Radon–Nikodým derivatives for vector measures belonging to Köthe function spaces

J.M. Calabuig \(^a,1\), P. Gregori \(^b\), E.A. Sánchez Pérez \(^a,*,2\)

\(^a\) Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, València 46022, Spain

\(^b\) Departament de Matemàtiques, Universitat Jaume I de Castelló, Campus Riu Sec, E-12071 Castelló de la Plana, Spain

1. Introduction

The aim of this paper is to obtain a specialized version of a Radon–Nikodým Theorem for vector measures. In [15], K. Musiał considers the following question: suppose that \(X\) is a locally convex space and take a couple of countably additive vector measures \(m, n\) with values in \(X\). When is it possible to obtain a scalar function \(f\), integrable in the sense of Bartle–Dunford–Schwartz, such that

\[
    n(A) = \int_A f \, dm
\]

for every measurable set \(A\)?

The author solves the problem by obtaining the requirements that \(m\) and \(n\) must fulfill to ensure that a relation as the one given by (1) holds. Two situations are considered; in the first one [15, Theorem 1] the Radon–Nikodým derivative of \(n\) with respect to \(m\)—the function \(f\) in (1)—is bounded; in the setting of vector measures defined on Banach spaces, this is equivalent to the fact that (the equivalence class of) \(f\) belongs to \(L^\infty(\mu)\), where \(\mu\) is a Rybakov control measure for \(m\) (i.e. a measure satisfying the conditions of Rybakov Theorem, see [4] for details). The second result gives less restrictive requirements, equivalent to the integrability of the function \(f\) in the sense of Bartle–Dunford–Schwartz [15, Theorem 2]. In this context, it can be considered as an element of \(L^1(m)\), the space of (classes of) integrable functions with respect to
the vector measure \( m \). The precise statements of these results that we will use several times through the paper and can be found in [15] are the following

**Theorem 1.** (See [15].) Let \( X \) be a normed space and let \( \nu \) and \( \kappa \) be \( X \)-valued measures. Then the following assertions are equivalent:

(1) There exists a bounded measurable function \( \theta \) such that
\[
\nu(E) = \int_E \theta \, d\kappa, \quad \text{for each} \ E \in \Sigma.
\]

(2) There exists \( M > 0 \) such that
\[
|\langle \nu, x' \rangle|_F \leq M |\langle \kappa, x' \rangle|_F, \quad \text{for all} \ E \in \Sigma, \ x' \in X'.
\]

**Theorem 2.** (See [15].) Let \( X \) be a normed space and let \( \nu \) and \( \kappa \) be \( X \)-valued measures. If for every \( x' \in X' \) the scalar measure \( \langle \nu, x' \rangle \) is absolutely continuous with respect to \( \langle \kappa, x' \rangle \), then the following assertions are equivalent:

(1') There exists a measurable function \( \theta \) such that
\[
\nu(E) = \int_E \theta \, d\kappa, \quad \text{for each} \ E \in \Sigma.
\]

(2') For each \( E \in \Sigma \) with \( |\kappa|_F (E) > 0 \) there is \( \Sigma \ni F \subseteq E \) such that \( |\kappa|_F (F) > 0 \) and the restrictions of the measures \( \nu, \kappa \) to \( F \) (namely \( \nu|_F, \kappa|_F \)) verify that there is a constant \( M_F > 0 \) such that
\[
|\langle \nu|_F, x' \rangle|_F \leq M_F |\langle \kappa|_F, x' \rangle|_F, \quad \text{for all} \ E \in \Sigma, \ x' \in X'.
\]

**Remark 1.** The results in [15] are given in a more general framework (being \( X \) a locally convex space). The condition (2) in Theorem 1 is the definition of vector measure \( \nu \) scalarly dominated by \( \kappa \) and the condition (2') in Theorem 2 is the definition of vector measure \( \nu \) locally scalarly dominated by \( \kappa \).

This couple of results are the extreme cases of a more general problem that can be formulated as follows. Is it possible to obtain a characterization of when the Radon–Nikodým derivative of \( n \) with respect to \( m \) belongs to a particular sublattice of \( L^1(m) \)? In this paper we provide such a result in the following sense. We will show that the Radon–Nikodým derivative belongs to the Köthe function subspace \( Z(\mu) \) of \( L^1(m) \) whenever a vector norm inequality associated to the norm of \( Z(\mu) \) is satisfied. This result can be found in Section 4. In order to prove it we present a general separation argument, that is developed in Section 3. Actually, we show that this result provides a general framework for the understanding of different arguments in the vector measure setting; our Radon–Nikodým type theorem is then obtained as a consequence of the separation result of Section 3. To finish the paper, some factorizations of operators regarding the results obtained in the other sections are given in Section 5.

2. Notation and preliminaries

Throughout this paper, \( (\Omega, \Sigma, \mu) \) will be a finite measure space, that is, \( \Omega \) is a set, \( \Sigma \) a \( \sigma \)-algebra on \( \Omega \) and \( \mu \) is a scalar positive finite measure unless otherwise indicated. If \( A \in \Sigma \) we denote by \( \mu|_A \) the restriction of \( \mu \) to the subset \( A \). Let \( X \) be a Banach space. We denote by \( X' \) the topological dual of \( X \) and by \( B_{X'} \) its closed unit ball. \( \mathcal{P}(\Omega) \) will represent the set of partitions \( \pi \) of \( \Omega \) into a finite number of disjoint measurable sets. If \( 1 \leq p \leq \infty \) then \( p' \in [1, \infty] \) is given by \( 1/p + 1/p' = 1 \).

Let \( m : \Sigma \to X \) be a (countably additive) vector measure. The semivariation of \( m \) is defined by
\[
\|m\| = \sup_{x' \in B_{X'}} \|(m, x')\| = \sup_{x' \in B_{X'}, \pi \in \mathcal{P}(\Omega)} \sup_{A \in \pi} \|(m, x')(A)\|
\]
where we have employed the usual notation \( \langle m, x' \rangle(A) = \langle m(A), x' \rangle \) for \( A \in \Sigma \). A set \( A \in \Sigma \) is called \( m \)-null if \( \|m\|(A) = 0 \). A property which holds outside an \( m \)-null set is said to hold \( m \)-almost everywhere (\( m \)-a.e. for short).

For a vector measure \( m \), we will consider a Rybakov measure \( \mu \); recall that a Rybakov measure for a vector measure \( m \) is a scalar measure \( \nu \) defined as the variation of a measure \( \langle m, x' \rangle \), where \( x' \in X' \), whenever \( m \) is absolutely continuous with respect to \( \nu \). Rybakov measures always exist for every vector measure \( m \) (see [4, IX.2.2]).

**Definition 2.** A function \( f : \Omega \to \mathbb{R} \) is said to be integrable with respect to the measure \( m \) if

(a) for each \( x' \in X' \) we have that \( f \in L^1(\|(m, x')\|) \),
(b) for each $A \in \Sigma$ there exists $x_A \in X$ such that

$$\langle x_A, x' \rangle = \int_A f \, d|m, x'|, \quad \text{for every } x' \in X'.$$

The space (of the equivalence classes with respect to $m$-almost everywhere equality) of these functions is denoted by $L^1(m)$. The expression

$$\|f\|_{L^1(m)} = \sup_{x' \in B_X' \Omega} \int f \, d|m, x'|, \quad \text{for each } f \in L^1(m),$$

defines a lattice norm on $L^1(m)$ for which $L^1(m)$ is an order continuous Banach lattice with weak unit the characteristic function $\chi_\Omega$ (see [16, Section 3.1]). The indefinite integral $m_f : \Sigma \to X$ of a function $f \in L^1(m)$ is defined by

$$m_f(A) = \int_A f \, dm, \quad A \in \Sigma.$$

The Orlicz–Pettis Theorem ensures that $m_f$ is again a countably additive vector measure. An equivalent norm for $L^1(m)$ is given by

$$\|f\|_{L^1(m)} = \sup_{A \in \Sigma} \|f\|_{A, X} \|f\|_{X}, \quad \text{for each } f \in L^1(m),$$

since $\|f\|_{L^1(m)} \leq \|f\|_{L^1(m)} \leq 2\|f\|_{L^1(m)}$ (see [4, Chapter 1.1.11]).

A function $f : \Omega \to \mathbb{R}$ is said to be scalarly integrable if only condition (a) in Definition 2 is satisfied. The space (of the equivalence classes with respect to $m$-almost everywhere equality) of these functions is denoted by $L^1_w(m)$. As in the case of $L^1(m)$ the expression

$$\|f\|_{L^1_w(m)} = \sup_{x' \in B_X' \Omega} \int f \, d|m, x'|, \quad \text{for each } f \in L^1_w(m),$$

defines a lattice norm on $L^1_w(m)$ for which $L^1(m)$ is a closed sublattice of $L^1_w(m)$. The reader is referred to [16, Chapter 3] for all the unexplained information about this subject.

**Definition 3.** Let $(Z(\mu), \|\cdot\|_Z(\mu))$ be a Banach space consisting of (equivalence classes with respect to $\mu$-almost everywhere equality of) measurable functions $f : \Omega \to \mathbb{R}$. We say that $Z(\mu)$ is a Köthe function space (over $\mu$) when the following conditions hold:

(a) If $f$ is a real measurable function defined on $\Omega$ and $|f| \leq |g|$ for some $g \in Z(\mu)$, then $f \in Z(\mu)$ and $\|f\|_Z(\mu) \leq \|g\|_Z(\mu)$.  
(b) $\chi_A \in Z(\mu)$ for each $A \in \Sigma$.  
(c) $Z(\mu) \subseteq L^1(\mu)$ and the inclusion is continuous.

Note that this definition implies that $\mu$ is finite; in the general case of a $\sigma$-finite measure we adopt the definition of [10, 1.b.17].

The space $L^1(m)$ is a Köthe function space over any Rybakov measure for $m$.

The corresponding Köthe dual of $Z(\mu)$, that is, $Z(\mu)^\ast$ is the vector space of all measurable functions $g$ on $\Omega$ such that $fg \in L^1(\mu)$ for all $f \in Z(\mu)$. The Köthe dual is often called the associated space of $Z(\mu)$. Observe that $Z(\mu)^\ast$ is also a Köthe function space. If $f \in Z(\mu)$ and $g \in Z(\mu)^\ast$, then $fg \in L^1(\mu)$ and (see notation $X'$ in [11, p. 27])

$$\left| \int_{\Omega} fg \, d\mu \right| \leq \|f\|_{Z(\mu)} \|g\|_{Z(\mu)^\ast}.$$

Given an (equivalence class of $\mu$-almost everywhere equal) measurable function(s) $h$ on $\Omega$ and a couple of Köthe function spaces $X(\mu), Y(\mu)$ let us consider

$$h \cdot X(\mu) = \{ hf \mid f \in X(\mu) \}.$$  

When $h \cdot X(\mu) \subseteq Y(\mu)$ we can define the associated multiplication operator $M_h : X(\mu) \to Y(\mu)$ by

$$M_h(f) = hf, \quad \text{for every } f \in X(\mu).$$

Note that, if $h$ and $g$ are equal $\mu$-a.e., then the multiplication operators $M_h$ and $M_g$ are equal. Therefore, we can define the vector space of (the classes of) all multiplication operators between $X(\mu)$ into $Y(\mu)$ as
Theorem we obtain that for every function

\[ M(X(\mu), Y(\mu)) = \{ M_h \mid h \cdot X(\mu) \subseteq Y(\mu) \} . \]

If \( h \cdot X(\mu) \subseteq Y(\mu) \) then the boundedness of \( M_h \) follows by an easy application of the Closed Graph Theorem. This means that the vector space of classes of multiplication operators \( M(X(\mu), Y(\mu)) \) equipped with the operator norm

\[ \| M_h \| = \sup \{ |hf|_{Y(\mu)} \mid f \in B_{X(\mu)} \} \]

is a subspace of \( L(X(\mu), Y(\mu)) \).

3. A general separation theorem

Consider a measurable space \((\Omega, \Sigma)\) and an index set \(I\). Let \(\{X_i \mid i \in I\}\) be a family of Banach spaces. Let \(\{\mu_i : \Sigma \rightarrow \mathbb{R} \mid i \in I\}\) a family of vector measures, and \(\{\mu_i : \Sigma \rightarrow \mathbb{R}_+ \mid i \in I\}\) a family of scalar measures. For the following result, we need the compatibility property between \(Z(\mu)\) and the family of measures \(\{\mu_i : \Sigma \rightarrow \mathbb{R}_+ \mid i \in I\}\) given by the following definition.

**Definition 4.** Let \(\mu\) be a \(\sigma\)-finite measure. Let \(Z(\mu)\) and \(\{\mu_i : \Sigma \rightarrow \mathbb{R}_+ \mid i \in I\}\) be as above. We say that \(Z(\mu)\) and \(\{\mu_i : \Sigma \rightarrow \mathbb{R}_+ \mid i \in I\}\) are compatible if \(Z(\mu)\) is \(\sigma\)-order continuous and, for every \(i \in I\), \(\mu_i\) is absolutely continuous with respect to \(\mu\) and the functional \(\Phi_i : Z(\mu)^\times \rightarrow \mathbb{R}\) given by \(\Phi_i(h) := \int_\Omega h \, d\mu_i, h \in Z(\mu)^\times\) is well defined (i.e. each function \(h \in Z(\mu)^\times\) is \(\mu_i\)-integrable) and can be identified with an element of \(Z(\mu)\).

Observe that if \(Z(\mu)\) and a set of measures are compatible, then in particular \(Z(\mu)' = Z(\mu)^\times\), since \(Z(\mu)\) is \(\sigma\)-order continuous.

**Remark 5.** Note that if \(\mu(\Omega) < \infty\)—that is the case that we are considering in the paper—\(\chi_A \in Z(\mu)\). Hence for each \(A \in \Sigma\) the functional (belonging to \((Z(\mu)^\times)'\)) \(\Phi_A : Z(\mu)^\times \rightarrow \mathbb{R}\) given by \(\Phi_A(h) = \int_\Omega h \, d\mu\) can be identified with \(\chi_A\) that belongs to \(Z(\mu)\). So when \(\mu(\Omega) < \infty\), \(Z(\mu)\) and \(\{\mu_i : \Sigma \rightarrow \mathbb{R}_+ \mid i \in I\}\) are compatible.

**Theorem 6.** Consider two sets of measures \(\{\mu_i : \Sigma \rightarrow X_i \mid i \in I\}\) and \(\{\mu_i : \Sigma \rightarrow \mathbb{R}_+ \mid i \in I\}\) as above. Suppose that \(Z(\mu)\) and \(\{\mu_i : \Sigma \rightarrow \mathbb{R}_+ \mid i \in I\}\) are compatible. Let \(K > 0\). Then the following assertions are equivalent:

(a) For every finite set of non-negative scalars \(\lambda_1, \ldots, \lambda_N\) and every finite set of indexes \(\{i_j \mid j = 1, \ldots, N\}\), the inequality

\[ \sum_{j=1}^N \lambda_j \|m_{i_j}(\Omega)\| \leq K \sup_{h \in B_{Z(\mu)^\times}} \int_\Omega h \, d\left( \sum_{j=1}^N \lambda_j \mu_{i_j} \right) \]

holds.

(b) There is a function \(0 \leq h_0 \in B_{Z(\mu)^\times}\) such that for every index \(i \in I\),

\[ \|m_i(\Omega)\| \leq K \int_\Omega h_0 \, d\mu_i. \]

**Proof.** Let \(N \in \mathbb{N}\). Consider a set of indexes \(i_1, \ldots, i_N \in I\) and a set of non-negative scalars \(\lambda_1, \ldots, \lambda_N\). Define the function \(\Phi_{i_1, \ldots, i_N, \lambda_1, \ldots, \lambda_N} : Z(\mu)^\times \rightarrow \mathbb{R}\) by

\[ \Phi_{i_1, \ldots, i_N, \lambda_1, \ldots, \lambda_N}(h) := \sum_{j=1}^N \lambda_j \|m_{i_j}(\Omega)\| - K \int_\Omega h \, d\left( \sum_{j=1}^N \lambda_j \mu_{i_j} \right). \]

The family of all such functions (for different sets of scalars and indexes) is a concave family of convex functions that, by the compatibility assumption, is weak* continuous.

Assuming (a), taking into account the weak* compactness of \(B_{Z(\mu)^\times}\) (since \(Z(\mu)' = Z(\mu)^\times\)) and using the Hahn–Banach Theorem we obtain that for every function \(\Phi_{i_1, \ldots, i_N, \lambda_1, \ldots, \lambda_N}\) there is a function \(h \in B_{Z(\mu)^\times}\) such that

\[ \Phi_{i_1, \ldots, i_N, \lambda_1, \ldots, \lambda_N}(h) \leq 0. \]

Then by Ky Fan Lemma (see [5, p. 190]) there exists \(h_0 \in B_{Z(\mu)^\times}\) such that \(\Phi_{i_1, \ldots, i_N, \lambda_1, \ldots, \lambda_N}(h_0) \leq 0\) for all \(i_1, \ldots, i_N, \lambda_1, \ldots, \lambda_N\).

This gives (a) just taking a family of indexes with a single element \(i\). The converse is trivial.

**Remark 7.** The same construction can be done for every convex and weak* compact subset of \(Z(\mu)^\times\) and not only for \(B_{Z(\mu)^\times}\).
A first simple application of Theorem 6 gives the following corollary. This result can be easily obtained by duality, but it shows that, for the scalar case, the theorem above generalizes this kind of arguments. In order to prove this using the theorem it is enough to take families of measures as \( \{ \tau_A \mid A \in \Sigma \} \), where \( \tau \) is a finite (scalar) measure.

**Corollary 8.** Let \((\Omega, \Sigma)\) be a measure space, and consider a couple of positive scalar (countably additive) finite measures \(\mu\) and \(v\). Let \(Z(\mu)\) be a \(\sigma\)-order continuous Köthe function space. For a constant \(K > 0\), the following statements are equivalent:

(a) For every family of non-negative scalars \(\lambda_1, \ldots, \lambda_N\) and \(A_1, \ldots, A_N \in \Sigma, N \in \mathbb{N}\),

\[
\sum_{i=1}^{N} \lambda_i \nu(A_i) \leq K \sup_{B \in B^{|\mu|}} \int_{\Omega} \left( \sum_{i=1}^{N} \lambda_i |\mu|_{A_i} \right) \, d\nu.
\]

(b) The natural (not necessarily injective) inclusion map \(i : Z(\mu) \hookrightarrow L^1(\nu)\) is continuous.

(c) There is a function \(h_0 \in B_{Z(\mu)^*}\) such that

\[
\nu(A) \leq K \int_{A} h_0 \, d\mu
\]

for every \(A \in \Sigma\).

(d) There is a function \(0 \leq g \in L^1(\mu)\) such that \(\nu(A) = \int_{A} g \, d\mu\) for every \(A \in \Sigma\) and \(g \leq K h_0\) \(-a.e.\) for a function \(h_0 \in B_{Z(\mu)^*}\).

(e) There is a function \(0 \leq g \in Z(\mu)^*\) with \(\|g\|_{Z(\mu)^*} \leq K\) such that \(\nu(A) = \int_{A} g \, d\mu\) for every \(A \in \Sigma\).

Given \(1 < p < \infty\) and \((\Omega, \Sigma, \mu)\) a finite measure space let us consider the \(\sigma\)-order continuous Köthe function space \(Z(\mu) = L^p(\mu)\). In this case the inequality given in Theorem 6(b), allows us to prove when a (countably additive) vector measure \(m : \Sigma \rightarrow X\) has bounded \(p\)-variation with respect to \(\mu\). Recall that a (countably additive) vector measure \(m : \Sigma \rightarrow X\) has bounded \(p\)-variation with respect to \(\mu\) if

\[
\|m\|_{V^p(\mu, X)} := \sup \left\{ \left( \sum_{i=1}^{N} \|m(A_i)\|^p_{\mu(A_i)^{p-1}} \right)^{\frac{1}{p}} : (A_i)_{i=1}^{N} \in \mathcal{P}(\Omega), \, \mu(A_i) > 0 \right\}
\]

\[
= \sup \left\{ \sum_{i=1}^{N} |\lambda_i| \|m(A_i)\| : \sum_{i=1}^{N} \lambda_i \chi_{A_i} \in B_{L^p(\mu)} \right\} < \infty.
\]

The set of all (countably additive) vector measures \(m : \Sigma \rightarrow X\) with bounded \(p\)-variation with respect to \(\mu\) equipped with the norm \(\|\cdot\|_{V^p(\mu, X)}\) is a Banach space denoted by \(V^p(\mu, X)\). The reader is referred to [6] for information about this space.

**Corollary 9.** Let \((\Omega, \Sigma, \mu)\) be a finite measure space. Let \(1 < p < \infty\) and \(m : \Sigma \rightarrow X\) be a (countably additive) vector measure. Then the following assertions are equivalent:

(a) \(m \in V^p(\mu, X)\).

(b) There is a constant \(K > 0\) such that for every set of non-negative scalars \(\lambda_1, \ldots, \lambda_N\) and \(A_1, \ldots, A_N \in \Sigma, N \in \mathbb{N}\), the inequality

\[
\sum_{i=1}^{N} \lambda_i \|m(A_i)\| \leq K \sup_{B \in B_{V^p(\mu)}} \int_{\Omega} \left( \sum_{i=1}^{N} \lambda_i |\mu|_{A_i} \right) \, dB
\]

holds.

(c) There is a constant \(K > 0\) and a function \(0 \leq \phi \in L^p(\mu)\), such that for every \(A \in \Sigma,\)

\[
\|m(A)\| \leq K \int_{A} \phi \, d\mu.
\]

**Proof.** Let us consider the \(\sigma\)-order continuous Köthe function space \(L^p(\mu)\).

(a) \(\Rightarrow\) (b). Assume that \(m \in V^p(\mu, X)\). Given \(N \in \mathbb{N}\) let us take a family of non-negative scalars \(\lambda_1, \ldots, \lambda_N\) and \(A_1, \ldots, A_N \in \Sigma\). Note that the class of measurable sets \(A_1, \ldots, A_N\) do not define a partition of \(\Omega\). Let us take a partition \(B_1, \ldots, B_s \in \Sigma\) associated to the family of measurable sets \(A_1, \ldots, A_N\) (i.e. a partition defined by disjoint finite intersections of the measurable sets \(A_1, \ldots, A_N, \Omega \setminus \{A_1 \cup \cdots \cup A_N\}\) and their complementary sets). For every \(1 \leq j \leq s\) take \(A_j = \{i \in \{1, \ldots, N\} : B_j \subseteq A_i\}\) and \(\beta_j = \sum_{i \in A_j} \lambda_i\), where \(\beta_j = 0\) if \(A_j = \emptyset\). Using Hölder’s inequality we obtain
\[
\sum_{i=1}^{N} \lambda_i \|m(A_i)\| \leq \sum_{j=1}^{s} \beta_j \|m(B_j)\| = \sum_{j=1}^{s} \beta_j \mu(B_j)^{1/p} \left(\sum_{j=1}^{s} \beta_j \mu(B_j)^{1/p}\right)^{1/p} \leq \left(\sum_{j=1}^{s} \beta_j \mu(B_j)^{1/p}\right)^{1/p} \left(\sum_{j=1}^{s} \|m(B_j)\|^{p/p}\right)^{1/p} \\
= \left\|\sum_{j=1}^{s} \beta_j 1_{B_j}\right\|_{L^p(\mu)} \|m\|_{V^p(\mu, X)} \sup_{h \in B_{L^p(\mu)}} \int_{\Omega} h \, d\left(\sum_{i=1}^{N} |\lambda_i| 1_{A_i}\right).
\]

(b) ⇒ (a). Just observe that for each partition of \(\Omega\), \(\pi = (A_i)_{i=1}^{N} \in \mathcal{P}(\Omega)\), and every simple function \(\sum_{i=1}^{N} \lambda_i 1_{A_i} \in B_{L^p(\mu)}\), we obtain that
\[
\sum_{i=1}^{N} |\lambda_i| \|m(A_i)\| \leq K\sup_{h \in B_{L^p(\mu)}} \int_{\Omega} h \, d\left(\sum_{i=1}^{N} |\lambda_i| 1_{A_i}\right) = K\|\sum_{i=1}^{N} \lambda_i 1_{A_i}\|_{L^p(\mu)}.
\]
Thus \(\|m\|_{V^P(\mu, X)} \leq K\).

For the equivalence between (b) and (c) just apply Theorem 6.

The extension of classical Lebesgue function spaces \(L^p(\mu)\) to Orlicz \((L^P(\mu),\) see [17]), Lorentz \((L^p,\) see [9]) and Köthe function spaces \((E(\mu),\) see [12–14]) has lead to the extension of the corresponding vector measures spaces (respectively \([18, 19, 2\) and \([3,7,8]\)). In each case, the simple functions taken in the supremum of Eq. (2) are taken in the unit ball of the associated space \(L^P(\mu)\) for the \(p\)-variation. The previous corollary can then be rephrased in terms of these vector measure spaces, using the same arguments.

It is well known—see for example [1]—that if \((\Omega, \Sigma, \mu)\) is a finite measure space then given \(1 < p < \infty\) the space \(V^p(\mu, X)\) is isometrically isomorphic to the space of the cone absolutely summing operators from \(L^p(\mu)\) into \(X\) denoted by \(A(L^p(\mu), X)\). This is the space consisting of all bounded linear operators \(T : L^p(\mu) \to X\) satisfying that there is a constant \(K > 0\) such that for each finite family of non-negative functions \(f_1, \ldots, f_N \in L^p(\mu), N \in \mathbb{N}\), the inequality
\[
\sum_{i=1}^{N} \|T(f_i)\| \leq K\sum_{i=1}^{N} \|f_i\|_{L^p(\mu)}
\]
holds. So given \(T \in A(L^p(\mu), X)\) we have that the vector measure associated to \(T\),
\[m_T : \Sigma \to X, \text{ given by } m_T(A) = T(1_A),\]
belongs to \(V^p(\mu, X)\). Reciprocally, given \(m \in V^p(\mu, X)\) the bounded linear operator, \(T_m\), defined from the set of the simple functions of \(L^P(\mu)\) into \(X\) by
\[
T_m\left(\sum_{i=1}^{N} \lambda_i 1_{A_i}\right) = \sum_{i=1}^{N} \lambda_i m(A_i)
\]
can be extended to \(L^p(\mu)\) and the resulting extension is an operator lying in \(A(L^p(\mu), X)\). Hence using Corollary 9 we obtain the following result.

**Corollary 10.** Let \((\Omega, \Sigma, \mu)\) be a finite measure space. Given \(1 < p < \infty\), let \(T \in L(L^p(\mu), X)\) be a bounded linear map. Then the following assertions are equivalent:

(a) \(T \in A(L^P(\mu), X)\).
(b) There is a constant \(K > 0\) such that for every family of non-negative scalars \(\lambda_1, \ldots, \lambda_N\) and \(A_1, \ldots, A_N \in \Sigma\), \(N \in \mathbb{N}\),
\[
\sum_{i=1}^{N} \lambda_i \|T(1_{A_i})\| \leq K\sum_{i=1}^{N} \lambda_i 1_{A_i}\|_{L^P(\mu)}
\]
holds.
(c) There is a constant \(K > 0\) and a function \(0 \leq \phi \in L^p(\mu)\), such that for every \(A \in \Sigma\),
\[
\|T(1_A)\| \leq K\int_A \phi \, d\mu.
\]
4. A Radon–Nikodým Theorem for vector measures

Let \((\Omega, \Sigma, \mu)\) be a measure space. Assume that \(m : \Sigma \to X\) is a countably additive vector measure such that \(L^1(m)\) is a Köthe function space over the measure space \((\Omega, \Sigma, \mu)\) (for instance, if \(\mu\) is a Rybakov measure for \(m\)). Let \(Z(\mu)\) be a \(\sigma\)-order continuous Köthe function space over \((\Omega, \Sigma, \mu)\) and assume that \(Z(\mu)^\times\) is (continuously) included in \(L^1(m)\).

In this framework, it is clear that each scalar measure \((m, x'), x' \in X',\) is absolutely continuous with respect to \(\mu\). Let us denote by \(f_{m,x'}\) the corresponding Radon–Nikodým derivative \(d(m, x')/d\mu\). The following direct argument shows that the set \(\mathcal{RN}(m, \mu) := \{f_{m,x'} : x' \in X'\}\) can be identified (not necessarily in an injective way) with a bounded subset of \((Z(\mu)^\times)'\).

If \(f_{m,x'} \in \mathcal{RN}(m, \mu)\), then for every \(f \in L^1(m)\),
\[
\int_\Omega ff_{m,x'} \, d\mu = \int_\Omega f \frac{d(m, x')}{d\mu} \, d\mu = \int_\Omega f d(m, x'),
\]
and so
\[
\left| \int_\Omega ff_{m,x'} \, d\mu \right| = \int_\Omega |f| \, d(m, x') \leq \|f\|_{L^1(\mu)}.
\]

Therefore, \(f_{m,x'} \in (L^1(m))'\), and then it can be also identified with an element of \((Z(\mu)^\times)'\) just by dualizing the inclusion scheme \(Z(\mu)^\times \hookrightarrow L^1(m)\).

Actually the inequality above allows us to prove that
\[
Z(\mu)^\times \hookrightarrow L^1_{\text{uf}}(m) \quad \Leftrightarrow \quad \mathcal{RN}(m, \mu) \text{ bounded } \subseteq (Z(\mu)^\times)'\.
\]

For the following proposition we need a bit more; \(Z(\mu)^\times \hookrightarrow L^1(m)\) and the set \(\mathcal{RN}(m, \mu)\) must be identified with a subset of \(Z(\mu)\).

Under these assumptions we can obtain the following specialized version of Radon–Nikodým Theorem for vector measures, that characterizes when the (vector valued version of the) Radon–Nikodým derivative of a vector measure \(n : \Sigma \to X\) with respect to the vector measure \(m : \Sigma \to X\) can be found to be an element of a Köthe function space \(Z(\mu)^\times\). The proof is a consequence of Theorem 6.

**Theorem 11.** Let \(m\) and \(n\) be countably additive vector measures on a Banach space \(X\) that are absolutely continuous with respect to a finite scalar measure \(\mu\). Let us consider a \(\sigma\)-order continuous Köthe function space \(Z(\mu)^\times\) satisfying that

(i) \(Z(\mu)^\times\) is (continuously) included in \(L^1(m)\),
(ii) \(\mathcal{RN}(m, \mu)\) can be identified with a (bounded) set of \(Z(\mu)\).

Then the following statements are equivalent:

(a) There is a constant \(K > 0\) such that for every finite set of non-negative scalars \(\lambda_1, \ldots, \lambda_N\), every finite set of vectors \(x'_1, \ldots, x'_N \in X'\) and every finite family of measurable sets \(A_1, \ldots, A_N \in \Sigma, N \in \mathbb{N}\), the inequality
\[
\sum_{i=1}^N \lambda_i \int_{A_i} \|n(x')\| \leq K \sup_{f \in Z(\mu)^\times} \sum_{i=1}^N \lambda_i \int_{A_i} \|f(m, x'_i)\|
\]
holds.
(b) There is a constant \(K > 0\) such that for every finite set of non-negative scalars \(\lambda_1, \ldots, \lambda_N\), every finite set of vectors \(x'_1, \ldots, x'_N \in X'\) and every finite family of measurable sets \(A_1, \ldots, A_N \in \Sigma, N \in \mathbb{N}\), the inequality
\[
\left\| \sum_{i=1}^N \lambda_i X_{A_i} |f_{m,x'_i}| \right\|_{L^1(\mu)} \leq K \left\| \sum_{i=1}^N \lambda_i X_{A_i} |f_{m,x'_i}| \right\|_{Z(\mu)}
\]
holds.
(c) There is a function \(f_0 \in Z(\mu)^\times\) such that
\[
n(A) = \int_A f_0 \, dm, \quad A \in \Sigma.
\]

**Proof.** Clearly, (a) and (b) are equivalent. To see that (c) implies (a) take a finite set of non-negative scalars \(\lambda_1, \ldots, \lambda_N\), of vectors \(x'_1, \ldots, x'_N \in X'\) and of measurable sets \(A_1, \ldots, A_N\). Then
Therefore with one of these conditions (a) or (b) we have that (i) implies (ii).

Example 12. Although (i) and (ii) may seem restrictive conditions there are many general situations for which one condition is enough.

(i) Assume that $Z(\mu)^{\times}$ is (continuously) included in $L^1(m)$.

(a) If $Z(\mu)$ is reflexive (and $\sigma$-order continuous), then $Z(\mu)^{\times} \hookrightarrow L^1(m) \subseteq L^1_{\mathfrak{w}}(m) \Rightarrow \mathcal{R}\mathcal{N}(m, \mu) \subseteq Z(\mu)^{\times} = Z(\mu)^{\times} = Z(\mu).$

(b) If $Z(\mu)$ verifies the Fatou Property (recall that this is equivalent to have $Z(\mu)^{\times \times} = Z(\mu)$, see for instance [11, p. 30]) and $Z(\mu)^{\times}$ is $\sigma$-order continuous then $Z(\mu)^{\times} \hookrightarrow L^1(m) \subseteq L^1_{\mathfrak{w}}(m) \Rightarrow \mathcal{R}\mathcal{N}(m, \mu) \subseteq Z(\mu)^{\times} = Z(\mu)^{\times} = Z(\mu).$

Therefore with one of these conditions (a) or (b) we have that (i) implies (ii).

(ii) Assume now that $\mathcal{R}\mathcal{N}(m, \mu)$ can be identified with a (bounded) set of $Z(\mu)$.

(a') If $X$ contains no copies of $c_0$ then $L^1(m) = L^1_{\mathfrak{w}}(m)$ (see for instance [16, pp. 131–132]). So

$$
\mathcal{R}\mathcal{N}(m, \mu) \subseteq Z(\mu) \subseteq Z(\mu)^{\times \times} \quad \Rightarrow \quad Z(\mu)^{\times} \hookrightarrow L^1_{\mathfrak{w}}(m) = L^1(m).
$$

We can obtain the same result also with $c_0$-valued vector measures.

(b') Consider, for instance, the $c_0$-valued vector measure $m(\tau) := (\int_A r_n(t) \, dt)$ (being $r_n(t)$ the sequence of Rademacher functions defined on $[0,1]$). In this case it is also true that $L^1(m) = L^1_{\mathfrak{w}}(m)$.

Therefore with one of these conditions (a') or (b') then (ii) implies (i).

Let us illustrate the result with other example.

Example 13. Let us take $\nu$ and $\mu$ finite positive scalar measures over the interval $[0,1]$ such that $\nu$ is absolutely continuous with respect to $\mu$. Let $(A_i)_{i=1}^\infty$ be a partition of $[0,1]$ into a countably disjoint class of measurable sets and let us define the $\ell_2$-valued vector measures

$$
m : \Sigma \to \ell_2, \quad \text{given by } m(A) = \sum_{i=1}^{\infty} \mu(A \cap A_i) e_i,
$$

$$
n : \Sigma \to \ell_2, \quad \text{given by } n(A) = \sum_{i=1}^{\infty} \nu(A \cap A_i) e_i.
$$

It can be easily shown that these measures are countably additive. Fixed $i \in \mathbb{N}$ and taking into account that $\nu|_{A_i}$ is absolutely continuous with respect to $\mu|_{A_i}$ then we can find a function $g_i \in L^1(\mu|_{A_i})$ such that $\frac{d\nu|_{A_i}}{d\mu|_{A_i}} = g_i$. Note that for all $(\tau_i)_{i=1}^\infty = x' \in (\ell_2)' = \ell_2$ we have that

$$
\frac{d|m(x')|}{d\mu} = \sum_{i=1}^{\infty} |\tau_i| \chi_{A_i}
$$

and

$$
\frac{d|n(x')|}{d\mu} = \sum_{i=1}^{\infty} |\tau_i| g_i \chi_{A_i}.
$$

In this case, the inequality given in Theorem 11(b) is
where \( \lambda_1, \ldots, \lambda_n \) are non-negative and \( (\tau_j^1)_{j=1}^\infty = e_2 \) for \( j = 1, \ldots, n \) and \( B_1, \ldots, B_n \) are measurable sets. If the previous inequality holds then Theorem 11(c) allows us to find a function \( f_0 \in Z(\mu)^\times \) such that

\[
\sum_{j=1}^n \sum_{i=1}^\infty \lambda_j |\tau_j^1| \int_{B_j \cap A_i} g_i \, d\mu \leq K \left\| \sum_{j=1}^n \sum_{i=1}^\infty \lambda_j |\tau_j^1| \chi_{B_j \cap A_i} \right\|_{Z(\mu)}.
\]

Then \( \mu \)-almost everywhere but

\[
\lim_{N \to \infty} \frac{\sum_{i=1}^N g_i - f_0}{\mu(A \cap \Sigma)} = \lim_{N \to \infty} \frac{\sum_{i=1}^N \chi_{A_i} - \chi_{\Sigma}}{\mu(\Sigma)} \neq 0.
\]

**Corollary 14.** Let \( Y(\mu) \) and \( Z(\mu) \) be \( \sigma \)-order continuous Köthe function spaces with weak unit. Let us consider a function \( g \in Y(\mu) \) defining a multiplication operator \( M_g \) from \( Y(\mu) \) into \( Z(\mu) \) and satisfying that there is a constant \( K > 0 \) such that

\[
\|h\|_{L^1(\mu)} \leq K \|gh\|_{Z(\mu)}, \quad \text{for all } h \geq 0.
\]

Then \( g^{-1} \in Z(\mu)^\times \).

**Proof.** Let us define the countably additive vector measures

\[
m : \Sigma \to Y(\mu), \quad \text{given by } m(A) = \chi_A,
\]

\[
n_g : \Sigma \to Y(\mu), \quad \text{given by } n_g(A) = g \chi_A.
\]

Let us take a finite set of positive scalars \( \lambda_1, \ldots, \lambda_n \), a finite set of functions \( h_1, \ldots, h_n \in Y(\mu) \) and a finite family of measurable sets \( A_1, \ldots, A_n \). Note that for each \( 1 \leq i \leq n \),

\[
\langle m, h_i \rangle(A_i) = \int_{A_i} dm(A_i, h_i) = \int_{A_i} \chi_{A_i} \, d\mu, \quad \langle n_g, h_i \rangle(A_i) = \int_{A_i} dn_g(A_i, h_i) = \int_{A_i} g \chi_{A_i} \, d\mu.
\]

This means that \( \frac{d(m, h_i)}{d\mu} = h_i \) and \( \frac{d(n_g, h_i)}{d\mu} = gh_i \) for all \( 1 \leq i \leq N \). Applying the inequality (3) with the positive function \( h = \sum_{i=1}^N \lambda_i \chi_{A_i} \) we obtain that

\[
\left\| \sum_{i=1}^N \lambda_i \chi_{A_i} \right\|_{L^1(\mu)} \leq K \left\| \sum_{i=1}^N \lambda_i \chi_{A_i} \right\|_{Z(\mu)} = K \left\| \sum_{i=1}^N \lambda_i \chi_{A_i} \right\|_{Z(\mu)}.
\]

Using Theorem 11 there is a function \( f_0 \in Z(\mu)^\times \) such that

\[
\chi_A = m(A) = \int_A f_0 \, dn_g = f_0 g \chi_A, \quad \text{for all } A \in \Sigma.
\]

So \( g^{-1} = f_0 \mu\text{-a.e. and } g^{-1} \in Z(\mu)^\times. \)

**5. Scalar factorizations for operators from \( L^1(m) \) into \( L^1(n) \)**

Let \( m : \Sigma \to X \) and \( n : \Sigma \to X \) be a pair of countably additive vector measures, and denote by \( N(m) \) and \( N(n) \), respectively, the family of \( m \)-null and \( n \)-null sets. In this section we are interested in the study of the relation between the subspace of continuous linear operators from \( L^1(m) \) to \( L^1(n) \) which are multiplication operators and the properties relating the measures \( m \) and \( n \). Note that in this case \( g \chi_{\Sigma} = g \in L^1(n) \), and then the expression
\[ ng(A) := \int_A g \, dn, \quad A \in \Sigma, \]
gives a (countably additive) vector measure \( ng : \Sigma \to X \). In all this section we will identify a multiplication operator \( M_g \) with the measurable function \( g \) that defines it. We will show that in the context of Theorem 11 a certain factorization scheme for the multiplication operator \( M_g \) is equivalent to the existence of a Radon–Nikodym derivative for the measure \( ng \) belonging to \( L^\infty(\mu) \). In this sense we show that, taking \( g := \chi_\Omega \) and \( Z(\mu) = L^1(\mu) \), our theorem gives directly the result of K. Musiał [15, Theorem 1] as a particular case, and that this result is equivalent to a factorization scheme. For the multiplication operator \( M_g \) to be well defined as a mapping it is compulsory that \( gN(\mu) \subseteq N(\mu) \), i.e. \( g \) should be null on the sets in \( N(\mu) \setminus N(\mu) \). In particular, if we want \( M_{X\Omega} \) to be a well-defined mapping—either continuous or not—we should have \( N(\mu) \subseteq N(\mu) \), hypothesis that we assume.

Then for a Rýbakov measure \( \mu \) for \( m \), we have that \( N(\mu) \subseteq N(\mu) \subseteq N(\mu) \) and we can consider the set of (classes of \( \mu \)-a.e. equal) measurable functions \( L^0(\mu) \) as the contextual space of functions. Let \( M(L^1(m),L^1(n)) = \{ g \in L^0(\mu) : g f \in L^1(n) \text{ for all } f \in L^1(m) \} \), the space of functions leading to multiplication mappings. For the purpose of this section we introduce the linear space \( M(m,n) \) consisting of all functions \( g \in L^0(\mu) \) that define a continuous map

\[ M_g : L^1(\{(m,x')\}) \to L^1(\{(n,x')\}), \quad \text{given by } M_g(f) = gf, \]
satisfying that there is \( K > 0 \) such that

\[ \|M_g\|_{L^1(\{(m,x')\}),L^1(\{(n,x')\})} \leq K, \quad (4) \]

for all \( x' \in X' \). The infimum of the constants \( K \) that occur in (4) will be denoted by \( \|\cdot\|_{M(m,n)} \). Note that we write \( M_g \) for the multiplication operator defined by \( g \in L^0(\mu) \) independently of the spaces between which it is defined. If \( g \) is a function in \( M(m,n) \) then for each \( x' \in X' \) we can consider the diagram

\[
\begin{array}{c}
L^1(m) \xrightarrow{M_g} L^1(n) \\
\uparrow i_{m,x'} \quad \uparrow i_{n,x'} \\
\downarrow \quad \downarrow \\
L^1(\{(m,x')\}) \rightarrow L^1(\{(n,x')\})
\end{array}
\]

where \( i_{m,x'} \) (analogously we define \( i_{n,x'} \)) is the identification map (not necessarily injective). Note that \( i_{m,x'} \) (respectively \( i_{n,x'} \)) is well defined as a consequence of the following well-known technical result whose proof is straightforward.

**Lemma 15.** Let \( m : \Sigma \to X \) a (countably additive) vector measure. If \( x' \in X' \), the linear map \( i_{m,x'} : L^1(m) \to L^1(\{(m,x')\}) \) defined by \( i_{m,x'}(f) = [f] \), where \([f]\) denotes the class of functions that are equal \( \{(m,x')\}\)-a.e. is well defined and continuous, and \( \|i_{m,x'}\| \leq 1 \).

We show in the following example that continuity of \( M_g \) in the upper part of the diagram does not lead automatically to continuity of \( M_g \) in the lower part.

**Example 16.** If \( \Omega = \mathbb{N} \), \( \Sigma = 2^\mathbb{N} \) and \( X = \mathbb{C}_0 \), consider the vector measures that are given respectively by \( m(\{i\}) = \frac{1}{2} e_{i+1} \) and \( n(\{i\}) = \frac{1}{2} e_i \) for \( i \in \mathbb{N} \). We have that, for any \( f = (f(i))_{i=1}^{\infty} \),

\[ \|f\|_{L^1(m)} = \|f\|_{L^1(n)} = \sup_{i \in \mathbb{N}} \left| \frac{f(i)}{i} \right|. \]

Thus, the multiplication \( M_{X\Omega} \) is continuous from \( L^1(m) \) to \( L^1(n) \) (in fact, this is an isometry). However, for fixed \( x' = e_1 \in (\mathbb{C}_0(\mathbb{N}))' \) we have that

\[ \langle m, e_1 \rangle(A) = 0 \quad \text{and} \quad \langle n, e_1 \rangle(A) = \begin{cases} 1, & 1 \in A, \\ 0, & 1 \notin A. \end{cases} \]

for \( A \in \Sigma \). However \( M_f \) is not even well defined from \( L^1(\{(m,e_1)\}) \) to \( L^1(\{(n,e_1)\}) \). Take \( f_1 = (1,0,0,\ldots) \) and \( f_2 = (0,1,0,\ldots) \). Then \( [f_1] = [f_2] \) in \( L^1(\{(m,e_1)\}) \) but the multiplication maps them to different (classes of) functions in \( L^1(\{(n,e_1)\}) \).

Let us give now a factorization theorem for multiplication operators in \( M(m,n) \).

**Proposition 17.** Let \( M_g \in M(L^1(m),L^1(n)) \). The following statements are equivalent:

1. \( M_g \in M(m,n) \),
2. \( \text{for all } f \in L^1(m) \text{ and } g \in L^0(\mu), \text{ we have } M_g(f) \in L^1(n) \text{, and } \|M_g\| \leq K < \infty \),
3. \( \|M_g\|_{L^1(\{(m,x')\}),L^1(\{(n,x')\})} \leq K \),
4. \( \|M_g\|_{L^1(m),L^1(n)} \leq K \).

The following diagram illustrates the factorization:

\[
\begin{array}{c}
L^1(m) \xrightarrow{M_g} L^1(n) \\
\uparrow i_{m,x'} \quad \uparrow i_{n,x'} \\
\downarrow \quad \downarrow \\
\downarrow \quad \downarrow \\
L^1(\{(m,x')\}) \rightarrow L^1(\{(n,x')\})
\end{array}
\]
(b) For each \( x' \in B_{X'} \), the operator \( M_g \) factorizes as

\[
L^1(m) \xrightarrow{M_g} L^1(n)
\]

\[
L^1(|\langle m, x' \rangle|) \xrightarrow{\psi} L^1(|\langle m, x' \rangle|)
\]

and \( \|M_g\|_{L^1(m), L^1(n)} \leq \|M_g\|_{M(n,m)} \).

(c) There is a function \( f_0 \in L^\infty(\mu) \) such that for every \( A \in \Sigma \),

\[
\int_A f_0 \, dm = \int_A g \, dn.
\]

The proof of the proposition is a direct consequence of the result of K. Musiał [15, Theorem 1]. Recall that the vector measure \( n_g : \Sigma \to X \) is scalarly dominated by \( m : \Sigma \to X \) whenever there is a constant \( M > 0 \) such that for every \( x' \in X' \) and \( A \in \Sigma \),

\[
\|n_g(x')\|_A \leq M \|m\|_A \|\langle m, x' \rangle\|_A.
\]

By [15, Theorem 1], this condition is equivalent to (c) in Theorem 11, and the (uniform) factorization of the operator given in (b) is also clearly equivalent to the scalar domination property. Thus, it is easy to see that this situation is recovered when \( Z(\mu) = L^1(\mu) \) in Theorem 11; statement (a) in this theorem can be written for this case as

\[
\sum_{i=1}^N \lambda_i \int_{\Omega_i} d\|n_g, x'_i\| = K \sup_{f \in L^\infty(\mu)} \sum_{i=1}^N \lambda_i \int_{\Omega_i} f d\|n, x'_i\| = K \sum_{i=1}^N \lambda_i \int_{\Omega_i} d\|n, x'_i\|.
\]

That is clearly equivalent to the scalar domination condition given above. Therefore, Theorem 11 can be considered a generalization of [15, Theorem 1], just taking \( Z(\mu) = L^1(\mu) \) and \( g = \chi_{\Omega} \).

References


