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Modular aspects of term graph rewriting [1]

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Abstract

Term rewriting is generally implemented using graph rewriting for efficiency reasons. Graph rewriting allows sharing of common structures thereby saving both time and space. This implementation is sound in the sense that computation of a normal form of a graph yields a normal form of the corresponding term. In this paper, we study modularity of the following properties in graph rewriting: (a) weak normalization, (b) strong normalization, (c) semi-completeness (confluence + weak normalization) and (d) completeness (confluence + strong normalization).
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1. Introduction

Term rewriting is often implemented using graph rewriting to cut down the evaluation costs. Graph (directed acyclic) representation of terms facilitates sharing of structures – unlike the tree representation – and hence saves space and avoids the repetition of computations. This implementation is both sound and complete in the following sense. If a graph G reduces to G' then the term corresponding to G rewrites to the term corresponding to G' (soundness) and two graphs are convertible if and only if the corresponding terms are convertible (completeness). One of the nice fallouts of this is that the computation of a normal form of a given graph yields a normal form of the corresponding term.

A graph rewriting step using a non-right-linear rewrite rule does not make multiple copies of the subgraphs corresponding to non-linear variables, but enforces sharing of these subgraphs. This improves both the space and time complexity of computations. Whereas the space required for term rewriting can grow *exponentially* with the number of rewrite steps (due to copying), it can grow only *linearly* as an evaluation step can only increase the size by a constant (maximum size of a right-hand side of a rule

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in the system). This improvement in space complexity leads to a very appreciable improvement in time complexity as well. However on the negative side, due to the above enforced sharing, certain properties of the given term rewrite system (TRS for short) are not reflected in the graph rewriting implementation. For example, the graph rewriting relation corresponding to a given confluent TRS need not be confluent. The following list indicates how subtle the relation between term rewriting and term graph rewriting is. Here, $\Rightarrow_{\mathscr{R}}$ denotes the term graph rewriting relation induced by TRS $\mathscr{R}$.

(1) Confluence of $\mathscr{R}$ does not imply confluence of $\Rightarrow_{\mathscr{R}}$ (see [8, 18]).
(2) Weak normalization (WN) of $\mathscr{R}$ does not imply WN of $\Rightarrow_{\mathscr{R}}$ (see [8, 16]).
(3) Termination of $\Rightarrow_{\mathscr{R}}$ does not imply termination of $\mathscr{R}$ (see [18]).
(4) Confluence is modular in term rewriting [20] but not modular in term graph rewriting (for direct sum).
(5) Termination is not modular in term rewriting [21] but modular in term graph rewriting (for direct sum).

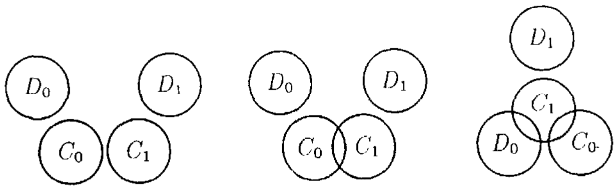
However the converses of 1, 2 and 3 hold. This subtle relation between term rewriting and graph rewriting is a boon as well as a bane. On the positive side, one can use results on confluence of $\Rightarrow_{\mathscr{R}}$ in proving confluence of nonterminating $\mathscr{R}$, which is otherwise a difficult task (see [19]). On the negative side, it is very difficult to use the general theory of term rewriting in studying graph implementations of TRSS. In view of this, it is very important to develop the theory of term graph rewriting. In this paper, we continue the research pursued earlier in [8, 12, 16–19] towards the above aim.

The study of properties which are preserved under combinations of systems (called modular properties) is of both theoretical and practical importance. In particular, modularity results facilitate (i) incrementality in the synthesis of systems and (ii) divide-and-conquer approach in the analysis of systems. The fragile relationship between term rewriting and term graph rewriting – in particular, the above statements 4 and 5 which show that modularity results in term graph rewriting for direct-sum are quite the opposite of the modularity results in term rewriting – makes it imperative to investigate modular aspects of term graph rewriting rigorously.

In this paper, we study modularity of the following properties in term graph rewriting: (a) weak normalization (WN), (b) strong normalization (SN), (c) semi-completeness (confluence + weak normalization) and (d) completeness (confluence + strong normalization).

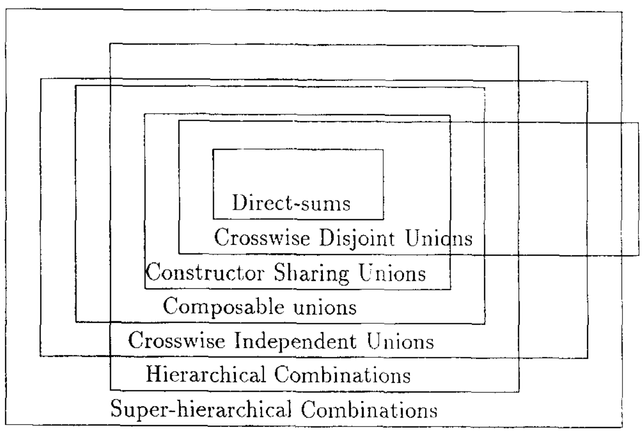
While studying modular aspects of term rewriting, the simplest combination one could consider is the *direct-sum*, where the two constituent systems (with non-empty sets of rules) do not share any function symbols. In studying modular aspects of graph rewriting, one has to first consider *signature extensions*, i.e., one of the two constituent systems has no rules. This is in particular so because the mere addition of function symbols can spoil certain properties in term graph rewriting. Confluence is one such property. In the following, we show that WN, SN, semi-completeness and completeness are preserved under signature extensions, unlike confluence.

To discuss various classes of combinations of TRSS, we explain the following notions of constructor/defined symbols. Function symbol f is a *constructor* in $\mathscr{R}$ if f does not occur as outermost symbol of left-hand-side of any rewrite rule in $\mathscr{R}$, otherwise, it is a *defined symbol*. In the following, $\mathscr{R}(D,C,R)$ denotes a TRS with the set of defined symbols D, the set of constructor symbols C and the set of rewrite rule R. A TRS $\mathscr{R}(D,C,R)$ can be seen as specification (or definition) of the symbols in D. The following diagram gives the pictorial view of three basic combinations. The sets of defined and constructor symbols of $\mathscr{R}_i$ are denoted by D_i and C_i, respectively.



(1) Direct sum (2) Sharing Constr. (3) Hier. Comb.

In a *direct sum*, two systems share no function symbols, In a *constructor sharing union*, two systems share (some) constructors but no defined symbol occurs in both the systems. In a *hierarchical combination*, some of the defined symbols of one system occur as constructors in the other system, but not vice versa. The following combinations are sporadically studied in the literature in addition to the above 3 basic combinations (see a.o. [3, 4, 7, 11, 13, 15]). A *composable union* $\mathscr{R}_0(D_0,C_0,R_0) \cup \mathscr{R}_1(D_1,C_1,R_1)$ allows sharing of some constructor as well as defined symbols in such a way that the shared defined symbols are defined precisely the same in both the systems, i.e., $R_0 \cap R_1 = \{l \to r \in R_0 \cup R_1 \mid root(l) \in D_0 \cap D_1\}$. In a *crosswise disjoint union* $\mathscr{R}_0 \cup \mathscr{R}_1$, no function symbol occurs in both the left-hand sides of $\mathscr{R}_i$ and the right-hand sides of $\mathscr{R}_{1-i}$ for $i = 0, 1$. In a *crosswise independent union* $\mathscr{R}_0(D \uplus D_0, C_0, R_0) \cup \mathscr{R}_1(D \uplus D_1, C_1, R_1)$, no function symbol in D_i occurs in the right-hand sides of $\mathscr{R}_{1-i}$ for $i = 0, 1$ and $R_0 \cap R_1 = \{l \to r \in R_0 \cup R_1 \mid root(l) \in D\}$. In a *super-hierarchical combination* $\mathscr{R}_0(D \uplus D_0, C_0, R_0) \cup \mathscr{R}_1(D \uplus D_1, C_1, R_1)$, no function symbol in D_1 occurs in the right-hand sides of $\mathscr{R}_0$ and $R_0 \cap R_1 = \{l \to r \in R_0 \cup R_1 \mid root(l) \in D\}$. The relationships among these classes of combinations are depicted in the following picture.



As can be seen in the above picture, the class of super-hierarchical combinations contains all the other classes of combinations, except the class of crosswise disjoint unions. In this paper, we mainly consider super-hierarchical combinations and study modularity of WN, SN, semi-completeness and completeness properties for these combinations in term graph rewriting. The two best results known about modularity of termination (SN) for super-hierarchical combinations in term rewriting need that (i) the two systems are confluent and terminating overlay systems [2, 9] or (ii) termination of both the systems is provable by simplification orderings (i.e., simple termination) [6]. In contrast, we show that for modularity of termination in term graph rewriting, neither confluence nor simple termination is needed and the systems need not be overlay systems. Similarly completeness is modular for super-hierarchical combinations of some non-overlay systems in term graph rewriting, whereas the modularity of completeness in term rewriting holds only for overlay systems.

2. Preliminaries

We assume that the reader is familiar with the basic terminology of term rewriting systems and give definitions only when they are required. The notations not defined in the paper can be found in Dershowitz and Jouannaud [1], Klop [5] or Plump [18].

In the following, $\mathscr{T}(\Sigma,X)$ denotes the set of terms constructed from a set of function symbols Σ and a countable set of variables X. For each variable x, $arity(x) = 0$. We use the (possibly with some subscripts) symbols a,b,c,d to denote constants, f,g,h to denote functions of arity at least one and x,y,z to denote variables. We recall the following definitions from Plump [17, 18].

A *hypergraph* G over Σ is a system $\langle V_G, E_G, s_G, t_G, l_G\rangle$, where V_G, E_G are finite sets of nodes and hyperedges, $s_G : E_G \rightarrow V_G$, $t_G : E_G \rightarrow V_G^*$ are mappings that assign a source node and a string of target nodes to each hyperedge, and $l_G : E_G \rightarrow \Sigma \cup X$ is a mapping that labels each hyperedge e such that $arity(l_G(e))$ is the length of $t_G(e)$.

A node v is a *predecessor* of a node v' if there is an edge e with source v such that v' occurs in $t_G(e)$. The relations $<_G$ and $\leqslant_G$ are the transitive and reflexive-transitive closures of the predecessor relation. We say v' is below v when $v \leqslant_G v'$. For each node v, G/v is the subhypergraph of G consisting of all the nodes v' with $v \leqslant_G v'$ and all the edges outgoing from these nodes.

A hypergraph G over Σ is a *collapsed tree* if (1) there is a node $root_G$ such that $root_G \leqslant_G v$ for each node v, (2) the predecessor relation of G is acyclic and (3) each node has a unique outgoing edge. A collapsed tree is also refered to as a *term graph* in the sequel.

Let G be a collapsed tree. Then the mapping $term_G : V_G \rightarrow T(\Sigma,X)$ is defined as $term_G(v) = l_G(e)$ if $t_G(e)$ is an empty string ε and $term_G(v) = l_G(e)(term_G(v_1), \ldots, term_G(v_n))$ if $t_G(e) = v_1 \ldots v_n$, where e is the unique edge with source v. In the following, $term(G)$ stands for $term_G(root_G)$.

For a collapsed tree G, the mapping $node_G : Pos(term(G)) \to V_G$ (relating positions/subterms of a term and the nodes/subgraphs in a term graph representing that term) is defined as (i) $node_G(\varepsilon) = root_G$ and (ii) $node_G(i.\pi) = node_{G/v_i}(\pi)$, where $v_1 \cdots v_n$ is the target string of the hyperedge with source $root_G$ and $Pos(t)$ is the set of positions in t.

A collapsed tree G is a *tree with shared variables* if (1) for each node v, $indegree_G(v) > 1$ implies $term_G(v) \in X$ and (2) for all nodes v, v', $term_G(v) = term_G(v') \in X$ implies $v = v'$. Here, $indegree_G(v)$ is the number of occurrences of v in the target strings.

In the following, $\Diamond t$ denotes a tree with shared variables such that $term_G(\Diamond t) = t$ and $\underline{G}$ denotes the hypergraph obtained from G by removing all the edges labeled with variables.

Let G, H be hypergraphs. A *hypergraph morphism* $g : G \to H$ is a pair of mappings $\langle g_v : V_G \to V_H, g_e : E_G \to E_H \rangle$ that preserve sources, targets and labels, i.e., $s_H \circ g_e = g_v \circ s_G$, $t_H \circ g_e = g_v^* \circ t_G$ and $l_H \circ g_e = l_G$.[2]

Definition 1 (*evaluation step*). Let G_1, G_2 be collapsed trees. Then there is an *evaluation step* from G_1 to G_2, denoted by $G_1 \Rightarrow_{\mathscr{E}} G_2$, if (i) there is a rule $l \to r \in \mathscr{R}$ and a hypergraph morphism $g : \underline{\Diamond l} \to G_1$ and (ii) G_2 is isomorphic to the collapsed tree constructed as follows:
(1) Remove the hyperedge outgoing from $g_v(root_{\Diamond l})$, yielding a hypergraph G'.
(2) Build the disjoint union $G' + \underline{\Diamond r}$ and
- identify $g_v(root_{\Diamond l})$ with $root_{\Diamond r}$ (i.e., merge the two nodes into one),
- for each pair $\langle u, u' \rangle \in V_{\Diamond l} \times V_{\Diamond r}$ with $term_{\Diamond l}(u) = term_{\Diamond r}(u') \in X$, identify $g_v(u)$ with u'.

Let G'' be the resulting hypergraph.
(3) Remove garbage, resulting in a collapsed tree $G''/root_G$.

For evaluations with non-left-linear rules, we need to 'fold' collapsed trees.

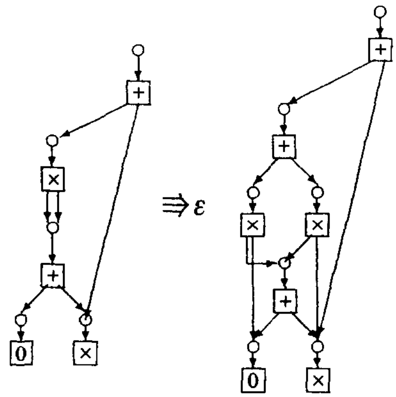
Definition 2 (*folding step*). Let G_1, G_2 be collapsed trees. Then there is a *folding step* $G_1 \Rightarrow_{\mathscr{F}} G_2$ if there are distinct edges $e, e' \in E_{G_1}$ with $l_{G_1}(e) = l_{G_1}(e')$, $t_{G_1}(e) = t_{G_1}(e')$, and G_2 is isomorphic to the collapsed tree obtained from G_1 by identifying e with e' and $s_{G_1}(e)$ with $s_{G_1}(e')$.

We denote the relation $\Rightarrow_{\mathscr{E}} \cup \Rightarrow_{\mathscr{F}}$ by $\Rightarrow_{\mathscr{R}}$ and omit the subscript if it does not lead to any confusion. The relation $\Rightarrow_{\mathscr{R}}$ is refered to as *the term graph rewriting relation of* $\mathscr{R}$.

Example 1. The following figure shows an evaluation step on a collapsed tree C representing term $(((0+x) \times (0+x)) + x)$ with rewrite rule $x \times (y+z) \to (x \times y) + (x \times z)$. Here, *the hyperedges are represented by boxes and the nodes are represented by cir-*

[2] Given a mapping $f : A \to B$, $f^* : A^* \to B^*$ sends ε to ε and $a_1 \ldots a_n$ to $f(a_1) \ldots f(a_n)$.

cles. The morphism locating the left-hand side $\underline{\Diamond(\mathtt{x}\times(\mathtt{y}+\mathtt{z}))}$ identifies the nodes representing x and $(\mathtt{y}+\mathtt{z})$ with the node representing $(0+\mathtt{x})$ in C.



In the examples below, we simplify the pictures of collapsed trees by deleting the nodes and replacing every hyperedge having n target nodes by n arcs, without introducing any confusion.

We need the following definition in the sequel.

Definition 3 (*dependency relation $\succeq_d$ over defined symbols*). The dependency relation of a rewrite system $\mathscr{R}(D,C,R)$ is the smallest quasi-order $\succeq_d$ over D satisfying the condition: *$f \succeq_d g$ if there is a rule $l \to r \in R$ such that $f = root(l)$ and g occurs in r.*

We say that a defined symbol $f \in D$ *depends on* a defined symbol $g \in D$ if $f \succeq_d g$. The set of symbols depending on a set of symbols S is defined as $\{f \mid f \succeq_d g$ and $g \in S\}$. Intuitively, $\mathtt{f} \succeq_d \mathtt{g}$ means that an evaluation of the defined function f for some arguments *may* involve an evaluation of the defined function g for some arguments (i.e., the definition of f depends in some sense on that of g). It also means that an appearance of f in a derivation might lead to a creation of g in the later part of the derivation.

Notions for modularity. In the following, we consider union of two term rewriting systems $\mathscr{R}_0(D_0 \uplus D, C_0, R_0)$ and $\mathscr{R}_1(D_1 \uplus D, C_1, R_1)$ sharing some function symbols and rules $R_0 \cap R_1 = \{l \to r \in R_0 \cup R_1 \mid root(l) \in D\}$. To define the notion of *rank*, we paint the symbols in black, white and transparent as follows. The symbols of D_0 are black, those of D_1 are white and others (including variables) are transparent. Each transparent symbol in a term takes the colour of its surroundings (i.e., the symbols above it) if there is any. A term t with $root(t) \in D_0$ (resp. D_1) is a top black (resp. white) term and a term with transparent root is a top transparent term.

Definition 4. Any term t can be uniquely written as $C[\![t_1,\cdots,t_n]\!], n \geqslant 0$ such that for each $i \in [1,n]$, (1) $root(t_i)$ and $root(t)$ are in different colour and (2) t_i is a maximal subterm of t with that property.

Definition 5. The *rank* of a term $t \equiv C[\![t_1, \cdots, t_n]\!], n \geqslant 0$ is defined as:

$$\begin{aligned} rank(t) &= 1 + max(\{rank(t_i) \mid 1 \leqslant i \leqslant n\}) && \text{if } root(t) \in D_0 \cup D_1, \\ &= max(\{rank(t_i) \mid 1 \leqslant i \leqslant n\}) && \text{otherwise.} \end{aligned}$$

The *rank* of a term graph G is defined as $rank(G) = rank(term(G))$.

Example 2. Consider the following two systems.

$$\begin{array}{lll} \mathscr{R}_0 : & \mathtt{mult(0,\ y)} \to \mathtt{0} & \mathscr{R}_1 : \mathtt{fib(0)} \to \mathtt{0} \\ & \mathtt{mult(s(x),\ y)} \to \mathtt{add(y,mult(x,\ y))} & \quad \mathtt{fib(s(0))} \to \mathtt{s(0)} \\ & & \quad \mathtt{fib(s(s(x)))} \to \\ & & \qquad \mathtt{add(fib(x),fib(s(x)))} \\ & \mathtt{add(0,\ y)} \to \mathtt{y} & \quad \mathtt{add(0,\ y)} \to \mathtt{y} \\ & \mathtt{add(s(x),\ y)} \to \mathtt{s(add(x,\ y))} & \quad \mathtt{add(s(x),\ y)} \to \mathtt{s(add(x,\ y))} \end{array}$$

$rank(\mathtt{add(s(0),s(s(0)))}) = 0$ and $rank(\mathtt{mult(s(x),\ y)}) = rank(\mathtt{add(y,mult(x,\ y))}) = 1$ and $rank(\mathtt{mult(fib(s(x)),\ fib(s(y)))}) = 2$. □

3. Term rewriting vs. term graph rewriting

In this section, we briefly review the relationship between term rewriting and term graph rewriting.

The following theorem is established in Plump [18].

Theorem 1 (Plump [18]). *If $\mathscr{R}$ is a term rewrite system and $\Rrightarrow_{\mathscr{R}}$ is its term graph rewriting relation then the following statements hold.*
(1) *$\Rrightarrow_{\mathscr{R}}$ is strongly normalizing if $\mathscr{R}$ is strongly normalizing.*
(2) *$\mathscr{R}$ is weakly normalizing if $\Rrightarrow_{\mathscr{R}}$ is weakly normalizing.*
(3) *$\mathscr{R}$ is confluent if $\Rrightarrow_{\mathscr{R}}$ is confluent.*
(4) *$\Rrightarrow_{\mathscr{R}}$ is confluent if $\mathscr{R}$ is confluent and $\Rrightarrow_{\mathscr{R}}$ is weakly normalizing.*

The converse of none of the first three statements of the above theorem hold.

Example 3. The following famous system $\mathscr{R}$ (from Toyama [21]) is nonterminating.

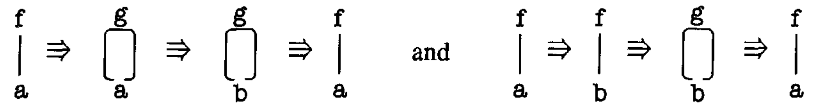
$$\begin{array}{l} \mathtt{f(0,1,x)} \to \mathtt{f(x,x,x)} \\ \mathtt{g(x,y)} \to \mathtt{x} \\ \mathtt{g(x,y)} \to \mathtt{y} \end{array}$$

But $\Rrightarrow_{\mathscr{R}}$ is terminating, showing that the converse of statement 1 of the above Theorem does not hold.

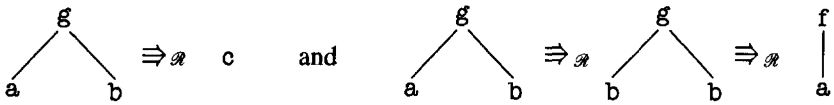
Example 4 (*Plump* [18]). The following system $\mathscr{R}$ is semi-complete, i.e., weakly normalizing and confluent.

$$\begin{aligned}&\mathtt{f(x)} \to \mathtt{g(x,x)}\\ &\mathtt{a} \to \mathtt{b}\\ &\mathtt{g(a,b)} \to \mathtt{c}\\ &\mathtt{g(b,b)} \to \mathtt{f(a)}\end{aligned}$$

But the relation $\Rrightarrow_{\mathscr{R}}$ is not weakly normalizing as the two possible $\Rrightarrow_{\mathscr{R}}$ derivations from a term graph representing f(a) loop, while f(a) has a normal form c.



The relation $\Rrightarrow_{\mathscr{R}}$ is neither confluent, as the following graph representing g(a,b) can be reduced to c and f(a), which are not joinable.



This demonstrates that, converses of statements 2 and 3 of the above Theorem do not hold.

4. Super-hierarchical combinations

In this section, we define a class of super-hierarchical combinations for which modularity of various properties is studied in later sections. Before defining this class, we show that WN, SN, semi-completeness and completeness are not modular for the whole class of (super) hierarchical combinations in general.

Example 5. It is easy to see that the following two systems $\mathscr{R}_0$ and $\mathscr{R}_1$ are *complete* and hence the graph rewriting relations $\Rrightarrow_{\mathscr{R}_0}$ and $\Rrightarrow_{\mathscr{R}_0}$ are complete as well. From completeness, it follows that $\Rrightarrow_{\mathscr{R}_0}$ and $\Rrightarrow_{\mathscr{R}_0}$ are WN, SN and semi-complete.

$$\mathscr{R}_0 \ : \ \mathtt{f(x)} \to \mathtt{x} \qquad \mathscr{R}_1 \ : \ \mathtt{h(a)} \to \mathtt{h(f(a))}$$

However, the term graph rewriting relation of the combined system is not weakly-normalizing (and hence not SN, semi-complete and complete as well). The term graph $\Diamond$h(a) representing h(a) has no normal form with respect to the combined system – note that the following cyclic derivation is the only possible derivation from $\Diamond$h(a).

$$\Diamond\mathtt{h(a)} \Rrightarrow_{\mathscr{R}_1} \Diamond\mathtt{h(f(a))} \Rrightarrow_{\mathscr{R}_0} \Diamond\mathtt{h(a)} \Rrightarrow_{\mathscr{R}_1} \cdots$$

In the following, we consider super-hierarchical combinations of two systems $\mathscr{R}_0(D_0 \uplus D, C_0, R_0)$ and $\mathscr{R}_1(D_1 \uplus D, C_1, R_1)$ sharing some symbols and rules $R_0 \cap R_1 = \{l \to r \in R_0 \cup R_1 \mid root(l) \in D\}$. In these combinations, defined symbols of system $\mathscr{R}_0$ can occur in both left- and right-hand sides of rules in $\mathscr{R}_1$ and defined symbols of $\mathscr{R}_1$ can occur in left-hand sides (but not right-hand sides) of rules in $\mathscr{R}_0$. Such combinations naturally arise while analyzing systems produced by completion procedures. See [9] for an example.

Notation. For discussions in the sequel, it is convenient to classify defined symbols in $D_1 \uplus D$ into two sets (i) $D_1^0 = \{f \mid f \in (D_1 \uplus D)$ and $f \succeq_d D_0\}$ consisting of function symbols depending on D_0 and (ii) $D_1^1 = (D_1 \uplus D) - D_1^0$ consisting of function symbols not depending on D_0. We denote the set of constructors $(C_0 \cup C_1) - (D_0 \cup D_1 \cup D)$ of the combined system by *Constr*.

The following definition from [9] characterizes the main class of super-hierarchical combinations we are interested in.

Definition 6. A term rewriting system $\mathscr{R}_1(D_1 \uplus D, C_1, R_1)$ is a *generalized nice-extension** of another term rewriting system $\mathscr{R}_0(D_0 \uplus D, C_0, R_0)$ if the following conditions are satisfied:
(1) $D_0 \cap D_1 = \phi$ and $R_0 \cap R_1 = \{l \to r \in R_0 \cup R_1 \mid root(l) \in D\}$.
(2) $\forall f \in (D_0 \uplus D)$, $\forall g \in D_1$, $f \not\succeq_d g$ (i.e., $\mathscr{R}_0 \cup \mathscr{R}_1$ is a super-hierarchical combination).
(3) Each rewrite rule $l \to r \in R_1$ satisfies the following condition:
 (H) : For every subterm s of r, if $root(s) \in D_1^0$, then s contains no function symbol (in $D_0 \cup D_1^0$) depending on D_0 except at $root(s)$.

The third (and the main) condition essentially says that the nesting of defined symbols from D_1^0 is not allowed in the right-hand side terms of rules and no symbol from D_0 occurs below D_1^0-symbols.

Example 6. The following system $\mathscr{R}_1$ is a generalized nice-extension* of $\mathscr{R}_0$.

$\mathscr{R}_0$: `add(0, x)` $\to$ `x`
 `add(S(x), y)` $\to$ `S(add(x, y))`
 `mult(0, x)` $\to$ `0`
 `mult(S(x), y)` $\to$ `add(y,mult(x, y))`

$\mathscr{R}_1$: `fact(0)` $\to$ `1`
 `fact(S(x))` $\to$ `mult(S(x), fact(x))`

The following lemma characterizes the rules in generalized nice-extension*s.

Lemma 1. *If $\mathscr{R}_1$ is a generalized nice-extension* of $\mathscr{R}_0$ then for rewrite each rule $l \to r \in R_1$, r is of the form $C[t_1,\ldots,t_n]$, where C is a context without any*

D_1^0-symbols, $root(t_i) \in D_1^0$ and no proper subterm of t_i contains any function symbol depending on D_0 for each $i \in [1,n]$.

Proof. Follows from the condition (H) of Definition 6. □

5. Weak normalization

In this section, we establish modularity of weak normalization (WN) for generalized nice-extension∗s in term graph rewriting. First we prove that weak normalization (WN) is preserved under signature extensions.

5.1. Signature extensions

In this subsection, we establish that weak normalization (WN) is preserved under signature extensions. In the following, we consider a TRS $\mathscr{R}(\mathscr{F},R)$ with weakly normalizing graph rewriting relation $\Rrightarrow_{\mathscr{R}}$ and a set of function symbols $\mathscr{F}'$ such that $\mathscr{F} \cap \mathscr{F}' = \phi$.

Definition 7. Let $\{X_1,\ldots,X_n\}$ be a set of variables and $S = \{H_1,\ldots,H_n\}$ be a set of term graphs such that for each $i \in [1,n]$, (a) H_i is a normal form of $\Rrightarrow_{\mathscr{R}}$, (b) $root(term(H_i)) \in \mathscr{F}'$ and (c) $H_i \neq H_j$ for any $i \neq j \in [1,n]$. If G is a term graph such that (1) the subgraph G/v is a member of $\{H_1,\ldots,H_n\}$ for each node $v \in V$, where $V = \{$node u in G | the hyperedge outgoing from u is labeled with a function symbol in $\mathscr{F}'\}$, (2) no H_i occurs more than once in G and (3) variables $X_1,\ldots,X_n$ do not occur in G, we define $\Phi_S(G)$ as the term graph obtained from G by applying the following two steps at each node $v \in V$:
– remove the hyperedge outgoing from v,
– identify v with $root_{\diamond X_j}$ if $G/v = H_j$
and removing the garbage.

If $t \equiv term(G)$ then $term(\Phi_S(G))$ is the term obtained from t by replacing each occurrence of subterm $term(H_j)$ in t by variable X_j. The following lemma shows that there is a derivation from G corresponding to any derivation from $\Phi_S(G)$ with the same evaluation and folding steps.

Lemma 2. *Let S, G and Φ_S be as in the above definition. If $\Phi_S(G) \Rrightarrow_{\mathscr{R}} G_1$ then there is a term graph G_2 such that $G \Rrightarrow_{\mathscr{R}} G_2$ and $\Phi_S(G_2) = G_1$.*

Proof. If $\Phi_S(G) \Rrightarrow_{\mathscr{R}} G_1$ is a folding step, the lemma is obvious because folding cannot be applied on hyperedges corresponding to the variables $X_1,\ldots,X_n$ (since no such variable occurs more than once) and all the other hyperedges in $\Phi_S(G)$ also occur in G.

Assume that $\Phi_S(G) \Rightarrow_{\mathscr{R}} G_1$ is an evaluation step involving a rewrite rule $l \to r$ and hypergraph morphism $g : \underline{\Diamond l} \to \Phi_S(G)$. It is clear that g is also a hypergraph morphism from $\underline{\Diamond l}$ to G as each hyperedge in $\Phi_S(G)$ labeled by a function symbol or a variable outside $\{X_1, \ldots, X_n\}$ also occurs in G and corresponding to each X_i in $\Phi_S(G)$ there is a hyperedge in G. It is easy to see that $\Phi_S(G_2) = G_1$ if G_2 is the graph obtained by applying this evaluation step on G. □

Now, we are in a position to prove the main result of this subsection.

Theorem 2. *If the graph rewriting relation $\Rightarrow_{\mathscr{R}}$ of a* TRS *$\mathscr{R}(\mathscr{F}, R)$ is weakly normalizing then the graph rewriting relation $\Rightarrow_{\mathscr{R}'}$ of the* TRS *$\mathscr{R}'(\mathscr{F} \uplus \mathscr{F}', R)$ is weakly normalizing too.*

Proof. We prove that every term graph G has a normal form[3] w.r.t. $\Rightarrow_{\mathscr{R}'}$ using induction on the number of occurrences of function symbols from $\mathscr{F}'$ in G (denoted by $\#'(G)$).

Basis: $\#'(G) = 1$. We have two cases: (a) $root(term(G)) = f \in \mathscr{F}'$ and (b) $root(term(G)) = f \in \mathscr{F}$.

Case (a): If $root(term(G)) = f \in \mathscr{F}'$ then all the subgraphs corresponding to the arguments of f have normal forms as they do not contain any symbols from $\mathscr{F}'$. Let G' be the term graph obtained from G by normalizing all the subgraphs corresponding to the arguments of f. It is clear that $term(G')$ is a normal form as $root(term(G')) = f \in \mathscr{F}'$. Therefore G' can be reduced to a normal form through folding steps.

Case (b): Let H be the subgraph in G such that $root(term(H)) \in \mathscr{F}'$. By case (a), H has a normal form. Let G' be the term graph obtained from G by normalizing H to one of its normal forms, say H' and let $S = \{H'\}$. The term graph $\Phi_S(G')$ has a normal form G_1 as $\Phi_S(G')$ does not have any symbol from $\mathscr{F}'$ in it. By repeated applications of the above lemma we get a derivation $G' \Rightarrow^*_{\mathscr{R}'} G_2$ such that $\Phi_S(G_2) = G_1$. It is easy to see that $term(G_2)$ is a normal form as G_1 and H' are normal forms. Therefore G_2 can be reduced to a normal form through folding steps. Hence G has a normal form.

Induction Hypothesis. *Assume that each term graph G' with $\#'(G') \leqslant k$ has a normal form w.r.t. $\Rightarrow_{\mathscr{R}'}$, for some $k \geqslant 1$.*

Induction Step. Now we prove that each term graph G with $\#'(G) = k + 1$ has a normal form w.r.t. $\Rightarrow_{\mathscr{R}'}$. Again we have two cases: (a) $root(term(G)) = f \in \mathscr{F}'$ and (b) $root(term(G)) = f \in \mathscr{F}$.

Case (a): If $root(term(G)) = f \in \mathscr{F}'$ then all the subgraphs corresponding to the arguments of f have normal forms as they contain at the most k occurrences of

[3] By 'G has a normal form', we mean that 'G has at least one (possibly more) normal form(s)'.

function symbols from $\mathscr{F}'$. The rest of the proof is the same as that of case (a) of **Basis**.

Case (b): Let $\{H_1, \dots, H_k\}$ be the set of all the subgraphs in G such that $root(term(H_i)) \in \mathscr{F}'$. It is clear that $\#'(H_i) \leqslant k+1$. By case (a), each H_i has normal forms. The subgraphs $H_1, \dots, H_k$ are distinct graphs (otherwise G can be folded into a graph G_1 such that $\#'(G_1) \leqslant k$ and by hypothesis G_1 has a normal form and hence G has a normal form too). Let G' be the term graph obtained from G by normalizing each H_i to H_i' and doing all the possible folding steps. Let S be the set of all the subgraphs in G' with root in $\mathscr{F}'$. The term graph $\Phi_S(G')$ has a normal form G_1 as $\Phi_S(G')$ does not have any symbol from $\mathscr{F}'$ in it. By repeated applications of the above lemma we get a derivation $G' \Rrightarrow^*_{\mathscr{R}'} G_2$ such that $\Phi_S(G_2) = G_1$. It is easy to see that $term(G_2)$ is a normal form as G_1 and the subgraphs in S are normal forms. Therefore G_2 can be reduced to a normal form through folding steps. Hence G has a normal form. $\square$

5.2. Weak normalization of nice-extensions

Now, we establish modularity of weak normalization for generalized nice-extension∗s. We basically have to show that every term graph has a normal form with respect to the graph rewriting relation of the combined system. We do more than required by giving an algorithm to compute a normal form of any given term graph. This algorithm is described by the following relation $\longrightarrow_a$ on term graphs. In the following, we consider two systems $\mathscr{R}_0(D_0 \uplus D, C_0, R_0)$ and $\mathscr{R}_1(D_1 \uplus D, C_1, R_1)$ such that (1) the graph rewrite relations $\Rrightarrow_{\mathscr{R}_0}$ and $\Rrightarrow_{\mathscr{R}_1}$ are weakly normalizing and (2) $\mathscr{R}_1$ is a generalized nice-extension∗ of $\mathscr{R}_0$. Let $\Rrightarrow_{\mathscr{R}_i'}$ be the graph rewrite relation of TRS $\mathscr{R}_i'(\Sigma, R_i)$, where $\Sigma = D_0 \cup C_0 \cup D_1 \cup C_1 \cup D$. Basically, $\mathscr{R}_i'$ is a signature extension of $\mathscr{R}_i$ and by the above result $\Rrightarrow_{\mathscr{R}_i'}$ is weakly normalizing.

Definition 8. The relation $\longrightarrow_a$ on term graphs is defined as follows:

$H \longrightarrow_a H'$ if and only if there exists a node v in H such that

(1) H/v is reducible by $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ and no proper subgraph of H/v is reducible by $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$.
(2) H' is the term graph obtained from H by normalizing H/v w.r.t. $\Rrightarrow_{\mathscr{R}_1'}$ if $root(term(H/v)) \in D_1$.
(3) H' is the term graph obtained from H by normalizing H/v w.r.t. $\Rrightarrow_{\mathscr{R}_0'}$ if $root(term(H/v)) \notin D_1$.

We show that this relation gives an algorithm for normalizing a given term graph by establishing strong normalization (termination) of $\longrightarrow_a$. The following lemmas are needed.

Lemma 3. *If $\mathscr{R}_1$ is a generalized nice-extension∗ of $\mathscr{R}_0$ and G is a term graph such that no subgraph (say, H) of G with $root(term(H)) \in D_1$ is reducible by $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$,* *then any normal form of G with respect to $\Rightarrow_{\mathscr{R}'_0}$ is also a normal form of G with respect to $\Rightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$.*

Proof. Follows from the fact that the symbols from D_1 do not occur in right-hand side terms of $\mathscr{R}_0$. □

The above lemma is useful in establishing that normalization of an innermost redex w.r.t. $\Rightarrow_{\mathscr{R}'_0}$ decreases the depth of innermost redexes. Now, we go about proving similar lemma about innermost redexes of $\Rightarrow_{\mathscr{R}'_1}$. For that purpose, we consider the following class of *single layered term graphs* where $(D_1^0 - D)$ symbols are not nested above any redex. That is, no term graph G in this class has two different nodes p and q such that (i) $root(term(G/p)) \in (D_1^0 - D)$ and G/p is reducible, (ii) $root(term(G/q)) \in (D_1^0 - D)$ and G/q is reducible and (iii) p is above q or q is above p.

Definition 9. Let $Topd10(G)$ denote the set of nodes $\{v \mid root(term(G/v)) \in (D_1^0 - D)$ and for each node u above v, $root(term(G/u)) \notin (D_1^0 - D)\}$. We define $\mathscr{S}$ as the set of all term graphs such that each $G \in \mathscr{S}$ satisfies the following: *no proper subgraph H of G/v with $root(term(H)) \in (D_0 \cup D_1^0)$ is reducible by $\Rightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ for any node $v \in Topd10(G)$.*

The following lemma establishes that $\mathscr{S}$ is closed under $\Rightarrow_{\mathscr{R}'_1}$.

Lemma 4. *If $G \Rightarrow_{\mathscr{R}'_1} G'$ and $G \in \mathscr{S}$ then, $G' \in \mathscr{S}$* too.

Proof. It is obvious that the lemma holds if $G \Rightarrow_{\mathscr{R}'_1} G'$ is a folding step. Let $l \to r$ be the rewrite rule applied in $G \Rightarrow_{\mathscr{R}_1} G'$. There are two cases: (a) $root(l) \in (D_1^0 - D)$ and (b) $root(l) \notin (D_1^0 - D)$.

Case (a): In this case, the reduction should take place at some $v \in Topd10(G)$. It follows from Lemma 1 that r is of the form $C[t_1, \ldots, t_n]$ such that C is a context without any D_1^0-symbols, $root(t_i) \in D_1^0$ and t_i does not contain any function symbol from $D_0 \cup D_1^0$ except at the root for each $i \in [1, n]$. It is easy to see that $Topd10(G') = Topd10(G) \cup \{u_i \mid u_i$ is the node in G' corresponding to t_i in $r\} - \{ v \}$. Since proper subgraphs of G/v with root in $(D_0 \cup D_1^0)$ are irreducible, it follows that the proper subgraphs of G'/u_i with root in $(D_0 \cup D_1^0)$ are irreducible as well. Therefore $G' \in \mathscr{S}$.

Case (b): In this case, no symbol from $D_1^0 - D$ occurs in r and hence the reduction $G \Rightarrow_{\mathscr{R}'_1} G'$ does not add any symbol from $D_1^0 - D$. It is easy to see that $G' \in \mathscr{S}$. □

From the above lemma, it follows that normal form of $G \in \mathscr{S}$ w.r.t. $\Rightarrow_{\mathscr{R}'_1}$ is in $\mathscr{S}$ as well.

Lemma 5. *If G' is a normal form of G w.r.t. $\Rightarrow_{\mathscr{R}'_1}$ and $G \in \mathscr{S}$ then, $G' \in \mathscr{S}$ too. Further, each subgraph H of G' with $root(term(H)) \in (D_1^0 - D)$ is irreducible by $\Rightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$.*

Now, we are in a position to establish strong normalization of $\rightarrow_a$.

Theorem 3. *If $\mathscr{R}_1$ is a generalized nice-extension* of $\mathscr{R}_0$ then the relation $\rightarrow_a$ defined in Definition 8 is strongly normalizing.*

Proof. We basically prove this using a terminating function f. For any given term graph G, $f(G)$ is an ordered pair $\langle f_1(G), f_2(G)\rangle$, where f_1 and f_2 are defined as follows: $f_1(G)$ is the set $\{\ v \mid root(term(G/v)) \in (D_1^0 - D)$, G/v is reducible by $\Rightarrow_{\mathscr{R}_0\cup\mathscr{R}_1}$ and no proper subgraph H of G/v with $root(term(H)) \in (D_1^0 - D)$ is reducible by $\Rightarrow_{\mathscr{R}_0\cup\mathscr{R}_1}\ \}$ of deepest reducible $(D_1^0 - D)$-nodes in G and $f_2(G)$ is the set $\{\ v \mid G/v$ is reducible by $\Rightarrow_{\mathscr{R}_0\cup\mathscr{R}_1}$ and no proper subgraph of G/v is reducible $\}$ of innermost redex nodes.

We use the lexicographic ordering $\prec$ induced by two orderings $\prec_1$ and $\prec_2$ on the two components as the well-founded ordering. The orderings $\prec_1$ and $\prec_2$ are multiset orderings induced by the predecessor relation $<_G$ over nodes in hypergraph G defined in Section 2.

To establish strong normalization of $\rightarrow_a$, it is enough to show that $f(G) \succ f(G')$ whenever $G \rightarrow_a G'$. Assume that the subgraph at node v in G is normalized in $G \rightarrow_a G'$. There are two cases: (a) $root(term(G/v)) \notin D_1$, and (b) $root(term(G/v)) \in D_1$.

Case (a): It is obvious that $f_2(G) \succ_2 f_2(G')$ as the normal form of G/v w.r.t. $\Rightarrow_{\mathscr{R}_0'}$ is a normal form w.r.t. $\Rightarrow_{\mathscr{R}_0\cup\mathscr{R}_1}$ by Lemma 3. Let w be the nearest $(D_1^0 - D)$ node above v in G. By definition, w is in $f_1(G)$. It is easy to see that no node below w can occur in $f_1(G')$ and in fact w itself *may* not be there in $f_1(G')$, i.e., G'/w may not be reducible. This clearly shows $f(G) \succ f(G')$.

Case (b): There are two subcases: (i) $root(term(G/v)) \in D_1^0$ and (ii) $root(term(G/v)) \in D_1^1$. In subcase (i) it's obvious that $f_1(G) \succ_1 f_1(G')$ by Lemma 5. In subcase (ii), the normal form of G/v w.r.t. $\Rightarrow_{\mathscr{R}_1'}$ is also a normal form of $\Rightarrow_{\mathscr{R}_0\cup\mathscr{R}_1}$ as D_1^1 symbols do not depend on D_0 and hence $f_2(G) \succ_2 f_2(G')$. Further, $f_1(G) \succeq_1 f_1(G')$ as in case (a), i.e., the nearest $(D_1^0 - D)$ node w above v in G may or may not be in $f_1(G')$. Therefore, $f(G) \succ f(G')$. This completes the proof. $\square$

From this result, it follows that weak normalization (WN) is modular for the class of generalized nice-extension*s.

Theorem 4. *Let $\mathscr{R}_0$ and $\mathscr{R}_1$ be two term rewriting systems such that $\mathscr{R}_1$ is a generalized nice-extension* of $\mathscr{R}_0$. Then $\Rightarrow_{\mathscr{R}_0\cup\mathscr{R}_1}$ is weakly normalizing (WN) if $\Rightarrow_{\mathscr{R}_0}$ and $\Rightarrow_{\mathscr{R}_1}$ are weakly normalizing (WN).*

6. Semi-completeness

In this section, we study modularity of semi-completeness in term graph rewriting for generalized nice-extension*s. In contrast to weak normalization, semi-completeness is not modular for the whole class of generalized nice-extension*s as shown by the following counterexample.

Example 7. The following TRSs are complete and hence the graph rewriting relations $\Rrightarrow_{\mathscr{R}_0}$ and $\Rrightarrow_{\mathscr{R}_1}$ are complete (and hence semi-complete). Further, $\mathscr{R}_1$ is a generalized nice-extension* of $\mathscr{R}_0$.

$\mathscr{R}_0$: $\mathtt{g(x, y)} \to \mathtt{y}$ $\quad\quad$ $\mathscr{R}_1$: $\mathtt{f(g(x, y))} \to \mathtt{x}$

However, $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ is not semi-complete — the term graph representing term f(g(x, y)) has two different normal forms x and f(y).

The following definition is neeeded in the sequel.

Definition 10. Let $l_1 \to r_1$ and $l_2 \to r_2$ be renamed versions of rewrite rules of a term rewriting system $\mathscr{R}$ such that they have no variables in common. Suppose $l_{1|p}$ is not a variable for some position p and $l_{1|p}$ unifies with l_2 through a most general unifier σ. *The pair of terms* $\langle l_1[r_2]_p\sigma,\ r_1\sigma\rangle$ *is called a critical pair of* $\mathscr{R}$. If $l_1 \to r_1$ and $l_2 \to r_2$ are renamed versions of the same rewrite rule, we do not consider the case $p = \varepsilon$. A critical pair $\langle l_1[r_2]_p\sigma,\ r_1\sigma\rangle$ with $p = \varepsilon$ is called an *overlay*.

In the following, we prove modularity of semi-completeness for a class of generalized nice-extension*s. The proof is based on Theorems 1 and 4 and the following result from [7].

Theorem 5 (Krishna Rao [7]). *Let $\mathscr{R}_0$ and $\mathscr{R}_1$ be two semi-complete* TRSs *such that*
(1) *$\mathscr{R}_1$ is a generalized nice-extension* of $\mathscr{R}_0$,*
(2) *if $l \to r \in \mathscr{R}_i$ and s is a subterm of l with root$(s) \in D_{1-i}$ then no subterm of s unifies with left-hand side term of any rule in $\mathscr{R}_0 \cup \mathscr{R}_1$ and*
(3) *if $l \to r \in \mathscr{R}_1$ and root$(l) \in D_1^0 - D$ then all the critical pairs involving this rule are overlays.*
Then the combined system $\mathscr{R}_0 \cup \mathscr{R}_1$ is semi-complete as well.

The following theorem establishes modularity of semi-completeness.

Theorem 6. *Let $\mathscr{R}_0$ and $\mathscr{R}_1$ be two* TRSs *satisfying the* 3 *conditions of Theorem* 5. *Then $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ is semi-complete if $\Rrightarrow_{\mathscr{R}_0}$ and $\Rrightarrow_{\mathscr{R}_1}$ are semi-complete.*

Proof. By statements (2) and (3) of Theorem 1, TRSs $\mathscr{R}_0$ and $\mathscr{R}_1$ are semi-complete. By Theorem 5, TRS $\mathscr{R}_0 \cup \mathscr{R}_1$ is semi-complete. By Theorem 4, $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ is weakly normalizing. Now, it follows from statement (4) of Theorem 1 that $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ is semi-complete. $\square$

7. Strong normalization

In this section, we study modularity of strong normalization (termination) in term graph rewriting for generalized nice-extension*s. We first prove that termination is modular for special subclasses of generalized nice-extension*s, namely, signature extensions and crosswise independent unions, and then extend the result for generalized nice-extension*s.

Termination of term graph rewriting is preserved under signature extensions.

Theorem 7. *If the graph rewriting relation $\Rightarrow_{\mathscr{R}}$ of a* TRS *$\mathscr{R}(\mathscr{F}, R)$ is noetherian then the graph rewriting relation $\Rightarrow_{\mathscr{R}'}$ of the* TRS *$\mathscr{R}'(\mathscr{F} \uplus \mathscr{F}', R)$ is noetherian too.*

Proof. Easy.[4] □

Notation. Exploiting the above theorem, we slightly abuse the notation in the sequel by using $\Rightarrow_{\mathscr{R}_i}$ to denote the graph rewriting relation of the signature-extension $\mathscr{R}_i'(D \cup D_0 \cup D_1 \cup C_0 \cup C_1, R_i)$, $i \in [0,1]$.

The following definition and lemma are needed in the sequel.

Definition 11. We define a *subgraph* relation $\Rightarrow_{sub}$ over term graphs as follows: $G \Rightarrow_{sub} G'$ if and only if $G' \equiv G/v$ for some non-root node $v \in V_G$.

Lemma 6. *Let S be a set of term graphs and $\mathscr{R}$ be a* TRS *with graph rewrite relation $\Rightarrow_{\mathscr{R}}$. Then, the relation $\Rightarrow_{\mathscr{R}} \cup \Rightarrow_{sub}$ is noetherian over S if and only if $\Rightarrow_{\mathscr{R}}$ is noetherian over S.*

Proof. The only-if part is obviously true and we prove the if part below.

It is clear that there must be infinitely many $\Rightarrow_{\mathscr{R}}$-steps in an infinite derivation of $\Rightarrow_{\mathscr{R}} \cup \Rightarrow_{sub}$. Since the subgraph-steps do not create new redex positions, for all G_1, G_2, G_3 with $G_1 \Rightarrow_{sub}^{+} G_3 \Rightarrow_{\mathscr{R}}^{+} G_2$ there exists a G_4 such that $G_1 \Rightarrow_{\mathscr{R}}^{+} G_4 \Rightarrow_{sub}^{+} G_2$. That is, the subgraph-steps can always be postponed and hence one can get an infinite derivation of $\Rightarrow_{\mathscr{R}}$ from any infinite derivation of $\Rightarrow_{\mathscr{R}} \cup \Rightarrow_{sub}$ by postponing the subgraph-steps. In other words, there cannot be any infinite derivation of $\Rightarrow_{\mathscr{R}} \cup \Rightarrow_{sub}$ if there is no infinite derivation of $\Rightarrow_{\mathscr{R}}$, i.e., the relation $\Rightarrow_{\mathscr{R}} \cup \Rightarrow_{sub}$ is noetherian if $\Rightarrow_{\mathscr{R}}$ is noetherian. □

7.1. Termination of crosswise independent unions

In this subsection, we study modularity of termination for crosswise independent unions in term graph rewriting.

[4] In fact this theorem follows from Plump's result on termination of crosswise disjoint union. However, the proof of this theorem is much simpler than the proof for crosswise disjoint union.

Definition 12. We say that two TRSs $\mathscr{R}_0(D_0 \uplus D, C_0, R_0)$ and $\mathscr{R}_1(D_1 \uplus D, C_1, R_1)$ are *crosswise independent* if (i) $R_0 \cap R_1 = \{l \to r \in R_0 \cup R_1 \mid root(l) \in D\}$ *and* (ii) $f_i \not\geq_d f_{1-i}$ *for each* $f_i \in D_i \cup D$ *and* $f_{1-i} \in D_{1-i}$, *where* $i \in \{0, 1\}$.

We say that $\mathscr{R}_0 \cup \mathscr{R}_1$ *is a crosswise independent union if* $\mathscr{R}_0$ *and* $\mathscr{R}_1$ *are crosswise independent.*

The notion of crosswise independent union is a generalization of constructor sharing (ao. [4, 12]) and composable unions (ao. [14, 15]). In the following, we consider two crosswise independent systems $\mathscr{R}_0(D_0 \uplus D, C_0, R_0)$ and $\mathscr{R}_1(D_1 \uplus D, C_1, R_1)$ such that $\Rightarrow_{\mathscr{R}_0}$ and $\Rightarrow_{\mathscr{R}_1}$ are *terminating*. We denote the combined system $\mathscr{R}_0 \cup \mathscr{R}_1$ by $\mathscr{R}$.

The following two lemmas are useful.

Lemma 7. *If* $G \Rightarrow_{\mathscr{R}} G'$ *then* $rank(G) \geqslant rank(G')$.

Definition 13. The relation $\Rightarrow_{sub_i}$, $i \in [0, 1]$ over term graphs is defined as $(\Rightarrow_{\mathscr{R}_i} \cup \Rightarrow_{sub})^+$. The multiset extension of $\Rightarrow_{sub_i}$ is denoted by $\Rightarrow^m_{sub_i}$.

By Lemma 6, it follows that $\Rightarrow_{sub_i}$ and $\Rightarrow^m_{sub_i}$ are noetherian.

Lemma 8. *The relations* $\Rightarrow_{sub_i}$ *and* $\Rightarrow^m_{sub_i}$ *are noetherian for* $i \in [0, 1]$.

Now, we prove that $\Rightarrow_{\mathscr{R}}$ is terminating. The proof is by induction on rank. A top transparent term can be uniquely written as $C[t_1, \ldots, t_n]$ such that C is a context of transparent symbols and t_i are top black/white terms.

Definition 14. For a top transparent graph G with $term(G) \equiv C[t_1, \ldots, t_n]$ such that C is a context of transparent symbols and each t_i is a top black/white term occurring at position p_i, we define

- (**top**): The top of G (denoted $top(G)$) is defined as *the term graph obtained from* G *by replacing the* (*unique*) *hyperedges outgoing from the nodes* $node_G(p_1), \ldots, node_G(p_n)$ *by* n *hyperedges representing* $\Box$ *and removing the garbage.*
- (**aliens**): The multiset of aliens in G is defined as $aliens(G) = \{G/v \mid v \in V\}$, where V is the set $\{node_G(p_i) \mid 1 \leqslant i \leqslant n$ and $node_G(p_i)$ is not strictly below $node_G(p_j)$ for any $j \neq i\}$. The multiset of top black and top white aliens of G are denoted by $A_0(G)$ and $A_1(G)$, respectively.
- (*inner/outer reductions*): We say an evaluation step $G \Rightarrow_{\mathscr{E}} G'$ is *inner* if the reduction took place in an element of $aliens(G)$ (i.e., below some node $v \in V$), otherwise it is an outer reduction. A folding step $G \Rightarrow_{\mathscr{F}} G'$ is *inner* if at least one of the two identified (merged) hyperedges is from $aliens(G)$, otherwise it is an outer reduction.

Note that $top(G)$ and the aliens $A_0(G)$ and $A_1(G)$ may share some parts of G.

The following lemma helps in ignoring folding steps in the proofs below.

Lemma 9. *If $G \Rrightarrow G'$ is an outer reduction and it is a folding step, then* (i) *$aliens(G') = aliens(G)$ and* (ii) *$top(G) \Rrightarrow top(G')$.*

Proof. A folding step identifies two hyperedges only when their target strings are identical and the graph remains the same except that two hyperedges are identified (merged). Since both the identified hyperedges are in $top(G)$, the aliens are not disturbed. Therefore $aliens(G') = aliens(G)$ and $top(G) \Rrightarrow top(G')$. □

The following characteristic lemma about graph reductions is repeatedly used in the sequel.

Lemma 10. *If $G \Rrightarrow G'$ is an outer reduction involving an evaluation step over top transparent graphs G and G', then $aliens(G) \succeq_{sub} aliens(G')$, where $\succ_{sub}$ is the multiset extension of $\Rrightarrow_{sub}$.*

Proof. The applicable rewrite rules are $\{l \to r \mid root(l) \in D\}$ and no function symbols from $D_0 \cup D_1$ occur in the right-hand sides of these rules. Therefore, no new subgraphs with root in $D_0 \cup D_1$ are created in any outer reduction. Further, some hyperedges in G corresponding to function symbols (e.g., b in the following Example) from $D_0 \cup D_1$ occurring in the left-hand side of the applied rule might be deleted in the garbage collection. The lemma follows from these observations and the fact that no proper subgraph of an alien is again an alien (see V in Definition 14). □

We first show that $\Rrightarrow_{\mathscr{R}}$ is noetherian over term graphs of rank 1. For this purpose, a noetherian relation $\succ$ is defined over the set of top transparent term graphs of rank 1.

Definition 15. The relation $\succ$ over the set of top transparent term graphs of rank at most 1 is defined as: $G \succ G'$ if and only if
(a) $A_0(G) \Rrightarrow^m_{sub_0} A_0(G')$ or
(b) $A_0(G) = A_0(G')$ and $A_1(G) \Rrightarrow^m_{sub_1} A_1(G')$ or
(c) $A_0(G) = A_0(G')$, $A_1(G) = A_1(G')$ and $G \Rrightarrow_{\mathscr{R}_0 \cap \mathscr{R}_1} G'$.

Lemma 11. *The relation $\succ$ over the set of top transparent term graphs of rank at most* 1 *is noetherian.*

Proof. By Lemma 8, $\Rrightarrow_{sub_0}$ and $\Rrightarrow_{sub_1}$ and their multiset-extensions are noetherian. Since the only rewrite rules in $\mathscr{R}_0 \cap \mathscr{R}_1$ are $\{l \to r \in R_0 \cup R_1 \mid root(l) \in D\}$ (a subsystem of $\mathscr{R}_i$, $i \in [0,1]$), the relation $\Rrightarrow_{\mathscr{R}_0 \cap \mathscr{R}_1}$ is noetherian. Therefore, $\succ$ is noetherian as it is a lexicographic extension of three noetherian relations $\Rrightarrow^m_{sub_0}$, $\Rrightarrow^m_{sub_1}$ and $\Rrightarrow_{\mathscr{R}_0 \cap \mathscr{R}_1}$. □

Theorem 8. *The graph rewriting relation $\Rrightarrow_{\mathscr{R}}$ is noetherian over term graphs of rank at most* 1.

Proof. The theorem obviously holds for term graphs of rank 0 as the only rewrite rules applicable on them are $\{l \to r \in R_0 \cup R_1 \mid root(l) \in D\}$ (a subsystem of $\mathscr{R}_i$, $i \in [0,1]$). We only have to consider term graphs of rank 1. Assume to the contrary that there is an infinite derivation $G_1 \Rrightarrow_{\mathscr{R}} G_2 \Rrightarrow_{\mathscr{R}} \cdots$ over term graphs of rank 1. If $root(term(G_k)) \in D_i$, $i \in [0,1]$ for some k, it is clear that each step after that is a $\Rrightarrow_{\mathscr{R}_i}$-step as any top black (white) term graph of rank 1 does not contain any white (resp. black) symbols. That is, if $root(term(G_k)) \in D_i$, there is an infinite derivation $G_k \Rrightarrow_{\mathscr{R}_i} G_{k+1} \Rrightarrow_{\mathscr{R}_i} G_{k+2} \Rrightarrow_{\mathscr{R}_i} \cdots$, contradicting the termination of $\Rrightarrow_{\mathscr{R}_i}$. Therefore, each $G_k, k \geqslant 0$ is top transparent.

We prove that there can be no infinite derivation $G_1 \Rrightarrow_{\mathscr{R}} G_2 \Rrightarrow_{\mathscr{R}} G_3 \Rrightarrow_{\mathscr{R}} \cdots$ over top transparent term graphs of rank 1 by showing that $G \Rrightarrow_{\mathscr{R}} G'$ implies $G \succ G'$. If the reduction took place in $A_0(G)$, it is obvious that $A_0(G) \Rrightarrow^m_{sub_0} A_0(G')$. Similarly, $A_1(G) \Rrightarrow^m_{sub_1} A_1(G')$ and $A_0(G) = A_0(G')$ if the reduction took place in $A_1(G)$ but not in (any part shared with) $A_0(G)$. In both these cases $G \succ G'$. If $G \Rrightarrow_{\mathscr{R}} G'$ is an outer reduction, it is clear that $G \Rrightarrow_{\mathscr{R}_0 \cap \mathscr{R}_1} G'$ and $aliens(G) \succeq_{sub} aliens(G')$ by Lemma 10. Now, it is easy to see that one of the 3 requirements of Def. 15 holds and hence $G \succ G'$. □

Now, we prove that $\Rrightarrow_{\mathscr{R}}$ is noetherian on the set of term graphs of arbitrary rank by induction. First, we prove that there is no infinite derivation from a top black term graph of rank k if $\Rrightarrow_{\mathscr{R}}$ is noetherian on the set of term graphs of rank less than k. The proof is similar for top white term graphs. We need the following definition and lemma.

Definition 16. The relation $\Rrightarrow_{sub_{01}}$ over term graphs is defined as $(\Rrightarrow_{\mathscr{R}} \cup \Rrightarrow_{sub})^+$. The multiset extension of $\Rrightarrow_{sub_{01}}$ is denoted by $\Rrightarrow^m_{sub_{01}}$.

Lemma 12. *The relations $\Rrightarrow_{sub_{01}}$ and $\Rrightarrow^m_{sub_{01}}$ are noetherian over term graphs of rank less than k if $\Rrightarrow_{\mathscr{R}}$ is noetherian over term graphs of rank less than k.*

Any term t of rank $k > 1$ such that $root(t) \notin D_1$ can be written as $C_b[t_1, \cdots, t_n]$ in a unique way, where C_b is a context of black and transparent symbols and $root(t_i) \in D_1$ for each $i \in [1,n]$. Notions of top_0 and $aliens_0$ can be defined for top black/transparent term graphs of rank k similar to the notions *top* and *aliens* defined in Definition 14. Note that the (top white) aliens of such a term graph are of rank less than k.

Theorem 9. *If the graph rewriting relation $\Rrightarrow_{\mathscr{R}}$ is noetherian over term graphs of rank less than k, there is no infinite $\Rrightarrow_{\mathscr{R}}$ derivation starting from a top black term graph of rank k.*

Proof. Since white-aliens of a top black term graph of rank k are of rank less than k, there cannot be a top white term graph in an infinite $\Rrightarrow_{\mathscr{R}}$ derivation starting from a top black term graph of rank k. So, the notions top_0 and $aliens_0$ are well-defined for each term graph in the derivation. Now, we show that there cannot be an infinite derivation by proving that $G \Rightarrow_{\mathscr{R}} G'$ implies: (i) $aliens_0(G) \Rightarrow^m_{sub_{01}} aliens_0(G')$ or (ii) $aliens_0(G') = aliens_0(G)$ and $G \Rightarrow_{\mathscr{R}_0} G'$. Since, $\Rightarrow_{sub_{01}}$ is noetherian over term graphs of rank less than k and $\Rightarrow_{\mathscr{R}_0}$ is noetherian, it follows that there cannot be an infinite $\Rightarrow_{\mathscr{R}}$ derivation starting with a top black term graph of rank k.

If $G \Rightarrow_{\mathscr{R}} G'$ took place in an alien of G, it is obvious that $aliens_0(G) \Rightarrow^m_{sub_{01}} aliens_0(G')$. In the other case, it is clear that $G \Rightarrow_{\mathscr{R}_0} G'$ and $aliens_0(G) \succeq_{sub} aliens_0(G')$. This implies that either (i) or (ii) above holds. □

Now, we prove the result for arbitrary term graphs of rank k.

Theorem 10. *If the graph rewriting relation $\Rightarrow_{\mathscr{R}}$ is noetherian over term graphs of rank less than k, then $\Rightarrow_{\mathscr{R}}$ is noetherian over term graphs of rank k as well.*

Proof. By the above lemma, $\Rightarrow_{\mathscr{R}}$ is noetherian over top black (white) term graphs of rank k. As in Theorem 8, every tree in an infinite $\Rightarrow_{\mathscr{R}}$ derivation starting with a top transparent term graph of rank k is top transparent. We can prove that there cannot be an infinite $\Rightarrow_{\mathscr{R}}$ derivation over top transparent term graphs of rank k by showing that $G \Rightarrow_{\mathscr{R}} G'$ implies (a) $A_0(G) \Rightarrow^m_{sub_{01}} A_0(G')$ or (b) $A_0(G) = A_0(G')$ and $A_1(G) \Rightarrow^m_{sub_{01}} A_1(G')$ or (c) $A_0(G) = A_0(G')$, $A_1(G) = A_1(G')$ and $G \Rightarrow_{\mathscr{R}_0 \cap \mathscr{R}_1} G'$. The proof of this fact runs on the same lines as in Theorem 8. □

From the above two theorems, we get the main result of this subsection.

Theorem 11. *Let $\mathscr{R}_0$ and $\mathscr{R}_1$ be two crosswise independent systems with graph rewrite relations $\Rightarrow_{\mathscr{R}_0}$ and $\Rightarrow_{\mathscr{R}_1}$. Then, $\Rightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ terminates if and only if both $\Rightarrow_{\mathscr{R}_0}$ and $\Rightarrow_{\mathscr{R}_1}$ terminate.*

7.2. Termination of nice extensions

In this subsection, we establish the modularity of termination for generalized nice-extension∗s. In the following, we consider two systems $\mathscr{R}_0(D_0 \uplus D, C_0, R_0)$ and $\mathscr{R}_1(D_1 \uplus D, C_1, R_1)$ such that (i) the graph rewrite relations $\Rightarrow_{\mathscr{R}_0}$ and $\Rightarrow_{\mathscr{R}_1}$ are terminating, (ii) $\mathscr{R}_1$ is a generalized nice-extension∗ of $\mathscr{R}_0$ and (iii) function symbols from D_1^0 occur in l only at the outermost level for any rule $l \to r$. We denote the combined system $\mathscr{R}_0 \cup \mathscr{R}_1$ by $\mathscr{R}$. Now, we define a measure, which does not increase after a reduction in generalized nice-extension∗s.

Definition 17. The *level* of a term t is defined inductively as follows:
(a) $level(t) = 0$ if t is a variable or a constant not in D_1^0.
(b) $level(t) = 1$ if t is a constant in D_1^0
(c) $level(f(t_1, \cdots, t_n)) = max(\{level(t_j) \mid j \in [1,n]\})$, if $f \notin D_1^0$.
(d) $level(f(t_1, \cdots, t_n)) = 1 + max(\{level(t_j) \mid j \in [1,n]\})$, if $f \in D_1^0$.

The *level* of a term graph G is defined as $level(term(G))$. The following two lemmas are easy.

Lemma 13. *$level(r) \leqslant level(l) \leqslant 1$ for each rule $l \to r$ in $\mathscr{R}_0 \cup \mathscr{R}_1$.*

Lemma 14. *If $G \Rightarrow_{\mathscr{R}} G'$, then $level(G) \geqslant level(G')$.*

7.2.1. Term graphs of level 0

Term graphs of level 0 contain function symbols from $Constr \cup D_0 \cup D_1^1$ and the function symbols D_1^0 are not reachable from these term graphs. Therefore the rules applicable on these term graphs are those in $\mathscr{R}_0 \cup \mathscr{R}_1'$, where $\mathscr{R}_1' = \{l \to r \in R_1 \mid root(l) \in D_1^1\}$. It is easy to see that $\mathscr{R}_0$ and $\mathscr{R}_1'$ are crosswise independent and hence $\Rightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1'}$ is terminating by Theorem 11.

Theorem 12. *The graph rewriting relation $\Rightarrow_{\mathscr{R}}$ is noetherian over term graphs of level 0.*

Definition 18. We define a relation $\Rightarrow_{sub_{01'}}$ over term graphs as $(\Rightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1'} \cup \Rightarrow_{sub})^+$. The multiset extension of $\Rightarrow_{sub_{01'}}$ is denoted by $\Rightarrow^m_{sub_{01'}}$.

By Lemma 6, the relation $\Rightarrow_{sub_{01'}}$ is noetherian.

7.2.2. Term graphs of level 1

We first consider term graphs of level 1 with root in D_1^0. A term t of level 1 with $root(t) \in D_1^0$ can be uniquely written as $C[t_1, \ldots, t_n]$, $n \geqslant 0$ such that C is a context of function symbols in $Constr \cup D \cup D_1$, t_i occurs at position p_i and $root(t_i) \in D_0$ for each $i \in [1, n]$. For a term graph G representing the above t, $D_0sub(G)$ denotes the multiset of subgraphs $\{G/v \mid v \in V\}$, where V is the set $\{node_G(p_i) \mid 1 \leqslant i \leqslant n$ and $node_G(p_i)$ is not below $node_G(p_j)$ for any $j \neq i\}$.

Definition 19. The relation $\succ_1$ over the set of term graphs of level 1 with outermost function symbol in D_1^0 is defined as: $G \succ_1 G'$ if and only if
(a) $D_0sub(G) \Rightarrow^m_{sub_{01'}} D_0sub(G')$ or
(b) $D_0sub(G) = D_0sub(G')$ and $G \Rightarrow_{sub_1} G'$.

The relation $\succ_1$ is noetherian as both $\Rightarrow_{sub_1}$ and $\Rightarrow_{sub_{01'}}$ are noetherian.

A term t of level 1 can be uniquely written as $C[t_1, \ldots, t_n]$, $n \geqslant 0$ such that C is a context of function symbols in $Constr \cup D_0 \cup D_1^1$ and $root(t_i) \in D_1^0$ for each $i \in [1, n]$. For a term graph G representing the above t, $D_1^0sub(G)$ denotes the multiset of subgraphs of G corresponding to the subterms $t_1, \ldots, t_n$. Note that D_1^0-symbols are not nested in any term of level 1 and we do not need to use V here unlike in defining $D_0sub(G)$.

Lemma 15. *If G and G' are two term graphs of level 1 such that $root(term(G)) \in D_1^0$ and $G \Rightarrow_{\mathscr{R}} G'$ then $D_1^0sub(G) \gg_1 D_1^0sub(G')$, where $\gg_1$ is the multiset-extension of $\succ_1$.*

Proof. First of all note that $D_1^0 sub(G) = \{G\}$. If the reduction took place in an element of $D_0 sub(G)$, it is obvious that (i) $D_1^0 sub(G') = \{G'\}$ and (ii) $D_0 sub(G) \Rrightarrow^m_{sub_{01}'} D_0 sub(G')$ and hence $G \succ_1 G'$. Therefore $D_1^0 sub(G) \gg_1 D_1^0 sub(G')$. If the reduction took place outside the subgraphs in $D_0 sub(G)$, we have two subcases: the reduction took place (1) below the root of G or (2) at the root.

Subcase (1): If the reduction took place below the root, $D_1^0 sub(G') = \{G'\}$ and either the reduction step is a folding step or the applied rule $l \to r \in \mathscr{R}_1'$. In either case, $G \Rrightarrow_{\mathscr{R}_1} G'$ and no hyperedges corresponding to functions in D_0 are added, i.e., $D_0 sub(G) = D_0 sub(G')$ or $D_0 sub(G) \Rrightarrow^m_{sub_{01}'} D_0 sub(G')$. Therefore, $G \succ_1 G'$ and hence $D_1^0 sub(G) = \{G\} \gg_1 \{G'\} = D_1^0 sub(G')$.

Subcase (2): If the reduction took place at the root of G, $D_1^0 sub(G') = \{G_1, \ldots, G_m\}$, $m \geqslant 0$ such that $D_0 sub(G_i) = D_0 sub(G')$ or $D_0 sub(G) \Rrightarrow^m_{sub_{01}'} D_0 sub(G_i)$ for each $i \in [1, m]$ (note: by condition (H) of Definition 6, D_0 symbols do not occur in right-sides of rules in $\mathscr{R}_1$ *below* D_1^0 *symbols*). Further, it is easy to see that $G \Rrightarrow_{sub_1} G_i$ for each $i \in [1, m]$. To sum up, $G \succ_1 G_i$ for each $i \in [1, m]$ and hence $D_1^0 sub(G) = \{G\} \gg_1 D_1^0 sub(G')$. □

Now, we are in a position to show that $\Rrightarrow_{\mathscr{R}}$ is noetherian over term graphs of level 1.

Theorem 13. *The graph rewriting relation* $\Rrightarrow_{\mathscr{R}}$ *is noetherian over term graphs of level* 1.

Proof. We prove this by showing that $G \Rrightarrow_{\mathscr{R}} G'$ implies (a) $D_1^0 sub(G) \gg_1 D_1^0 sub(G')$ or (b) $D_1^0 sub(G) = D_1^0 sub(G')$ and $G \Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1'} G'$. Since both $\gg_1$ and $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1'}$ are noetherian, it follows that $\Rrightarrow_{\mathscr{R}}$ is noetherian.

There are two cases: (1) the reduction took place in an element of $D_1^0 sub(G)$ or (2) the reduction took place outside the subgraphs in $D_1^0 sub(G)$. By the above lemma $D_1^0 sub(G) \gg_1 D_1^0 sub(G')$ in case (1). Consider case (2). If the reduction took place outside the subgraphs in $D_1^0 sub(G)$, either the reduction step is a folding step or the applied rule $l \to r \in \mathscr{R}_0 \cup \mathscr{R}_1'$. In either case, $G \Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1'} G'$ and no hyperedges corresponding to functions in D_1^0 are added, i.e., $D_1^0 sub(G') = D_0 sub(G)$ or $D_1^0 sub(G) \gg_1 D_1^0 sub(G')$. This completes the proof. □

7.2.3. Term graphs of level greater than 1

Definition 20. A term t of level k can be uniquely written as $C[t_1, \ldots, t_n]$, $n \geqslant 0$ such that C is a context of level 1, t_i occurs at position p_i, $root(t_i) \in D_1^0$ and $rank(t_i) < k$ for each $i \in [1, n]$. For a term graph G representing the above t,

- $D_1^0 sub_k(G)$ denotes the multiset of subgraphs $\{G/v \mid v \in V\}$, where V is the set $\{node_G(p_i) \mid 1 \leqslant i \leqslant n$ and $node_G(p_i)$ is not below $node_G(p_j)$ for any $j \neq i\}$.
- $cap_k(G)$ denotes the term graph obtained from G by replacing the (unique) hyperedges outgoing from the nodes $node_G(p_1)$, …, $node_G(p_n)$ by n hyperedges representing □ and removing the garbage.

We prove that $\Rightarrow_{\mathscr{R}}$ is noetherian over term graphs of level k assuming that $\Rightarrow_{\mathscr{R}}$ is noetherian over term graphs of level less than k. We define a noetherian relation $\succ_k$ over term graphs of level $< k$ with the outermost symbol in D_1^0 as $G \succ_k G'$ if and only if there is a G'' such that (i) $G \Rightarrow_{\mathscr{R}} G''$, (ii) $G' \equiv G''/v$ for some node in G'' and (iii) $root(term(G')) \in D_1^0$. The multiset extension of $\succ_k$ is denoted by $\gg_k$.

Theorem 14. *If the graph rewriting relation $\Rightarrow_{\mathscr{R}}$ is noetherian over term graphs of level less than k, then $\Rightarrow_{\mathscr{R}}$ is noetherian over term graphs of level k as well.*

Proof. We show that a reduction $G \Rightarrow_{\mathscr{R}} G'$ over term graphs of level k implies (i) $D_1^0 sub_k(G) \gg_k D_1^0 sub_k(G')$ or (ii) $D_1^0 sub_k(G) = D_1^0 sub_k(G')$ and $cap_k(G) \Rightarrow_{\mathscr{R}} cap_k(G')$. Since $\gg_k$ is noetherian over term graphs of level less than k with the outermost symbol in D_1^0 and $\Rightarrow_{\mathscr{R}}$ is noetherian over term graphs of level 1, it follows that $\Rightarrow_{\mathscr{R}}$ is noetherian over term graphs of level k as well.

Consider a reduction $G \Rightarrow_{\mathscr{R}} G'$ over term graphs of level k. If the reduction took place in an element of $D_1^0 sub_k(G)$ it is obvious that $D_1^0 sub_k(G) \gg_k D_1^0 sub_k(G')$. Since D_1^0 symbols do not occur in left-sides of rules except at the outermost level and nesting of D_1^0 symbols is not allowed in right-sides, it follows that $D_1^0 sub_k(G') \subseteq D_1^0 sub_k(G)$ and $cap_k(G) \Rightarrow_{\mathscr{R}} cap_k(G')$ if the reduction took place outside the subgraphs in $D_1^0 sub_k(G)$. That is, $D_1^0 sub_k(G) \gg_k D_1^0 sub_k(G')$ or $D_1^0 sub_k(G) = D_1^0 sub_k(G')$ and $cap_k(G) \Rightarrow_{\mathscr{R}} cap_k(G')$. This completes the proof. □

Now, we are in a position to establish one of the main results of the paper. This theorem extends Theorem 11 to generalized nice-extension*s.

Theorem 15. *Let $\mathscr{R}_0$ and $\mathscr{R}_1$ be two term rewriting systems such that* (i) *$\Rightarrow_{\mathscr{R}_0}$ and $\Rightarrow_{\mathscr{R}_1}$ are terminating,* (ii) *$\mathscr{R}_1$ is a generalized nice extension* of $\mathscr{R}_0$ and* (iii) *function symbols from D_1^0 occur in l only at the outermost level for any rule $l \to r \in R_1$. Then the graph rewriting relation $\Rightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ is terminating as well.*

Proof. Follows by induction from the above two theorems. □

The condition (iii) above is only used in Theorem 14. In our proof, this condition is very essential to establish that $cap_k(G) \Rightarrow_{\mathscr{R}} cap_k(G')$ if the reduction took place in $cap_k(G)$. Without this condition, $cap_k(G')$ can possibly contain some function symbols from $D_1^0 sub_k(G)$ when a D_1^0 symbol occurs in the left-hand side of the rewrite rule applied.

8. Completeness

In this section, we study modularity of completeness in graph rewriting for generalized nice-extension*s. The proof of our result is based on Theorems 6 and 15.

Theorem 16. *Let $\mathscr{R}_0$ and $\mathscr{R}_1$ be two* TRSS *such that $\Rrightarrow_{\mathscr{R}_0}$ and $\Rrightarrow_{\mathscr{R}_1}$ are complete and*
(1) *$\mathscr{R}_1$ is a generalized nice-extension∗ of $\mathscr{R}_0$,*
(2) *function symbols from $D_1^0 - D$ occur in l only at the outermost level for any rule $l \to r \in R_1$,*
(3) *if $l \to r \in \mathscr{R}_i$ and s is a subterm of l with $root(s) \in D_{1-i}$ then no subterm of s unifies with left-hand side term of any rule in $\mathscr{R}_0 \cup \mathscr{R}_1$ and*
(4) *if $l \to r \in \mathscr{R}_1$ and $root(l) \in D_1^0 - D$ then all the critical pairs involving this rule are overlays.*
Then, the graph rewriting relation $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ of the combined system $\mathscr{R}_0 \cup \mathscr{R}_1$ is complete as well.

Proof. Since $\Rrightarrow_{\mathscr{R}_0}$ and $\Rrightarrow_{\mathscr{R}_1}$ are semi-complete (in fact, complete), $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ is semi-complete (hence confluent) by Theorem 6. By Theorem 15, $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ is terminating. Therefore $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ is complete. □

9. Comparison and crosswise disjoint unions

In this section, we compare our results of the previous sections with other known modularity results in term graph rewriting. We also study modular aspects for the class of crosswise disjoint unions, which is not a proper subclass of the class of super hierarchical combinations.

The following example demonstrates the fact that the class of crosswise disjoint unions is not a proper subclass of the class of super hierarchical combinations.

Example 8. The following two systems are crosswise disjoint.

$\mathscr{R}_0 \ : \ \mathsf{a} \to \mathsf{b}$ $\qquad\qquad$ $\mathscr{R}_1 \ : \ \mathsf{a} \to \mathsf{c}$

However, their union is not a super hierarchical combination as the shared defined symbol a is defined differently in the two systems.

Plump [17] proved modularity of termination in term graph rewriting for crosswise disjoint systems.

Theorem 17 (Plump [17]). *Let $\mathscr{R}_0$ and $\mathscr{R}_1$ be two crosswise disjoint systems. Then $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ terminates if and only if both $\Rrightarrow_{\mathscr{R}_0}$ and $\Rrightarrow_{\mathscr{R}_1}$ terminate.*

Kurihara and Ohuchi [12] proved modularity of termination in term graph rewriting for constructor sharing systems (which share constructors but not defined symbols).

Theorem 18 (Kurihara and Ohuchi [12]). *Let $\mathscr{R}_0$ and $\mathscr{R}_1$ be two* TRSS *sharing constructors. Then $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ terminates if and only if both $\Rrightarrow_{\mathscr{R}_0}$ and $\Rrightarrow_{\mathscr{R}_1}$ terminate.*

Note that Theorem 18 is not a generalization of Theorem 17 as crosswise disjoint systems do not forbid sharing of defined symbols. To have a theorem which is generalization of Theorem 17, Kurihara and Ohuchi allow a set $\mathscr{B}$ of defined symbols to be shared provided those symbols (in $\mathscr{B}$) occur only in left-sides of $\mathscr{R}_0 \cup \mathscr{R}_1$.

Theorem 19 (Kurihara and Ohuchi [12]). *Let* TRS $\mathscr{R}_0(D_0 \uplus \mathscr{B}, C_0, R_0)$ *and* $\mathscr{R}_1(D_1 \uplus \mathscr{B}, C_1, R_1)$ *be two* TRSs *such that* (i) $D_0 \cap D_1 = D_0 \cap C_1 = C_0 \cap D_1 = \phi$ *and* (ii) *symbols in* $\mathscr{B}$ *occur only left-sides of rules in* $R_0 \cup R_1$. *Then* $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ *terminates if and only if both* $\Rrightarrow_{\mathscr{R}_0}$ *and* $\Rrightarrow_{\mathscr{R}_1}$ *terminate.*

The proof of this theorem can be sketched as follows. Since graph rewriting does not copy subgraphs and $\mathscr{B}$-symbols do not occur in right-sides of the rewrite rules, $G \Rrightarrow G'$ implies $\#(G) \geqslant \#(G')$, where $\#(G)$ denotes the number of occurrences of $\mathscr{B}$-symbols in G. Therefore, in any derivation $G_1 \Rrightarrow G_2 \Rrightarrow \cdots$, the number of such occurrences remains constant after a finite number of steps, i.e., $\#(G_{i+j}) = \#(G_i)$ for some i and each j. In other words, after a finite number of steps, the rewrite rules in $S = \{l \to r \mid root(l) \in \mathscr{B}\}$ are not applied. Since $(\mathscr{R}_0 \cup \mathscr{R}_1) - S$ is a union of two constructor sharing TRSs, its graph rewrite relation terminates by Theorem 18. Hence there cannot be an infinite $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ derivation.

This idea works for our results on modularity of termination as well, leading to the following theorem.

Theorem 20. *Let* $\mathscr{R}_0(D_0 \uplus D \uplus \mathscr{B}, C_0, R_0)$ *and* $\mathscr{R}_1(D_1 \uplus D \uplus \mathscr{B}, C_1, R_1)$ *be two term rewriting systems such that* (1) $\Rrightarrow_{\mathscr{R}_0}$ *and* $\Rrightarrow_{\mathscr{R}_1}$ *are terminating,* (2) $\mathscr{R}_1 - \{l \to r \mid root(l) \in \mathscr{B}\}$ *is a generalized nice-extension∗ of* $\mathscr{R}_0 - \{l \to r \mid root(l) \in \mathscr{B}\}$, (3) *function symbols from* D_1^0 *occur in* l *only at the outermost level for any rule* $l \to r \in R_1$ *and* (4) *symbols in* $\mathscr{B}$ *occur only in left-sides of rules in* $R_0 \cup R_1$. *Then* $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ *is terminating as well.*

This theorem is not only a generalization of Theorems 17–19 but also more powerful as hierarchical combinations are very natural and occur in practice very often compared to the combinations considered in those theorems.

Our main results on modularity of completeness, weak normalization and semi-completeness can also be extended similarly.

9.1. Weak normalization

In this subsection, we consider two systems $\mathscr{R}_0(D_0 \uplus D \uplus \mathscr{B}, C_0, R_0)$ and $\mathscr{R}_1(D_1 \uplus D \uplus \mathscr{B}, C_1, R_1)$ such that (1) the graph rewrite relations $\Rrightarrow_{\mathscr{R}_0}$ and $\Rrightarrow_{\mathscr{R}_1}$ are weakly normalizing and (2) $\mathscr{R}_1 - \{l \to r \mid root(l) \in \mathscr{B}\}$ is a generalized nice-extension∗ of $\mathscr{R}_0 - \{l \to r \mid root(l) \in \mathscr{B}\}$ and (3) symbols in $\mathscr{B}$ occur only in left-sides of rules in $R_0 \cup R_1$.

For proving the main result of this subsection, we consider the following class of term graphs which cannot be reduced by the rules in $\{l \to r \mid root(l) \in \mathscr{B}\}$.

Definition 21. Let $\mathscr{S}_1$ be the set of all term graphs $\{G \mid G \Rrightarrow^*_{\mathscr{R}_0 \cup \mathscr{R}_1} G'$ implies no rule in $\{l \to r \mid root(l) \in \mathscr{B}\}$ is applicable on $G'\}$.

The following lemma establishes that each term graph in $\mathscr{S}_1$ has normal forms w.r.t. $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$.

Lemma 16. *Every term graph $G \in \mathscr{S}_1$ has at least one normal form w.r.t. $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$.*

Proof. Follows from the fact that $\mathscr{R}_1 - \{l \to r \mid root(l) \in \mathscr{B}\}$ is a generalized nice-extension∗ of $\mathscr{R}_0 - \{l \to r \mid root(l) \in \mathscr{B}\}$. □

Now, we are in a position to prove the main result of this subsection.

Theorem 21. *Let $\mathscr{R}_0(D_0 \uplus D \uplus \mathscr{B}, C_0, R_0)$ and $\mathscr{R}_1(D_1 \uplus D \uplus \mathscr{B}, C_1, R_1)$ be two term rewriting systems such that (1) $\Rrightarrow_{\mathscr{R}_0}$ and $\Rrightarrow_{\mathscr{R}_1}$ are weakly normalizing, (2) $\mathscr{R}_1 - \{l \to r \mid root(l) \in \mathscr{B}\}$ is a generalized nice-extension∗ of $\mathscr{R}_0 - \{l \to r \mid root(l) \in \mathscr{B}\}$ and (3) symbols in $\mathscr{B}$ occur only in left-sides of rules in $R_0 \cup R_1$. Then $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ is weakly normalizing as well.*

Proof. Proof by contradiction.

Let k be the smallest natural number such that a graph G with $\#(G) = k$ has no normal form w.r.t. $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$. That is, every graph with less than k occurrences of $\mathscr{B}$-symbols has at least one normal form w.r.t. $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$. We have two cases: (a) $G \in \mathscr{S}_1$ and (b) $G \notin \mathscr{S}_1$.

Case (a): By the above lemma, G has at least one normal form w.r.t. $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$, a contradiction.

Case (b): If $G \Rrightarrow^*_{\mathscr{R}_0 \cup \mathscr{R}_1} G'$ and a rule in $\{l \to r \mid root(l) \in \mathscr{B}\}$ is applicable on G', we get a term graph G_2 from G' by applying an evaluation step with that rule. It is clear that $\#(G_2) < k$ as such a step removes the root symbol of l which is in $\mathscr{B}$ and r does not contain any symbol from $\mathscr{B}$. Therefore G_2 has at least one normal form w.r.t. $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ and hence G has at least one normal form w.r.t. $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$, a contradiction. □

9.2. Completeness

As can be seen from Example 8, no confluence property is modular for any superclass of crosswise disjoint unions, as crosswise disjoint unions allow rule $\mathsf{a} \to \mathsf{b}$ in $\mathscr{R}_0$ and rule $\mathsf{a} \to \mathsf{c}$ in $\mathscr{R}_1$. It is natural to impose the following restriction to ensure modularity of confluence properties such as completeness and semi-completeness.

Definition 22. Two term rewriting systems $\mathscr{R}_0$ and $\mathscr{R}_1$ are *non-interfering* if all the critical pairs between any pair of rules $l_0 \to r_0 \in \mathscr{R}_0$ and $l_1 \to r_1 \in \mathscr{R}_1$ are trivial.

Theorem 22. *Let $\mathscr{R}_0(D_0 \uplus D \uplus \mathscr{B}, C_0, R_0)$ and $\mathscr{R}_1(D_1 \uplus D \uplus \mathscr{B}, C_1, R_1)$ be two non-interfering term rewriting systems such that* (1) $\Rrightarrow_{\mathscr{R}_0}$ *and* $\Rrightarrow_{\mathscr{R}_1}$ *are complete,* (2) $\mathscr{R}_1 - \{l \to r \mid root(l) \in \mathscr{B}\}$ *is a generalized nice-extension* of* $\mathscr{R}_0 - \{l \to r \mid root(l) \in \mathscr{B}\}$ *satisfying conditions* (2)–(4) *of Theorem* 16 *and* (3) *symbols in* $\mathscr{B}$ *occur only in left-sides of rules in* $R_0 \cup R_1$. *Then* $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ *is complete as well.*

Proof. By Theorem 20, $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ is terminating. Since $\Rrightarrow_{\mathscr{R}_0}$ and $\Rrightarrow_{\mathscr{R}_1}$ are complete, they are locally confluent. It is clear that $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ is locally confluent as $\mathscr{R}_0$ and $\mathscr{R}_1$ are non-interfering. Therefore, $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ is complete. □

We believe that Theorem 6 can also be similarly extended, i.e., the above theorem holds even if we replace ‘complete’ by ‘semi-complete’. However, the proof of that theorem can be given neither on the lines of the above theorem (as weak-normalization and local-confluence do not imply semi-completeness) nor on the lines of the proof of Theorem 6 (as we do not know whether Theorem 5 in term rewriting holds for the above combinations). Though, we have not completely worked out the details yet, it is our conviction that one can prove this theorem from the first principles by establishing that $\Rrightarrow_{\mathscr{R}_0 \cup \mathscr{R}_1}$ is a compatible refinement of $\to_a$ defined in Section 6 as in [7].

10. Conclusion

In this paper, the modular aspects in graph implementations of term rewriting are investigated. Various combinations of rewrite systems are discussed and modularity of (a) weak normalization, (b) strong normalization, (c) semi-completeness (confluence + weak normalization) and (d) completeness (confluence + strong normalization) properties is studied for the most general combinations (namely super-hierarchical combinations). A comparison with related works is provided. In sharp contrast to the case of pure term rewriting, modularity of termination in term graph rewriting needs neither confluence nor simple termination. In contrast to confluence, weak normalization and semi-completeness are preserved under signature extensions (it is already known that termination and completeness enjoy this property) in term graph rewriting.

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