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GROUP EXPLICIT METHODS FOR HYPERBOLIC EQUATIONS

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Abstract—Here the strategy of the group explicit (GE) methods is applied to the numerical solution of hyperbolic partial differential equations. Theoretical aspects of the stability, consistency, convergence and truncation errors of this new class of methods are presented with supporting numerical evidence.

1. INTRODUCTION

In this paper, *the group explicit (GE) methods* which were first introduced by Evans and Abdullah [1] to solve parabolic problems will be extended to hyperbolic equations of first and second order. The development of these methods stems from the general observation that the *alternate* use of different algorithms with truncation errors of opposite signs can lead to the *cancellations* of the error terms at most points on the mesh lines. Although this alternating strategy does not necessarily amount to the upgrading of the order of the approximation, it is, however, expected to provide a better accuracy than the individual algorithms themselves as well as other schemes that are traditionally employed to solve the given differential equation.

The GE techniques involve the utilisation of *asymmetric* approximations which when coupled in groups of two adjacent points on the mesh result in *implicit* equations. These equations will then be converted to *explicit* ones which produce the numerical solutions of the differential equation thus exhibiting the simple nature of the methods.

This section will deal with the construction of two different schemes in which the GE procedure is used to solve the simple hyperbolic equation of first order of the form,

$$\frac{\partial U}{\partial t} = -\frac{\partial U}{\partial x}, \quad 0 \leqslant x \leqslant 1, \quad t \geqslant 0. \tag{1}$$

The application of the procedure can be extended (in a later paper), to the wave equation,

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2}, \quad 0 \leqslant x \leqslant 1, \quad t \geqslant 0. \tag{2}$$

An analysis of the local truncation errors will also be performed followed by an investigation of the stability requirements of the various schemes.

2. GE METHODS FOR THE GENERALIZED WEIGHTED APPROXIMATION TO THE FIRST ORDER EQUATION

The generalized weighted finite-difference analogue for equation (1) at the point

$$(x_i, t_{j+\theta}) = [i\Delta x, (j+\theta)\Delta t]$$

is given by

$$-\frac{1}{\Delta x}\{\theta[(1-w)u_{i+1,j+1} + (2w-1)u_{i,j+1} - wu_{i-1,j+1}] + (1-\theta)[(1-w)u_{i+1,j} + (2w-1)u_{ij} - wu_{i-1,j}]\} = \frac{(u_{i,j+1} - u_{ij})}{\Delta t}, \tag{3}$$

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or

$$-\lambda\{\theta[(1-w)u_{i+1,j+1}+(2w-1)u_{i,j+1}-wu_{i-1,j+1}] \\ +(1-\theta)[(1-w)u_{i+1,j}+(2w-1)u_{ij}-wu_{i-1,j}]\}=u_{i,j+1}-u_{ij}, \quad (4)$$

where $\lambda=\Delta t/\Delta x$, the mesh ratio.

With $w=1$, this equation reduces to

$$(1+\lambda\theta)u_{i,j+1}-\lambda\theta u_{i-1,j+1}=[1-\lambda(1-\theta)]u_{i,j}+\lambda(1-\theta)u_{i-1,j} \quad (5)$$

and for $w=0$ equation (4) becomes,

$$(1-\lambda\theta)u_{i,j+1}+\lambda\theta u_{i+1,j+1}=[1+\lambda(1-\theta)]u_{i,j}-\lambda(1-\theta)u_{i+1,j}. \quad (6)$$

The local truncation error representations can be obtained by expanding the terms $U_{i,j+1}$, $U_{i-1,j+1}$, $U_{i-1,j}$ and U_{ij} about the point $[i\Delta x,(j+1/2)\Delta t]$ using the Taylor series. The expansion for equation (5) leads to,

$$\begin{aligned}T_5=&\left(\frac{\partial U}{\partial x}+\frac{\partial U}{\partial t}\right)_{i,j+1/2}+\Delta x\left[-\frac{1}{2}\frac{\partial^2 U}{\partial x^2}-\frac{(\Delta t)^2}{16}\frac{\partial^4 U}{\partial x^2\partial t^2}\right]_{i,j+1/2}\\&+\Delta t\left[-\frac{1}{2}(1-2\theta)\frac{\partial^2 U}{\partial x\partial t}-\frac{(\Delta x)^2}{12}(1-2\theta)\frac{\partial^4 U}{\partial x^3\partial t}\right]_{i,j+1/2}+(\Delta x)(\Delta t)\left[\frac{1}{4}(1-2\theta)\frac{\partial^3 U}{\partial x^2\partial t}\right]_{i,j+1/2}\\&+(\Delta x)^2\left[\frac{1}{6}\frac{\partial^3 U}{\partial x^3}-\frac{\Delta x}{24}\frac{\partial^4 U}{\partial x^4}\right]_{i,j+1/2}+(\Delta t)^2\left[\frac{1}{8}\frac{\partial^3 U}{\partial x\partial t^2}+\frac{1}{24}\frac{\partial^3 U}{\partial t^3}-\frac{\Delta t}{48}(1-2\theta)\frac{\partial^4 U}{\partial x\partial t^3}\right]_{i,j+1/2}\\&+\left[\frac{(\Delta x)^4}{5!}\frac{\partial^5 U}{\partial x^5}+\frac{5(\Delta x)^3(\Delta t)}{5!2}(1-2\theta)\frac{\partial^5 U}{\partial x^4\partial t}+\frac{5(\Delta x)^2(\Delta t)^2}{5!2}\frac{\partial^5 U}{\partial x^3\partial t^2}\right.\\&\left.+\frac{5(\Delta x)(\Delta t)^3}{5!4}(1-2\theta)\frac{\partial^5 U}{\partial x^2\partial t^3}+\frac{5(\Delta t)^4}{5!16}\frac{\partial^5 U}{\partial x\partial t^4}+\frac{(\Delta t)^4}{5!16}\frac{\partial^5 U}{\partial t^5}\right]_{i,j+1/2}+\cdots,\end{aligned}$$

i.e.

$$\begin{aligned}T_5=&\Delta x\left[-\frac{1}{2}\frac{\partial^2 U}{\partial x^2}-\frac{(\Delta t)^2}{16}\frac{\partial^4 U}{\partial x^2\partial t^2}\right]_{i,j+1/2}+\Delta t\left[-\frac{1}{2}(1-2\theta)\frac{\partial^2 U}{\partial x\partial t}-\frac{(\Delta x)^2}{12}(1-2\theta)\frac{\partial^4 U}{\partial x^3\partial t}\right]_{i,j+1/2}\\&+(\Delta x)(\Delta t)\left[\frac{1}{4}(1-2\theta)\frac{\partial^3 U}{\partial x^2\partial t}\right]_{i,j+1/2}+(\Delta x)^2\left[\frac{1}{6}\frac{\partial^3 U}{\partial x^3}-\frac{\Delta x}{24}\frac{\partial^4 U}{\partial x^4}\right]_{i,j+1/2}\\&+(\Delta t)^2\left[\frac{1}{8}\frac{\partial^3 U}{\partial x\partial t^2}+\frac{1}{24}\frac{\partial^3 U}{\partial t^3}-\frac{\Delta t}{48}(1-2\theta)\frac{\partial^4 U}{\partial x\partial t^3}\right]_{i,j+1/2}+0[(\Delta x)^{\alpha_1}(\Delta t)^{\alpha_2}], \quad (7)\end{aligned}$$

with $\alpha_1+\alpha_2=4$ and $0\leqslant\theta\leqslant 1$. A similar expansion for the terms $U_{i,j+1}$, $U_{i+1,j+1}$, U_{ij} and $U_{i+1,j}$ about the point $[i\Delta x,(j+1/2)\Delta t]$ provides the following truncation error expression for formula (6):

$$\begin{aligned}T_6=&\left(\frac{\partial U}{\partial x}+\frac{\partial U}{\partial t}\right)_{i,j+1/2}+\Delta x\left[\frac{1}{2}\frac{\partial^2 U}{\partial x^2}+\frac{(\Delta t)^2}{16}\frac{\partial^4 U}{\partial x^2\partial t^2}\right]_{i,j+1/2}\\&+\Delta t\left[-\frac{1}{2}(1-2\theta)\frac{\partial^2 U}{\partial x\partial t}-\frac{1}{12}(\Delta x)^2(1-2\theta)\frac{\partial^4 U}{\partial x^3\partial t}\right]_{i,j+1/2}\\&+(\Delta x)(\Delta t)\left[-\frac{1}{4}(1-2\theta)\frac{\partial^3 U}{\partial x^2\partial t}\right]_{i,j+1/2}+(\Delta x)^2\left[\frac{1}{6}\frac{\partial^3 U}{\partial x^3}-\frac{\Delta x}{24}\frac{\partial^4 U}{\partial x^4}\right]_{i,j+1/2}\\&+(\Delta t)^2\left[\frac{1}{8}\frac{\partial^3 U}{\partial x\partial t^2}+\frac{1}{24}\frac{\partial^3 U}{\partial t^3}-\frac{\Delta t}{48}(1-2\theta)\frac{\partial^4 U}{\partial x\partial t^3}\right]_{i,j+1/2}\end{aligned}$$

$$+\left[\frac{(\Delta x)^4}{5!}\frac{\partial^5 U}{\partial x^5}+\frac{5(\Delta x)^3(\Delta t)}{5!2}(1-2\theta)\frac{\partial^5 U}{\partial x^4\partial t}+\frac{5(\Delta x)^2(\Delta t)^2}{5!2}\frac{\partial^5 U}{\partial x^3\partial t^2}\right.$$

$$\left.-\frac{5(\Delta x)(\Delta t)^3(1-2\theta)}{5!4}\frac{\partial^5 U}{\partial x^2\partial t^3}+\frac{5(\Delta t)^4}{5!16}\frac{\partial^5 U}{\partial x\partial t^4}+\frac{(\Delta t)^4}{5!16}\frac{\partial^5 U}{\partial t^2}\right]_{i,j+1/2}+\cdots,$$

i.e.

$$T_6=\Delta x\left[\frac{1}{2}\frac{\partial^2 U}{\partial x^2}-\frac{(\Delta t)^2}{16}\frac{\partial^4 U}{\partial x^2\partial t^2}\right]_{i,j+1/2}+\Delta t\left[-\frac{1}{2}(1-2\theta)\frac{\partial^2 U}{\partial x\partial t}-\frac{1}{12}(\Delta x)^2(1-2\theta)\frac{\partial^4 U}{\partial x^3\partial t}\right]_{i,j+1/2}$$

$$+(\Delta x)(\Delta t)\left[-\frac{1}{4}(1-2\theta)\frac{\partial^3 U}{\partial x^2\partial t}\right]_{i,j+1/2}+(\Delta x)^2\left[\frac{1}{6}\frac{\partial^3 U}{\partial x^3}-\frac{\Delta x}{24}\frac{\partial^4 U}{\partial x^4}\right]_{i,j+1/2}$$

$$+(\Delta t)^2\left[\frac{1}{8}\frac{\partial^3 U}{\partial x\partial t^2}+\frac{1}{24}\frac{\partial^3 U}{\partial t^3}-\frac{\Delta t}{48}(1-2\theta)\frac{\partial^4 U}{\partial x\partial t^3}\right]_{i,j+1/2}+0[(\Delta x)^{\alpha_1}(\Delta t)^{\alpha_2}]; \tag{8}$$

with $\alpha_1+\alpha_2=4$ and $0\leqslant\theta\leqslant 1$.

Now, at the point $[(i-1)\Delta x,(j+\theta)\Delta t]$, equation (6) takes the form

$$\lambda\theta u_{i,j+1}+(1-\lambda\theta)u_{i-1,j+1}=-\lambda(1-\theta)u_{i,j}+[1+\lambda(1-\theta)]u_{i-1,j}. \tag{9}$$

By coupling the equations (5) and (9) the two formulae can be written simultaneously in matrix form as

$$\begin{bmatrix}-\lambda\theta & (1+\lambda\theta)\\(1-\lambda\theta) & \lambda\theta\end{bmatrix}\begin{bmatrix}u_{i-1,j+1}\\u_{i,j+1}\end{bmatrix}=\begin{bmatrix}\lambda(1-\theta) & 1-\lambda(1-\theta)\\1+\lambda(1-\theta) & -\lambda(1-\theta)\end{bmatrix}\begin{bmatrix}u_{i-1,j}\\u_{ij}\end{bmatrix}, \tag{10}$$

i.e.

$$A\mathbf{u}_{j+1}=B\mathbf{u}_j, \tag{11}$$

where

$$A=\begin{bmatrix}-\lambda\theta & (1+\lambda\theta)\\(1-\lambda\theta) & \lambda\theta\end{bmatrix},\quad B=\begin{bmatrix}\lambda(1-\theta) & 1-\lambda(1-\theta)\\1+\lambda(1-\theta) & -\lambda(1-\theta)\end{bmatrix}$$

and

$$\mathbf{u}_j=(u_{i-1,j},u_{i,j})^{\mathrm{T}}.$$

The (2×2) matrix A can be easily inverted. Hence from equation (11), we have,

$$\mathbf{u}_{j+1}=A^{-1}B\mathbf{u}_j, \tag{12}$$

with

$$A^{-1}=\begin{bmatrix}-\lambda\theta & (1+\lambda\theta)\\(1-\lambda\theta) & \lambda\theta\end{bmatrix}\quad\text{and}\quad A^{-1}B=\begin{bmatrix}(1+\lambda) & -\lambda\\\lambda & (1-\lambda)\end{bmatrix}.$$

From equation (12), this gives rise to the following *set of explicit* equations:

$$u_{i-1,j+1}=(1+\lambda)u_{i-1,j}-\lambda u_{i,j} \tag{13}$$

and

$$u_{i,j+1}=\lambda u_{i-1,j}+(1-\lambda)u_{ij}, \tag{14}$$

whose computational molecules are shown in Fig. 1.

Equations (13) and (14) are for adjacent points which are grouped two at a time on the mesh line. Special formulae are needed to cope with the possibility of the existence of *ungrouped* points near the boundaries. The solution at the ungrouped point near *the right boundary* at the advanced time level can be computed from the equation (6) by putting $i=m-1$. This leads to,

$$u_{m-1,j+1}=\{[1+\lambda(1-\theta)]u_{m-1,j}-\lambda(1-\theta)u_{m,j}-\lambda\theta u_{m,j+1}\}/(1-\lambda\theta), \tag{15}$$

where $\lambda\theta\neq 1$.

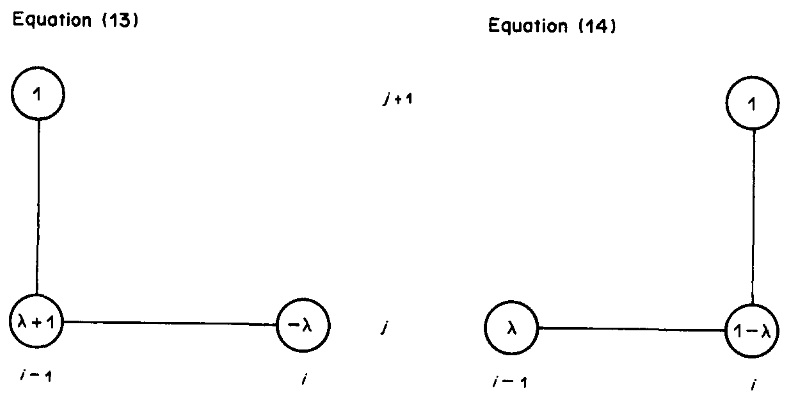

Fig. 1

Equations (5) with $i = 1$, deals with the value of u at the ungrouped point near *the left boundary*. Thus, we have,

$$u_{1,j+1} = [\lambda(1-\theta)u_{0,j} + (1 - \lambda(1-\theta))u_{1,j} + \lambda\theta u_{0,j+1}]/(1+\lambda\theta). \tag{16}$$

Since the initial line $0 \leqslant x \leqslant 1$ is uniformly divided with a spacing or increment Δx, the manner in which the above points are grouped very much depends on whether the number m of intervals of the line segment is *even* or *odd*. On this basis, a variety of group explicit schemes can be devised—as we will presently see.

Even Number of Intervals

When m is even, we will have an odd number $(m-1)$ of internal points (i.e. points that do not include the left and right boundaries whose values are given by u_0 and u_m respectively at every time level). Consequently, the single ungrouped point will be located near either boundary.

(*i*) *The GER scheme*

This refers to the group explicit with right ungrouped point (GER) scheme. It results in the consecutive application for $(1/2)(m-2)$ times of the equations (13) and (14) for the first $(m-2)$ points grouped two at a time. This is followed by a final use of equation (15) for the $(m-1)$th point at every time level as shown in Fig. 2. Thus, we have the following set of equations:

$$\left.\begin{aligned} -\lambda\theta u_{i-1,j+1} + (1+\lambda\theta)u_{i,j+1} &= \lambda(1-\theta)u_{i-1,j} + [1-\lambda(1-\theta)]u_{i,j}, \\ (1-\lambda\theta)u_{i-1,j+1} + \lambda\theta u_{i,j+1} &= [1+\lambda(1-\theta)]u_{i-1,j} - \lambda(1-\theta)u_{i,j}, \end{aligned}\right\}, \quad i = 2, 4, \ldots, (m-2)$$

and

$$(1-\lambda\theta)u_{m+1,j+1} = -\lambda\theta u_{m,j+1} - \lambda(1-\theta)u_{m,j} + [1+\lambda(1-\theta)]u_{m-1,j}, \quad \lambda\theta \neq 1,$$

[*see equation (17) on facing page*]

which can be written in the more compact, *implicit* matrix form as,

$$
\left[\begin{array}{cc|cc|c|cc|c}
-\lambda\theta & (1+\lambda\theta) & & & & & & \\
(1-\lambda\theta) & \lambda\theta & & & & & & \\
\hline
 & & -\lambda\theta & (1+\lambda\theta) & & & & \\
 & & (1-\lambda\theta) & \lambda\theta & & & 0 & \\
\hline
 & & & & \ddots & & & \\
\hline
 & & & & & -\lambda\theta & (1+\lambda\theta) & \\
 & & 0 & & & (1-\lambda\theta) & \lambda\theta & \\
\hline
 & & & & & & & (1-\lambda\theta)
\end{array}\right]
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{m-3} \\ u_{m-2} \\ u_{m-1} \end{bmatrix}_{j+1}
$$

$$
= \left[\begin{array}{cc|cc|c|cc|c}
\lambda(1-\theta) & [1-\lambda(1+\theta)] & & & & & & \\
[1+\lambda(1-\theta)] & -\lambda(1-\theta) & & & & & & \\
\hline
 & & \lambda(1-\theta) & [1-\lambda(1-\theta)] & & & & \\
 & & [1+\lambda(1-\theta)] & -\lambda(1-\theta) & & & 0 & \\
\hline
 & & & & \ddots & & & \\
\hline
 & & & & & \lambda(1-\theta) & [1-\lambda(1-\theta)] & \\
 & & 0 & & & [1+\lambda(1-\theta)] & -\lambda(1-\theta) & \\
\hline
 & & & & & & & [1+\lambda)1-\theta)]
\end{array}\right]
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{m-3} \\ u_{m-2} \\ u_{m-1} \end{bmatrix}_{j}
+ \mathbf{b}_1, \tag{17}
$$

where $\mathbf{b}_1 = [0, 0, \ldots, -\lambda(1-\theta)u_{m,j} - \lambda\theta u_{m,j+1}]^T$ which consists of known boundary values. Now, if we define,

$$E_1 = \left[\begin{array}{cc|cc|c|cc|c} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ \hline & & 0 & 1 & & & & \\ & & 1 & 0 & 0 & & & \\ \hline & & & & \ddots & & & \\ \hline & & & 0 & & 0 & 1 & \\ & & & & & 1 & 0 & \\ \hline & & & & & & & 1 \end{array}\right]_{[(m-1)\times(m-1)]} \tag{18}$$

and

$$G_1 = \left[\begin{array}{cc|cc|c|cc|c} -1 & 1 & & & & & & \\ -1 & 1 & & & & & & \\ \hline & & -1 & 1 & & & & \\ & & -1 & 1 & 0 & & & \\ \hline & & & & \ddots & & & \\ \hline & & & & 0 & -1 & 1 & \\ & & & & & -1 & 1 & \\ \hline & & & & & & & -1 \end{array}\right]_{[(m-1)\times(m-1)]}, \tag{19}$$

then we have

$$(E_1 + \lambda\theta G_1)\mathbf{u}_{j+1} = [E_1 - \lambda(1-\theta)G_1]\mathbf{u}_j + \mathbf{b}_1.$$

On premultiplying this equation by E_1^{-1} provides us with

$$E_1^{-1}(E_1 + \lambda\theta G_1)\mathbf{u}_{j+1} = E_1^{-1}[E_1 - \lambda(1-\theta)G_1]\mathbf{u}_j + E_1^{-1}\mathbf{b}_1,$$

i.e.

$$(E_1^{-1}E_1 + \lambda\theta E_1^{-1}G_1)\mathbf{u}_{j+1} = [E_1^{-1}E_1 - \lambda(1-\theta)E_1^{-1}G_1]\mathbf{u}_j + \mathbf{b}_1.$$

But $E_1^{-1}E_1 = I$ and $E_1^{-1}G_1 = G_1$ which implies that,

$$(I + \lambda\theta G_1)\mathbf{u}_{j+1} = [I - \lambda(1-\theta)G_1]\mathbf{u}_j + \mathbf{b}_1, \tag{20}$$

where I is the identity matrix of order $[(m-1)\times(m-1)]$. Hence, we obtain

$$\mathbf{u}_{j+1} = (I + \lambda\theta G_1)^{-1}[I - \lambda(1-\theta)G_1]\mathbf{u}_j + \hat{\mathbf{b}}_1, \tag{21}$$

where, $\hat{\mathbf{b}}_1 = (I + \lambda\theta G_1)^{-1}\mathbf{b}_1$.

The *explicit* equation (21) is the governing equation for the computation of the GER scheme.

(ii) The GEL scheme

This is an abbreviation for the group explicit with left (GER) ungrouped point scheme and it is in fact a reverse of the GER scheme. It is obtained by the use of equation (16) for the first internal

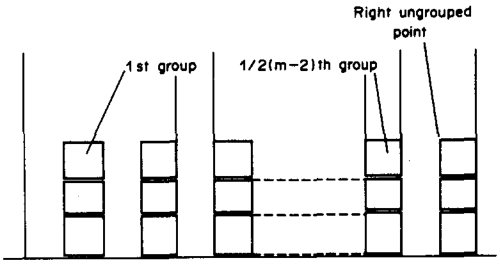

Fig. 2. The GER scheme (even number of intervals).

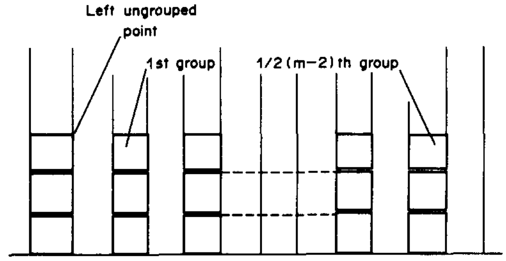

Fig. 3. The GEL scheme (even number of intervals).

point followed by the application of equations (13) and (14) for $(1/2)(m-2)$ times for the remaining points on the mesh line. The scheme is displayed diagrammatically in Fig. 3 and is determined by the following set of linear equations,

$$(1+\lambda\theta)u_{1,j+1} = [1-\lambda(1-\theta)]u_{1j} + \lambda(1-\theta)u_{0j} + \lambda\theta u_{0,j+1},$$

$$\left.\begin{aligned} -\lambda\theta u_{i-1,j+1} + (1+\lambda\theta)u_{i,j+1} &= \lambda(1-\theta)u_{i-1,j} + [1-\lambda(1-\theta)]u_{ij}, \\ \text{and} \qquad & \\ (1-\lambda\theta)u_{i-1,j+1} + \lambda\theta u_{i,j+1} &= [1+\lambda(1-\theta)]u_{i-1,j} - \lambda(1-\theta)\,u_{ij}, \end{aligned}\right\} i = 3, 5, \ldots, m-1; \quad \lambda\theta \neq 1.$$

In the *implicit* matrix form, these equations can be written as,

$$\begin{bmatrix} 1+\lambda\theta & & & & & & & \\ & -\lambda\theta & (1+\lambda\theta) & & & & & \\ & (1-\lambda\theta) & \lambda\theta & & & & & \\ & & & -\lambda\theta & (1+\lambda\theta) & & & \\ & & & (1-\lambda\theta) & \lambda\theta & & & \\ & & & & & \ddots & & 0 \\ & & & & 0 & & -\lambda\theta & (1+\lambda\theta) \\ & & & & & & (1-\lambda\theta) & \lambda\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ \vdots \\ u_{m-2} \\ u_{m-1} \end{bmatrix}_{j+1}$$

$$= \begin{bmatrix} [1-\lambda(1-\theta)] & & & & & \\ & \lambda(1-\theta) & [1-\lambda(1-\theta)] & & & \\ & [1+\lambda(1-\theta)] & -\lambda(1-\theta) & & 0 & \\ & & & \ddots & & \\ & & 0 & & \lambda(1-\theta) & [1-\lambda(1-\theta)] \\ & & & & [1+\lambda(1-\theta)] & -\lambda(1-\theta) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{m-2} \\ u_{m-1} \end{bmatrix}_j + \mathbf{b}_2, \tag{22}$$

where

$$\mathbf{b}_2 = [\lambda(1-\theta)u_{0,j} + \lambda\theta u_{0,j+1}, 0, \ldots, 0]^{\mathrm{T}}.$$

If we define,

$$E_2 = \left[\begin{array}{c|cc|cc|c|cc}
1 & & & & & & & \\ \hline
 & 0 & 1 & & & & & \\
 & 1 & 0 & & & & & \\ \hline
 & & & 0 & 1 & 0 & & \\
 & & & 1 & 0 & & & \\ \hline
 & & & & & \ddots & & \\
 & & 0 & & & & & \\ \hline
 & & & & & & 0 & 1 \\
 & & & & & & 1 & 0
\end{array}\right] \tag{23}$$

and

$$G_2 = \left[\begin{array}{c|cc|cc|c|cc}
1 & & & & & & & \\ \hline
 & -1 & 1 & & & & & \\
 & -1 & 1 & & & & & \\ \hline
 & & & -1 & 1 & 0 & & \\
 & & & -1 & 1 & & & \\ \hline
 & & & & & \ddots & & \\ \hline
 & & & & 0 & & -1 & 1 \\
 & & & & & & -1 & 1
\end{array}\right] \tag{24}$$

then the GEL scheme takes the form

$$(E_2 + \lambda\theta G_2)\mathbf{u}_{j+1} = [E_2 - \lambda(1-\theta)G_2]\mathbf{u}_j + \mathbf{b}_2.$$

If we premultiply this equation by E_1^{-1}, we get

$$(I + \lambda\theta G_2)\mathbf{u}_{j+1} = [I + \lambda(1-\theta)G_2]\mathbf{u}_j + \mathbf{b}_2, \tag{25}$$

and this leads to the following explicit formula for the computation of the GEL scheme:

$$\mathbf{u}_{j+1} = (I + \lambda\theta G_2)^{-1}[I - \lambda(1-\theta)G_2]\mathbf{u}_j + \hat{\mathbf{b}}_2, \tag{26}$$

where $\hat{\mathbf{b}}_2 = (1 + \lambda\theta G_2)^{-1}\mathbf{b}_2$.

(iii) The (S)AGE scheme

As the name suggests, this (single) alternating group explicit [(S)AGE] scheme entails the *alternate* use of the GER and the GEL formulae [i.e. equations (20) and (25)] as we march our solutions forward with respect to time as illustrated in Fig. 4. Thus, the *two time-level process* of the (S)AGE scheme is given by,

$$\left.\begin{aligned}
&(I + \lambda\theta G_1)\mathbf{u}_{j+1} = [I - \lambda(1-\theta)G_1]\mathbf{u}_j + \mathbf{b}_1 \\
&\text{and} \\
&(I + \lambda\theta G_2)\mathbf{u}_{j+2} = [I - \lambda(1-\theta)G_2]\mathbf{u}_{j+1} + \mathbf{b}_2
\end{aligned}\right\} j = 0, 2, 4, \ldots. \tag{27}$$

(iv) The (D)AGE scheme

The (double) alternating group explicit scheme [(D)AGE] is a *four-time level process.* A single application of the algorithm requires the utilisation of the GER and GEL schemes for the first two time levels followed by the employment of the same formulae but now in reverse order for the second two time levels. This alternating procedure is repeated as the solutions are progressed vertically. Thus, we see that the (D)AGE scheme is basically a periodic rotation of the (S)AGE scheme with the second half cycle implemented in opposite order to complete the four time level process and is given by

$$\left.\begin{aligned}(I+\lambda\theta G_1)\mathbf{u}_{j+1}&=[I-\lambda(1-\theta)G_1]\mathbf{u}_j+\mathbf{b}_1\\(I+\lambda\theta G_2)\mathbf{u}_{j+2}&=[I-\lambda(1-\theta)G_2]\mathbf{u}_{j+1}+\mathbf{b}_2\\(I+\lambda\theta G_2)\mathbf{u}_{j+3}&=[I-\lambda(1-\theta)G_2]\mathbf{u}_{j+2}+\mathbf{b}_2\\(I+\lambda\theta G_1)\mathbf{u}_{j+4}&=[I-\lambda(1-\theta)G_1]\mathbf{u}_{j+3}+\mathbf{b}_1\end{aligned}\right\}j=0,4,8,\ldots. \tag{28}$$

The scheme is represented diagrammatically by Fig. 5.

Before we proceed to establish the corresponding finite-difference analogues for the case when the line segment $0\leqslant x\leqslant 1$ has an *odd* number of intervals, we observe that if we denote

$$G^{(i)}=\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix},\quad i=1,2,\ldots,1/2(m-2), \tag{29}$$

then from equations (19) and (24), we have,

$$G_1=\begin{bmatrix}G^{(1)} & & & & & \\ & G^{(2)} & & & & \\ & & \ddots & 0 & & \\ & & & G^{[(1/2)(m-2)-1]} & & \\ & & & & G^{[(1/2)(m-2)]} & \\ & 0 & & & & -1\end{bmatrix}_{[(m-1)\times(m-1)]} \tag{30}$$

and

$$G_2=\begin{bmatrix}1 & & & & & \\ & G^{(1)} & & & & \\ & & G^{(2)} & & 0 & \\ & 0 & & \ddots & G^{[(1/2)(m-2)-1]} & \\ & & & & & G^{[(1/2)(m-2)]}\end{bmatrix}_{[(m-1)\times(m-1)]}. \tag{31}$$

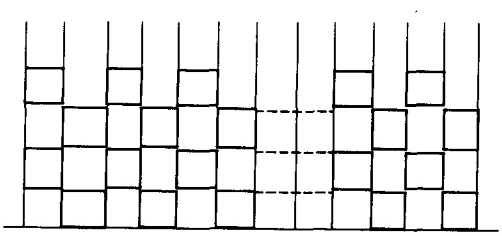

Fig. 4. The (S)AGE scheme (even number of intervals).

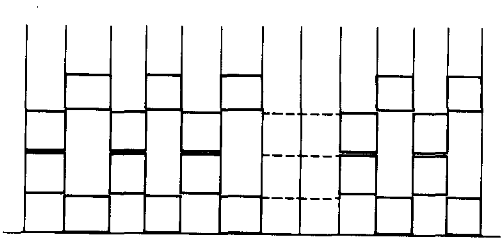

Fig. 5. The (D)AGE scheme (even number of intervals).

Odd Number of Intervals

We will have an odd number of intervals when m is odd. Therefore, at every time level, the number of internal points is even. Accordingly, against there are two possibilities when determine the manner in which the points are grouped on the mesh line. In the first possibility, we will have $(1/2)(m-1)$ *complete* groups of two points. In the second possibility, however, we are led to $[(m-3)/2]$ groups of two points and one point which is ungrouped adjacent to *each* boundary. Based on these observations, the following group explicit schemes can be constructed in an analogous fashion as in the *even* case.

(*i*) *The GEU scheme*

In this scheme, there are two points which are ungrouped, one each which is adjacent to the left and right boundary. Thus, for the *left* ungrouped point (the second point), we use equation (16) whilst the solution at the *right* ungrouped point [the $(m-1)$th point] is determined by equation (15). For the grouped points in between, we apply equations (13) and (14) in succession for $(1/2)(m-3)$ times to give the solutions at these points. This is repeated for progressive time levels and the whole procedure is known as the group explicit with ungrouped (GEU) ends method. Thus, the GEU method requires the solution of

$$(1+\lambda\theta)u_{1,j+1}=[1-\lambda(1-\theta)]u_{1,j}+\lambda(1-\theta)u_{0,j}+\lambda\theta u_{0,j+1},$$

$$\left.\begin{aligned}-\lambda\theta u_{i-1,j+1}+(1+\lambda\theta)u_{i,j+1}&=\lambda(1-\theta)u_{i-1,j}+[1-\lambda(1-\theta)]u_{i,j},\\(1-\lambda\theta)u_{i-1,j+1}+\lambda\theta u_{i,j+1}&=[1+\lambda(1-\theta)]u_{i-1,j}-\lambda(1-\theta)u_{i,j},\end{aligned}\right\}i=3,5,\ldots,m-2;\quad\lambda\theta\neq1,$$

$$(1-\lambda\theta)u_{m-1,j+1}=-\lambda\theta u_{m,j+1}-\lambda(1-\theta)u_{m,j}+[1+\lambda(1-\theta)]u_{m-1,j},$$

which can be written in the implicit matrix form as,

[*see equation (32) on facing page*]

$$\begin{bmatrix} (1+\lambda\theta) & & & & & & \\ & -\lambda\theta & (1+\lambda\theta) & & & & \\ & (1-\lambda\theta) & \lambda\theta & & & & \\ & & & \ddots & & 0 & \\ & & & & -\lambda\theta & (1+\lambda\theta) & \\ & 0 & & & (1-\lambda\theta) & \lambda\theta & \\ & & & & & & 1-\lambda\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{m-3} \\ u_{m-2} \\ u_{m-1} \end{bmatrix}_{j+1}$$

$$= \begin{bmatrix} 1-\lambda(1-\theta) & & & & & & \\ & \lambda(1-\theta) & [1-\lambda(1-\theta)] & & & 0 & \\ & [1+\lambda(1-\theta)] & -\lambda(1-\theta) & & & & \\ & & & \ddots & & & \\ & 0 & & & \lambda(1-\theta) & [1-\lambda(1-\theta)] & \\ & & & & [1+\lambda(1-\theta)] & -\lambda(1-\theta) & \\ & & & & & & [1+\lambda(1-\theta)] \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{m-3} \\ u_{m-2} \\ u_{m-1} \end{bmatrix}_{j} + \mathbf{b}_3, \quad (32)$$

where $\mathbf{b}_3 = [\lambda(1-\theta)u_{0,j} + \lambda\theta u_{0,j+1}, 0, \ldots, 0, -\lambda(1-\theta)u_{m,j} - \lambda\theta u_{m,j+1}]^T$. Now if we let

$$\hat{G}_1 = \begin{bmatrix} 1 & & & & & & \\ & G^{(1)} & & & & & \\ & & G^{(2)} & & 0 & & \\ & & & \ddots & & & \\ & & & & G^{[(1/2)(m-2)-1]} & & \\ & & 0 & & & G^{[(1/2)(m-3)]} & \\ & & & & & & -1 \end{bmatrix} \tag{33}$$

and

$$\hat{G}_1 = \begin{bmatrix} G^{(1)} & & & & \\ & G^{(2)} & & 0 & \\ & & \ddots & & \\ & & & G^{[(1/2)(m-3)]} & \\ & 0 & & & G^{[(1/2)(m-1)]} \end{bmatrix} \tag{34}$$

where the (2×2) matrices $G^{(i)}$, $i = 1, 2, \ldots, (1/2)(m-1)$ are defined as in equation (27), then the GEU scheme is given by

$$(I + \lambda\theta\hat{G}_1)\mathbf{u}_{j+1} = [I - \lambda(1-\theta)\hat{G}_1]\mathbf{u}_j + \mathbf{b}_3, \tag{35}$$

and is described by Fig. 6.

(ii) The GEC scheme

This scheme, known as the group explicit complete (GEC) method is obtained by applying successively $(1/2)(m-1)$ times equation (13) and (14) for the first to $(m-1)$th point along each progressive mesh line as displayed in Fig. 7. Thus, the relevant implicit equations are,

$$\left.\begin{aligned} -\lambda\theta u_{i-1,j+1} + (1+\lambda\theta)u_{i,j+1} &= \lambda(1-\theta)u_{i-1,j} + [1-\lambda(1-\theta)]u_{i,j} \\ (1-\lambda\theta)u_{i-1,j+1} + \lambda\theta u_{i,j+1} &= [1+\lambda(1-\theta)]u_{i-1,j} - \lambda(1-\theta)u_{i,j} \end{aligned}\right\} i = 2, 4, \ldots, (m-1); \quad \lambda\theta \neq 1,$$

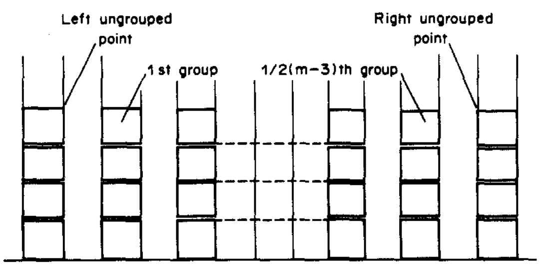

Fig. 6. The GEU scheme (odd number of intervals).

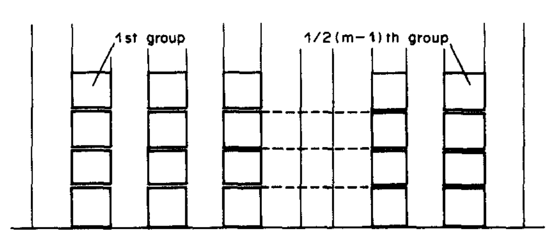

Fig. 7. The GEC scheme (odd number of intervals).

which in the matrix form are written as

$$\left[\begin{array}{cc|c|cc} -\lambda\theta & (1+\lambda\theta) & & & \\ (1-\lambda\theta) & \lambda\theta & & & \\ \hline & & \ddots & 0 & \\ \hline & 0 & & -\lambda\theta & (1+\lambda\theta) \\ & & & (1-\lambda\theta) & \lambda\theta \end{array}\right]\left[\begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_{m-2} \\ u_{m-1} \end{array}\right]_{j+1}$$

$$= \left[\begin{array}{cc|c|cc} \lambda(1-\theta) & [1-\lambda(1-\theta)] & & & \\ [1+\lambda(1-\theta)] & -\lambda(1-\theta] & & & \\ \hline & & \ddots & 0 & \\ \hline & 0 & & \lambda(1-\theta) & [1-\lambda(1-\theta)] \\ & & & [1+\lambda(1-\theta)] & -\lambda(1-\theta) \end{array}\right]\left[\begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_{m-2} \\ u_{m-1} \end{array}\right]_{j}. \tag{36}$$

Therefore, by using equation (34), the GEC scheme can be expressed as

$$(I + \lambda\theta\hat{G}_2)\mathbf{u}_{j+1} = [I - \lambda(1-\theta)\hat{G}_2)\mathbf{u}_j. \tag{37}$$

(iii) The (S)AGE and (D)AGE scheme

The alternating schemes corresponding to the ones that we have developed for the even case are given by

$$\left.\begin{array}{l} (I + \lambda\theta\hat{G}_1)\mathbf{u}_{j+1} = [I - \lambda(1-\theta)\hat{G}_1]\mathbf{u}_j + \mathbf{b}_3 \\ (I + \lambda\theta\hat{G}_2)\mathbf{u}_{j+2} = [I - \lambda(1-\theta)\hat{G}_2]\mathbf{u}_{j+1} \end{array}\right\}, \tag{38}$$

for (S)AGE, Fig. 8, and

$$\left.\begin{array}{l} (I + \lambda\theta\hat{G}_1)\mathbf{u}_{j+1} = [I - \lambda(1-\theta)\hat{G}_1]\mathbf{u}_j + \mathbf{b}_3 \\ (I + \lambda\theta\hat{G}_2)\mathbf{u}_{j+2} = [I - \lambda(1-\theta)\hat{G}_2]\mathbf{u}_{j+1} \\ (I + \lambda\theta\hat{G}_2)\mathbf{u}_{j+3} = [I - \lambda(1-\theta)\hat{G}_2]\mathbf{u}_{j+2} \\ (I + \lambda\theta\hat{G}_1)\mathbf{u}_{j+4} = [I - \lambda(1-\theta)\hat{G}_1]\mathbf{u}_{j+3} + \mathbf{b}_3 \end{array}\right\}, \tag{39}$$

for (D)AGE, Fig. 9.

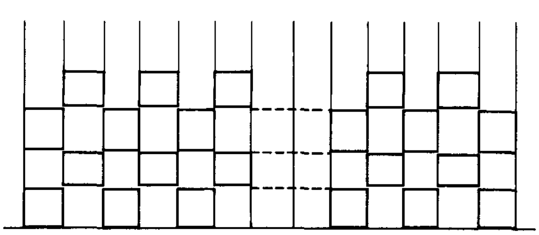

Fig. 8. The (S)AGE scheme (odd number of intervals).

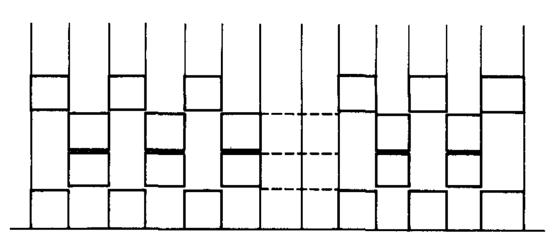

Fig. 9. The (D)AGE scheme (odd number of intervals).

Notice that the above GE formulae are slightly different from those obtained by Evans and Abdullah [1] for parabolic problems in the sense that the same group $\hat{G}_i$ ($i = 1$ or 2) appears in both sides of the equations. The following conclusions may therefore be drawn:

(a) the GE schemes can be derived from the class of locally one-dimensional methods (LOD);
(b) there is no overlapping of the grouping of points. They are disjoint as shown in Figs 2–9;

and

(c) there is no longer a need for the commutativity of the matrices $\hat{G}_1$ and $\hat{G}_2$.

3. TRUNCATION ERROR ANALYSIS FOR THE GE METHODS

(i) Truncation error for the GER scheme

The set of explicit equations obtained by coupling equations (5) and (6) are

$$u_{i-1,j+1} + \lambda u_{ij} - (1+\lambda)u_{i-1,j} = 0 \tag{40}$$

$$u_{i,j+1} - \lambda u_{i-1,j} - (1-\lambda)u_{ij} = 0. \tag{41}$$

The truncation errors for any two grouped points are given by the truncation errors of equations (40) and (41) for $i = 2, 4, \ldots, m-2$. By expanding the terms $U_{i-1,j+1}$, U_{ij}, $U_{i-1,j}$ in equation (40) about the point $[(i-1)\Delta x, (j+1/2)\Delta t]$, we get

$$\begin{aligned}
T_{40} = {} & \left(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t}\right)_{i-1,j+1/2} + \Delta x\left[\frac{1}{2}\frac{\partial^2 U}{\partial x^2} + \frac{(\Delta t)^2}{16}\frac{\partial^4 U}{\partial x^2 \partial t^2}\right]_{i-1,j+1/2} \\
& + \Delta t\left[-\frac{1}{2}\frac{\partial^2 U}{\partial x \partial t} - \frac{1}{12}(\Delta x)^2\frac{\partial^4 U}{\partial x^3 \partial t}\right]_{i-1,j+1/2} + (\Delta x)(\Delta t)\left(-\frac{1}{4}\frac{\partial^3 U}{\partial x^2 \partial t}\right)_{i-1,j+1/2} \\
& + (\Delta x)^2\left(\frac{1}{6}\frac{\partial^3 U}{\partial x^3} + \frac{\Delta x}{24}\frac{\partial^4 U}{\partial x^4}\right)_{i-1,j+1/2} + (\Delta t)^2\left(\frac{1}{8}\frac{\partial^3 U}{\partial x \partial t^2} + \frac{1}{24}\frac{\partial^3 U}{\partial t^3} - \frac{\Delta t}{48}\frac{\partial^4 U}{\partial x \partial t^3}\right)_{i-1,j+1/2} \\
& + \left[\frac{(\Delta x)^4}{120}\frac{\partial^5 U}{\partial x^5} - \frac{(\Delta t)(\Delta x)^3}{48}\frac{\partial^5 U}{\partial x^4 \partial t} + \frac{1}{48}(\Delta x)^2(\Delta t)^2\frac{\partial^5 U}{\partial x^3 \partial t^2}\right. \\
& \left. - \frac{1}{96}(\Delta x)(\Delta t)^3\frac{\partial^5 U}{\partial x^2 \partial t^3} + \frac{1}{384}(\Delta t)^4\frac{\partial^5 U}{\partial x \partial t^4} + \frac{1}{1920}(\Delta t)^4\frac{\partial^5 U}{\partial t^5}\right]_{i-1,j+1/2} + \cdots,
\end{aligned}$$

i.e.

$$\begin{aligned}
T_{40} = {} & \Delta x\left[\frac{1}{2}\frac{\partial^2 U}{\partial x^2} + \frac{(\Delta t)^2}{16}\frac{\partial^4 U}{\partial x^2 \partial t^2}\right]_{i-1,j+1/2} + \Delta t\left[-\frac{1}{2}\frac{\partial^2 U}{\partial x \partial t} - \frac{1}{12}(\Delta x)^2\frac{\partial^4 U}{\partial x^3 \partial t}\right]_{i-1,j+1/2} \\
& + (\Delta x)(\Delta t)\left(-\frac{1}{4}\frac{\partial^3 U}{\partial x^2 \partial t}\right)_{i-1,j+1/2} + (\Delta x)^2\left(\frac{1}{6}\frac{\partial^3 U}{\partial x^3} + \frac{\Delta x}{24}\frac{\partial^4 U}{\partial x^4}\right)_{i-1,j+1/2} \\
& + (\Delta t)^2\left(\frac{1}{8}\frac{\partial^3 U}{\partial x \partial t^2} + \frac{1}{24}\frac{\partial^3 U}{\partial t^3} + \frac{\Delta t}{48}\frac{\partial^4 U}{\partial x \partial t^3}\right)_{i-1,j+1/2} + 0[(\Delta x)^{\alpha_1}(\Delta t)^{\alpha_2}]; \quad \alpha_1 + \alpha_2 = 4.
\end{aligned} \tag{42a}$$

Similarly by expanding the terms $U_{i,j+1}$, $U_{i-1,j}$ and U_{ij} about the point $[i\Delta x, (j+1/2)\Delta t]$ leads to

$$\begin{aligned}
T_{41} = {} & \left(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t}\right)_{i,j+1/2} + \Delta x\left[-\frac{1}{2}\frac{\partial^2 U}{\partial x^2} + \frac{(\Delta t)^2}{16}\frac{\partial^4 U}{\partial x^2 \partial t^2}\right]_{i,j+1/2} \\
& + \Delta t\left[-\frac{1}{2}\frac{\partial^2 U}{\partial x \partial t} - \frac{(\Delta x)^2}{12}\frac{\partial^4 U}{\partial x^3 \partial t}\right]_{i,j+1/2} + (\Delta t)(\Delta x)\left(\frac{1}{4}\frac{\partial^3 U}{\partial x^2 \partial t}\right)_{i,j+1/2}
\end{aligned}$$

$$+(\Delta x)^2\left(\frac{1}{6}\frac{\partial^3 U}{\partial x^3}-\frac{\Delta x}{24}\frac{\partial^4 U}{\partial x^4}\right)_{i,j+1/2}+(\Delta t)^2\left(\frac{1}{8}\frac{\partial^3 U}{\partial x\partial t^2}+\frac{1}{24}\frac{\partial^3 U}{\partial t^3}-\frac{\Delta t}{48}\frac{\partial^4 U}{\partial x\partial t^3}\right)_{i,j+1/2}$$

$$+\left[\frac{(\Delta x)^4}{120}\frac{\partial^5 U}{\partial x^5}+\frac{(\Delta t)(\Delta x)^3}{48}\frac{\partial^5 U}{\partial x^4\partial t}+\frac{(\Delta t)^2(\Delta x)^2}{48}\frac{\partial^5 U}{\partial x^3\partial t^2}+\frac{(\Delta t)^3\Delta x}{96}\frac{\partial^5 U}{\partial x^2\partial t^3}\right.$$

$$\left.+\frac{1}{384}(\Delta t)^4\frac{\partial^5 U}{\partial x\partial t^4}+\frac{(\Delta t)^4}{1920}\frac{\partial^5 U}{\partial t^5}\right]_{i,j+1/2}+\cdots,$$

i.e.

$$T_{41}=\Delta x\left[-\frac{1}{2}\frac{\partial^2 U}{\partial x^2}+\frac{(\Delta t)^2}{16}\frac{\partial^4 U}{\partial x^2\partial t^2}\right]_{i,j+1/2}+\Delta t\left[-\frac{1}{2}\frac{\partial^2 U}{\partial x\partial t}-\frac{(\Delta x)^2}{12}\frac{\partial^4 U}{\partial x^3\partial t}\right]_{i,j+1/2}$$

$$+(\Delta t)(\Delta x)\left(\frac{1}{4}\frac{\partial^3 U}{\partial x^2\partial t}\right)_{i,j+1/2}+(\Delta x)^2\left(\frac{1}{6}\frac{\partial^3 U}{\partial x^3}-\frac{\Delta x}{24}\frac{\partial^4 U}{\partial x^4}\right)_{i,j+1/2}$$

$$+(\Delta t)^2\left(\frac{1}{8}\frac{\partial^3 U}{\partial x\partial t^2}+\frac{1}{24}\frac{\partial^3 U}{\partial t^3}-\frac{\Delta t}{48}\frac{\partial^4 U}{\partial x\partial t^3}\right)_{i,j+1/2}+0[(\Delta x)^{\alpha_1}(\Delta t)^{\alpha_2}];\quad \alpha_1+\alpha_2=4. \tag{42b}$$

The truncation error for the single ungrouped point near the right end is given by the truncation error incurred for equation (6). This is obtained directly by putting $i = m - 1$ in equation (8) which gives

$$T_R=\Delta x\left[\frac{1}{2}\frac{\partial^2 U}{\partial x^2}+\frac{(\Delta t)^2}{16}\frac{\partial^4 U}{\partial x^2\partial t^2}\right]_{m-1,j+1/2}$$

$$+\Delta t\left[-\frac{1}{2}(1-2\theta)\frac{\partial^2 U}{\partial x\partial t}-\frac{1}{12}(\Delta x)^2(1-2\theta)\frac{\partial^4 U}{\partial x^3\partial t}\right]_{m-1,j+1/2}$$

$$+(\Delta x)(\Delta t)\left[-\frac{1}{4}(1-2\theta)\frac{\partial^3 U}{\partial x^2\partial t}\right]_{m-1,j+1/2}+(\Delta x)^2\left(\frac{1}{6}\frac{\partial^3 U}{\partial x^3}+\frac{\Delta x}{24}\frac{\partial^4 U}{\partial x^4}\right)_{m-1,j+1/2}$$

$$+(\Delta t)^2\left[\frac{1}{8}\frac{\partial^3 U}{\partial x\partial t^2}+\frac{1}{24}\frac{\partial^3 U}{\partial t^3}-\frac{\Delta t}{48}(1-2\theta)\frac{\partial^4 U}{\partial x\partial t^3}\right]_{m-1,j+1/2}+0[(\Delta x)^{\alpha_1}(\Delta t)^{\alpha_2}];\quad \alpha_1+\alpha_2=4. \tag{42c}$$

(*ii*) *Truncation error for the GEL scheme*

The truncation error for the single ungrouped point near the left boundary is given by the truncation error for equation (5). Hence with $i = 1$, the expression (7) gives

$$T_L=\Delta x\left[-\frac{1}{2}\frac{\partial^2 U}{\partial x^2}-\frac{(\Delta t)^2}{16}\frac{\partial^4 U}{\partial x^2\partial t^2}\right]_{1,j+1/2}+\Delta t\left[-\frac{1}{2}(1-2\theta)\frac{\partial^2 U}{\partial x\partial t}-\frac{(\Delta x)^2}{12}(1-2\theta)\frac{\partial^4 U}{\partial x^3\partial t}\right]_{1,j+1/2}$$

$$+(\Delta x)(\Delta t)\left[\frac{1}{4}(1-2\theta)\frac{\partial^3 U}{\partial x^2\partial t}\right]_{1,j+1/2}+(\Delta x)^2\left[\frac{1}{6}\frac{\partial^3 U}{\partial x^3}-\frac{\Delta x}{24}\frac{\partial^4 U}{\partial x^4}\right]_{1,j+1/2}$$

$$+(\Delta t)^2\left[\frac{1}{8}\frac{\partial^3 U}{\partial x\partial t^2}+\frac{1}{24}\frac{\partial^3 U}{\partial t^3}-\frac{\Delta t}{48}(1-2\theta)\frac{\partial^4 U}{\partial x\partial t^3}\right]_{1,j+1/2}+0[(\Delta x)^{\alpha_1}(\Delta t)^{\alpha_2}];$$

$$\alpha_1+\alpha_2=4,\quad 0\leqslant\theta\leqslant 1. \tag{43}$$

We note that the truncation errors for any two grouped points of the GEL scheme are given by T_{40} and T_{41} of the equations (42a) and (42b), respectively.

(*iii*) *Truncation error for the GEU scheme*

As indicated by Fig. 6 for the case when an odd number of intervals is used, the truncation error of the scheme at the left ungrouped point is given by T_L of equation (43) whilst the error at the single ungrouped point near the right end is T_R of equation (42c). For the points in between the boundaries which are grouped two at a time, the truncation errors are given by T_{40} and T_{41} of equations (42a) and (42b), respectively.

(iv) Truncation error for the GEC scheme

In this scheme, the grouping of two points at a time along each mesh line is complete as shown by Fig. 7. Hence, the truncation error for this scheme is given by T_{40} and T_{41} respectively for $i = 1, 3, \ldots, m-4, m-2$ when m is odd.

(v) Truncation error for the S(AGE) scheme

If we assume that the number of intervals is even, then as we know from Fig. 4, this scheme entails the alternate use of the GER and the GEL schemes along the vertical direction. Accordingly, its truncation error is given by the truncation errors of the GER and the GEL schemes along the alternate time level. This produces the possible effect of the cancellation of the component error terms at most internal points. A more accurate solution with this scheme is therefore expected than any of the previous GE methods. A similar argument holds when m is odd.

(vi) Truncation error for the D(AGE) scheme

The GER, GEL, GEL and GER methods, in that order, are employed at each of every four time levels. By the same reasoning as above, we will expect this four-step process to be as accurate if not better than the S(AGE) method. In fact, our numerical experiment will show that the D(AGE) procedure can be more superior than the S(AGE) scheme or any of the other GE methods.

4. STABILITY ANALYSIS FOR THE GE METHODS

We shall now proceed to establish the stability requirement of the GE methods. From the formulae (21), (26), (35) and (37), we present below the explicit expressions for the GER, GEL, GEU and the GEC schemes

$$\mathbf{u}_{j+1} = (I + \lambda\theta G_1)^{-1}[I - \lambda(1-\theta)G_1]\mathbf{u}_j + \hat{\mathbf{b}}_1, \tag{44}$$

$$\mathbf{u}_{j+1} = (I + \lambda\theta G_2)^{-1}[I - \lambda(1-\theta)G_2]\mathbf{u}_j + \hat{\mathbf{b}}_2, \tag{45}$$

$$\mathbf{u}_{j+1} = (I + \lambda\theta \hat{G}_1)^{-1}[I - \lambda(1-\theta)\hat{G}_1]\mathbf{u}_j + \hat{\mathbf{b}}_3, \tag{46}$$

and

$$\mathbf{u}_{j+1} = (I + \lambda\theta \hat{G}_2)^{-1}[I - \lambda(1-\theta)\hat{G}_2]\mathbf{u}_j. \tag{47}$$

These processes may be written in a single form as

$$\mathbf{u}_{j+1} = \Gamma\mathbf{u}_j + \hat{\mathbf{b}}, \tag{48}$$

where Γ, the amplification matrix, corresponds to the scheme employed and $\hat{\mathbf{b}}$ is the relevant column vector of order $(m-1)$ as indicated in the formulae above.

(i) Stability of the GER scheme

From the equations (44) and (48), the GER amplification matrix is given by

$\Gamma_{\text{GER}} =$

$$\left[\begin{array}{cc|cc|c|cc|c}
(1+\lambda) & -\lambda & & & & & & \\
\lambda & (1-\lambda) & & & & & & \\
\hline
 & & (1+\lambda) & -\lambda & & & & \\
 & & \lambda & (1-\lambda) & & \multicolumn{2}{c|}{0} & \\
\hline
 & & & & \ddots & & & \\
\hline
 & & & & & (1+\lambda) & -\lambda & \\
 & & \multicolumn{2}{c|}{0} & & \lambda & (1-\lambda) & \\
\hline
 & & & & & & & 1+\dfrac{\lambda}{(1-\lambda\theta)}
\end{array}\right]_{[(m-1)\times(m-1)]} \tag{49}$$

with $\lambda\theta \neq 1$. It can be easily shown that Γ_{GER} possesses the eigenvalues 1, of multiplicity $(m-2)$, and

$$\left[1+\frac{\lambda}{(1-\lambda\theta)}\right].$$

If we denote $\rho(\Gamma_{GER})$ as the spectral radius of Γ_{GER}, then for the stability of the GER scheme, we require $\rho(\Gamma_{GER}) \leqslant 1$. This implies that

$$\left|1+\frac{\lambda}{(1-\lambda\theta)}\right| \leqslant 1, \tag{50}$$

which gives

$$-2 \leqslant \frac{\lambda}{(1-\lambda\theta)} \leqslant 0. \tag{51}$$

Since λ is non-negative, then $(1-\lambda\theta) < 0$ or

$$\lambda\theta > 1 \quad \text{and} \quad \lambda \leqslant -2(1-\lambda\theta). \tag{52}$$

Different cases of θ are now treated to investigate the condition of stability of the GER scheme.

(a) For $\theta = 0$, we have

$$\left|1+\frac{\lambda}{(-\lambda\theta)}\right| = 1+\lambda$$

for all positive values of λ. Therefore $\rho(\Gamma_{GER}) > 1$, which shows that GER scheme is always stable.

(b) For $0 < \theta < 1/2$, condition (52) gives

$$\lambda\theta > 1 \quad \text{and} \quad \lambda \leqslant \frac{2}{(2\theta-1)}.$$

The second inequality can never be satisfied since λ is non-negative whilst $(2\theta - 1)$ is always negative. Hence, for this particular case of θ, the GER method is always unstable.

(c) for $\theta = 1/2$, we obtain

$$\left|1+\frac{\lambda}{(1-\lambda\theta)}\right| = \left|1+\frac{\lambda}{(1-(1/2)\lambda)}\right| > 1,$$

and as in case (a), the GER scheme is absolutely unstable.

(d) For $1/2 < \theta \leqslant 1$, condition (52) becomes

$$\lambda\theta > 1 \quad \text{and} \quad \lambda \geqslant \frac{2}{(2\theta-1)}$$

or

$$\lambda > \frac{1}{\theta} \quad \text{and} \quad \lambda \geqslant \frac{1}{(\theta-1/2)}.$$

We deduce that, the scheme is conditionally stable for

$$\lambda \geqslant \frac{2}{(2\theta-1)}.$$

We conclude from cases (a)–(d) that the GER scheme is:

(1) always unstable for $0 \leqslant \theta \leqslant 1/2$,

and

(2) it is conditionally stable for

$$\lambda \geqslant \frac{2}{(2\theta-1)}$$

when $\theta \in (1/2, 1]$.

It may therefore be summarized that none of the cases above can really be considered useful either because of their unconditional instability (when $0 \leqslant \theta < 1/2$) or due to their "inverse" conditional stability (when $1/2 < \theta < 1$ and which could lead to excessively large time steps).

(ii) Stability of the GEL scheme

From equations (45) and (48), the GEL amplification matrix is given by

$$\Gamma_{\text{GEL}} = \left[\begin{array}{c|cc|c|cc} 1-\dfrac{\lambda}{(1+\lambda\theta)} & & & & & \\ \hline & (1+\lambda) & -\lambda & & & \\ & \lambda & (1-\lambda) & 0 & & \\ \hline & & & \ddots & & \\ \hline & & 0 & & (1+\lambda) & -\lambda \\ & & & & \lambda & (1-\lambda) \end{array}\right]_{[(m-1)\times(m-1)]} . \qquad (53)$$

The eigenvalues of Γ_{GEL} are 1 (of multiplicity $(m-2)$] and

$$\left[1-\frac{\lambda}{(1+\lambda\theta)}\right]$$

and the GEL is stable if $\rho(\Gamma_{\text{GEL}}) \leqslant 1$. This requires that

$$\left|1-\frac{\lambda}{(1+\lambda\theta)}\right| \leqslant 1, \qquad (54)$$

giving,

$$0 \leqslant \frac{\lambda}{(1+\lambda\theta)} \leqslant 2. \qquad (55)$$

Since λ is non-negative then from condition (55), we must have $(1+\lambda\theta) > 0$. Hence,

$$\lambda \leqslant 2(1+\lambda\theta). \qquad (56)$$

(a) For $\theta = 0$, we have

$$\left|1-\frac{\lambda}{(1+\lambda\theta)}\right| = |1-\lambda|.$$

In order that $\rho(\Gamma_{\text{GEL}}) \leqslant 1$, we must have $|1-\lambda| \leqslant 1$ which is satisfied for $\lambda \leqslant 2$. Therefore, for this particular case of θ, the condition of stability is $\lambda \leqslant 2$.

(b) If $0 < \theta < 1/2$, then from condition (56) we obtain $\lambda(1-2\theta) \leqslant 2$ which leads to the following condition of stability:

$$\lambda \leqslant \frac{2}{(1-2\theta)}.$$

(c) For $\theta = 1/2$, we get,

$$\left|1-\frac{\lambda}{(1+\lambda\theta)}\right| = \left|1-\frac{\lambda}{(1+(1/2)\lambda)}\right| < 1$$

for every positive value of λ. This implies that the scheme is always stable for $\theta = 1/2$.

(d) For $1/2 < \theta \leqslant 1$, inequality (56) leads to,

$$\lambda \geqslant \frac{2}{(1-2\theta)}.$$

Now, the quantity $2/(1-2\theta)$ is always negative whilst λ is non-negative. Hence, the scheme is absolutely stable for all values of λ. From all the cases above, we conclude that the GEL scheme is

(1) conditionally stable for $\lambda \leqslant 2/(1-2\theta)$ with $0 \leqslant \theta < 1/2$.

and

(2) it is absolutely stable for all values of λ when $1/2 \leqslant \theta \leqslant 1$.

(iii) Stability of the GEU scheme

From equations (46) and (47), the GEU amplification matrix is given by

$$\Gamma_{\mathrm{GEU}} = \left[\begin{array}{c|cc|c|cc|c} 1-\dfrac{\lambda}{(1+\lambda\theta)} & & & & & & \\ \hline & (1+\lambda) & -\lambda & & & & \\ & \lambda & (1-\lambda) & & & & \\ \hline & & & \ddots & 0 & & \\ \hline & & & & (1+\lambda) & -\lambda & \\ & & & & \lambda & (1-\lambda) & \\ \hline & & & & & & 1-\dfrac{\lambda}{(1+\lambda\theta)} \end{array}\right]_{[(m-1)\times(m-1)]} \tag{57}$$

whose eigenvalues are

$$1-\frac{\lambda}{(1+\lambda\theta)},$$

$$1 \quad \text{[of multiplicity } (m-3)]$$

and

$$1+\frac{\lambda}{(1-\lambda\theta)}.$$

Hence we can easily deduce from the conclusions drawn on the stability analysis of the GER and the GEL schemes that the GEU method is conditionally stable for

$$\lambda \geqslant \frac{2}{(2\theta-1)}$$

when $\theta \in (1/2, 1)$.

(iv) Stability of the GEC scheme

From the equations (47) and (48), the GEC amplification matrix is given by

$$\Gamma_{\text{GEC}} = \left[\begin{array}{cc|cc|c|cc|cc} (1+\lambda) & -\lambda & & & & & & & \\ \lambda & (1-\lambda) & & & & & & & \\ \hline & & (1+\lambda) & -\lambda & & & & & \\ & & \lambda & (1-\lambda) & & 0 & & & \\ \hline & & & & \ddots & & & & \\ \hline & & & & & (1+\lambda) & -\lambda & & \\ & & & 0 & & \lambda & (1-\lambda) & & \\ \hline & & & & & & & (1+\lambda) & -\lambda \\ & & & & & & & \lambda & (1-\lambda) \end{array}\right]_{[(m-1)\times(m-1)]} \tag{58}$$

Since Γ_{GEC} has $(m-1)$ eigenvalues, each equals to 1, then clearly the GEC scheme is always stable with no restrictions on λ and $\theta \in [0, 1]$.

(v) Stability of the (S)AGE scheme

We shall first consider the case when m is even. By means of equations (27), we obtain

$$\begin{aligned} \mathbf{u}_{j+2} &= (I+\lambda\theta G_2)^{-1}[I-\lambda(1-\theta)G_2]\mathbf{u}_{j+1} + (I+\lambda\theta G_2)^{-1}\mathbf{b}_2, \\ &= (I+\lambda\theta G_2)^{-1}[I-\lambda(1-\theta)G_2](I+\lambda\theta G_1)^{-1}[I-\lambda(1-\theta)G_1]\mathbf{u}_j + \mathbf{b}_2', \end{aligned} \tag{59}$$

i.e.

$$\mathbf{u}_{j+2} = \Gamma_{\text{SAGE}}\mathbf{u}_j + \mathbf{b}_2', \tag{60}$$

where

$$\begin{aligned} \Gamma_{\text{SAGE}} &= \{(I+\lambda\theta G_2)^{-1}[I-\lambda(1-\theta)G_2]\}\{(I+\lambda\theta G_1)^{-1}[I-\lambda(1-\theta)G_1]\} \\ &= \Gamma_1\Gamma_2 \end{aligned} \tag{61}$$

and $\mathbf{b}_2'$ is the appropriate column vector of order $(m-1)$. We observe that Γ_1 and Γ_2 are exactly the amplification matrices of the GEL equation (53) and the GER equation (49) schemes, respectively. Hence, by multiplying these matrices, we obtain

$$\Gamma_{\text{SAGE}} = \begin{bmatrix} a & b & & & & & & & & \\ c & d & -c & e & & & & & & \\ e & f & d & -f & & & & & & \\ & & c & d & -c & e & & 0 & & \\ & & e & f & d & -f & & & & \\ & & & & & \ddots & & & & \\ & & & & & & c & d & -c & e \\ & & 0 & & & & e & f & d & -f \\ & & & & & & & & c & d & g \\ & & & & & & & & e & f & h \end{bmatrix}_{[(m-1)\times(m-1)]} \tag{62}$$

where

$$a = (1+\lambda)[1-\lambda/(1+\lambda\theta)],$$
$$b = -\lambda[1-\lambda/(1+\lambda\theta)],$$
$$c = \lambda(1+\lambda),$$
$$d = 1-\lambda^2,$$
$$e = \lambda^2,$$
$$f = \lambda(1-\lambda),$$
$$g = -\lambda[1+\lambda/(1-\lambda\theta)]$$

and

$$h = (1-\lambda)[1+\lambda/(1-\lambda\theta)]. \tag{63}$$

Note that $\operatorname{diag}(\Gamma_{\text{SAGE}}) = (a, d, d, \ldots, d, h)$ with d occurring $(m-3)$ times. It is difficult to evaluate directly the eigenvalues of Γ_{SAGE} in a closed form. However, we know from matrix theory that if the eigenvalues of Γ_{SAGE} are denoted by μ_i, $i = 1, 2, \ldots, m-1$, then

$$\sum_{i=1}^{m-1} \mu_i = \operatorname{tr}(\Gamma_{\text{SAGE}}),$$

where $\operatorname{tr}(\Gamma_{\text{SAGE}})$ is the trace of Γ_{SAGE} which is the sum of the diagonal elements of Γ_{SAGE}, i.e.

$$\mu_1 + \mu_2 + \cdots + \mu_{m-1} = a + (m-3)d + h.$$

Now if we insist $\rho(\Gamma_{\text{SAGE}}) \leqslant 1$ it follows that

$$|\mu_1 + \mu_2 + \cdots + \mu_{m-1}| = |a + (m-3)d + h| \leqslant |\mu_1| + |\mu_2| + \cdots + |\mu_{m-1}| \leqslant (m-1).$$

Hence we seek the values of λ such that,

$$|a + (m-3)d + h| \leqslant |a| + (m-3)|d| + |h| \leqslant (m-1),$$

i.e.

$$(1+\lambda)|[1-\lambda/(1+\lambda\theta)]| + (m-3)|(1-\lambda^2)| + |(1-\lambda)|[1+\lambda/(1-\lambda\theta)]| \leqslant (m-1).$$

Let

$$\phi(\lambda) = (1+\lambda)|[1-\lambda/(1+\lambda\theta)]| + (m-3)|(1-\lambda^2)| + |(1-\lambda)||[1+\lambda/(1-\lambda\theta)]|.$$

$\phi(\lambda)$ is non-negative and if $\lambda \leqslant 1$ for $\theta \in [0, 1]$, $\lambda\theta \neq 1$ we find that $\phi(\lambda)$ will be a continuous function of λ, i.e.

$$\phi(\lambda) = (1+\lambda)[1-\lambda/(1+\lambda\theta)] + (m-3)(1-\lambda^2) + (1-\lambda)[1+\lambda/(1-\lambda\theta)].$$

$\phi(\lambda)$ attains its greatest value of $(m-1)$ at $\lambda = 0$ and $\phi(\lambda) < m-1$ in the range $0 < \lambda \leqslant 1$.

Therefore, if, $\rho(\Gamma_{\text{SAGE}}) \leqslant 1$, then $\lambda \leqslant 1$, for $\theta \in [0, 1]$, $\lambda\theta \neq 1$.

Now suppose that we form the sequence $\Gamma^2_{\text{SAGE}}, \Gamma^3_{\text{SAGE}}, \ldots$. It is observed that the entries of the product Γ^k_{SAGE} contain combinations of powers in λ, $(1-\lambda)$ and $[1-\lambda/(1+\theta)]$. Hence if $\lambda \leqslant 1$,

$$\lim_{k\to\infty} \Gamma^k_{\text{SAGE}} = 0$$

which implies that Γ_{SAGE} is convergent. A necessary and sufficient condition for this to be so is $\rho(\Gamma_{\text{SAGE}}) < 1$. We conclude that the (S)AGE scheme is stable for $\lambda \leqslant 1$.

When m is odd, the equations constituting the (S)AGE procedure are given by equation (38). This time, however, the amplification matrix Γ_{SAGE} is the product of the amplification matrices of the GEC and the GEU schemes and it takes the form,

$$
\Gamma_{\mathrm{SAGE}} = \begin{bmatrix}
a & -c & e & & & & & & & & & \\
-b & d & -f & & & & & & & & & \\
 & c & d & -c & e & & & & & & & \\
 & e & f & d & -f & & & & & & & \\
 & & & c & d & -c & e & & & 0 & & \\
 & & & e & f & d & -f & & & & & \\
 & & & & & & \ddots & & & & & \\
 & & 0 & & & & & c & d & -c & e & \\
 & & & & & & & e & f & d & -f & \\
 & & & & & & & & & c & d & g \\
 & & & & & & & & & e & f & h
\end{bmatrix}_{[(m-1)\times(m-1)]} \tag{64}
$$

The matrix has the same diagonal elements as in equation in (62) and clearly the (S)AGE scheme for the odd case is also conditionally stable for $\lambda \leqslant 1$.

(vi) Stability of the D(AGE) scheme

The four-step process of equation (28) can also be written in the form

$$
\mathbf{u}_{j+4} = \Gamma_{\mathrm{DAGE}}\mathbf{u}_j + \mathbf{b}_1', \tag{65}
$$

where the amplification matrix Γ_{DAGE} is given by

$$
\begin{aligned}
\Gamma_{\mathrm{DAGE}} &= \{(I+\lambda\theta G_1)^{-1}[I-\lambda(1-\theta)G_1]\}\{(I+\lambda\theta G_2)^{-1}[I-\lambda(1-\theta)G_2]\} \\
&\quad \times \{(I+\lambda\theta G_2)^{-1}[I-\lambda(1-\theta)G_2](I+\lambda\theta G_1)^{-1}[I-\lambda(1-\theta)G_1]\} \\
&= \Gamma_{\mathrm{GER}}\Gamma_{\mathrm{GEL}}\Gamma_{\mathrm{SAGE}}
\end{aligned} \tag{66}
$$

and $\mathbf{b}_1'$ is the relevant column vector of order $(m-1)$. By multiplying the matrices in equation (60), we find that

$$
\Gamma_{\mathrm{DAGE}} = \begin{bmatrix}
p & q & r & s & & & & & & & & \\
-q & t & u & v & & & & & & & & \\
r & -u & w & x & r & s & & & & & & \\
-s & v & -x & z & u & v & & & & & & \\
 & & r & -u & w & x & r & s & & & & \\
 & & -s & v & -x & z & u & v & & 0 & & \\
 & & & & & & \ddots & & & & & \\
 & & & & & & r & -u & w & x & r & s \\
 & & & 0 & & & -s & v & -x & z & u & v \\
 & & & & & & & & r & -u & w & x & y \\
 & & & & & & & & -s & v & -x & z & q_1 \\
 & & & & & & & & & & y & -q_1 & p_1
\end{bmatrix}_{[(m-1)\times(m-1)]} \tag{67}
$$

where

$$
\begin{aligned}
p &= (1+\lambda)^2\left[1-\frac{\lambda}{(1+\lambda\theta)}\right]^2-\lambda^2(1+2\lambda),\\
q &= -\{\lambda(1+\lambda)[1-\lambda/(1+\lambda\theta)]^2+\lambda(1-\lambda)(1+2\lambda)\},\\
r &= 2(1+\lambda)\lambda^2,\\
s &= -2\lambda^3,\\
t &= -\lambda^2[1-\lambda/(1+\lambda\theta)]^2+(1-\lambda)^2(1+2\lambda),\\
u &= -2\lambda(1-\lambda^2),\\
v &= 2(1-\lambda)\lambda^2,\\
w &= (1+\lambda)^2(1-2\lambda)-\lambda^2(1+2\lambda),\\
x &= 2\lambda(2\lambda^2-1),\\
y &= 2\lambda^2[1+\lambda/(1-\lambda\theta)],\\
z &= (1-\lambda)^2(1+2\lambda)-\lambda^2(1-2\lambda),\\
q_1 &= -2\lambda(1-\lambda)[1+\lambda/(1-\lambda\theta)],\\
p_1 &= (1-2\lambda)[1+\lambda/(1-\lambda\theta)]^2.
\end{aligned}
\tag{68}
$$

Now, $\text{diag}(\Gamma_{\text{DAGE}}) = (p, t, w, z, w, z, \ldots, w, z, p_1)$ with w, z each occurring alternately for $(m-4)/2$ times.

If u_i, $i = 1, \ldots, m-1$ represents the eigenvalues of Γ_{DAGE} then

$$\sum_{i=1}^{m-1} \mu_i = \text{tr}(\Gamma_{\text{DAGE}}),$$

i.e.

$$
\begin{aligned}
\mu_1+\mu_2+\cdots+\mu_{m-1} &= p+t+(1/2)(m-4)w+(1/2)(m-4)z+p_1\\
\therefore\ |\mu_1+\mu_2+\cdots+\mu_{m-1}| &= |p+t+(1/2)(m-4)w+(1/2)(m-4)z+p_1|\\
&\leqslant |\mu_1|+|\mu_2|+\cdots+|\mu_{m-1}|.
\end{aligned}
$$

If we require that $p(\Gamma_{\text{DAGE}}) \leqslant 1$ then we have

$$|p+t+(1/2)(m-4)w+(1/2)(m-4)z+p_1| \leqslant |p| + |t| + (1/2)(m-4)|w| + (1/2)(m-4)|z| + |p_1|$$

and we seek values of λ such that

$$|p|+|t|+(1/2)(m-4)|w|+(1/2)(m-4)|z|+|p_1| \leqslant m-1.$$

Now,

$$
\begin{aligned}
&|p|+|t|+(1/2)(m-4)|w|+(1/2)(m-4)|z|+|p_1|\\
&= \left|(1+\lambda)^2\left[1-\frac{\lambda}{(1+\lambda\theta)}\right]^2-\lambda^2(1+2\lambda)\right| + |-\lambda^2[1-\lambda(1+\lambda\theta)]^2+(1-\lambda)^2(1+2\lambda)|\\
&\quad + (1/2)(m-4)|(1+\lambda)^2(1-2\lambda)-\lambda^2(1+2\lambda)| + (1/2)(m-4)|(1-\lambda)^2(1+2\lambda)\\
&\quad -\lambda^2(1-2\lambda)| + |(1-2\lambda)[1+\lambda(1-\lambda\theta)]^2|,\\
&\leqslant |(1+\lambda)^2[1-\lambda/(1+\lambda\theta)]^2| + |\lambda^2(1+2\lambda)| + |-\lambda^2[1-\lambda/(1+\lambda\theta)]^2| + |(1-\lambda)^2(1+2\lambda)|\\
&\quad + (1/2)(m-4)\{|(1+\lambda)^2(1-2\lambda)| + |\lambda^2(1+2\lambda)|\}\\
&\quad + (1/2)(m-4)\{|(1-\lambda)^2(1+2\lambda)| + |\lambda^2(1-2\lambda)|\} + |(1-2\lambda)[1+\lambda/(1-\lambda\theta)]^2|,\\
&= \psi(\lambda).
\end{aligned}
$$

$\psi(\lambda)$ is non-negative and if $\lambda \leqslant 1/2$ for $\theta \in [0, 1]$ we observe that $\psi(\lambda)$ is a continuous function of λ, i.e.

$$\begin{aligned}\psi(\lambda) &= (1+\lambda)^2[1-\lambda/(1+\lambda\theta)]^2 + \lambda^2(1+2\lambda) + \lambda^2[1-\lambda/(1+\lambda\theta)]^2 + (1-\lambda)^2(1+2\lambda) \\ &\quad + (1/2)(m-4)\{(1+\lambda)^2(1-2\lambda) + \lambda^2(1+2\lambda)\} + (1/2)(m-4)\{(1-\lambda)^2(1+2\lambda) \\ &\quad + \lambda^2(1-2\lambda)\} + (1-2\lambda)[1+\lambda/(1-\lambda\theta)]^2 \\ &= [\lambda^2 + (1+\lambda)^2][1-\lambda/(1+\lambda\theta)]^2 + [\lambda^2 + (1-\lambda)^2](1+2\lambda) + (m-4)(1-2\lambda^2) \\ &\quad + (1-2\lambda)[1+\lambda/(1-\lambda\theta)]^2.\end{aligned}$$

Our problem is now reduced to seeking λ such that $\psi(\lambda) \leqslant (m-1)$. $\psi(\lambda)$ achieves its greatest value of $(m-1)$ at $\lambda = 0$ and in the range $0 < \lambda \leqslant 1/2$, $\psi(\lambda) < m-1$. Therefore, if $\rho(\Gamma_{DAGE}) \leqslant 1$ then $\lambda \leqslant 1/2$ for $\theta \in [0, 1]$. For convenience, let us replace Γ_{GEL}, Γ_{GER}, Γ_{SAGE} and Γ_{DAGE} by Γ_1, Γ_2, Γ_3 and Γ_4, respectively. We now construct the sequence of matrices $\Gamma_4, \Gamma_4^2, \Gamma_4^3, \ldots, \Gamma_4^r, \ldots$. Consider,

$$\begin{aligned}\Gamma_4 &= \Gamma_2\Gamma_1\Gamma_3, \quad \text{[from equation (66)]} \\ &= \Gamma_2\Gamma_1(\Gamma_1\Gamma_2), \\ &= \Gamma_2(\Gamma_1^2)\Gamma_2.\end{aligned}$$

Hence,

$$\begin{aligned}\Gamma_4^2 &= (\Gamma_2\Gamma_1^2\Gamma_2)(\Gamma_2\Gamma_1\Gamma_3), \\ &= \Gamma_2\Gamma_1^2\Gamma_2^2\Gamma_1(\Gamma_1\Gamma_2), \\ &= \Gamma_2(\Gamma_1^2\Gamma_2^2)\Gamma_1^2\Gamma_2.\end{aligned}$$

Similarly, we also have

$$\begin{aligned}\Gamma_4^3 &= \Gamma_2(\Gamma_1^2\Gamma_2^2)^2\Gamma_1^2\Gamma_2, \\ \Gamma_4^4 &= \Gamma_2(\Gamma_1^2\Gamma_2^2)^3\Gamma_1^2\Gamma_2\end{aligned}$$

and continuing in this manner, we find that,

$$\Gamma_4^r = \Gamma_2(\Gamma_1^2\Gamma_2^2)^{r-1}\Gamma_1^2\Gamma_2. \tag{69}$$

Combinations of powers in 2λ, $(1-2\lambda)$ and

$$\left[1 - \frac{\lambda}{(1+\lambda\theta)}\right]$$

appear in the entries of $(\Gamma_1^2\Gamma_2^2)^r$. Therefore, if $\lambda \leqslant 1/2$ then

$$\lim_{r\to\infty} (\Gamma_1^2\Gamma_2^2) = 0$$

and from equation (69),

$$\lim_{r\to\infty} \Gamma_4^r = 0,$$

the null matrix.

Hence the sequence $\Gamma_4, \Gamma_4^2, \Gamma_4^3, \ldots$ converges and a necessary and sufficient condition for this to be so is $\rho(\Gamma_4) < 1$.

We conclude that the DAGE scheme is conditionally stable for $\lambda < 1/2$.

The corresponding amplification matrix for the (D)AGE scheme when m is odd is found to be,

$$\Gamma_{\mathrm{DAGE}} =$$

$$\left[\begin{array}{cccccccccccc}
p_2 & q_2 & r_1 & & & & & & & & & \\
c_1 & t_1 & x & r & s & & & & & & & \\
r_2 & -x & w_1 & u & v & & & & 0 & & & \\
 & r & -u & t_1 & x & r & s & & & & & \\
 & -s & v & -x & w_1 & u & v & & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\
 & 0 & & & r & -u & t_1 & x & r & s & & \\
 & & & & -s & v & -x & w_1 & u & v & & \\
 & & & & & & r & -u & t_1 & x & y & \\
 & & & & & & -s & v & -x & w_1 & q_1 & \\
 & & & & & & & & y & -q_1 & p_1 &
\end{array}\right]_{[(m-1)\times(m-1)]} \tag{70}$$

where

$$\begin{aligned}
p_2 &= (1+2\lambda)\left[1-\frac{\lambda}{(1+\lambda\theta)}\right]^2,\\
q_2 &= -2\lambda(1+\lambda)\left[1-\frac{\lambda}{(1+\lambda\theta)}\right],\\
r_1 &= 2\lambda^2\left[1-\frac{\lambda}{(1+\lambda\theta)}\right],\\
c_1 &= 2\lambda(1+\lambda)\left[1-\frac{\lambda}{(1+\lambda\theta)}\right],\\
t_1 &= 1-4\lambda^2(1+\lambda),\\
w_1 &= 1+4\lambda^2(\lambda+1),\\
r_2 &= 2\lambda^2\left[1-\frac{\lambda}{(1+\lambda\theta)}\right]
\end{aligned} \tag{71}$$

and the other entries of the matrix take on the same values as in equation (68). As previously, it can be established that the scheme has conditional stability for $\lambda \leqslant 1/2$.

5. APPLICATION OF THE GE METHODS TO A MORE GENERAL FIRST ORDER EQUATION

Our discussion on the GE methods easily carries over to the case of solving a more general equation of the form,

$$\frac{\partial U}{\partial t}+\frac{\partial U}{\partial x}=k(x,t). \tag{72}$$

The basic implicit formulae defining the GE schemes now become,

$$(1+\lambda\theta)u_{i,j+1}-\lambda\theta u_{i-1,j+1}=[1-\lambda(1-\theta)]u_{i,j}+\lambda(1-\theta)u_{i-1,j}+\Delta t k_{i,j+\theta} \tag{73}$$

and

$$\lambda\theta u_{i\,j+1}+(1-\lambda\theta)u_{i-1,j+1}=-\lambda(1-\theta)u_{ij}+[1+\lambda(1-\theta)]u_{i-1,j}+\Delta t k_{i-1,j+\theta}, \tag{74}$$

from which the following set of explicit equations determining the solutions at the grouped points are derived:

$$u_{i-1,j+1} = (1+\lambda)u_{i-1,j} - \lambda u_{ij} + \Delta t[(1+\lambda\theta)k_{i-1,j+\theta} - \lambda\theta k_{i,j+\theta}] \tag{75}$$

and

$$u_{i,j+1} = \lambda u_{i-1,j} + (1-\lambda)u_{ij} + \Delta t[\lambda\theta k_{i-1,j+\theta} + (1-\lambda\theta)k_{i,j+\theta}]. \tag{76}$$

The equations describing the u-values at the left and right ungrouped points are given respectively by

$$u_{1,j+1} = \{\lambda(1-\theta)u_{0,j} + [1+\lambda(1-\theta)]u_{1j} + \lambda\theta u_{0,j+1} + \Delta t k_{1,j+\theta}\}/(1+\lambda\theta) \tag{77}$$

and

$$u_{m-1,j+1} = \{[1+\lambda(1-\theta)]u_{m-1,j} - \lambda(1-\theta)u_{mj} - \lambda\theta u_{m,j+1} + \Delta t k_{m-1,j+\theta}\}/(1-\lambda\theta), \quad \lambda\theta \neq 1. \tag{78}$$

The one-, two- and four-step processes are then developed in exacly the same manner as before.

6. GE METHODS FOR THE SPATIALLY-CENTRED APPROXIMATION TO THE FIRST ORDER EQUATION

Let us now consider the hyperbolic equation of first order of the form

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} = 0, \tag{79}$$

If we approximate the time and spatial derivatives by the foward and central different formulae, respectively at the point (x_i, t_j), we obtain

$$\frac{\partial U}{\partial t} = \frac{U_{i,j+1} - U_{i,j}}{\Delta t} + 0(\Delta t), \tag{80}$$

and

$$\frac{\partial U}{\partial x} = \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x} + 0([\Delta x]^2), \tag{81}$$

Now, by using the Taylor's series about the point (x_{i+1}, t_j) we have,

$$U_{i+1,j+1} = U_{i+1,j} + (\Delta t)\left(\frac{\partial U}{\partial x}\right)_{i+1,j} + 0([\Delta t]^2) \tag{82}$$

or

$$U_{i+1,j} = U_{i+1,j+1} - (\Delta t)\left(\frac{\partial U}{\partial x}\right)_{i+1,j} + 0([\Delta t]^2). \tag{83}$$

If we substitute this expression into equation (81), we get

$$\frac{\partial U}{\partial x} = \frac{\left[U_{i+1,j+1} - (\Delta t)\left(\frac{\partial U}{\partial t}\right)_{i+1,j} - U_{i-1,j}\right]}{2\Delta x} + 0\left(\frac{[\Delta t]^2}{\Delta x}\right) + 0([\Delta x]^2). \tag{84}$$

By virtue of equation (79), equation (84) together with equation (80) leads to the following finite-difference analogue:

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{-(u_{i+1,j+1} - u_{i-1,j})}{2\Delta x},$$

or

$$u_{i,j+1} + ru_{i+1,j+1} = u_{ij} + ru_{i-1,j}, \tag{85}$$

where we have assumed the consistency relation

$$\frac{\Delta t}{\Delta x} \to 0$$

as $\Delta t, \Delta x \to 0$ and

$$r = (1/2)\lambda = (1/2)\frac{\Delta t}{\Delta x}.$$

Figure 10 shows the computational molecule for formula (85) and the approximation is known as the RL (right to left) type since its computation at the mesh points proceeds from the right boundary. Similarly, if we reverse the above procedure, we obtain the following LR approximation,

$$ru_{i-1,j+1} + u_{i,j+1} = u_{ij} + ru_{i+1\,j} \tag{86}$$

and its computational molecule is given by Fig. 11.

The local truncation error for equation (85) is obtained from the following Taylor series expansion about the point $(x_i, t_{j+1/2})$

$$\begin{aligned} T_{85} = \Bigg[&\left(\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x}\right) + 1/2\frac{\Delta t}{\Delta x}\frac{\partial U}{\partial t} + \frac{1}{8.3!}\frac{(\Delta t)^3}{\Delta x}\frac{\partial^3 U}{\partial t^3} + \frac{1}{3!}(\Delta x)^2\frac{\partial^3 U}{\partial x^3} + \frac{3}{2.3!}(\Delta t)(\Delta x)\frac{\partial^3 U}{\partial x^2 \partial t} \\ &- \frac{3}{4.3!}(\Delta t)^2\frac{\partial^3 U}{\partial x \partial t^2} + \frac{1}{4.3!}(\Delta t)^2\frac{\partial^3 U}{\partial t^3} + \frac{1}{5!}(\Delta x)^4\frac{\partial^5 U}{\partial x^5} + \frac{5}{2.5!}(\Delta x)^3(\Delta t)\frac{\Delta^5 U}{\partial x^4 \partial t} \\ &+ \frac{5}{2.5!}(\Delta x)^2(\Delta t)^2\frac{\partial^5 U}{\partial x^3 \partial t^2} + \frac{5}{4.5!}(\Delta x)(\Delta t)\frac{\partial^5 U}{\partial x^2 \partial t^3} + \frac{5}{16.5!}(\Delta t)^4\frac{\partial^5 U}{\partial x \partial t^4} \\ &+ \frac{1}{32.5!}\frac{(\Delta t)^5}{\Delta x}\frac{\partial^5 U}{\partial t^5} + \cdots\Bigg]_{i,j+1/2}, \end{aligned}$$

i.e.

$$\begin{aligned} T_{85} = &\frac{\Delta t}{\Delta x}\left[\frac{1}{2}\frac{\partial U}{\partial t} + \frac{1}{48}(\Delta t)^2\frac{\partial^3 U}{\partial t^3} + \frac{1}{4}(\Delta x)^2\frac{\partial^3 U}{\partial x^2 \partial t}\right]_{i,j+1/2} + \frac{1}{6}(\Delta x)^2\left(\frac{\partial^3 U}{\partial x^3}\right)_{i,j+1/2} \\ &+ (\Delta t)^2\left(\frac{1}{24}\frac{\partial^3 U}{\partial t^3} - \frac{1}{8}\frac{\partial^3 U}{\partial x \partial t^2}\right)_{i,j+1/2} + \frac{1}{(\Delta x)}0[(\Delta x)^{\alpha_1}(\Delta t)^{\alpha_2}], \quad \alpha_1 + \alpha_2 = 5; \end{aligned} \tag{87}$$

and the truncation error for approximation (86) is given by

$$\begin{aligned} T_{86} = \Bigg\{&\left(\frac{\partial U}{\partial t} - \frac{\partial U}{\partial x}\right) + \frac{1}{2}\frac{\Delta t}{\Delta x}\frac{\partial U}{\partial t} - \frac{1}{3}(\Delta t)^2\frac{\partial^3 U}{\partial x^3} + \frac{3}{2.3!}(\Delta t)(\Delta x)\frac{\partial^3 U}{\partial x^2 \partial t} - \frac{3}{4.3!}(\Delta t)^2\frac{\partial^3 U}{\partial x \partial t^2} \\ &+ \left[\frac{1}{4.3!}(\Delta t)^2 + \frac{1}{8.3!}\frac{(\Delta t)^3}{\Delta x}\right]\frac{\partial^3 U}{\partial t^3} - \frac{1}{5!}(\Delta x)^4\frac{\partial^5 U}{\partial x^5} + \frac{5}{2.5!}(\Delta x)^3(\Delta t)\frac{\partial^5 U}{\partial x^4 \partial t} \\ &- \frac{5}{2.5!}(\Delta x)^2(\Delta t)^2\frac{\partial^5 U}{\partial x^3 \partial t^2} + \frac{5}{4.5!}(\Delta x)(\Delta t)^3\frac{\partial^5 U}{\partial x^2 \partial t^3} \\ &- \left[\frac{1}{16.5!}(\Delta t)^4 + \frac{1}{32.5!}\frac{(\Delta t)^5}{\Delta x}\right]\frac{\partial^5 U}{\partial t^5} + \cdots\Bigg\}_{i,j+1/2}, \end{aligned}$$

i.e.

$$\begin{aligned} T_{86} = &\left(\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x}\right)_{i,j+1/2} + \frac{\Delta t}{\Delta x}\left[\frac{1}{2}\frac{\partial U}{\partial t} + \frac{1}{48}(\Delta t)^2\frac{\partial^3 U}{\partial t^3} + \frac{1}{4}(\Delta x)^2\frac{\partial^3 U}{\partial x^2 \partial t}\right]_{i,j+1/2} \\ &- \frac{1}{6}(\Delta x)^2\left(\frac{\partial^3 U}{\partial x^3}\right)_{i,j+1/2} + (\Delta t)^2\left(\frac{1}{24}\frac{\partial^3 U}{\partial t^3} - \frac{1}{8}\frac{\partial^3 U}{\partial x \partial t^2}\right)_{i,j+1/2} + \frac{1}{\Delta x}0[(\Delta x)^{\alpha_1}(\Delta t)^{\alpha_2}]; \\ &\alpha_1 + \alpha_2 = 5. \end{aligned} \tag{88}$$

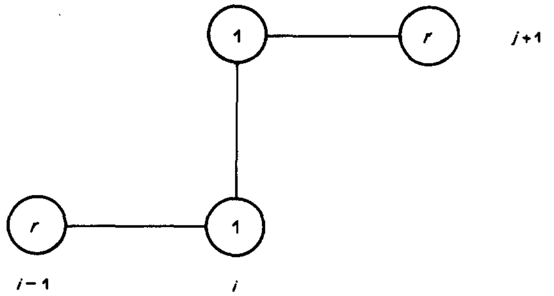

Fig. 10

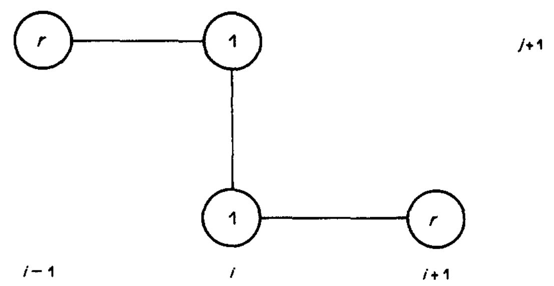

Fig. 11

As previously, the two equations in implicit form, i.e. equations (85) and (86) can be coupled to produce the following set of explicit equations:

$$\begin{bmatrix} u_{i-1,j+1} \\ u_{i,j+1} \end{bmatrix} = \frac{1}{(1-r^2)} \begin{bmatrix} u_{i-1,j} - ru_{ij} + ru_{i-2,j} - r^2u_{i+1,j} \\ -ru_{i-1,j} + u_{ij} - r^2u_{i-2,j} + ru_{i+1,j} \end{bmatrix}$$

or

$$u_{i-1,j+1} = r_1u_{i-1,j} + r_2(u_{ij} - u_{i-2,j}) - r_3u_{i+1,j} \tag{89}$$

and

$$u_{i,j+1} = r_1u_{ij} + r_2(u_{i-1,j} - u_{i+1,j}) - r_3u_{i-2,j}, \tag{90}$$

where

$$r_1 = \frac{1}{(1-r^2)}, \quad r_2 = \frac{-r}{(1-r^2)} \quad \text{and} \quad r_3 = \frac{r^2}{(1-r^2)}$$

with $r \neq 1$. The corresponding computational molecules are shown in Fig. 12.

The solution at the right ungrouped point is given by equation (85) with $i = m - 1$, i.e.

$$u_{m-1,j+1} = -ru_{m,j+1} + u_{m-1,j} + ru_{m-2,j}, \tag{91}$$

whilst the solution at the ungrouped point near the left boundary is determined by equation (86) with $i = 1$, i.e.

$$u_{1,j+1} = -ru_{0,j+1} + u_{1j} + ru_{2j}. \tag{92}$$

Without loss of generality, we shall consider only the GER, GEL, S(AGE) and D(AGE) schemes of the GE class of methods. The truncation errors and stability of these methods will be investigated in some detail.

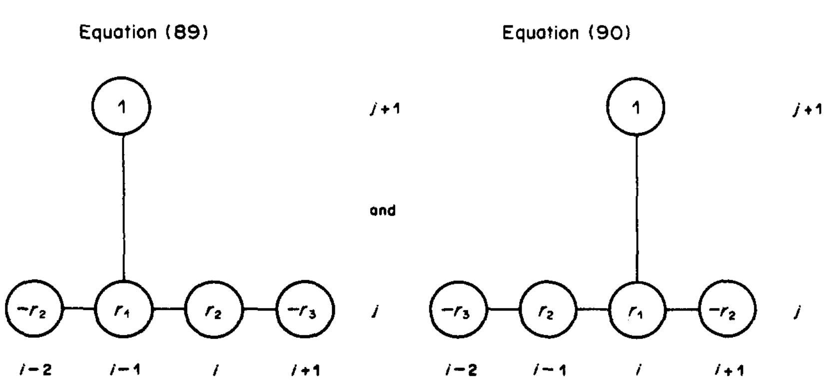

Fig. 12

(i) The GER scheme

From equations (85), (86) and (91), the implicit equations constituting the GER scheme are given by

$$\left.\begin{aligned} u_{i-1,j+1} + ru_{i,j+1} &= u_{i-1,j} + ru_{i-2,j}, \\ ru_{i-1,j+1} + u_{i,j+1} &= u_{i,j} + ru_{i+1,j}, \end{aligned}\right\} i = 2, 4, \ldots, (m-2)$$

and

$$u_{m-1,j+1} = -ru_{m,j+1} + u_{m-1,j} + ru_{m-2,j},$$

which, in matrix form, can be written as,

$$\left[\begin{array}{cc|cc|c|cc|c} 1 & r & & & & & & \\ r & 1 & & & & & & \\ \hline & & 1 & r & & & & \\ & & r & 1 & 0 & & & \\ \hline & & & & \ddots & & & \\ \hline & & 0 & & & 1 & r & \\ & & & & & r & 1 & \\ \hline & & & & & & & 1 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ | \\ | \\ | \\ u_{m-3} \\ u_{m-2} \\ u_{m-1} \end{bmatrix}_{j+1} = \left[\begin{array}{c|cc|c|cc} 1 & & & & & \\ \hline & 1 & r & & & \\ & r & 1 & & & \\ \hline & & & \ddots & 0 & \\ & & & 0 & & \\ \hline & & & & 1 & r \\ & & & & r & 1 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ | \\ | \\ | \\ | \\ u_{m-2} \\ u_{m-1} \end{bmatrix}_{j} + \mathbf{b}_1, \qquad (93)$$

where $\mathbf{b}_1 = (ru_{0j}, 0, \ldots, -ru_{m,j+1})^{T}$. The matrix equation (93) can be represented in the more compact form by

$$(I + G_1)\mathbf{u}_{j+1} = (I + rG_2)\mathbf{u}_j + \mathbf{b}_1, \qquad (94)$$

where

$$G_1 = \left[\begin{array}{cc|cc|c|cc|c} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ \hline & & 0 & 1 & & & & \\ & & 1 & 0 & 0 & & & \\ \hline & & & & \ddots & & & \\ \hline & & & & & 0 & 1 & \\ & & & 0 & & 1 & 0 & \\ \hline & & & & & & & 0 \end{array}\right] = \left[\begin{array}{c|c|c|c|c} G^{(1)} & & & & \\ \hline & G^{(2)} & & 0 & \\ \hline & & \ddots & & \\ \hline & & & G^{(1/2)(m-2)} & \\ & 0 & & & \\ \hline & & & & 0 \end{array}\right],$$

$$G_2 = \left[\begin{array}{c|c|c|c|c} 0 & & & & \\ \hline & G^{(1)} & & & \\ \hline & & G^{(2)} & 0 & \\ \hline & & & \ddots & \\ \hline & & 0 & & G^{(1/2)(m-2)} \end{array}\right] \qquad (95)$$

and

$$G^{(i)}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},$$

$i=1,2,\ldots,(1/2)(m-2)$; m being even.

(ii) The GEL scheme

The implicit equations for the GEL scheme are given by

$$\left.\begin{aligned} u_{1,j+1}&=u_{1j}+ru_{2j}-ru_{0,j+1},\\ u_{i-1,j+1}+ru_{i,j+1}&=u_{i-1,j}+ru_{i-2,j},\\ ru_{i-1,j+1}+u_{i,j+1}&=u_{ij}+ru_{i+1,j},\end{aligned}\right\} i=3,5,\ldots,(m-1),$$

$$\left[\begin{array}{c|cc|cc|c|cc} 1 & & & & & & & \\ \hline & 1 & r & & & & & \\ & r & 1 & & & & & \\ \hline & & & 1 & r & & & \\ & & & r & 1 & & 0 & \\ \hline & & & & & \ddots & & \\ \hline & & & 0 & & & 1 & r \\ & & & & & & r & 1 \end{array}\right]\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ \vdots\\ u_{m-2}\\ u_{m-1}\end{bmatrix}_{j+1}$$

$$=\left[\begin{array}{cc|cc|c|cc|c} 1 & r & & & & & & \\ r & 1 & & & & & & \\ \hline & & 1 & r & & & & \\ & & r & 1 & 0 & & & \\ \hline & & & & \ddots & & & \\ \hline & & 0 & & & 1 & r & \\ & & & & & r & 1 & \\ \hline & & & & & & & 1 \end{array}\right]\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\\ \vdots\\ u_{m-3}\\ u_{m-2}\\ u_{m-1}\end{bmatrix}_j+\mathbf{b}_2, \quad (96)$$

where

$$\mathbf{b}_2=(-ru_{0,j+1},0,0,\ldots,ru_{mj})^{\mathrm{T}},$$

i.e.

$$(I+rG_2)\mathbf{u}_{j+1}=(I+rG_1)\mathbf{u}_j+\mathbf{b}_2. \quad (97)$$

(iii) The (S)AGE and (D)AGE schemes

The two-step (S)AGE scheme is given by the following equations:

$$\left.\begin{aligned}(I+rG_1)\mathbf{u}_{j+1}&=(I+rG_2)\mathbf{u}_j+\mathbf{b}_1,\\ &\text{and}\qquad j=0,2,4,\ldots\\ (I+rG_2)\mathbf{u}_{j+2}&=(I+rG_1)\mathbf{u}_{j+1}+\mathbf{b}_2,\end{aligned}\right\}, \quad (98)$$

while the four-step (D)AGE method is computed from

$$\left.\begin{aligned}(I+rG_1)\mathbf{u}_{j+1}&=(I+rG_2)\mathbf{u}_j+\mathbf{b}_1\\(I+rG_2)\mathbf{u}_{j+2}&=(I+rG_1)\mathbf{u}_{j+1}+\mathbf{b}_2\\(I+rG_2)\mathbf{u}_{j+3}&=(I+rG_1)\mathbf{u}_{j+2}+\mathbf{b}_2\\ \text{and}\qquad (I+rG_1)\mathbf{u}_{j+4}&=(I+rG_2)\mathbf{u}_{j+3}+\mathbf{b}_1\end{aligned}\right\}j=0,4,8,\ldots \tag{99}$$

7. TRUNCATION ERROR ANALYSIS FOR THE GE METHODS

(*i*) *Truncation error for the GER scheme*

From equation (94), the explicit form of the GER scheme is given by

$$\mathbf{u}_{j+1}=(I+rG_1)^{-1}(I+rG_2)\mathbf{u}_j+\hat{\mathbf{b}}_1, \tag{100}$$

where $\hat{\mathbf{b}}_1=(I+rG_1)^{-1}\mathbf{b}_1$. The matrix presentation (100) consists of the following equations:

$$u_{i-1,j+1}=r_1u_{i-1,j}+r_2(u_{ij}-u_{i-2,j})-r_3u_{i+1,j}, \tag{101}$$

$$u_{i,j+1}=r_1u_{ij}+r_2(u_{i-1,j}-u_{i+1,j})-r_3u_{i-2,j},\quad \text{for}\quad i=2,4,\ldots,(m-2), \tag{102}$$

and

$$u_{m-1,j+1}=-ru_{m,j+1}+u_{m-1,j}+ru_{m-2,j}. \tag{103}$$

The truncation error for equation (101) is obtained by expanding the terms $U_{i-1,j+1}$, U_{ij}, $U_{i-2,j}$, $U_{i-1,j}$ and $U_{i+1,j}$ using the Taylor's series about the point $[(i-1)\Delta x,(j+1/2)\Delta t]$

$$\begin{aligned}T_{101}={}&(\Delta t)\left(\frac{\partial U}{\partial t}\right)_{(i-1,j+1/2)}+\frac{(\Delta x)(\Delta t)}{(\Delta t-2\Delta x)}\left[-2\frac{\partial U}{\partial x}-\frac{1}{4}(\Delta t)^2\frac{\partial^3U}{\partial x\partial t^2}+\frac{1}{24}(\Delta t)^3\frac{\partial^4U}{\partial x\partial t^3}\right]_{(i-1,j+1/2)}\\&+\frac{(\Delta x)(\Delta t)^2}{(2\Delta x+\Delta t)}\left(\frac{\partial^2U}{\partial x\partial t}\right)_{(i-1,j+1/2)}+\frac{(\Delta t)^3}{24}\left(\frac{\partial^3U}{\partial t^3}\right)_{(i-1,j+1/2)}+\frac{(\Delta t)^2(\Delta x)^2}{(4(\Delta x)^2-(\Delta t)^2)}\\&\times\left[2\frac{\partial^2U}{\partial x^2}-(\Delta t)\frac{\partial^3U}{\partial x^2\partial t}+\frac{1}{4}(\Delta t)^2\frac{\partial^4U}{\partial x^2\partial t^2}+\frac{2}{3}(\Delta x)^2\frac{\partial^4U}{\partial x^4}\right]_{(i-1,j+1/2)}\\&+\frac{2}{3}\frac{(\Delta x)^3(\Delta t)(\Delta x+2\Delta t)}{(4(\Delta x)^2+(\Delta t)^2)}\left[-\frac{\partial^3U}{\partial x^3}+\frac{(\Delta t)}{2}\frac{\partial^4U}{\partial x^3\partial t}\right]_{(i-1,j+1/2)}+\cdots.\end{aligned} \tag{104}$$

In the same way, a Taylor series expansion for $U_{i,j+1}$, $U_{i-1,j}$, $U_{i+1,j}$, $U_{i,j}$ and $U_{i-2,j}$ about the point $[i\Delta x,(j+1/2)\Delta t]$ gives the following truncation error expression for equation (102):

$$\begin{aligned}T_{102}={}&(\Delta t)\left(\frac{\partial U}{\partial t}\right)_{(i,j+1/2)}+\frac{(\Delta x)(\Delta t)}{(\Delta t-2\Delta x)}\left[2\frac{\partial U}{\partial x}+\frac{1}{4}(\Delta t)^2\frac{\partial^3U}{\partial x\partial t^3}-\frac{1}{24}(\Delta t)^3\frac{\partial^4U}{\partial x\partial t^3}\right]_{(i,j+1/2)}\\&+\frac{(\Delta x)(\Delta t)^2}{(2\Delta x+\Delta t)}\left(\frac{\partial^2U}{\partial x\partial t}\right)_{(i,j+1/2)}+\frac{1}{24}(\Delta t)^3\left(\frac{\partial^3U}{\partial t^3}\right)_{(i,j+1/2)}+\frac{(\Delta x)^2(\Delta t)^2}{(4(\Delta x)^2-(\Delta t)^2)}\\&\times\left[2\frac{\partial^2U}{\partial x^2}-(\Delta t)\frac{\partial^3U}{\partial x^2\partial t}+\frac{1}{4}(\Delta t)^2\frac{\partial^4U}{\partial x^2\partial t^2}+\frac{2}{3}(\Delta x)^2\frac{\partial^4U}{\partial x^4}\right]_{(i,j+1/2)}\\&+\frac{2}{3}\frac{(\Delta x)^3(\Delta t)(\Delta x+2\Delta t)}{(4(\Delta x)^2+(\Delta t)^2)}\left[\frac{\partial^3U}{\partial x^3}-\frac{1}{2}(\Delta t)\frac{\partial^4U}{\partial x^3\partial t}\right]_{(i,j+1/2)}+\cdots.\end{aligned} \tag{105}$$

The truncation error for the single ungrouped point near the right end is given by the equation (87) with $i = m - 1$, i.e.

$$T_R = \frac{\Delta t}{\Delta x}\left[\frac{1}{2}\frac{\partial U}{\partial t} + \frac{1}{48}(\Delta t)^2\frac{\partial^3 U}{\partial t^3} + \frac{1}{4}(\Delta x)^2\frac{\partial^3 U}{\partial x^2 \partial t}\right]_{(m-1,j+1/2)}$$
$$+ \frac{1}{6}(\Delta x)^2\left(\frac{\partial^3 U}{\partial x^3}\right)_{(m-1,j+1/2)} + (\Delta t)^2\left(\frac{1}{24}\frac{\partial^3 U}{\partial t^3} - \frac{1}{8}\frac{\partial^3 U}{\partial x \partial t^2}\right)_{(m-1,j+1/2)} + \cdots. \tag{106}$$

(ii) Truncation error for the GEL scheme

From equation (97), the GEL scheme takes the explicit matrix form,

$$\mathbf{u}_{j+1} = (I + rG_2)^{-1}(I + rG_1)\mathbf{u}_j + \hat{\mathbf{b}}_2, \tag{107}$$

where $\hat{\mathbf{b}}_2 = (I + rG_2)^{-1}\mathbf{b}_2$. When written component-wise, equation (107) becomes,

$$u_{1,j+1} = -ru_{0,j+1} + u_{1j} + ru_{2j}, \tag{108}$$

$$u_{i-1,j+1} = r_1 u_{i-1,j} + r_2(u_{ij} - u_{i-2,j}) - r_3 u_{i+1,j}, \quad i = 3, 5, \ldots, m-1, \tag{109}$$

and

$$u_{i,j+1} = r_1 u_{ij} + r_2(u_{i-1,j} - u_{i+1,j}) - r_3 u_{i-2,j}. \tag{110}$$

The truncation error for the equation (108) can be obtained directly from equation (88) by putting $i = 1$, to give,

$$T_L = \left(\frac{\partial U}{\partial t} - \frac{\partial U}{\partial x}\right)_{1,j+1/2} + \frac{\Delta t}{\Delta x}\left[\frac{1}{2}\frac{\partial U}{\partial t} + \frac{1}{48}(\Delta t)^2\frac{\partial^3 U}{\partial t^3} + \frac{1}{4}(\Delta x)^2\frac{\partial^3 U}{\partial x^2 \partial t}\right]_{1,j+1/2}$$
$$- \frac{1}{6}(\Delta x)^2\left(\frac{\partial^3 U}{\partial x^3}\right)_{1,j+1/2} + (\Delta t)^2\left(\frac{1}{24}\frac{\partial^3 U}{\partial t^3} - \frac{1}{8}\frac{\partial^3 U}{\partial x \partial t^2}\right)_{1,j+1/2} + \cdots. \tag{111}$$

The truncation errors for any two grouped points [equations (109) and (110)] are given by T_{101} and T_{102} of the expressions (104) and (105), respectively for $i = 3, 5, \ldots, m-1$.

(iii) Truncation errors for the (S)AGE and (D)AGE schemes

Based on the truncation errors of the GER and the GEL schemes, the truncation errors of the (S)AGE and (D)AGE methods are analysed in exactly the same manner as for the alternating schemes of the generalized weighted approximation of Section 3.

8. STABILITY ANALYSIS FOR THE GE METHODS

Before we proceed to investigate the stability condition of the GE schemes, we shall first of all establish the following results which will be used in our stability analysis.

From equation (95), we find that the matrices G_1 and G_2 have the same set of eigenvalues which consists of 0, 1 [of multiplicity $(1/2)(m-2)$] and -1 [of multiplicity $(1/2)(m-2)$]. Hence, we observe that although these matrices are symmetric, they are not positive definite. Furthermore, each of the matrices $(I + rG_1)$ and $(I + rG_2)$ has the eigenvalues $0, 1 + r$ [of multiplicity $(1/2)(m-2)$] and $1 - r$ [of multiplicity $(1/2)(m-2)$]. Hence, the spectral radius of $(I + rG_k)$ $(k = 1, 2)$ is given by

$$\rho(I + rG_k) = ||(I + rG_k)||_2 \tag{112}$$
$$= 1 + r \quad \text{for all positive values of } r.$$

The inverses of $(I + rG_k)$ take the form,

$$(I + rG_1)^{-1} = \frac{1}{(1-r^2)} \left[\begin{array}{cc|cc|c|cc|c} 1 & -r & & & & & & \\ -r & 1 & & & & & & \\ \hline & & 1 & -r & & & & \\ & & -r & 1 & & 0 & & \\ \hline & & & & \ddots & & & \\ \hline & & & & 0 & 1 & -r & \\ & & & & & -r & 1 & \\ \hline & & & & & & & (1-r^2) \end{array}\right]_{[(m-1)\times(m-1)]} \tag{113}$$

and

$$(I + rG_2)^{-1} = \frac{1}{(1-r^2)} \left[\begin{array}{c|cc|c|cc} (1-r^2) & & & & & \\ \hline & 1 & -r & & & \\ & -r & 1 & & & \\ \hline & & & \ddots \quad 0 & & \\ & & & 0 \quad \ddots & & \\ \hline & & & & 1 & -r \\ & & & & -r & 1 \end{array}\right]_{[(m-1)\times(m-1)]} \tag{114}$$

A direct evaluation of the eigenvalues of $(I + rG_k)^{-1}$ shows that these eigenvalues are 1, $1/(1-r)$ of multiplicity $(1/2)(m-2)$ and $1/(1+r)$ of multiplicity $(1/2)(m-2)$.

(i) Stability of the GER scheme

From the equation (100), the amplification matrix of the GER scheme is known to be

$$\Gamma_{\text{GER}} = (I + rG_1)^{-1}(I + rG_2). \tag{115}$$

Hence,

$$\begin{aligned} \rho(\Gamma_{\text{GER}}) &= \|\Gamma_{\text{GER}}\|_2, \\ &= \|(I + rG_1)^{-1}(I + rG_2)\|_2, \\ &\leqslant \|(I + rG_1)^{-1}\|_2 \|(I + rG_2)\|_2, && (116) \\ &= \alpha_1. && (117) \end{aligned}$$

To enable us to find the spectral radius of Γ_{GER}, we shall now consider expression (116) for different cases of r.

(*a*) For $0 < r < 1$, we have

$$\begin{aligned} \rho(rG_1) &= \|rG_1\|_2 \\ &= r. \end{aligned}$$

Therefore, if $\|rG_1\|_2 < 1$, then it follows that,

$$\frac{1}{1 + \|rG_1\|_2} \leqslant \|(I + rG_1)^{-1}\|_2 \leqslant \frac{1}{1 - \|rG_1\|_2}. \tag{118}$$

Hence, using expressions (116) and (118) we find that

$$\rho(\Gamma_{\mathrm{GER}}) \leqslant \frac{||(I+rG_2)||_2}{1-||rG_1||_2} = \alpha_1.$$

It is clear that $\rho(\Gamma_{\mathrm{GER}}) \leqslant \alpha_1$ with

$$\alpha_1 = \frac{1+r}{1-r} > 1.$$

(*b*) For $r > 1$, we have

$$(I+rG_1)^{-1}(I+rG_1) = I$$

and

$$||I||_2 \leqslant ||(I+rG_1)^{-1}||_2||(I+rG_1)||_2,$$

which implies,

$$||(I+rG_1)^{-1}||_2 \geqslant \frac{1}{||(I+rG_1)||_2}. \tag{119}$$

Hence, we get from expressions (116) and (119) that $\rho(\Gamma_{\mathrm{GER}}) \leqslant \alpha_1$ with

$$\alpha_1 \geqslant \frac{||(I+rG_2)||_2}{||(I+rG_1)||_2} = \frac{1+r}{1+r} = 1.$$

We deduce from the cases (a) and (b) that for all values of r, the GER scheme is always unstable.

Alternatively, this condition of stability can also be established by first considering the eigenvalues of $(I+rG_k)^{-1}$ which are 1, $1/(1-r)$ of multiplicity $(1/2)(m-2)$ and $1/(1+r)$ of multiplicity $(1/2)(m-2)$ for $k = 1, 2$. It is seen that,

$$\rho[(I+rG_k)^{-1}] = \begin{cases} \dfrac{1}{(1-r)}, & \text{if} \quad 0 < r < 1, \\[2ex] \dfrac{1}{|(1-r)|}, & \text{if} \quad 1 < r \leqslant 2, \\[2ex] 1, & \text{if} \quad r > 2. \end{cases} \tag{120}$$

Now, using expressions (116), (117) and (120) we find that:

(a) for $0 < r < 1$,

$$\alpha_1 = \frac{(1+r)}{(1-r)} > 1 \quad \text{and} \quad ||\Gamma_{\mathrm{GER}}||_2 \leqslant \alpha_1 \quad \text{with} \quad \alpha_1 > 1;$$

(b) for $1 < r \leqslant 2$,

$$\alpha_1 = \frac{(1+r)}{|(1-r)|} > 1 \quad \text{and} \quad ||\Gamma_{\mathrm{GER}}||_2 \leqslant \alpha_1 \quad \text{with} \quad \alpha_1 > 1;$$

(c) for $r > 2$,

$$\alpha_1 = 1 + r > 1 \quad \text{and} \quad ||\Gamma_{\mathrm{GER}}||_2 \leqslant \alpha_1 \quad \text{with} \quad \alpha_1 > 1.$$

From (a)–(c) we deduce that the GER scheme is always unstable.

(*ii*) *Stability of the GEL scheme*

From the equation (107), the amplification matrix of the GEL scheme is given by

$$\Gamma_{\mathrm{GEL}} = (I+rG_1)^{-1}(I+rG_1). \tag{121}$$

and

$$\rho(\Gamma_{\rm GEL}) = ||\Gamma_{\rm GEL}||_2,$$
$$= ||(I+rG_2)^{-1}(I+rG_1)||_2,$$
$$\leqslant ||(I+rG_2)^{-1}||_2||(I+rG_1)||_2. \tag{122}$$

Since $||(I+rG_2)^{-1}||_2 = ||(I+rG_1)^{-1}||_2$ and $||(I+rG_1)||_2 = ||(I+rG_2)||_2$, the analysis of the stability of the GEL method will be the same as that of the GER scheme and we therefore conclude that the GEL scheme is also absolutely unstable.

(iii) Stability of the (S)AGE and (D)AGE schemes

The second equation of condition (98) gives us

$$\mathbf{u}_{j+2} = (I+rG_2)^{-1}(I+rG_1)\mathbf{u}_{j+1} + (I+rG_2)^{-1}\mathbf{b}_2. \tag{123}$$

By inserting $\mathbf{u}_{j+1}$ obtained from the first equation leads to

$$\mathbf{u}_{j+2} = (I+rG_2)^{-1}(I+rG_1)\{(I+rG_1)^{-1}(I+rG_2)\}\mathbf{u}_j + \mathbf{b}_2',$$
$$= (I+rG_2)^{-1}I(I+rG_2)\mathbf{u}_j + \mathbf{b}_2',$$
$$= I\mathbf{u}_j + \mathbf{b}_2'. \tag{124}$$

Hence, the amplification matrix of the (S)AGE scheme is $\Gamma_{\rm SAGE} = I$ with eigenvalues equal to 1 of multiciplicity $(m-1)$. (S)AGE is therefore stable (weakly) for whatever choice of r or λ.

Similarly, from the last two equations of equations (99) we obtain

$$\mathbf{u}_{j+4} = (I+rG_1)^{-1}(I+rG_2)\{(I+rG_2)^{-1}(I+rG_1)\mathbf{u}_{j+2} + (I+rG_2)^{-1}\mathbf{b}_2\} + (I+rG_1)^{-1}\mathbf{b}_1,$$
$$= (I+rG_1)^{-1}\{(I+rG_2)(I+rG_2)^{-1}\}(I+rG_1)\mathbf{u}_{j+2} + \mathbf{b}_2''',$$
$$= (I+rG_1)^{-1}I(I+rG_1)\mathbf{u}_{j+2} + \mathbf{b}_2''',$$
$$= (I+rG_1)^{-1}(I+rG_1)\mathbf{u}_{j+2} + \mathbf{b}_2''',$$
$$= I\mathbf{u}_{j+2} + \mathbf{b}_2'''. \tag{125}$$

The vector $\mathbf{u}_{j+2}$ of equation (124) is then inserted into equation (125) to give

$$\mathbf{u}_{j+4} = I\mathbf{u}_j + \mathbf{b}_2'''. \tag{126}$$

Again, the amplification matrix $\Gamma_{\rm DAGE}$ is the identity matrix with $(m-1)$ eigenvalues, each equal to 1 implying that the (D)AGE scheme is also weakly stable.

9. NUMERICAL EXAMPLES AND COMPARATIVE RESULTS

To demonstrate the application of the GE schemes on hyperbolic problems, four numerical experiments were conducted.

Experiment 1

The weighted GE algorithms of Section 2 were implemented on the following two first-order hyperbolic problems:

(*a*) *Problem 1:*

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} = 0,$$

subject to

$$U(x,0) = \cos x,$$
$$U(0,t) = \cos t,$$

and

$$U(1,t) = \cos(1-t). \tag{127}$$

The analytical solution is given by

$$U(x, t) = \cos(x - t). \tag{128}$$

(*b*) *Problem 2:*

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} = k(x, t),$$
$$[k(x, t) = -2\sin(x - t)\,e^{-2t}],$$

subject to

$$U(x, 0) = \sin x,$$
$$U(0, t) = -\sin t\, e^{-2t},$$

and

$$U(1, t) = \sin(1 - t)\,e^{-2t}. \tag{129}$$

The analytical solution is given by

$$U(x, t) = \sin(x - t)\,e^{-2t}. \tag{130}$$

The GE solutions to Problems 1 and 2 are compared with the solutions obtained from some of the standard methods, such as the classical explicit scheme (EXP) and the schemes of Lax–Wendroff (L–W), Roberts-Weiss (R–W) and Crank–Nicolson (C–N) [or the centred-in-distance, centred-in-time (CD–CT) scheme].

A comparison of their accuracies is obtained by computing the absolute error (A.E.)

$$\text{A.E.} = |e_{i,j}| = |u_{ij} - U_{ij}|, \tag{131}$$

or the percentage error (P.E.)

$$\text{P.E.} = \frac{|e_{ij}|}{|U_{i,j}|} \times 100, \tag{132}$$

at each point along the mesh line where u and U are the numerical and the analytical (exact) solutions respectively. Tables 1 and 2 provide the absolute errors of the numerical solutions to Problem 1 at $t = 0.4$ and $t = 1.0$ for $\lambda = 0.5$ and $\theta = 0.5$. Similarly, the A.E. for the numerical solutions to Problem 2 are shown by Tables 3 and 4. The average of all the absolute errors along the time levels $t = 0.4$ and $t = 1.0$ for each of the schemes involved is also entered in the tables.

Experiment 2

Several runs were made on the implementation of the (S)AGE and (D)AGE schemes for a range of values of θ in [0, 1] and for $t = 0.2(0.2)1.0$. For each particular value of θ, the entries in Tables 5 (Problem 1) and 6 (Problem 2) give the average of the absolute errors along each of the chosen time levels.

Experiment 3

The (S)AGE and (D)AGE schemes of the spatially-centred approximations of Section 6 were applied on Problem 1 and the A.E. and P.E. calculated. Table 7 displays these errors at each mesh point on the time level $t = 1.0$ for $\lambda = 0.1$.

10. DISCUSSION OF NUMERICAL RESULTS

It is clear from Tables 1–4 that the (S)AGE and (D)AGE schemes are more accurate than the GEL method in solving Problems 1 and 2. This result is expected because of the cancellation of error terms at most points of the grid system when the GER and the GEL schemes are applied in their appropriate order of alternation for the (S)AGE and (D)AGE processes. We also find that at some of the mesh points (along $t = 0.4$ and $t = 1.0$), the (S)AGE and (D)AGE schemes can have about the same magnitude of absolute errors as that of the high-order Lax–Wendroff,

Table 1. A.E.'s of the numerical solutions to Problem 1

Method	x: 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Average of all absolute errors
GEL	4.75×10^{-3}	9.89×10^{-2}	5.93×10^{-2}	9.5×10^{-2}	5.65×10^{-2}	8.72×10^{-2}	5.15×10^{-2}	7.6×10^{-2}	4.44×10^{-2}	6.37×10^{-2}
(S)AGE	6.38×10^{-4}	9.92×10^{-4}	2.95×10^{-4}	7.87×10^{-4}	2.42×10^{-4}	9.86×10^{-4}	5.98×10^{-4}	1.25×10^{-3}	6.99×10^{-4}	7.21×10^{-4}
(D)AGE	1.02×10^{-3}	1.02×10^{-3}	1.0×10^{-3}	2.11×10^{-4}	2.04×10^{-4}	2.04×10^{-4}	5.43×10^{-4}	5.43×10^{-4}	1.04×10^{-3}	6.43×10^{-4}
EXP	2.41×10^{-3}	4.83×10^{-3}	7.0×10^{-3}	8.6×10^{-3}	9.45×10^{-3}	9.66×10^{-3}	9.5×10^{-3}	9.17×10^{-3}	8.74×10^{-3}	7.71×10^{-3}
L-W	3.29×10^{-5}	3.87×10^{-5}	1.76×10^{-5}	2.18×10^{-5}	6.91×10^{-5}	1.17×10^{-4}	1.69×10^{-4}	1.93×10^{-4}	3.39×10^{-4}	1.11×10^{-4}
R-W	9.49×10^{-5}	1.11×10^{-4}	6.28×10^{-4}	2.58×10^{-5}	1.36×10^{-4}	2.42×10^{-4}	4.28×10^{-4}	2.75×10^{-4}	1.09×10^{-3}	2.74×10^{-4}
C-N(CD-CT)	5.77×10^{-5}	7.2×10^{-5}	4.18×10^{-5}	5.06×10^{-6}	9.73×10^{-5}	9.82×10^{-5}	3.44×10^{-4}	5.78×10^{-5}	7.15×10^{-4}	1.65×10^{-4}
Exact solution	0.9553365	0.9800666	0.9950042	1.0	0.9950042	0.9800666	0.9553365	0.9210610	0.8775826	—

$t = 0.4$, $\lambda = 0.5$, $\Delta t = 0.05$, $\Delta x = 0.1$, $\theta = 0.5$.

Table 2. A.E.'s of the numerical solutions to Problem 1

Method	x: 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Average of all absolute errors
GEL	3.33×10^{-3}	5.31×10^{-1}	4.38×10^{-1}	5.31×10^{-1}	4.35×10^{-1}	5.09×10^{-1}	4.14×10^{-1}	4.86×10^{-1}	3.78×10^{-1}	4.12×10^{-1}
(S)AGE	2.18×10^{-4}	1.58×10^{-3}	2.46×10^{-4}	2.07×10^{-3}	4.82×10^{-4}	2.27×10^{-3}	5.0×10^{-4}	1.89×10^{-3}	1.18×10^{-4}	1.04×10^{-3}
(D)AGE	3.27×10^{-4}	1.12×10^{-3}	1.87×10^{-4}	1.36×10^{-3}	1.66×10^{-4}	1.50×10^{-3}	1.55×10^{-4}	1.88×10^{-3}	2.39×10^{-4}	7.71×10^{-4}
EXP	1.65×10^{-3}	3.64×10^{-3}	5.94×10^{-3}	8.47×10^{-3}	1.11×10^{-2}	1.39×10^{-2}	1.66×10^{-2}	1.90×10^{-2}	2.11×10^{-2}	1.13×10^{-2}
L-W	9.52×10^{-5}	1.73×10^{-4}	2.31×10^{-4}	2.68×10^{-4}	2.8×10^{-4}	2.72×10^{-4}	2.15×10^{-4}	1.80×10^{-4}	4.89×10^{-6}	1.91×10^{-4}
R-W	2.38×10^{-4}	4.7×10^{-4}	5.29×10^{-4}	9.27×10^{-4}	3.53×10^{-4}	1.39×10^{-3}	1.44×10^{-4}	1.12×10^{-3}	1.93×10^{-4}	5.97×10^{-4}
C-N(CD-CT)	1.14×10^{-6}	4.58×10^{-4}	4.97×10^{-5}	8.29×10^{-4}	7.91×10^{-6}	9.04×10^{-4}	5.1×10^{-6}	5.75×10^{-4}	3.55×10^{-5}	3.18×10^{-4}
Exact solution	0.6216100	0.6967067	0.7648422	0.8253356	0.8775826	0.9210610	0.9553365	0.9800666	0.9950042	—

$t = 1.0$, $\lambda = 0.5$, $\Delta t = 0.05$, $\Delta x = 0.1$, $\theta = 0.5$.

Table 3. A.E.'s of the numerical solutions to Problem 2

Method	x: 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Average of all absolute errors
GEL	5.94×10^{-4}	9.06×10^{-3}	6.99×10^{-3}	2.39×10^{-2}	1.64×10^{-2}	3.78×10^{-2}	2.52×10^{-2}	5.01×10^{-2}	3.3×10^{-2}	1.85×10^{-2}
(S)AGE	9.13×10^{-4}	7.71×10^{-4}	8.33×10^{-4}	9.11×10^{-4}	8.14×10^{-4}	6.27×10^{-4}	7.08×10^{-4}	1.37×10^{-4}	6.84×10^{-4}	5.82×10^{-4}
(D)AGE	5.04×10^{-4}	2.74×10^{-4}	4.41×10^{-4}	1.65×10^{-5}	3.57×10^{-5}	5.19×10^{-5}	5.08×10^{-4}	4.69×10^{-4}	3.68×10^{-4}	2.42×10^{-4}
EXP	3.76×10^{-3}	9.7×10^{-3}	1.71×10^{-2}	2.46×10^{-2}	3.05×10^{-2}	3.47×10^{-2}	3.76×10^{-2}	3.99×10^{-2}	4.17×10^{-2}	2.18×10^{-2}
L-W	3.02×10^{-4}	7.62×10^{-4}	1.26×10^{-3}	1.61×10^{-3}	1.75×10^{-3}	1.77×10^{-3}	1.77×10^{-3}	1.62×10^{-3}	2.04×10^{-3}	1.17×10^{-3}
R-W	1.11×10^{-4}	3.35×10^{-4}	5.72×10^{-4}	7.7×10^{-4}	9.25×10^{-4}	1.02×10^{-3}	1.24×10^{-3}	8.2×10^{-4}	2.18×10^{-3}	7.25×10^{-4}
C-N(CD-CT)	7.0×10^{-5}	2.08×10^{-4}	3.58×10^{-4}	4.56×10^{-4}	5.61×10^{-4}	5.05×10^{-4}	7.85×10^{-4}	2.9×10^{-4}	1.17×10^{-3}	4.89×10^{-4}
Exact solution	−0.1327158	−0.0890597	−0.044858	0	0.0448580	0.0892679	0.1327858	0.1749769	0.2154198	—

$t = 0.4$, $\lambda = 0.5$, $\Delta t = 0.05$, $\Delta x = 0.1$. $\theta = 0.5$.

Table 4. A.E.'s of the numerical solutions to Problem 2

Method	x: 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Average of all absolute errors
GEL	5.93×10^{-4}	1.12×10^{-3}	4.96×10^{-3}	6.06×10^{-2}	5.61×10^{-2}	1.18×10^{-1}	1.05×10^{-1}	1.70×10^{-1}	1.5×10^{-1}	6.06×10^{-2}
(S)AGE	1.38×10^{-3}	2.48×10^{-4}	1.23×10^{-3}	2.53×10^{-4}	1.16×10^{-3}	4.85×10^{-4}	1.01×10^{-3}	4.15×10^{-4}	8.9×10^{-4}	6.43×10^{-4}
(D)AGE	7.06×10^{-4}	2.99×10^{-4}	6.61×10^{-4}	2.83×10^{-5}	5.24×10^{-4}	1.82×10^{-4}	4.28×10^{-4}	3.34×10^{-4}	3.48×10^{-4}	3.19×10^{-4}
EXP	3.64×10^{-4}	1.99×10^{-4}	6.98×10^{-4}	2.57×10^{-3}	5.68×10^{-3}	1.02×10^{-2}	1.63×10^{-2}	2.35×10^{-2}	3.15×10^{-2}	8.28×10^{-3}
L–W	5.24×10^{-5}	1.38×10^{-4}	2.54×10^{-4}	3.9×10^{-4}	5.93×10^{-4}	8.71×10^{-4}	1.43×10^{-3}	1.57×10^{-3}	3.16×10^{-3}	7.69×10^{-4}
R–W	6.86×10^{-6}	6.5×10^{-5}	9.85×10^{-5}	4.22×10^{-4}	8.38×10^{-4}	1.01×10^{-3}	2.06×10^{-3}	8.27×10^{-4}	2.48×10^{-3}	7.1×10^{-4}
C–N(CD–CT)	2.21×10^{-4}	2.63×10^{-4}	4.99×10^{-4}	5.96×10^{-4}	8.46×10^{-4}	6.46×10^{-4}	1.17×10^{-3}	3.34×10^{-4}	1.44×10^{-3}	6.68×10^{-4}
Exact solution	−0.1060118	−0.0970836	−0.0871854	−0.0764160	−0.0648832	−0.0527020	−0.03999431	−0.0268870	−0.0135110	—

$t = 1.0$, $\lambda = 0.5$, $\Delta t = 0.05$, $\Delta x = 0.1$, $\theta = 0.5$.

Table 5. Average of A.E.'s for Problem 1

	Method (θ)									
	(S)AGE					(D)AGE				
t	0	0.25	0.5	0.75	1	0	0.25	0.5	0.75	1
0.2	3.80×10^{-4}	3.5×10^{-4}	3.48×10^{-4}	3.71×10^{-4}	4.39×10^{-4}	2.43×10^{-4}	2.09×10^{-4}	2.88×10^{-4}	3.48×10^{-4}	3.88×10^{-4}
0.4	7.66×10^{-4}	7.23×10^{-4}	7.21×10^{-4}	7.88×10^{-4}	1.02×10^{-3}	5.72×10^{-4}	4.63×10^{-4}	6.43×10^{-4}	7.62×10^{-4}	1.01×10^{-3}
0.6	1.01×10^{-3}	9.43×10^{-4}	9.43×10^{-4}	1.12×10^{-3}	1.62×10^{-3}	8.55×10^{-4}	6.43×10^{-4}	8.19×10^{-4}	1.03×10^{-3}	1.38×10^{-3}
0.8	1.18×10^{-3}	1.07×10^{-3}	1.02×10^{-3}	1.27×10^{-3}	2.13×10^{-3}	1.01×10^{-3}	7.90×10^{-4}	8.55×10^{-4}	1.29×10^{-3}	1.67×10^{-3}
1.0	1.34×10^{-3}	1.17×10^{-3}	1.04×10^{-3}	1.3×10^{-3}	2.45×10^{-3}	1.21×10^{-3}	9.49×10^{-4}	7.71×10^{-4}	1.38×10^{-3}	2.09×10^{-3}

$\lambda = 0.5$, $\Delta t = 0.05$.

Table 6. Average of A.E.'s for Problem 2

	Method (θ)									
	(S)AGE					(D)AGE				
t	0	0.25	0.5	0.75	1	0	0.25	0.5	0.75	1
0.2	1.60×10^{-2}	7.81×10^{-3}	3.82×10^{-4}	7.56×10^{-3}	1.47×10^{-2}	1.67×10^{-2}	8.32×10^{-3}	1.41×10^{-4}	8.4×10^{-3}	1.69×10^{-2}
0.4	2.05×10^{-2}	9.8×10^{-3}	5.82×10^{-4}	9.88×10^{-3}	2.16×10^{-2}	2.29×10^{-2}	1.13×10^{-2}	2.43×10^{-4}	1.15×10^{-2}	2.34×10^{-2}
0.6	1.66×10^{-2}	8.19×10^{-3}	6.68×10^{-4}	1.07×10^{-2}	2.72×10^{-2}	2.18×10^{-2}	1.07×10^{-2}	3.19×10^{-4}	1.13×10^{-2}	2.64×10^{-2}
0.8	1.58×10^{-2}	9.07×10^{-3}	6.77×10^{-4}	1.57×10^{-2}	4.31×10^{-2}	1.82×10^{-2}	9.62×10^{-3}	2.87×10^{-4}	1.38×10^{-2}	3.56×10^{-2}
1.0	2.09×10^{-2}	1.2×10^{-2}	6.43×10^{-4}	2.05×10^{-2}	5.73×10^{-2}	2.11×10^{-2}	1.20×10^{-2}	3.19×10^{-4}	1.8×10^{-2}	4.65×10^{-2}

$\lambda = 0.5$, $\Delta t = 0.05$.

Table 7. A.E. and P.E. of the numerical solutions to Problem 1

Method	x: 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Average of all errors
(S)AGE [A.E.]	3.96×10^{-1}	2.83×10^{-1}	1.9×10^{-1}	9.57×10^{-2}	3.16×10^{-12}	9.57×10^{-2}	1.9×10^{-1}	2.83×10^{-1}	3.73×10^{-1}	1.73×10^{-1}
(S)AGE [P.E. (%)]	63.77	40.67	24.91	11.6	3.6×10^{-10}	10.39	19.94	28.91	37.53	21.61
(D)AGE [A.E.]	3.85×10^{-1}	2.83×10^{-1}	1.9×10^{-1}	9.57×10^{-2}	2.89×10^{-12}	9.57×10^{-2}	1.9×10^{-1}	2.83×10^{-1}	3.85×10^{-1}	1.74×10^{-1}
(D)AGE [P.E. (%)]	61.88	40.67	24.91	11.6	3.3×10^{-10}	10.39	19.94	28.91	38.66	21.54
Exact solution	0.6216100	0.6967067	0.7648422	0.8253356	0.8775826	0.9210610	0.9553365	0.9800666	0.9950042	—

$t = 1.0$, $\lambda = 0.1$, $\Delta t = 0.01$, $\Delta x = 0.01$.

Table 8

Method	Number of multiplications	Number of divisions	Number of additions (subtractions)
EXP	m	—	$2m$
L-W	$2m$	m	$5m$
GER/GEL/(S)AGE/(D)AGE	$m+1$	1	$2m+1$
C-N(CD-CT)	$8m-1$	$3m-2$	$7m-3$

Roberts–Weiss and the Crank–Nicolson methods. In fact, an examination of the average of A.E. for Problem 2 (Tables 3 and 4) clearly shows that the (S)AGE and (D)AGE schemes are more superior than the other methods that we have considered. Furthermore, the computational complexity incurred in solving the first-order hyperbolic equation (1) is also considerably less than that of, say, the Crank–Nicolson method. Table 8 gives us a comparison of the amount of arithmetic involved at m internal mesh points *along each time row* where the solutions of the various difference schemes are determined. It is seen that the (S)AGE and (D)AGE schemes even compare well with the explicit, second-order accurate Lax–Wendroff formula.

We observe from the entries in Tables 5 and 6 that the (S)AGE and (D)AGE schemes are most accurate along the time rows $t = 0.2(0.2)1.0$ for $\lambda = 0.5$ when the time weighting θ takes the value of about 0.5. A possible explanation of this result is that, the terms involving the coefficients $(1-2\theta)$ in the truncation errors in equations (42) and (43) vanish when θ is exactly 0.5. This leads to a considerable increase in the accuracy of the solutions at the ungrouped points and the overall effect of the cancellation of errors due to the alternate use of the GER and GEL algorithms is the improvement in the solutions as they progress forward in time.

Table 7 obviously shows that the stability advantage of the (S)AGE and (D)AGE schemes is clearly overridden by their very poor accuracy when applied to the spatially-centred approximations of Section 6. This stems from the consistency difficulty of the two asymmetric formulae [equations (85) and (86)] which when coupled together determine the basic equations of the GE schemes. From equation (87) we see that in order for $T_{85} \to 0$ as $\Delta x, \Delta t \to 0$, it is essential $\Delta t \to 0$ *faster* than $\Delta x \to 0$. Even if we assume that this consistency requirement is accomplished, we still find from equation (88) that the difference equation (86) would be consistent with the differential equation

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} - \frac{\partial U}{\partial x} = 0,$$

i.e.

$$2\frac{\partial U}{\partial t} = 0,$$

rather than with the hyperbolic equation (1). The truncation error expressions for the GER and GEL further confirm the above consistency problem.

REFERENCE

1. D. J. Evans and A. R. B. Abdullah, Group explicit methods for parabolic equations. *Int. J. Comput. Math.* **14**, 73–105 (1983).