chi_l(G) \) of \( G \) and the list edge chromatic number (or list chromatic index) \( \chi'_l(G) \) of \( G \) are defined similarly in terms of coloring vertices alone, or edges alone, respectively; and so are the concepts of vertex-\( f \)-choosability and edge-\( f \)-choosability.

The ordinary vertex, edge and total chromatic number of \( G \) are denoted by \( \chi(G) \), \( \chi'(G) \) and \( \chi''(G) \), respectively.

Part (a) of the following conjecture was formulated independently by Vizing, by Gupta, by Albertson and Collins, and by Bollobás and Harris (see [4] or [6]), and it is well known as the List Coloring Conjecture and part (b) was formulated by Borodin et al. [2].

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Conjecture A. If \( G \) is a multigraph, then
(a) \( \chi'_1(G) = \chi'(G) \), (b) \( \chi''_1(G) = \chi''(G) \).

List Coloring Conjecture has been proved for a few special cases, such as bipartite multigraphs [3], complete graphs of odd order [5], multicircuits [11], line-perfect multigraphs [8], graphs with \( \Delta \geq 12 \) which can be embedded in a surface of nonnegative characteristic [2] and outerplanar graphs [9].

Part (b) of Conjecture A has been proved for outerplanar graphs [9], graphs with \( \Delta \geq 12 \) which can be embedded in a surface of nonnegative characteristic [2].

For planar graphs, we also obtained several related results of Conjecture A by adding grith restrictions [2]. Note that the added grith requirement prohibits the appearance of triangles. The forbidden cycle or the grith restriction plays an important role in considering list coloring planar graphs. For example, Kratochvil and Tuza showed that every triangle free planar graph is 4-choosable and Thomassen observed that a planar graph is 3-choosable if the grith of the graph is at least 5 (both results can be found in Section 2.13 of [6]). Lam et al. [7] proved that if planar graph is free of \( k \)-cycles for some \( k \in \{4, 5, 6\} \), then \( G \) is 4-choosable. We shall adopt a similar approach and prove the following theorem, which partly shows Conjecture A is true for planar graphs. Note that triangles are allowed in the graph \( G \) in our theorem.

**Theorem 1.1.** Let \( G \) be a planar graph with maximum degree \( \Delta \) such that \( G \) has no cycle of length from 4 to \( k \), where \( k \geq 4 \). If
(1) \( \Delta \geq 7 \) and \( k \geq 4 \), or
(2) \( \Delta \geq 6 \) and \( k \geq 5 \), or
(3) \( \Delta \geq 5 \) and \( k \geq 8 \), then \( \chi'_1(G) = \Delta \) and \( \chi''_1(G) = \Delta + 1 \). Furthermore, if
(4) \( \Delta \geq 4 \) and \( k \geq 14 \), then \( \chi'_1(G) = \Delta \).

2. **Proof of Theorem 1.1**

Let us introduce some notations and definitions. Let \( G = (V, E, F) \) be a planar graph. A vertex \( v \) is called a \( k \)-vertex or \( k^+ \)-vertex if \( d(v) = k \) or \( d(v) \geq k \), respectively. For \( f \in F \), we use \( b(f) \) to denote the boundary of \( f \) and write \( f = [u_1 u_2 \ldots u_n] \) if \( u_1, u_2, \ldots, u_n \) are the vertices of \( f \) in a clockwise order. The degree of a face \( f \), denoted by \( d(f) \), is the number of edges incident with it, where each cut-edge is counted twice. A \( k \)-face or a \( k^+ \)-face is a face of degree \( k \) or of degree at least \( k \), respectively. Let \( \delta(f) \) denote the minimum degree of vertices incident with \( f \). A face \( f \) of degree \( k \) is called simple if \( b(f) \) forms a cycle of length \( k \). Obiously, if \( G \) has no pendant edges every face of degree at most 5 is simple. If \( f = [u_1 u_2 \ldots u_n] \) is not simple, then \( f \) contains at least one cut-vertex \( v \). Let \( m_v(f) \) denote the number of times passing through \( v \) of \( f \) in clockwise order. Thus if \( m_v(f) = t \), then there exists a \( t \)-subset \( \{i_1, i_2, \ldots, i_t\} \) of \( \{1, 2, \ldots, n\} \) such that \( u_{i_1} = u_{i_2} = \cdots = u_{i_t} = v \).

**Proof of Theorem 1.1.** Let \( G = (V, E, F) \) be a minimal counterexample to any of (1)–(4) in theorem. If \( G \) is not edge-\( \Delta \)-choosable, then there is a edge assignment \( L \) for \( G \) with \( |L(e)| = \Delta \) for any edge \( e \in E \) such that \( G \) is not edge-\( \Delta \)-colorable. If \( G \) is not total-(\( \Delta + 1 \))-choosable, then there is a total assignment \( L \) for \( G \) with \( L(x) = \Delta + 1 \) for any \( x \in V \cup E \), such that \( G \) is not total-\( \Delta \)-colorable.

By the minimality of \( G \), \( G \) has the following proprieties:
(a) \( G \) is connected, and
(b) any vertex \( v \) is incident with at most \( \lfloor \frac{d(v) + 1}{2} \rfloor \) 3-faces, and
(c) \( G \) contains no even cycle \( v_1 v_2 \ldots v_2 v_1 \) such that \( d(v_1) = d(v_3) = \cdots = d(v_{2t-1}) = 2 \), and
(d) \( G \) contains no edge \( uv \) with \( \min\{d(u), d(v)\} \leq \lfloor \frac{d(G)/2}{2} \rfloor \) and \( d(u) + d(v) \leq \Delta + 1 \).

(a) and (b) are obvious. The proofs of (c) and (d) can be found in [2].

Let \( G_2 \) be the subgraph induced by the edges incident with the 2-vertices of \( G \). Wang and Wu [10] proved that \( G_2 \) contains a matching \( M \) such that all 2-vertices in \( G_2 \) are saturated. If \( uv \in M \) and \( d(u) = 2 \), then \( v \) is called the 2-master of \( u \) and \( u \) is called the dependent of \( v \). Each 2-vertex has a 2-master and each vertex of degree \( \Delta \) can be the 2-master of at most one 2-vertex.
We begin the proof of (1) in theorem. Since $G$ is a planar graph, by Euler’s formula, we have

$$\sum_{v \in V} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0.$$ 

Now we define the initial charge function $w(x)$ for each $x \in V \cup F$. Let $w(v) = 2d(v) - 6$ if $v \in V$ and $w(f) = d(f) - 6$ if $f \in F$. It follows that $\sum_{x \in V \cup F} w(x) < 0$. The discharging method distributes the positive charge to neighbors so as to leave as little positive charge remaining as possible. This leads to $\sum_{x \in V \cup F} w(x) > 0$. A contradiction follows.

Claim 2.1. If $\Delta \geq 7$, then $G$ does not contain a 3-face $f = uvw$ such that $d(u) = d(v) = d(w) = 4$.

Proof. Suppose it does contain such a 3-face. Let $G' = G - \{uv, uw, vw\}$. By the minimality of $G$, $G'$ is edge-$\Delta$-choosable, or $G'$ is total-$(\Delta + 1)$-choosable. If $G'$ is edge-$\Delta$-choosable, then $G'$ is edge-$L$-colorable with $L'(e) = L(e)$ for any edge $e \in E(G)$ and every edge in $b(f)$ has at least three colors available in $L$. It follows from $\chi'_1(f) = \chi'(f) = 3$ that $f$ can be properly colored with available colors in $L$. Thus $G$ is edge-$L$-colorable, a contradiction. If $G'$ is total-$(\Delta + 1)$-choosable, then $G'$ is total-$L'$-colorable with $L'(x) = L(x)$ for any $x \in V(G') \cup E(G')\{u, v, w\}$.

Let $L'$ be total assignment for $f$ with available colors in $L$. Then we have $|L'(uv)| \geq 4$, $|L'(uw)| \geq 4$, $|L'(vw)| \geq 4$, $|L'(u)| \geq 4$, $|L'(v)| \geq 4$ and $|L'(w)| \geq 4$. It follows from $\chi''_1(f) = \chi''(f) = 3$ that $f$ is total-$L''$-colorable. Thus $G$ is total-$L$-colorable, a contradiction. □

To prove (1) in theorem, we are ready to construct a new charge $w^*(x)$ on $G$ as follows:

1. Each 2-vertex receives 2 from its 2-master.
2. Each 3-face receives $\frac{1}{2}$ from its incident vertices of degree at least 5.
3. Each 3-face receives $\frac{2}{3}$ from its incident vertices of degree 4.
4. Each 5-face receives $\frac{1}{3}$ from its incident vertices of degree at least 5.
5. Each 5-face receives $\frac{1}{3}$ from its incident vertices of degree 4.

Clearly, $w^*(f) = w(f) \geq 0$ if $d(f) \geq 6$. Assume that $d(f) = 3$. If $d(f) \leq 3$, then $f$ is incident with two $6^+$-vertices by (d). So $w^*(f) = w(f) + 2 \times \frac{3}{2} = 0$. Otherwise, $f$ is incident with at least one $5^+$-vertex by Claim 2.1. So $w^*(f) \geq w(f) + \frac{3}{2} + 2 \times \frac{3}{2} = 0$. Let $d(f) = 5$. If $d(f) \leq 3$, then $f$ is incident with at most one 3-face of degree less than 3 and, if $f$ is incident with two vertices of degree less than 3, then $f$ is incident with three $6^+$-vertices. Thus $w^*(f) \geq w(f) \min\{\frac{3}{2} + 2 \times \frac{3}{2}, 3 \times \frac{3}{2}\} = 0$. Otherwise, $w^*(f) \geq w(f) + 5 \times \frac{3}{2} > 0$. Let $v$ be a vertex of $G$. Clearly, $w^*(v) = w(v) + 2 = 0$ if $d(v) = 2$, and $w^*(v) = w(v) = 0$ if $d(v) = 3$. If $d(v) = 4$, then $v$ is incident with at most two 3-faces by (b). So $w^*(v) \geq w(v) - 2 \times \frac{3}{2} - 2 \times \frac{1}{2} = 0$. If $d(v) = 5$, then $v$ is incident with at most two 3-faces by (b). So $w^*(v) \geq w(v) - 2 \times \frac{3}{2} - 3 \times \frac{1}{2} = 0$. If $d(v) = 6$, then $w^*(v) \geq w(v) - 3 \times \frac{3}{2} - 3 \times \frac{1}{2} > 0$. If $d(v) \geq 7$, then $v$ can be the 2-master of at most one 2-vertex. So $w^*(v) \geq w(v) - 2 - \left\lfloor \frac{d(v)}{2} \right\rfloor \times \frac{3}{2} - (d(v) - \left\lfloor \frac{d(v)}{2} \right\rfloor) \times \frac{1}{2} > 0$. It follows that $\sum_{x \in V \cup F} w^*(x) = \sum_{x \in V \cup F} w^*(x) > 0$, a contradiction. This completes the proof.

Note that (1) implies that (2) is true if $\Delta \geq 7$. Hence it is sufficient to prove (2) by assuming $\Delta = 6$. Similarly, we may assume that $\Delta = 5$ in the proof of (3) and $\Delta = 4$ in the proof of (4).

To prove (2), we define a new charge function $w(x)$ for each $x \in V \cup F$. Let $w(x) = d(x) - 4$ for each $x \in V \cup F$. It follows from Euler’s formula that $\sum_{x \in V \cup F} w(x) = -8 < 0$. We construct a new charge $w^*(x)$ on $G$ as follows:

1. Each $r^*(6)$-face $f$ gives $(1 - \frac{4}{7})m_{v}(f)$ to its incident vertex $v$ if $v$ is cut-vertex, and gives $1 - \frac{4}{7}$ otherwise.
2. Each 2-vertex receives $\frac{11}{7}$ from its 2-master if it is incident with a 3-face and receives $\frac{4}{3}$ from its 2-master otherwise.
3. Each 3-vertex $v$ receives $\frac{1}{3}$ from $u$ if $v$ is incident with 3-face $f$ and $u$ is a neighbor of $v$ but not incident with $f$.
4. Each 3-face receives $\frac{1}{3}$ from its incident vertex $v$ if $d(v) \geq 5$ and receives $\frac{1}{3}$ if $d(v) = 4$.

Clearly, we have $w^*(f) \geq 0$ for any face $f$. Let $v$ be an arbitrary vertex of $G$. Consider the case of $d(v) = 2$. If it is incident with a 3-face, then its other incident face must have degree at least 7 since $G$ is a $C_4$-free and $C_5$-free graph. It follows that $v$ receives at least $1 - \frac{4}{7} = \frac{3}{7}$ from the incident face and $\frac{11}{7}$ from its 2-master; that is, $w^*(v) \geq w(v) + \frac{3}{7} + \frac{11}{7} = 0$. Otherwise if $v$ is not incident with any 3-face, then it receives at least $2 \times (1 - \frac{4}{6}) = \frac{2}{3}$ from its incident faces of degree at least 6 and $\frac{4}{3}$ from its 2-master. Hence, $w^*(v) \geq w(v) + \frac{2}{3} + \frac{4}{3} = 0$. Suppose $d(v) = 3$. If $v$ is incident with a 3-face, then $v$ receives at least $\frac{2}{7}$ from its incident faces and $\frac{1}{3}$ from its incident vertex.
not lying on the same 3-face. Hence $w^*(v) \geq w(v) + \frac{2}{3} + \frac{1}{3} = 0$. Otherwise, $v$ receives at least $3 \times \frac{1}{3} = 1$ from its incident faces. So $w^*(v) \geq w(v) + 1 = 0$. Note that $v$ gives either $\frac{1}{4}$ if $d(v) = 4$ or $\frac{1}{4}$ if $d(v) \geq 5$ to an incident 3-face, say $uvw$ where $u, w, v \in N(v)$, or gives $\frac{1}{4}$ to $u$ and $\frac{1}{4}$ to $w$ by $R_{2.3}$ but $v$ will then receive at least $1 - \frac{4}{6} = \frac{1}{3}$ from the face whose partial boundary contains $u, v, w$ sequentially if $uw \notin E(G)$. In the evaluation of the lower bound of $w^*(v)$, it suffices to consider the case when $v$ gives either $\frac{1}{4}$ or $\frac{1}{4}$ to its incident 3-faces. If $d(v) = 4$, then it receives at least $\frac{1}{4}$ from its incident 6+0-faces and gives at most $\frac{5}{3}$ to its incident 3-faces since any 4-vertex is incident with at most two 3-faces. It follows that $w^*(v) \geq w(v) + \frac{2}{3} - \frac{2}{3} = 0$. If $d(v) = 5$, then $v$ is incident with at most two triangles. If four neighbors of $v$ form two 3-faces and a 3-vertex is pending on the remaining neighbor of $v$, then $v$ discharges at $2 \times \frac{1}{4} + \frac{1}{3}$ via $R_{2.3}$. This implies that $w^*(v) \geq w(v) + 3 \times \frac{1}{3} - 2 \times \frac{1}{3} = 0$. Suppose $d(v) = 6$. It follows that $v$ can be the 2-master of some vertex $u$. In this case, if $u$ is not incident with a 3-face, then $v$ is incident with at most two 3-faces. So $w^*(v) \geq w(v) + 4 \times \frac{1}{3} - 2 \times \frac{1}{2} = \frac{5}{3} > 0$. Otherwise, $u$ is incident with a 3-face. The other case $u$ is incident with 7+face because $G$ is $C_4$-free and $C_5$-free. If $v$ gives $\frac{1}{4}$ to some 3-vertex, then $v$ gives at most $2 \times \frac{1}{4}$ to its incident 3-vertices and $\frac{1}{4}$ to another 3-face. Thus $w^*(v) \geq w(v) + \frac{2}{3} - \frac{1}{3} - \frac{1}{2} - 2 \times \frac{1}{2} > 0$. Otherwise, $v$ is incident with at most three 3-faces. Hence, $w^*(v) \geq w(v) + \frac{1}{4} - \frac{1}{2} > 0$. If $v$ is not a 2-master of some 2-vertex, then $v$ is incident with at most three 3-faces. So $w^*(v) \geq w(v) + 3 \times \frac{1}{3} - \frac{3}{4} > 0$. It follows that $\sum_{x \in V \cup F} x(x) = \sum_{x \in V \cup F} w^*(x) > 0$, a contradiction. This completes the proof of (2).

Before proof (3), we need the following claim.

Claim 2.2. If $\Delta \geq 5$, and $G$ contains a triangle $f = uvw$ such that $d(u) = d(v) = 3$, then $d(w) \geq 5$.

Proof. On the contrary, suppose that $d(w) \leq 4$. If $d(w) \leq 3$, then the proof is the same as Claim 2.1. So we only consider the case $d(w) = 4$. Let $G' = G - uvw$. By the minimality of $G$, $\Delta(G') = \Delta$ or $\Delta(G') = \Delta + 1$. If $\Delta(G') = \Delta$, $G'$ is edge-L×0-colorable with $L'(e) = L(e)$ for any edge $e \in E(G')$. Let $L''$ be edge assignment for $f$ with available colors in $L$. Then $|L''(uvw)| \geq 2$, $|L''(uvw)| \geq 2$ and $|L''(uw)| \geq 3$. We color $uw, vw, uv$ successively. Thus $f$ is edge-L×0-colorable and $G$ is edge-L-colorable, a contradiction. If $G'$ is total-(\Delta + 1)-choosable, then $G'$ is total-L×0-colorable with $L''(x) = L(x)$ for any $x \in V(G') \cup E(G')\{u, v, w\}$. Let $L''$ be the total assignment for $f$ with available colors in $L$. Then we have $|L''(u)| \geq 2$, $|L''(uv)| \geq 3$, $|L''(uw)| \geq 3$, $|L''(v)| \geq 4$ and $|L''(uv)| \geq 4$. We only consider the case which the equalities hold. If there is a color $x \in L''(uw) \cap L''(v)$, then $uvw$ with $x$ and color $w, uv, wuv, uv$ successively. So $L''(uw) \subseteq L''(v)$. Similarly, $L''(uw) \subseteq L''(v)$. If there is a color $\beta \in L''(uv), L''(u)$, then color $u$ with $\beta$, and color $w, uv, wuv, uv$ successively. So $L''(u) = L''(v)$. Thus there is a color $\gamma \in L''(uw) \cap L''(v)$. We color $uw$ and $v$ with $\gamma$, and color $w, uv, wuv$ and $uv$ successively. In any case, $f$ is total-L×0-colorable, and so $G$ is total-L-colorable, a contradiction. This complete the proof. □

To prove (3), we need a new charge function $w(x)$ for each $x \in V \cup F$. Let $w(v) = 4d(v) - 10$ if $v \in V$ and $w(f) = d(f) - 10$ if $f \in F$. It follows from Euler’s formula that $\sum_{x \in V \cup F} x(x) < 0$. We construct charge $w^*(x)$ on $G$ as follows:

$R_{3.1}$: Each 2-vertex receives 2 from its 2-master.

$R_{3.2}$: Each $(\geq 9)$-face receives $\frac{m(f)}{8}$ from its incident 3-vertex $v$ if $v$ is a cut-vertex, and receives $\frac{1}{8}$ otherwise.

$R_{3.3}$: Each $(\geq 9)$-face receives $\frac{m(f)}{4}$ from its incident 4×-vertex $v$ if $v$ is a cut-vertex, and receives $\frac{1}{4}$ otherwise.

$R_{3.4}$: Each 3-face receives $\frac{7}{3}$ from its incident vertex $v$ if $d(v) = 5$, receives $\frac{11}{4}$ if $d(v) = 4$ and receives $\frac{7}{3}$ if $d(v) = 3$.

Clearly, $w^*(f) = w(f) \geq 0$ if $d(f) \geq 10$. Let $f = uvw$ be a 3-face. If $\delta(f) = 2$, then $f$ is incident with exactly two 5-vertices by (d). So $w^*(f) = w(f) + 2 \times \frac{1}{3} = 0$. Consider the case $\delta(f) = 3$. If $f$ is incident with two 3-vertices, without loss of generality, $d(u) = d(v) = 3$, then $d(w) = 5$ by Claim 2.2. So $w^*(f) \geq w(f) + 2 \times \frac{1}{4} + \frac{7}{3} = 0$. Otherwise, $f$ is incident with two 4×-vertices. So $w^*(f) = w(f) + 2 \times \frac{1}{4} + \frac{7}{3} > 0$. If $\delta(f) \geq 4$, then $w^*(f) \geq w(f) + 3 \times \frac{1}{11} > 0$. Let $f$ be a 9-face. Consider $\delta(f) = 2$. If $f$ is incident with at least three 2-vertices, then $f$ is incident with at least four 5-vertices by (d). So $w^*(f) \geq w(f) + 4 \times \frac{1}{4} = 0$. Otherwise, $w^*(f) \geq w(f) + \min\{2 \times \frac{1}{4} + (9 - 3) \times \frac{1}{8}, 3 \times \frac{1}{4} + (9 - 5) \times \frac{1}{8}\} > 0$. If $\delta(f) \geq 3$, then $w^*(f) \geq w(f) + 9 \times \frac{1}{8} > 0$. 


Let $v$ be a $k$-vertex. Clearly, $w^*(v) = w(v) + 2 = 0$ if $k = 2$, and $w^*(v) \geq w(v) - \frac{7}{4} - 2 \times \frac{1}{4} = 0$ if $k = 3$. If $k = 4$, then $v$ is incident with at most two 3-faces by (b). So $w^*(v) \geq w(v) - 2 \times \frac{11}{4} - 2 \times \frac{1}{4} = 0$. If $k = 5$, then $v$ is incident with at most two 3-faces by (b), and $v$ is the 2-master of at most one 2-vertex. So $w^*(v) \geq w(v) - 2 \times \frac{7}{4} - 3 \times \frac{1}{4} > 0$. It follows that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) > 0$, a contradiction. This completes the proof of (3).

We will prove the following claim before we prove (4).

**Claim 2.3.** If $\Delta = 4$, then $G$ contains no 4-vertex $z$ where $z$ is incident with two 3-faces $zu, zv$ and $d(x) = d(y) = 2$.

**Proof.** Suppose, on the contrary, such vertex $z$ does exist. Let $G' = G - \{ux, uz, xz, vz, vy, yz\}$, and let $H$ be the subgraph induced by $\{x, y, z, u, v\}$. By the minimality of $G$, $G'$ is edge-$\Delta$-choosable. So $G'$ is edge-$L'$-colorable with $L'(e) = L(e)$ for any edge $e \in E(G')$. Let $L''$ be the edge assignment for $H$ with available colors in $L$. Then we have $|L''(ux)| \geq 2$, $|L''(uz)| \geq 2$, $|L''(xz)| \geq 4$, $|L''(vy)| \geq 2$, $|L''(vy)| \geq 2$ and $|L''(yz)| \geq 4$. We only consider the case which the equalities hold. If there is a color $\gamma \in L''(uz) \setminus L''(ux)$. Color $uz$ with $\gamma$ and color $zu, vy, yz, xz \text{ and } ux$ successively. So $L''(ux) = L''(uz)$. Similarly, $L''(uz) = L''(vy)$. If there is a color $\beta \in L''(uz) \cap L''(uv)$, then color $uv$ and $ux$ with $\beta$, and color $uz, vy, yz$ and $xz$ successively. So we assume that $L''(uz) = L''(ux) = \{1, 2\}$ and $L''(uv) = L''(vy) = \{3, 4\}$. If there is a color $\gamma \in (L''(uz) \cup L''(vu)) \setminus L''(xv)$, without loss of generality, $\gamma \in L''(uz) \setminus L''(vx)$. Color $uz$ with $\gamma$, and color $ux, vz, vy, yz$ and $xz$ successively. So $L''(xz) = \{1, 2, 3, 4\}$. Similarly, $L''(yz) = \{1, 2, 3, 4\}$. Color $uz$ with color 1, color $ux$ and $zy$ with color 2, color $zv$ with color 3 and color $xz$ and $vy$ with color 4. We get a proper coloring. In any case, $H$ is edge-$L''$-colorable. It implies that $G$ is edge-$L$-colorable, a contradiction. That completes the proof.

To prove (4), we define the initial charge function $w(x)$ for each $x \in V \cup F$. Let $w(x) = d(x) - 4$ for all $x \in V \cup F$. It follows from Euler’s formula that $\sum_{x \in V \cup F} w(x) < 0$. We construct charge $w^*(x)$ on $G$ as follows:

$R_{4.1}$: Each $r(\geq 15)$-face $f$ gives $(1 - \frac{4}{7})m(v)$ to its incident vertex $v$ if $v$ is cut-vertex, and gives $1 - \frac{4}{7}$ otherwise.

$R_{4.2}$: Each 2-vertex receives $\frac{19}{24}$ from its neighbors if it is incident with a 3-face and receives $\frac{8}{13}$ from its 2-master otherwise.

$R_{4.3}$: Each 3-face receives $\frac{1}{7}$ from its incident vertices.

It is obvious that $w(v) = 0$ for any face $f$. Let $u$ be an arbitrary vertex of $G$. First consider the case of $d(v) = 2$. If it is incident with a 3-face, then its other incident face $f$ must have degree at least 16. From (d), any neighbor of $u$ should be of degree at least $(\Delta + 2) - 2 = 4$. Hence, they cannot be 2-vertices. It follows that $v$ receives at least $1 - \frac{4}{13} = \frac{9}{13}$ from $f$ and $2 \times \frac{19}{24} = \frac{19}{12}$ from its neighbors, and gives $\frac{1}{7}$ to its incident 3-face. Otherwise $v$ receives at least $2 \times \frac{11}{13} = \frac{22}{13}$ from its incident faces and $\frac{8}{13}$ from its 2-master. Hence, $w^*(v) \geq w(v) + \min \left\{ \frac{9}{13}, \frac{19}{12}, \frac{1}{3}, \frac{22}{13}, \frac{8}{13} \right\} = 0$. Now consider the case of $d(v) = 3$. $v$ receives at least $2 \times \frac{11}{13} = \frac{22}{13}$ from its incident faces. Hence, $w^*(v) = w(v) + \frac{22}{13} - \frac{1}{3} = \frac{2}{13} > 0$.

If $d(v) = 4$ and it is incident with two 3-faces, then $v$ is adjacent to at most one 2-vertex by Claim 2.3. It follows that $w^*(v) \geq w(v) + \frac{22}{13} - \left( \frac{2}{3} + \frac{19}{24} \right) = \frac{1}{720} > 0$. Otherwise it receives at least $3 \times \frac{11}{13}$ from its incident faces, and gives at most $\frac{1}{7}$ to its incident 3-face and $\frac{19}{24} + \frac{8}{13}$ to its adjacent 2-vertices. It follows that $w^*(v) \geq w(v) + \frac{33}{13} - \left( \frac{1}{7} + \frac{19}{24} + \frac{8}{13} \right) = \frac{11}{24} > 0$. This implies that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) > 0$, a contradiction. This completes the proof of (4).

In the proof of theorem, we showed that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) > 0$. It implies the following corollary.

**Corollary.** Let $G$ be a graph with maximum degree $\Delta$ embedded in a surface of nonnegative characteristic, and $G$ has no cycle of length from 4 to $k$, where $k \geq 4$. Then $\chi'_1(G) = \Delta + 1$ if $(\Delta, k) \in \{(7, 4), (6, 5), (5, 8)\}$. Furthermore, $\chi'_1(G) = \Delta$ if $(\Delta, k) \in \{(4, 14)\}$.

**References**


