J. Math. Anal. Appl. 351 (2009) 607–615

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Distributional chaos for backward shifts ☆

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ARTICLE INFO

Article history:
Received 2 July 2008
Available online 29 October 2008
Submitted by Richard M. Aron

Keywords:
Distributional chaos
Hypercyclic operators
Devaney chaos
Backward shifts
Köthe sequence spaces

ABSTRACT

We provide sufficient conditions which give uniform distributional chaos for backward shift operators. We also compare distributional chaos with other well-studied notions of chaos for linear operators, like Devaney chaos and hypercyclicity, and show that Devaney chaos implies uniform distributional chaos for weighted backward shifts, but there are examples of backward shifts which are uniformly distributionally chaotic and not hypercyclic.

1. Introduction

The study of different chaotic properties for backward shift operators on Köthe sequence spaces, and more generally on Banach or Fréchet sequence spaces, has been widely treated (see, e.g., [5,11,12,27–29,40,47,48]). Backward shift operators serve as a test for different properties related to chaotic dynamics and, for instance, allow the existence of hypercyclic operators on arbitrary separable infinite dimensional Fréchet spaces [1,10,18]. This fact contrasts with the existence of infinite dimensional separable Banach spaces without chaotic operators in the sense of Devaney [17]. On the other hand backward shifts are connected to semigroups of operators (see, e.g., [22,38,46]), which have applications in the asymptotic behavior of solutions to certain linear PDEs. Semigroups of operators represent the continuous version of the iteration of a single operator, although it has been shown recently that, in fact, there is no difference between the hypercyclic behavior, either in the continuous or in the discrete case [20].

In the paper [36] Li and Yorke observed complicated dynamical behavior for the class of interval maps with period 3. This phenomena is currently known under the name of chaos in the sense of Li and Yorke. It was soon realized that Devaney-chaotic interval maps, or more generally, maps with positive topological entropy exhibit chaos in the sense of Li and Yorke [32] and there are also interval maps with Li–Yorke chaos but with entropy zero [50] (it is also known that interval map with zero entropy does not have nondegenerate subsystems which are chaotic in the sense of Devaney [35]). This motivated Schweizer and Smítal to extend in [49] the definition of Li and Yorke in a way that the definition equivalent to positive topological entropy is obtained for mappings from the compact interval into itself. This new property is currently known under the name of distributional chaos. The equivalence of different kinds of chaos does not usually hold when the space

☆ The research of Piotr Oprocha was supported by the Polish Ministry of Science and Higher Education grant No. NN201272333 for years 2007–2009, an annual national scholarship for young scientists from the Foundation for Polish Science and AGH grant No. 10.420.03. The research of F. Martínez-Giménez and A. Peris was supported by MEC and FEDER, Project MTM2007-64222. A. Peris received the support of Generalitat Valenciana, Project PROMETEO/2008/101.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.10.049

is no longer one-dimensional (e.g. see [2] for a discussion). Distributional chaos always implies chaos in the sense of Li and Yorke, as it requires more complicated statistical dependence between orbits than the existence of points which are proximal but not asymptotic. The converse implication is not true in general. However in practice, even in the simple case of Li and Yorke chaos, it might be quite difficult to prove chaotic behavior from the very definition. Such attempts have been made in the context of linear operators (see [24,25]). Further results of [24] were extended in [44] to distributional chaos for the annihilation operator of a quantum harmonic oscillator.

Our framework will be linear and continuous operators $T: E \to E$ on separable Fréchet spaces E, i.e. vector spaces E which have an increasing sequence $\{\|.\|_n\}_{n\geqslant 1}$ of seminorms that define a metric

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1+\|x-y\|_n}, \quad x, y \in E, \tag{1}$$

under which E is complete and separable.

An operator $T: E \to E$ is called *hypercyclic* if there is a vector $x \in E$ such that its orbit $\mathrm{Orb}(T,x) := \{x, Tx, T^2x, \ldots\}$ is dense in E. In this case x is a *hypercyclic vector* for T. Hypercyclity is then equivalent to transitivity in our context. Devaney [23] defined a continuous mapping f on a metric space X to be chaotic (*Devaney-chaotic*) if it is *topologically transitive*, i.e. for each pair U, V of non-empty open sets there is $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$; the periodic points of f form a dense subset of X; and f possesses the following *sensitivity to initial conditions*: There is an $\varepsilon > 0$ such that for all $\delta > 0$ and $x \in X$, there are $y \in X$ and $n \in \mathbb{N}$ such that $d(x,y) < \delta$ but $d(f^n x, f^n y) > \varepsilon$. When X is separable, complete and has no isolated points, transitivity is equivalent to the existence of a dense orbit. It is known that T is Devaney-chaotic if and only if T is hypercyclic and its periodic points form a dense subset of E (see, e.g., [4]).

The unilateral backward shift B on a sequence space is defined by

$$B(x_1, x_2, x_3, \ldots) := (x_2, x_3, x_4, \ldots).$$

Rolewicz [47] showed that, on the Hilbert space ℓ^2 of 2-summable sequences, any multiple of the backward shift by a scalar of modulus larger than 1 is hypercyclic. MacLane [39] showed that the operator of differentiation, acting on the space of entire functions of one complex variable, is hypercyclic. Salas [48] extended the study of backward shift operators on ℓ^2 to weighted backward shifts and bilateral weighted shift operators. The representation of certain operators on function spaces as weighted backward shift operators on Köthe echelon sequence spaces motivated us to study chaos of weighted shifts. MacLane's example can be easily represented as a Devaney-chaotic weighted backward shift on a Köthe echelon space. The differentiation operator acting on other spaces of entire functions also admits such representation on certain Hilbert sequence spaces [19], or more general spaces of entire functions [15,16]. There exist many sufficient conditions for transitivity of operators [5,7–9,11,14,21,26,27,30,33]. No general method exists to verify if a given operator is distributionally chaotic.

One of our main purposes is to give several conditions ensuring uniform distributional chaos for backward shifts. We also compare distributional chaos with Devaney chaos and hypercyclicity, and show that Devaney chaos implies uniform distributional chaos for weighted backward shifts, but there are examples of backward shifts which are uniformly distributionally chaotic and not hypercyclic.

2. Preliminaries

2.1. Köthe sequence spaces

Our notation for Köthe sequence spaces and Fréchet spaces is standard and we refer to [34,42].

An infinite matrix $A = (a_{j,k})_{j,k\in\mathbb{N}}$ is called a Köthe matrix if, for each $j \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ with $a_{j,k} > 0$, and $0 \leqslant a_{j,k} \leqslant a_{j,k+1}$ for all $j, k \in \mathbb{N}$. For $1 \leqslant p < \infty$, we consider the (separable) Fréchet spaces

$$\lambda_p(A) := \left\{ x \in \mathbb{K}^{\mathbb{N}}\colon\ \|x\|_k := \left(\sum_{j=1}^{\infty} |x_j a_{j,k}|^p \right)^{1/p} < \infty,\ \forall k \in \mathbb{N} \right\},$$

and for $p = 0$

$$\lambda_0(A) := \left\{ x \in \mathbb{K}^{\mathbb{N}}\colon\ \lim_{j\to\infty} x_j a_{j,k} = 0,\ \|x\|_k := \sup_{j\in\mathbb{N}} |x_j| a_{j,k},\ \forall k \in \mathbb{N} \right\},$$

which are the corresponding Köthe sequence spaces.

Köthe spaces constitute a natural class of Fréchet sequence spaces in which many typical examples of weighted shifts are chaotic in some of the senses considered in this article. The easiest example corresponds to the matrix with entries $a_{j,k} = 1$ for all $j, k \in \mathbb{N}$. Such a Köthe matrix gives $\lambda_p(A) = \ell^p$ ($\lambda_0(A) = c_0$), the space of p-summable sequences (null sequences). A diagonal transform of the previous example yields the weighted ℓ^p-spaces. That is $a_{j,k} = a_j$ for all $j, k \in \mathbb{N}$.

The derivative D exhibits chaotic behavior in many spaces X of C^∞-functions. $D: X \to X$ can be represented as a weighted backward shift if the Taylor representation around 0 of functions $f \in X$ allows an isomorphism of X with a Köthe space. The most important case corresponds to $X = \mathcal{H}(\mathbb{C})$, the space of entire functions on the complex plane, where the map $f \mapsto (f^{(j-1)}(0)/(j-1)!)_{j\in\mathbb{N}}$ gives the equality $\mathcal{H}(\mathbb{C}) = \lambda_1(A)$ for the Köthe matrix A with entries $a_{j,k} = k^{j-1}$, $j,k \in \mathbb{N}$.

Gulisashvili and MacCluer [31] studied the quantum harmonic oscillator and observed that the annihilation operator is Devaney-chaotic within the natural context of the Fréchet space of rapidly decreasing functions. This space can be identified with $\lambda_1(A)$ for the matrix A with entries $a_{j,k} = j^k$, $j,k \in \mathbb{N}$, and the annihilation operator becomes a weighted shift. Oprocha [44] showed that the annihilation operator is distributionally chaotic.

2.2. Distributional chaos

The notion of distributional chaos was introduced by Schweizer and Smítal in [49]. The definition was stated for interval maps, and the main motivation was to extend the definition of Li and Yorke chaos in such a way that a condition equivalent to positive topological entropy is obtained (for interval maps). But this definition does not depend on the dimension of the space and can be formulated in any metric space. Various variants of this definition were developed later (e.g. see [3]), and it was also observed that in general this notion is independent from other definitions of chaos (e.g., there are systems with positive topological entropy which do not exhibit distributional chaos [45] and vice-versa [37]).

We will consider in this paper only the definition of uniform distributional chaos, which is one of the strongest possibilities [43]. This property can be defined in the following way:

Definition 1. Let f be a continuous self map on a metric space (X,d). If there exists an uncountable set $D \subset X$ and $\varepsilon > 0$ such that for every $t > 0$ and every distinct $x, y \in D$ the following conditions hold:

$$\liminf_{n\to\infty} \frac{1}{n}\left|\left\{i:\ d\left(f^i(x), f^i(y)\right) < \varepsilon,\ 0 \leqslant i < n\right\}\right| = 0,$$
$$\limsup_{n\to\infty} \frac{1}{n}\left|\left\{i:\ d\left(f^i(x), f^i(y)\right) < t,\ 0 \leqslant i < n\right\}\right| = 1,$$

then we say that f exhibits *uniform distributional chaos* (where $|A|$ denotes the cardinality of the set A). The set D is called a distributionally ε-scrambled set.

Generally speaking, for any distinct $x, y \in D$ the iterations of these points are arbitrarily close and ε separated alternately, but additionally there are time intervals where any of these two excluding possibilities is much more frequent than the other.

If we introduce the following notation (where $x, y \in X$, $n \in \mathbb{N}$ and $t \in \mathbb{R}$)

$$\Phi^{(n)}_{x,y}(t) = \frac{1}{n}\left|\left\{i:\ d\left(f^i(x), f^i(y)\right) < t,\ 0 \leqslant i < n\right\}\right|,$$
$$\Phi_{x,y}(t) = \liminf_{n\to\infty} \Phi^{(n)}_{x,y}, \qquad \Phi^*_{x,y}(t) = \limsup_{n\to\infty} \Phi^{(n)}_{x,y}$$

and additionally denote $\Psi_\alpha = \chi_{(\alpha,+\infty)}$, where χ_A is the characteristic function of the set A, then we can state the definition of uniform distributional chaos in a more compact way. Namely, a set D is distributionally ε-scrambled if $\Phi^*_{x,y} = \Psi_0$ and $\Phi_{x,y} \leqslant \Psi_\varepsilon$ for any distinct $x, y \in D$. A pair which fulfills the above property (i.e. $\Phi^*_{x,y} = \Psi_0$ and $\Phi_{x,y} \leqslant \Psi_\varepsilon$) is called *a distributionally chaotic pair* (*of modulus* ε).

Theorem 2. *Let* (f, X), (g, Y) *be two dynamical systems* (*we denote by* d, ρ *the metric on* X *and* Y *respectively*) *and let* $\pi : X \to Y$ *be a conjugacy between* f *and* g (*i.e.* π *is a homeomorphism and* $\pi \circ f = g \circ \pi$). *Assume additionally that* π *is uniformly continuous. In that case,* f *exhibits uniform distributional chaos if and only if* g *does.*

Proof. The proof follows the same lines as the proof of [51, Theorem 2]. The original proof is for maps acting on compact metric spaces, however only uniform continuity is needed in the argument. □

Remark 3. It directly follows from the definition of the metric (1), that a linear map $\pi : E_1 \to E_2$ between Fréchet spaces E_1, E_2 which is continuous at $0 \in E_1$ is uniformly continuous.

Corollary 4. *Let* $T_i : E_i \to E_i$ *be an operator acting on separable Fréchet spaces* E_i; $i = 1, 2$; *and let* T_1 *be conjugate with* T_2 *by an operator* $\phi : E_1 \to E_2$. *In that case,* T_1 *exhibits uniform distributional chaos if and only if* T_2 *exhibits uniform distributional chaos.*

3. Backward shift operators on Köthe sequence spaces

In order to apply notions from topological dynamics, the backward shift

$$B(x_1, x_2, x_3, \ldots) := (x_2, x_3, x_4, \ldots)$$

must be continuous and well defined on the Köthe echelon space $\lambda_p(A)$. This is equivalent to the following condition on the matrix A:

$$\forall n \in \mathbb{N},\ \exists m > n:\quad \sup_{j\in\mathbb{N}} \frac{a_{j,n}}{a_{j+1,m}} < +\infty \tag{2}$$

where in the case $a_{j+1,m} = 0$, we have $a_{j,n} = 0$ and we consider $\frac{0}{0}$ as 1.

The easiest case is when $A = (1)$, the constant matrix equal to 1, that gives $\lambda_p(A) = \ell^p$. In this case $B : \ell^p \to \ell^p$ is not distributionally chaotic since, for each $x, y \in \ell^p$, we have that $\lim_n d(B^n x, B^n y) = 0$.

Theorem 5. *Let A be a Köthe matrix satisfying* (2) *and $1 \leqslant p < \infty$ or $p = 0$. If there exists a pair $x, y \in \lambda_p(A)$ such that $\Phi_{x,y} \leqslant \Psi_\varepsilon$ for some $\varepsilon > 0$, then $B : \lambda_p(A) \to \lambda_p(A)$ exhibits uniform distributional chaos.*

Proof. We may assume that $x = 0$. Otherwise we redefine, by the translation invariance of the metric, $\hat{x} = 0$ and $\hat{y} = x - y$.

Since $y \in \lambda_p(A)$ and there is $\varepsilon > 0$ such that $\Phi_{0,y} \leqslant \Psi_\varepsilon$, we can find increasing sequences $(m_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$\sum_{j=m_k}^{\infty} |y_j a_{j,k}|^p < \frac{1}{2^k}, \tag{3}$$

$$\frac{n_k - m_k}{n_k} > \frac{k-1}{k}, \quad \text{and} \tag{4}$$

$$\left|\left\{n \leqslant n_k\colon\ d\big(0, B^n y\big) < \varepsilon\right\}\right| < \frac{n_k}{k}, \quad k \in \mathbb{N}. \tag{5}$$

We define $z = (z_j)_{j\in\mathbb{N}}$ by the formula:

$$z_j = \begin{cases} k y_j, & m_k \leqslant j < m_{k+1}, \\ y_j, & 1 \leqslant j < m_0. \end{cases}$$

Observe that $z \in \lambda_p(A)$, because

$$\sum_{j=m_k}^{\infty} |z_j a_{j,k}|^p = \sum_{l\geqslant k} \sum_{j=m_l}^{m_{l+1}-1} |z_j a_{j,k}|^p = \sum_{l\geqslant k} \sum_{j=m_l}^{m_{l+1}-1} l^p |y_j a_{j,k}|^p \leqslant \sum_{l\geqslant k} \sum_{j=m_l}^{m_{l+1}-1} l^p |y_j a_{j,l}|^p \leqslant \sum_{l\geqslant k} \frac{l^p}{2^l} < +\infty.$$

Fix $N \in \mathbb{N}$ and $\delta > 0$ such that

$$\|x\|_N := \left(\sum_{j=1}^{\infty} |x_j a_{j,N}|^p\right)^{1/p} < \delta \quad \text{implies} \quad d(0, x) < \varepsilon, \quad \forall x \in \lambda_p(A). \tag{6}$$

Let $0 < \varepsilon' < 2^{-N-1}$ such that $\frac{2^{N+1}\varepsilon'}{1-2^{N+1}\varepsilon'} < \delta$.

We can find a sufficiently fast increasing sequence $(q_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that, for $\bar{z} = (\bar{z}_j)_{j\in\mathbb{N}}$ defined by the formula

$$\bar{z}_j = \begin{cases} z_j, & q_{2k-1} \leqslant j < q_{2k}, \\ 0, & \text{otherwise}, \end{cases}$$

on one hand we have that each $k \in \mathbb{N}$ admits $j = j(k) \in \mathbb{N}$ so that $[m_j, n_j] \subset [q_{2k-1}, q_{2k}]$ and

$$d\big(B^i \bar{z}, B^i z\big) < \varepsilon', \quad \forall i \in [m_j, n_j], \tag{7}$$

and, on the other hand, we have introduced in $\bar{z}$ sufficiently large intervals of 0's so that $\Phi^*_{0,\alpha\bar{z}} \equiv \Psi_0$ for all α with $|\alpha| \leqslant 1$.

Let us then define $S = \{z_\alpha = \alpha\bar{z}\colon\ \alpha \in [0, 1]\}$. We will show that S is distributionally ε'-scrambled. Note that $\Phi_{z_\alpha, z_\beta} = \Phi_{0, z_{|\alpha-\beta|}}$ for every $\alpha \neq \beta$. In other words, it is enough to prove that $\Phi_{0,z_\alpha} \leqslant \Psi_{\varepsilon'}$ for each $z_\alpha \in S$, $\alpha \neq 0$. Indeed, given $k \in \mathbb{N}$, if $i \in [m_j, n_j] \subset [q_{2k-1}, q_{2k}]$ is such that $d(0, B^i(\alpha\bar{z})) < \varepsilon'$, by (7) we easily get that $d(0, B^i(\alpha z)) < 2\varepsilon'$. The definition of this distance yields $\alpha\|B^i z\|_N < \frac{2^{N+1}\varepsilon'}{1-2^{N+1}\varepsilon'} < \delta$ since, otherwise, $2\varepsilon' \leqslant \frac{1}{2^N}\big(\frac{\|B^i(\alpha z)\|_N}{1+\|B^i(\alpha z)\|_N}\big) \leqslant d(0, B^i(\alpha z))$. We additionally assume that k is big enough so that $\|B^i y\|_N \leqslant \alpha\|B^i z\|_N$ for all $i \in [m_j, n_j]$. Therefore, by (6), we get $d(0, B^i y) < \varepsilon$. That is, we have shown

$$\left\{i \in [m_j, n_j]\colon\ d\big(0, B^i(\alpha\bar{z})\big) < \varepsilon'\right\} \subset \left\{i \in [m_j, n_j]\colon\ d\big(0, B^i y\big) < \varepsilon\right\}.$$

The inequalities (4) and (5) conclude the proof. □

We recall that *the upper density* $\mathscr{D}(A)$ of a set $A \subset \mathbb{N}$ is defined by:

$$\mathscr{D}(A) = \limsup_{n\to\infty} \frac{|A \cap \{1, \dots, n\}|}{n}.$$

Theorem 6. *If there exists a decreasing sequence of sets $\mathbb{N} \supset S_1 \supset S_2 \supset \cdots$ such that for any $n \in \mathbb{N}$ it holds that $\sum_{j \in S_n} a_{j,n}^p < +\infty$ and $\mathscr{D}(S_n) = 1$ then $B : \lambda_p(A) \to \lambda_p(A)$, $1 \leqslant p < \infty$, is uniformly distributionally chaotic.*

Proof. There exists an increasing sequence $(m_n)_{n \in \mathbb{N}}$ such that $\frac{|D_n|}{m_n} > \frac{n-1}{n}$ where $D_n = S_n \cap [0, m_n]$. If we denote $D = \bigcup_{n \in \mathbb{N}} D_n$ then $\mathscr{D}(D) = 1$ and $D \subset [0, m_k] \cup S_k$ for every $k \in \mathbb{N}$. In particular $\sum_{j \in D} a_{j,k}^p < +\infty$ for every $k \in \mathbb{N}$.

By the definition of Köthe matrix, there is K such that $a_{0,K} > 0$. This implies that there is an $\varepsilon > 0$ such that $d(y, 0) > \varepsilon$ provided that $y \in \lambda_p(A)$ and $y_0 = 1$.

If we set $x = \sum_{i \in D} e_i$ then by the definition of the set D we obtain that $x \in \lambda_p(A)$. But $(B^j(x))_0 = 1$ for every $j \in D$; in particular

$$\Phi_{0,x}^{(m_n)}(\varepsilon) \leqslant \frac{m_n - |D_n|}{m_n} \leqslant 1 - \frac{n-1}{n} \longrightarrow 0.$$

We have just proved that $\Phi_{0,x} \leqslant \Psi_\varepsilon$ and by Theorem 5 the proof is finished. □

Given a sequence $\{w_i\}_{i \geqslant 2}$ of strictly positive scalars we may consider its associated weighted backward shift $B_w(x_1, x_2, \ldots) := (w_2 x_2, w_3 x_3, \ldots)$. In our context of Köthe echelon spaces, this class of operators can be reduced to the unweighted case in the following way. Set

$$v_1 := 1, \qquad v_i := \frac{1}{w_2 \ldots w_i}, \quad i > 1,$$
$$\bar{A} := (\bar{a}_{i,k})_{i,k \in \mathbb{N}}: \quad \bar{a}_{i,k} := v_i a_{i,k}, \ \forall i, k \in \mathbb{N}.$$

Via the diagonal transform $\phi_v(x_1, x_2, x_3, \ldots) := (v_1 x_1, v_2 x_2, v_3 x_3, \ldots)$ we construct the (commutative) diagram

$$\begin{array}{ccc} \lambda_p(\bar{A}) & \xrightarrow{B} & \lambda_p(\bar{A}) \\ \phi_v \downarrow & & \downarrow \phi_v \\ \lambda_p(A) & \xrightarrow{B_w} & \lambda_p(A) \end{array}$$

which clearly transfers the dynamics of B to B_w and vice-versa (see [40] for the details). Accordingly, the operator B_w is continuous if and only if

$$\forall n \in \mathbb{N}, \ \exists m > n: \quad \sup_{i \in \mathbb{N}} w_{i+1} \frac{a_{i,n}}{a_{i+1,m}} < \infty. \tag{8}$$

If condition (8) is fulfilled then, by Corollary 4, the study of distributional chaos for weighted backward shifts can be reduced to the unweighted case, with the suitable Köthe matrix. In particular, the stronger condition given in [40, Corollary 3.4] characterizing chaos in the sense of Devaney was

$$\sum_{j=1}^{\infty} \frac{a_{j,n}^p}{(w_2 \ldots w_j)^p} < \infty.$$

This condition implies distributional chaos, since the hypothesis of Theorem 6 are satisfied for $\mathbb{N} = S_1 = S_2 = \cdots$.

Corollary 7. *If there exists a decreasing sequence of sets*

$$\mathbb{N} \supset S_1 \supset S_2 \supset \cdots$$

such that for any $n \in \mathbb{N}$ it holds that

$$\sum_{j \in S_n} \frac{a_{j,n}^p}{(w_2 \ldots w_j)^p} < \infty$$

and $\mathscr{D}(S_n) = 1$ then $B_w : \lambda_p(A) \to \lambda_p(A)$ presents uniform distributional chaos. In particular, if B_w is Devaney-chaotic then it is uniformly distributionally chaotic.

As a consequence we can give a collection of examples that are uniformly distributionally chaotic since they are actually chaotic in the sense of Devaney.

Examples 8. (1) If A is such that for each $k \in \mathbb{N}$ there is $i_k \in \mathbb{N}$ with $a_{i,k} = 0$ for all $i > i_k$ then $\lambda_p(A) := \mathbb{K}^{\mathbb{N}}$. It turns out that B_w exhibits chaos in the sense of Devaney on $\mathbb{K}^{\mathbb{N}}$ for any weight w.

(2) The derivative operator D acting on the space of entire functions $\mathcal{H}(\mathbb{C})$ endowed with its natural compact open topology may be represented as the backward shift B acting on the Köthe space $\lambda_1(A)$ where $A = (a_{i,k})_{i\in\mathbb{N}_0,k\in\mathbb{N}} = (\frac{e^{ik}}{i!})_{i\in\mathbb{N}_0,k\in\mathbb{N}}$, via the isomorphism $f \overset{\Phi}{\mapsto} (f^{(i)}(0))_{i\geqslant 0}$. By [27] we know that D is Devaney-chaotic.

(3) Consider the following subspace of $L^2(\mathbb{R})$:

$$\Phi = \left\{\phi \in L^2(\mathbb{R}):\ \phi = \sum_{n=0}^{\infty} x_n\psi_n,\ \sum_{n=0}^{\infty} |x_n|^2(n+1)^r < \infty \text{ for } r = 0, 1, \ldots\right\}$$

together with the family of seminorms

$$p_m\left(\sum_{n=0}^{\infty} x_n\psi_n\right) = \left(\sum_{n=0}^{\infty} |x_n|^2(n+1)^m\right)^{1/2}, \quad m \geqslant 0,$$

where vectors $\{\psi_n\}_{n\in\mathbb{N}}$ form an orthonormal basis for $L^2(\mathbb{R})$. If we endow Φ with the metric

$$\rho(\phi,\psi) = \sum_{m=0}^{\infty} \frac{1}{2^m} \cdot \frac{p_m(\phi-\psi)}{1+p_m(\phi-\psi)}$$

then it is a separable Fréchet space and the operator $\hat{a} : \Phi \to \Phi$ defined by

$$\hat{a}(\psi_1) = 0, \qquad \hat{a}(\psi_n) = \sqrt{n}\psi_{n-1}, \quad n > 1,$$

is continuous. For a suitable choice of the basis $\{\psi_n\}_{n\in\mathbb{N}}$ operator $\hat{a}$ becomes the annihilation operator for a quantum harmonic oscillator [13]. It was proved in [44] that the operator $\hat{a}$ exhibits uniform distributional chaos. But the system $(\Phi, \hat{a})$ may also be represented as the weighted backward shift operator:

$$B_w(x_1, x_2, \ldots) := (\sqrt{2}x_2, \sqrt{3}x_3, \sqrt{4}x_4, \ldots)$$

defined on Köthe echelon space $\lambda_2(A)$ where

$$a_{j,k} = (j+1)^{\frac{k}{2}}.$$

So assumptions of Corollary 7 are fulfilled (in fact B_w is Devaney-chaotic) and by Corollary 4 we obtain a simple proof of the main result of [44].

(4) The Bessel operator $\Delta_\mu = z^{-2\mu-1}Dz^{2\mu+1}D$ ($\mu > -1/2$), acting on the space of even entire functions $\mathcal{H}_e(\mathbb{C})$ endowed with the compact open topology may also be represented as a weighted backward shift with weights $\{w_i\}_{i\geqslant 1} = \{(i+1+\mu)/(i+1/2)\}_{i\geqslant 1}$ acting on the Köthe space $\lambda_1(A)$, where $A = (a_{i,k})_{i\in\mathbb{N}_0,k\in\mathbb{N}} = (e^{2ik}/(2i)!)_{i\in\mathbb{N}_0,k\in\mathbb{N}}$, and the linking isomorphism is $g \mapsto (g^{(2i)}(0))_{i\geqslant 0}$ (see [6,41] for more details). The series

$$\sum_{j\in\mathbb{N}} \frac{\frac{e^{2jk}}{(2j)!}}{\prod_{i=1}^{j} \frac{i+1+\mu}{i+1/2}}$$

is convergent for any $k \in \mathbb{N}$, and hence Δ_μ is Devaney-chaotic on $\mathcal{H}_e(\mathbb{C})$.

4. Weighted ℓ^p-spaces

In this section we will assume that B (respectively, B_w) is the (weighted) backward shift operator acting on a weighted ℓ^p-space $\ell^p((a_j)_j)$ defined by a sequence of weights $(a_j)_{j\in\mathbb{N}}$. Note that, in this case, conditions (2) and (8) for the continuity of the respective operators simplify to the existence of $\gamma > 0$ such that $\gamma a_{j+1} > a_j$ (respectively, $\gamma a_{j+1} > w_{j+1}a_j$) for each $j \in \mathbb{N}$. Theorem 6 and Corollary 7 also adopt a simplified expression in this case.

Corollary 9. *If there exists a set $S \subset \mathbb{N}$ such that $\sum_{i\in S} a_i^p < +\infty$ and $\mathcal{D}(S) = 1$ then $B : \ell^p((a_j)_j) \to \ell^p((a_j)_j)$ exhibits uniform distributional chaos.*

If there exists a set $S \subset \mathbb{N}$ such that $\sum_{i\in S} \frac{a_i^p}{(w_2 \ldots w_i)^p} < +\infty$ and $\mathcal{D}(S) = 1$ then $B_w : \ell^p((a_j)_j) \to \ell^p((a_j)_j)$ is uniformly distributionally chaotic.

With the above corollary in mind, we provide an example of a hypercyclic and uniformly distributionally chaotic operator that is not Devaney-chaotic.

Example 10. For $k = 1, 2, \dots$ denote $n_k = 2^{k^2}$ and observe that

$$n_1 + \cdots + n_k \leqslant 2 + 4 + \cdots + 2^{k^2} \leqslant 2^{k^2+1}$$

and so

$$\frac{n_{k+1}}{n_1 + \cdots + n_{k+1}} \geqslant \frac{2^{(k+1)^2}}{2^{(k+1)^2} + 2^{k^2+1}} \geqslant 1 - \frac{2^{k^2+1}}{2^{(k+1)^2} + 2^{k^2+1}} \geqslant 1 - \frac{1}{4^k + 1}. \tag{9}$$

Now we take

$$a := \left\{ \mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2^2}}, \dots, \mathbf{\frac{1}{2^{n_1}}}, \frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^{n_1}}, \mathbf{\frac{1}{2^{n_1+1}}}, \dots, \mathbf{\frac{1}{2^{n_2}}}, \dots, \frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^{n_k}}, \mathbf{\frac{1}{2^{n_k+1}}}, \dots, \mathbf{\frac{1}{2^{n_{k+1}}}}, \dots \right\}$$

and S the set of indices associated to weights that are written in **bold** type.

Clearly we have $\sum_{i=1}^{\infty} a_i^p = \infty$ and therefore $B : \ell^p((a_j)_j) \to \ell^p((a_j)_j)$ cannot be chaotic in the sense of Devaney. On the other hand we have that

$$\sum_{i \in S} a_i^p < \infty.$$

It remains to show that S has upper density equal to 1. To this aim it is enough to prove that there is a sequence $\{m_k\}_{k\in\mathbb{N}}$ for which $\lim_{k\to\infty} M_k = 1$ where

$$M_k = \frac{|S \cap \{1, \dots, m_k\}|}{m_k}.$$

Set $m_k := n_1 + \cdots + n_k$ for each $k \in \mathbb{N}$. The following lines show the process of counting the indices from $\{1, \dots, m_k\}$ that belong to S in order to compute the required limit.

$$M_1 = \frac{n_1}{n_1} = 1,$$
$$M_2 = \frac{n_1 + (n_2 - n_1)}{n_1 + n_2} = \frac{n_2}{n_1 + n_2},$$
$$M_3 = \frac{n_1 + (n_2 - n_1) + (n_3 - n_2)}{n_1 + n_2 + n_3} = \frac{n_3}{n_1 + n_2 + n_3},$$
$$\dots.$$

Finally by (9) we have that $\lim_{k\to\infty} \frac{n_k}{n_1 + \cdots + n_k} = 1$.

Given positive integers $i < j$ and a number $\alpha > 0$ we denote

$$S_{i,j}(\alpha) = \{k \in [i, j]:\ a_k \geqslant \alpha\}.$$

Theorem 11. *If there exist a sequence $(\alpha_n)_{n\in\mathbb{N}} \subset (0, +\infty)$ and increasing functions $j_0, j_1 : \mathbb{N} \to \mathbb{N}$ such that $j_1(n) - j_0(n) \geqslant n$ for all $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} \frac{a_{j_1(n)}}{\alpha_n} = 0, \tag{10}$$

$$\lim_{n\to\infty} \frac{|S_{j_0(n), j_1(n)}(\alpha_n)|}{j_1(n) - j_0(n)} = 1 \tag{11}$$

then B exhibits uniform distributional chaos.

Proof. The condition (10) implies that there is an increasing sequence n_k such that:

$$\sum_{k=1}^{\infty} \frac{a_{j_1(n_k)}}{\alpha_{n_k}} < +\infty. \tag{12}$$

Let us define $x = (x_i)_{i\in\mathbb{N}}$ by

$$x_i = \begin{cases} \frac{1}{\alpha_{n_k}}, & i = j_1(n_k), \\ 0, & \text{otherwise}. \end{cases}$$

Note that for $i \in S_{j_0(n_k), j_1(n_k)}(\alpha_{n_k})$ the following condition holds:

$$\left\| B^{j_1(n_k) - i}(x) \right\| \geqslant x_{j_1(n_k)} a_i \geqslant \frac{a_i}{\alpha_{n_k}} \geqslant 1.$$

But $j_1(n_k) - i \in [0, j_1(n_k) - j_0(n_k)]$ and so

$$\Phi_{0,x}^{(j_1(n_k)-j_0(n_k))}(1) < 1 - \frac{|S_{j_0(n_k),j_1(n_k)}(\alpha_{n_k})|}{j_1(n_k) - j_0(n_k)} \longrightarrow 0.$$

We have just proved that $\Phi_{0,x} \leqslant \Psi_1$ and so the result follows by Theorem 5. □

Corollary 12. *Let us set any $r \in (0,1)$. If for any integer $N > 0$ there exists an integer i such that*

$$a_{j+1} \leqslant r a_j, \quad j = i, \ldots, i+N,$$

then B exhibits uniform distributional chaos.

Proof. By the assumptions, for every $n \in \mathbb{N}$ there are $j_0(n)$, $j_1(n)$ such that $j_1(n) - j_0(n) > 2^n$ and $a_{i+1} \leqslant r a_i$ for every $j_0(n) \leqslant i < j_1(n)$. Let us set $\alpha_n = r^{-n} a_{j_1(n)}$.

Observe that when $j_0(n) \leqslant i < j_1(n) - n$ then $a_i \geqslant r^{i-j_1(n)} a_{j_1(n)} \geqslant \alpha_n$ and so

$$\left|S_{j_0(n),j_1(n)}(\alpha_n)\right| \geqslant j_1(n) - j_0(n) - n \geqslant 2^n - n.$$

Additionally, $\lim_{n\to\infty} \frac{a_{j_1(n)}}{\alpha_n} = \lim_{n\to\infty} r^n = 0$ which, by Theorem 11, finishes the proof. □

We can give now an easy example of a uniformly distributionally chaotic backward shift which is not hypercyclic.

Example 13. Let $a = (a_j)_{j\in\mathbb{N}} := (1, 2, 1, 2^2, 2, 1, 2^3, 2^2, 2, 1, \ldots)$. The backward shift $B : \ell^p((a_j)_j) \to \ell^p((a_j)_j)$ is uniformly distributionally chaotic since the hypothesis of Corollary 12 are satisfied for $r = 1/2$. But B is not hypercyclic because the sequence of weights is bounded away from 0 [40, Proposition 3.1].

The following corollary shows that in some cases we may limit our considerations to a special subclass of sequences $(\alpha_n)_{n\in\mathbb{N}}$. This may be useful in applications, where given a sequence of weights we have to choose $(\alpha_n)_{n\in\mathbb{N}}$. In such cases these additional conditions may simplify the search.

Corollary 14. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of weights such that conditions* (10) *and* (11) *are fulfilled by some sequence $(\alpha_n)_{n\in\mathbb{N}}$ and increasing functions j_0, j_1.*

(1) *If $(a_n)_{n\in\mathbb{N}}$ is bounded, then* (*equivalently*) *there exist a sequence $(\hat{\alpha}_n)_{n\in\mathbb{N}} \subset (0, +\infty)$ and increasing functions $\hat{j}_0, \hat{j}_1 : \mathbb{N} \to \mathbb{N}$ such that $\hat{j}_1(n) - \hat{j}_0(n) > n$ and*

$$\lim_{n\to\infty} \hat{\alpha}_n = 0, \qquad \lim_{n\to\infty} \frac{a_{\hat{j}_1(n)}}{\hat{\alpha}_n} = 0,$$
$$\lim_{n\to\infty} \frac{|S_{\hat{j}_0(n),\hat{j}_1(n)}(\hat{\alpha}_n)|}{\hat{j}_1(n) - \hat{j}_0(n)} = 1.$$

(2) *If $(a_n)_{n\in\mathbb{N}}$ is bounded away from* 0, *then* $\lim_{n\to\infty} \alpha_n = +\infty$.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be bounded. If $\liminf_{n\to\infty} \alpha_n = 0$ then it is enough to take a subsequence of $(\alpha_n)_{n\in\mathbb{N}}$ and modify functions j_0, j_1 accordingly. Otherwise, suppose that the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is bounded from below by some $M > 0$. Since the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded so is the sequence $(\alpha_n)_{n\in\mathbb{N}}$, if not $S_{i,j}(\alpha_{m_k})$ would be empty for some increasing sequence $(m_k)_{k\in\mathbb{N}}$. By condition (10) we obtain that $\lim_{n\to\infty} a_n = 0$. There exists $N > 0$ such that $a_n < M^2$ for all $n > N$. It is enough to put $\hat{j}_i = j_i$ for $i = 0, 1$ and $\hat{\alpha}_n = \alpha_n$ if $n \leqslant N$, $\hat{\alpha}_n = \sqrt{a_n}$ otherwise. To finish the proof, observe that $\hat{\alpha}_n < M \leqslant \alpha_n$ and so $S_{i,j}(\alpha_n) \subset S_{i,j}(\hat{\alpha}_n)$.

For the second case, let $(a_n)_{n\in\mathbb{N}}$ be bounded away from 0. There exists $M > 0$ such that $\frac{a_{j_1(n)}}{\alpha_n} \geqslant \frac{M}{\alpha_n}$ which, by condition (10), implies $\lim_{n\to\infty} \alpha_n = +\infty$. □

Acknowledgment

We would like to thank the referee whose comments produced an improvement of the presentation of the paper.

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