The Pełczyński Property for Tight Subspaces

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We show that if $X$ is a tight subspace of $C(K)$ then $X$ has the Pełczyński property and $X^*$ is weakly sequentially complete. We apply this result to the space $U$ of uniformly convergent Taylor series on the unit circle and using a minimal amount of Fourier theory prove that $U$ has the Pełczyński property and $U^*$ is weakly sequentially complete. Using separate methods, we prove $U$ and $U^*$ have the Dunford–Petits property. Some results concerning pointwise bounded approximation are proved for tight uniform algebras. We use tightness and the Pełczyński property to make a remark about inner functions on strictly pseudoconvex domains in $\mathbb{C}^n$.

1. INTRODUCTION AND BACKGROUND

The Pełczyński property, whose concept was influenced by the work of Orlicz, was introduced by Pełczyński in [20]. We say a sequence $\{x_n\}$ in a Banach space $X$ is a weakly unconditionally Cauchy series (w.u.C. series) if $\sum |x^*(x_n)| < \infty$ for every $x^* \in X^*$ and $\{x_n\}$ is an unconditionally converging series if $\sum x_{\pi(n)}$ converges in norm for every permutation $\pi$ of the natural numbers. If $X$ and $Y$ are Banach spaces and $T: X \to Y$ is a continuous linear operator we say $T$ is an unconditionally converging operator if $T$ takes every weakly unconditionally Cauchy series in $X$ to an unconditionally converging series in $Y$. It follows from the work of Orlicz that every weakly compact operator is an unconditionally converging operator.

The Pełczyński property for a Banach space is the realization of a converse to the result of Orlicz. We say $X$ has the Pełczyński property if every unconditionally converging operator on $X$ is weakly compact. It is a theorem of Bessaga and Pełczyński in [1] that a continuous linear operator $T: X \to Y$ is unconditionally converging if and only if $T$ is never an isomorphism on a copy of $c_0$ in $X$.

We say a sequence $\{x_n\}$ is a $c_0$-sequence if it is a basic sequence which is equivalent to the unit vector basis of $c_0$ and similarly for $l^1$-sequences.

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Given a bounded subset \( E \subseteq X^* \) we will be interested in knowing when there exists a weakly unconditionally Cauchy series \( \{x_n\} \) in \( X \) that fails to tend to zero uniformly on \( E \); that is, \( \lim_{n \to \infty} \sup_{x \in E} |x^*(x_n)| > 0 \). It follows from the result of Bessaga and Pelczyński mentioned above that this is equivalent to the existence of a \( c_0 \) sequence \( \{x_n\} \) that fails to tend to zero uniformly on \( E \) (just consider the operator \( T : X \to l^\infty(E) \) by \( Tx(x) = x^*(x) \)). We say a sequence \( \{x_n\} \) is a weak-Cauchy sequence if \( \lim x^*(x_n) \) exists for every \( x^* \in X^* \) and we say a Banach space \( X \) is weakly sequentially complete if every weak-Cauchy sequence in \( X \) is weakly convergent.

The following are some more or less well-known characterizations of the Pelczyński property.

**Proposition 1.1.** If \( X \) is a Banach space then the following are equivalent.

a) \( X \) has the Pelczyński property.

b) If \( T : X \to Y \) is a continuous linear operator which fails to be weakly compact then \( T \) is an isomorphism on some copy of \( c_0 \) in \( X \).

c) If \( E \subseteq X^* \) and the weak closure of \( E \) fails to be weakly compact then there exists a weakly unconditionally Cauchy series \( \{x_n\} \) in \( X \) which fails to tend to zero uniformly on \( E \).

d) The following hold: (i) \( X^* \) is weakly sequentially complete (ii) If \( \{x_n^*\} \) is an \( l^1 \)-sequence in \( X^* \) then there exists a \( c_0 \)-sequence in \( X \) such that \( |x_n^*(x_k)| > \delta > 0 \) for all \( k \) for some sequence \( \{n_k\} \).

The equivalence of (a) and (b) follows from the theorem of Bessaga and Pelczyński mentioned above, while the equivalence of (a) and (c) is well known. The equivalence of (a) and (d) is less popular, but can be deduced from (c) and the now ubiquitous result of Rosenthal and Dor: if \( X \) is any Banach space and \( \{x_n\} \) is a bounded sequence in \( X \) which has no weak-Cauchy subsequence then \( \{x_n\} \) has an \( l^1 \)-subsequence.

All \( C(K) \) spaces were shown to have the Pelczyński property in [20]. Every infinite-dimensional \( L^1 \)-space fails to have this property since these spaces do not contain a copy of \( c_0 \). Delbaen and Kisliakov independently showed the disk algebra has the Pelczyński property in [10] and [15] respectively. Delbaen extended these results to \( R(K) \) for special classes of planar sets \( K \) in [11] as did Wojtaszczyk in [23]. It was shown that \( R(K) \) has the Pelczyński property for every compact planar set \( K \) in [22]. It was also shown in [22] that every so-called T-invariant uniform algebra on a compact planar set has the Pelczyński property. The T-invariant class includes \( R(K) \) as well as \( A(K) \) for all compact planar sets \( K \). However, it is not known if any of these planar uniform algebras fail to be linearly...
isomorphic to the disk algebra. Bourgain showed the ball-algebras and the polydisk-algebras have the Pelczyński property in [3]. This result was extended in [22] to $A(D)$ for strictly pseudoconvex domains $D$ in $\mathbb{C}^n$.

Not all uniform algebras have the Pelczyński property. In fact, it is a result of Milne in [18] that every Banach space $X$ is isomorphic to a complemented subspace of a uniform algebra $A$, where $A$ can be taken to be the uniform algebra on $B_X^*$ generated by $X$. However, the author is not presently aware of any uniform algebras on compact subspaces of $\mathbb{R}^n$ which fail to have the Pelczyński property.

The Pelczyński property holds for a special class of spaces which includes many examples of uniform algebras of analytic functions. If $K$ is a compact space and $X \subseteq C(K)$ is a closed subspace then we say $X$ is a \textit{tight subspace} if the operator $S_g : X \to C(K)/X$ by $f \mapsto fg + X$ is weakly compact for every $g \in C(K)$. We say a uniform algebra $A$ on $K$ is a \textit{tight uniform algebra} if it is a tight subspace. The concept of tightness was introduced by B. Cole and T. W. Gamelin in [6] as the ability to solve an abstract $\varepsilon$-problem with a mild gain in smoothness.

Although the authors in [6] were mainly interested in weakly compact Hankel-type operators, in many of the examples the operators $S_g$ were proven to be compact. We say $X \subseteq C(K)$ is a \textit{strongly tight subspace} if $S_g$ is compact for every $g$, and similarly we define strongly tight uniform algebras. It was proven in [6] that $R(K)$ is strongly tight for every compact planar set $K$, and also $A(D)$ is strongly tight for every strictly pseudoconvex domain $D$ in $\mathbb{C}^n$ with $C^2$ boundary. More generally, $A(D)$ will be strongly tight whenever the $\varepsilon$-problem can be solved in $D$ with Hölder estimates on the solutions.

Currently there is no known example of a tight uniform algebra which fails to be strongly tight. However there are examples of tight, nonstrongly tight subspaces. We say an operator $T : X \to Y$ between Banach spaces is \textit{completely continuous} is $T$ takes weakly null sequences to norm null sequences. We say a Banach space $X$ has the Dunford–Pettis property if every weakly compact operator $T : X \to Y$ is completely continuous. It follows from the work of Bourgain in [4] that any strongly tight subspace has the Dunford–Pettis property. By considering the identity operator, we see that every infinite-dimensional reflexive space fails to have the Dunford–Pettis property. Hence, any infinite-dimensional reflexive space $X$ will be tight in any $C(K)$ space it is embedded in, but can never be realized as a strongly tight subspace.

Our main result is Theorem 2.1 which states that every tight subspace of $C(K)$ has the Pelczyński property. This result generalizes a theorem from [22], namely that every strongly tight uniform algebra has the Pelczyński property (actually, the proof in [22] does not use any algebraic structure and would work for strongly tight subspaces of $C(K)$).
An application of this theorem was kindly forwarded to the author by the referee. Let $U$ be the space of continuous functions on the unit circle which extend to be analytic in the unit disk and whose Taylor series converge uniformly on the closed disk. We define a norm on $U$ by taking the supremum of sup-norms the partial sums of the Taylor series. Included in the referee's report was a fairly short proof that $U$ embeds into some $C(K)$ space as a tight subspace. Using a shortcut that allows us to check that $S_g$ is weakly compact for a only small collection of functions $g$, we give an even simpler proof of this result in Proposition 4.2. It now follows from Theorem 2.1 that $U$ has the Pełczyński property. This is a result that has been established by Bourgain in [2]. Bourgain's proof uses a fair amount of hard analysis, including Carleson's theorem on the pointwise almost everywhere convergence of Fourier series in $L^2$. Interestingly enough, our proof uses little more than the Plancherel theorem.

The results on the space $U$ are in Section 4. In addition to proving $U$ has the Pełczyński property, we show that $U$ and $U^*$ have the Dunford–Pettis property. The main ingredient is a theorem of Bourgain which concerns the operators $S_g$. Bourgain proves in [4] that a closed subspace $X \subseteq C(K)$ will have a dual space with the Dunford–Pettis property whenever $S_g^{**}$ is completely continuous for every $g \in C(K)$. It is well-known that a Banach space $Y$ has the Dunford–Pettis property whenever $Y^*$ does. We show that $U$ embeds into a $C(K)$ space (the same $K$ indicated above) as a subspace $X$ satisfying Bourgain's criteria. As in the tight subspace case, the proof is quite simple, and uses very little Fourier theory. Our results on the Banach space structure of $U$ are summarized in Theorem 4.1. Using the same proof used for the disk algebra (see [21]), we prove the known result that $U$ is not isomorphic to a quotient of $C(G)$ for any compact space $G$.

We noted above that $U$ is isomorphic to a tight subspace $X$ of $C(K)$ for some space $K$. We prove that $X$ is not strongly tight and give a characterization of those $g$ for which $S_g$ is compact. Hence, in addition to the reflexive spaces, $X$ yields a new example of a tight, non-strongly tight space. As we noted above, an example of a tight algebra of functions which is not strongly tight has not yet been produced.

In addition to the Pełczyński property, we investigate properties of tightness which are more commonly studied in the context of function algebras. Let $A$ be a uniform algebra and let $\mathcal{M}_A$ be the maximal ideal space of $A$. If $\varphi$ and $\psi$ are elements of $\mathcal{M}_A$ then we say $\varphi$ and $\psi$ are in the same *Gleason part* if $\|\varphi - \psi\|_A < 2$. This is an equivalence relation where the classes are called the Gleason parts of $A$. We say a part is trivial if it consists of one point. It was shown in [6] that every tight uniform algebra on a compact metric space $K$ possesses at most countably many nontrivial Gleason parts. We give a simple proof of this fact. We will need the theory of bands of measures (for more information on bands and related ideas see [6] or [8]).
Let $K$ be a compact space. If $\mathcal{B} \subseteq M(K)$ we say $\mathcal{B}$ is a band of measures if $\mathcal{B}$ is a closed subspace of $M(K)$ and when $\mu, \nu \in M(K)$, and $\nu \ll \mu$ then $\nu \in \mathcal{B}$. The Lebesgue Decomposition Theorem says that if $\mu \in M(K)$ then $\mu$ can be uniquely written as $\mu = \mu_s + \mu_a$, where $\mu_s \in \mathcal{B}$ and $\mu_a$ is singular to every element of $\mathcal{B}$. If $\mathcal{B}$ is a band, the complementary band $\mathcal{B}'$ of $\mathcal{B}$ is the collection of measures singular to every measure in $\mathcal{B}$. It follows that $M(K) = \mathcal{B} \oplus \mathcal{B}'$. It is a well-known fact that if $\mathcal{B}$ is a band then $\mathcal{B} \cong L^1(\mu)$ for some abstract measure $\mu$.

If $\mathcal{B}$ is a band, we define $L^\infty(\mathcal{B})$ to be the space of uniformly bounded families of functions $F = \{F_v\}_{v \in \mathcal{B}}$ where $F_v \in L^\infty(\nu)$ and $F_v = F_{\nu}$ a.e. [dv] whenever $\nu \ll \mu$. The norm in $L^\infty(\mathcal{B})$ is given by $\|F\| = \sup_{v \in \mathcal{B}} \|F_v\|_{L^\infty(\nu)}$. The pairing $\langle v, F \rangle = \int F_v \, dv$ defines an isometric isomorphism between $L^\infty(\mathcal{B})$ and $\mathcal{B}^*$. If $X$ is a subspace of $C(K)$ let $H^\infty(\mathcal{B})$ and $H^\infty(\mu)$ be the weak-star closure of $X$ in $L^\infty(\mathcal{B})$ and $L^\infty(\mu)$ respectively. If $\mu \in \mathcal{B}$, there is a natural projection $H^\infty(\mathcal{B}) \rightarrow H^\infty(\mu)$ defined by $F \mapsto F_{\mu}$. We define $\mathcal{B}_X$ to be the band generated by the measures in $X^\perp$ and $\mathcal{F}$ to be the band complement to $\mathcal{B}_X$. It follows from the Lebesgue decomposition that $X^* \cong \mathcal{B}_X \cap X^\perp$.

We say a band $\mathcal{B}$ is a reducing band for $X$ if for any measure $\nu \in X^\perp$ the projection $\nu_\mu$ of $\nu$ into $\mathcal{B}$ by the Lebesgue decomposition is also in $X^\perp$. We say $\mathcal{B}$ is a minimal reducing band if $\mathcal{B} \neq \{0\}$ while $\{0\}$ is the only reducing band properly contained in $\mathcal{B}$.

Suppose, for now, the subspace $X$ is a uniform algebra $A$. The following version of the Abstract F. and M. Riesz Theorem can be found in [6]. Let $\varphi \in A_{\text{ad}}$. Then the band generated by the representing measures for $\varphi$ is a minimal reducing band. The band generated by the representing measures for $\varphi$ is equal to the band generated by the representing measures for all the points in the same Gleason part as $\varphi$. Hence every Gleason part of a uniform algebra corresponds to a distinct minimal reducing band.

If $A$ is a uniform algebra we say a point $z \in K$ is a peak point for $A$ if there exists an element $f \in A$ such that $f(z) = 1$ and $|f(w)| < 1$ for $w \neq z$. We say $z$ is a generalized peak point if the only complex representing measure for $z$ is the point mass at $z$. The Choquet boundary of $A$ is the collection of all generalized peak points. The point-evaluations for the points off the Choquet boundary lie in $\mathcal{B}_X \cap A^\perp$ while those for the points on the Choquet boundary lie in $\mathcal{F}$.

If $\mathcal{B}$ is a minimal reducing band for $A$ and $\mathcal{B} \subseteq \mathcal{F}$ then it can be seen that $\mathcal{B}$ is all multiples of a point mass $\delta_z$ at some generalized peak point $z \in K$. We call these trivial minimal reducing bands and the others nontrivial minimal reducing bands. Note that a minimal reducing band $\mathcal{B}$ is trivial if and only if $\mathcal{B} \cap A^\perp = 0$ (this implies every subband of $\mathcal{B}$ is a reducing band).
Note that the intersection of two reducing bands is a reducing band and so two minimal reducing bands either coincide or are singular. If we let \( \mathcal{R} \) be the collection of all the non-trivial minimal reducing bands for \( A \) then \( \bigoplus \alpha \mathcal{R} \) is a reducing band contained in \( \mathcal{B}_A \). However, this may not be all of \( \mathcal{B}_A \). For more information, see [6]. The sum \( \bigoplus \alpha \mathcal{R} / \mathcal{B}_A \) is now isometric to a closed subspace of \( A^* \) which is contained in \( \mathcal{B}_A \).

We show in Section 3 that if \( X \) is a tight subspace of \( C(K) \) for a metric space \( K \) then \( \mathcal{B}_A / X \) is separable. It then follows that \( X^* \) is a separable distortion of an \( L^1 \)-space; that is, the dual of \( X \) can be written as the direct sum of an \( L^1 \)-space and a separable space. When \( A \) is a tight uniform algebra on a metric space \( K \) then we see that \( A \) can have at most countably many nontrivial Gleason parts and at most countably many nontrivial minimal reducing bands. This conclusion is easily deduced once we see that \( \mathcal{B}_A / A^\perp \) is separable. In fact we show that \( A \) will have at most countably many nontrivial Gleason parts whenever \( A^* \) is merely embedded in a separable distortion of an \( L^1 \)-space. The proof is an adaptation of a method of Henkin which can be found in [21].

The separability of \( \mathcal{B}_A / A^\perp \) has some interesting consequences. For example, when \( K \) is a metric space this implies that there exists an \( m \in \mathcal{B}_A \) such that every non-peak point of \( A \) has a representing measure absolutely continuous with respect to \( m \). Furthermore, \( m \) will have some other special properties concerning pointwise bounded approximation. The prototypical example is the following. Let \( K \) be a compact subspace of \( C \). Let \( Q \subseteq K \) be the non-peak points of \( R(K) \) and let \( \lambda_0 \) be Lebesgue planar measure restricted to \( Q \). Let \( H^\infty(\lambda_0) \) be the weak-star closure of \( R(K) \) in \( L^\infty(\lambda_0) \). It is a theorem of A.M. Davie in [9] that given \( f \in H^\infty(\lambda_0) \) there exists a sequence \( f_n \in R(K) \) with \( \|f_n\| \leq \|f\| \) such that \( f_n \to f \) pointwise a.e. \( [\lambda_0] \). The measure \( m \) will possess the same property as \( \lambda_0 \).

Section 5 uses the theory of tight uniform algebras and the Pełczyński property to deduce a result about inner functions on strictly pseudoconvex domains in \( C^n \). The background is as follows. We say a subalgebra \( B \subseteq L^\infty(m) \), where \( m \) is Lebesgue measure on the unit circle, is a Douglas algebra if \( B \) contains \( H^\infty \). Recall the Chang–Marshall Theorem which states that every Douglas algebra on the unit circle is generated by \( H^\infty \) and a collection of conjugates of inner functions. In contrast to this result we prove the following. If \( n > 1 \) then there are no nonconstant inner functions whose conjugate is in \( H^\infty(m) + C \) where \( m \) is surface area measure on the unit ball in \( C^n \). It is well-known that \( H^\infty(m) + C \) is a closed subalgebra of \( L^\infty(m) \). The proof is identical when the unit ball is replaced by a strictly pseudoconvex domain \( D \) which has \( C^1 \) boundary. The proof is quite soft and relies mainly on the fact that \( A(D) \) is a strongly tight subalgebra of \( C(D) \).
2. THE PEŁCZYŃSKI PROPERTY

In this section we prove our main result concerning tight subspaces of \( C(K) \).

**Theorem 2.1.** Let \( X \subseteq C(K) \) be a tight subspace. Then \( X \) has the Pełczyński property.

The following well-known theorem on weak compactness will be essential. Recall that a set \( E \) is relatively weakly compact if the weak closure of \( E \) is weakly compact.

**Theorem 2.2 (R. C. James).** Let \( X \) be a Banach space and let \( E \subseteq X \) be bounded subset. Then the following are equivalent:

(a) \( E \) fails to be relatively weakly compact.

(b) There exists a sequence \( \{x_n\} \) in \( E \) and a \( \rho > 0 \) such that if

\[
V_n = \text{co} \{x_1, x_2, ..., x_n\},
\]

and

\[
W_n = \text{co} \{x_{n+1}, x_{n+2}, ...\},
\]

then \( \text{dist}(V_n, W_n) > \rho \) for all \( n \).

(c) There exist sequences \( \{\varphi_n\} \) in \( B_{X^*} \) and \( \{x_n\} \) in \( E \) and a \( \rho > 0 \) such that

\[
\varphi_k(x_k) = 0 \quad \text{for} \quad 1 \leq k \leq n
\]

\[
\text{Re} \varphi_k(x_k) > \rho \quad \text{for} \quad k > n + 1.
\]

The following lemmas deal with non-weakly compact sets in arbitrary Banach spaces. The second lemma is an integral part of the gliding hump construction used to prove Theorem 2.1.

**Lemma 2.3.** If \( T: X \to Z \) is a continuous linear operator and \( S: X \to Y \) is weakly compact and \( x^{**} \in B_{X^{**}} \) with \( \|T^{**}x^{**}\| > \rho > 0 \) and \( \|S^{**}x^{**}\| < \varepsilon \) then there exists an \( x \in B_X \) with \( \|Tx\| > \rho \) and \( \|Sx\| < \varepsilon \).

**Proof.** Choose \( z^* \in Z^* \) with \( \|z^*\| = 1 \) and \( \langle T^{**}x^{**}, z^* \rangle > \rho \). Let \( x^* = T^*z^* \) and define \( \Omega = \{x \in B_X \mid \text{Re} \langle x, x^* \rangle > \rho \} \). Then \( \Omega \) is convex with \( x^{**} \in \Omega^{**} \) and \( \|Tx\| > \rho \) for all \( x \in \Omega \). Since \( S \) is weakly compact we have \( S(\Omega)^{**} = S(\Omega)^{***} \) and so \( S^{**}x^{**} \in S(\Omega)^{***} \). Since \( \|S^{**}x^{**}\| < \varepsilon \) we may find \( x \in \Omega \) with \( \|Sx\| < \varepsilon \). \[\square\]
Lemma 2.4. Let $X$ be a Banach space and suppose $E \subseteq X^*$ is bounded and fails to be relatively weakly compact. Then there exists a $\rho > 0$ and an infinite subset $F \subseteq E$ such that for any infinite subset $F' \subseteq F$ and any weakly compact linear operator $S: X \to Z$ there exist sequences $\{x_n\} \subseteq B_X$ and $\{\xi_n^*\} \subseteq F'$ with $|\xi_n^*(x_n)| > \rho$ and $\|Sx_n\| \to 0$.

Proof. Assume $E \subseteq X^*$ is bounded and fails to be relatively weakly compact. Then by the R. C. James theorem there exists a $\rho' > 0$ and a sequence $\{x_n^*\} \subseteq E$ and $\{\varphi_n\} \subseteq B_{X^*}$ so that
\[
\varphi_n(x_n^*) = 0 \quad \text{for} \quad 1 \leq k \leq n
\]
\[
\text{Re} \varphi_n(x_n^*) > \rho' \quad \text{for} \quad k \geq n + 1.
\]

Let $F = \{x_n^*\}$. Suppose $F' \subseteq F$ is infinite. Without loss of generality we may assume $F' = F$; that is, $F'$ will be a sequence satisfying (2.1) with a subsequence of the $\{\varphi_n\}$ but with the same constant $\rho'$.

Let $Y = l^\infty(F')$ and let $T: X \to Y$ be the canonical map. If $x^* \in F'$ define $\delta_{x^*} \in l^1(F')$ to be the point mass at $x^*$ so $(T^*\varphi_n)(\delta_{x^*}) = \varphi_n(x_n^*)$. Let $y_n^* = T^*\varphi_n$ and let $y_n^* = \delta_{x_n^*}$. Then
\[
\langle y_n^*, y_k^* \rangle = 0 \quad \text{for} \quad n \geq k
\]
\[
\text{Re} \langle y_n^*, y_k^* \rangle > \rho' \quad \text{for} \quad 1 \leq n \leq k - 1
\]
for $n \geq 1$. Therefore
\[
\text{dist} (\text{co}\{y_1^*, y_2^*, \ldots, y_k^*, \ldots\}, \text{co}\{y_k^*, y_{k+1}^*, \ldots\}) > \rho'
\]
for $k \geq 2$.

Choose
\[
u_n^* \in \text{co}\{\varphi_1, \varphi_2, \ldots, \varphi_n\}
\]
and
\[
\nu_n^* \in \text{co}\{\varphi_{n+1}, \varphi_{n+2}, \ldots\}
\]
so that $\|T^*(\nu_n^* - \nu_k^*)\| > \rho'$. Since $S^{**}$ is weakly compact we have $\|S^{**}(\nu_n^* - \nu_k^*)\| \to 0$. Let $x_n^* = \frac{1}{2}(\nu_n^* - \nu_k^*)$ so $x_n^* \in B_{X^*}$ and $\|S^{**}x_n^*\| \to 0$ and $\|T^*(x_n^*)\| > \rho'/2$. Let $\rho = \rho'/2$ and choose $x_n \in B_X$ by Lemma 2.3 so that $\|Sx_n\| \to 0$ and $|\xi_n^*(x_n)| > \rho$. By definition of $T$ we may find $x_n^* \in F'$ so that $|\xi_n^*(x_n)| > \rho$. This completes the proof.

We now return to the Pełczyński property.

Proof of Theorem 2.1. Assume $S_g$ is weakly compact for every $g \in C(K)$. Suppose $E \subseteq X^*$ is a bounded subset which fails to be relatively weakly compact.
compact. We must show there exists a $c_0$-sequence which fails to tend to zero uniformly on $E$. Without loss of generality we may assume $E = \{ x^*_n \}$ for some sequence $\{ x^*_n \}$ and there exists some $\rho > 0$ such that $E$ satisfies the conclusion of Lemma 2.4 with respect to $\rho$.

Let $\mu_n \in M(K)$ be a Hahn–Banach extension of $x^*_n$ and let $v_n = |\mu_n|$. Let $v$ be a weak-star accumulation point of $\{ v_n \}$ so that $v \geq 0$. Let $C = \sup |\mu_n|$. Choose $\delta > 0$ so $\sum \delta_n < \rho/2$.

Let $U : X \to L^1(\nu)$ be the natural inclusion. Then $U$ is weakly compact by the uniform integrability criterion for weak compactness in $L^1(\nu)$. It now follows from Lemma 2.4 that there exists a sequence $\{ h_n \}$ in $X$ with $|h_n| \leq 1$ such that $\int |h_n| \, dv \to 0$ and

$$|x^*_n(h_n)| > \rho$$

for all $n$ and some sequence $\{ f_n \}$. Choose $n_1$ so that

$$\int |h_{n_1}| \, dv < \frac{\delta_1}{2}.$$ We may now find an increasing sequence $\{ k_1 \}$ with $k_1 = j_{n_1}$ so that

$$\int |h_{n_1}| \, dv_{k_1} < \delta_1 \quad \text{for } l \geq 2$$

and

$$\left| \int h_{n_1} \, d\mu_{k_1} \right| > \rho.$$

Define $f_1 = h_{n_1}$. After renumbering we may now assume we have

1. $f_1 \in B_X$.
2. $\int f_1 \, d\mu_1 > \rho$.
3. $\int |f_1| \, d|\mu_1| < \delta_1$ for $k > 1$.

Let $g_1 = 1 - |f_1|$ and redefine $v$ to be a weak-star accumulation point of the new sequence $\{ v_n \} = \{ |\mu_n| \}$ which is now a subsequence of the sequence we started with. Define $T : X \to C/X \otimes L^1(\nu)$ by $T = S_\mu \otimes U$ where $U$ is the operator defined above. Since $S_\mu$ is weakly compact by assumption it follows that $T$ is weakly compact. By Lemma 2.4 there exists a new sequence $\{ h_n \}$ in $X$ with $|h_n| \leq 1$ such that $\|Th_n\| \to 0$ and

$$|x^*_n(h_n)| > \rho.$$
for all $n$ for some sequence $\{j_n\}$. Note that these elements $\{x^n_j\}$ are now being chosen from a subsequence of the original set $E$. It is critical here that Lemma 2.4 allows us to use the same constant $\rho$ that we used for the set $E$.

We now have

$$\text{dist}(h_n(1-|f_1|), X) \to 0$$

and

$$\left| \int |h_n| \, dv \right| \to 0.$$ 

Choose $n_2$ so that $\int |h_{n_2}| \, dv < \delta_2/2$ and

$$\text{dist}(h_{n_2}(1-|f_1|), X) < \frac{\delta_2}{2C}.$$ 

We may now find an increasing sequence $\{k_l\}$ with $k_0 = 1$ and $k_1 = j_{n_1}$ so that

$$\int |h_{n_2}| \, d|\mu_{k_l}| < \delta_2 \quad \text{for } l \geq 2$$

and

$$\left| \int h_{n_2} \, d\mu_{k_l} \right| > \rho.$$ 

Define $f_2 = h_{n_2}$. After renumbering we may assume we have

\begin{enumerate}
  \item[(1')] $f_n \in B_X$ for $n = 1, 2$.
  \item[(2')] $\| f_n \, d\mu_n \| > \rho$ for $n = 1, 2$.
  \item[(3')] $\int |f_n| \, d|\mu_k| < \delta_n$ for $n = 1, 2$ and $k > n$.
  \item[(4')] $\text{dist}(f_2(1-|f_1|), X) < \delta_2/2C$.
\end{enumerate}

Now let $g_2 = (1-|f_1|)(1-|f_2|)$ and repeat the process. At the $N$th step we will have the following.

\begin{enumerate}
  \item[(1'')] $f_n \in B_X$ for $1 \leq n \leq N$.
  \item[(2'')] $\| f_n \, d\mu_n \| > \rho$ for $1 \leq n \leq N$.
  \item[(3'')] $\int |f_n| \, d|\mu_k| < \delta_n$ for $k > n$ and $1 \leq n \leq N$.
  \item[(4'')] $\text{dist}(f_n \prod_{j=1}^{n-1} (1-|f_j|), X) < \delta_n/2C$ for $2 \leq n \leq N$.
\end{enumerate}
We now proceed as in [3], whose proof was elucidated in [24]. At the $N$th step define $\omega_N = \mu_N$. Let $\varphi_1 = f_1$ and $\varphi_n = f_n \prod_{j=1}^{n-1} (1 - |f_j|)$ for $n > 1$ so

$$|\varphi_n| = \prod_{j=1}^{n-1} (1 - |f_j|) - \prod_{j=1}^{n} (1 - |f_j|)$$

and $\sum |\varphi_n| \leq 2$. Hence $\{\varphi_n\}$ is a w.u.C. series.

In general if $0 \leq \xi_j \leq 1$ then $1 - \prod_{j=1}^{n} (1 - \xi_j) \leq \sum_{j=1}^{n} \xi_j$. Therefore for $n \geq 2$ we have

$$\left| \int \varphi_n \, d\omega_n \right| = \left| \int f_n \prod_{j=1}^{n-1} (1 - |f_j|) \, d\omega_n \right|$$

$$\geq \left| \int f_n \, d\omega_n \right| - \left| \int f_n \left( 1 - \prod_{j=1}^{n-1} (1 - |f_j|) \right) \, d\omega_n \right|$$

$$\geq \mu - \left| \sum_{j=1}^{n-1} |f_j| \, d|\omega_n| \right|$$

$$\geq \mu - \sum_{j=1}^{n-1} \delta_j$$

$$\geq \mu - \frac{\mu}{2}$$

Choose $\psi_n \in X$ with $\psi_1 = \varphi_1$ and

$$\|\varphi_n - \psi_n\| \leq \frac{\delta_n}{2C}$$

for $n \geq 1$. Then $\sum \psi_n$ is a w.u.C. series and furthermore

$$\left| \int \psi_n \, d\omega_n \right| \geq \mu - \frac{\delta_n}{2} \geq \frac{\mu}{2}$$

for $n \geq 1$. Since the sequence $\{\omega_n\}$ consists of Hahn–Banach extensions of some sequence in $E$, it now follows from the notes at the beginning of Section 1 that there exists a $C_0$-sequence in $X$ failing to tend to zero uniformly on $E$. □
3. TIGHT UNIFORM ALGEBRAS AND SEPARABLE DISTORTIONS

We will now discuss tightness and some of its connections to separably distorted dual spaces, Gleason parts, reducing bands and pointwise bounded approximation.

**Lemma 3.1.** Let $K$ be a compact space and let $X$ be a closed subspace of $C(K)$. If $B$ is a reducing band for $X$ with $B \subseteq B_X$ then

$$
\mathcal{B} \cap X^\perp = \bigcup_{\nu \in \nu C(K)} S^*_\nu(B \cap X^\perp).
$$

**Proof.** We claim that

$$
\mathcal{B} = \{ g \, dv \mid v \in \mathcal{B} \cap X^\perp, g \in C(K) \}.
$$

Let $E$ be the right-hand side of (3.1) so $E \subseteq \mathcal{B}$. Let $\mu \in \mathcal{B}$. Then, since $\mathcal{B} \subseteq B_X$, it follows from a result of Chaumat (see Proposition V.17.11 in [8]) that there exists some $v \in X^\perp$ such that $\mu \ll v$. Let $v_\nu + v_\perp$ be the Lebesgue decomposition of $v$ with respect to $\mathcal{B}$. Since $\mathcal{B}$ is a reducing band the measure $v_\nu$ lies in $\mathcal{B} \cap X^\perp$. We now have $\mu \ll v_\nu + v_\perp$, and $\mu \perp v_\perp$. Therefore $\mu \ll v_\nu$. Write $dv = Fd\nu$ for some $F \in L^1(v_\nu)$ and let $\{ g_n \}$ be a sequence in $C(K)$ so that $g_n \to F$ in norm. Evidently $\mu$ is in $E$ which implies $\mathcal{B} = E$.

Let $E$ be the right-hand side of (3.1) so $E \subseteq \mathcal{B}$. Let $\mu \in \mathcal{B}$. Then, since $\mathcal{B} \subseteq B_X$, it follows from a result of Chaumat (see Proposition V.17.11 in [8]) that there exists some $v \in X^\perp$ such that $\mu \ll v$. Let $v_\nu + v_\perp$ be the Lebesgue decomposition of $v$ with respect to $\mathcal{B}$. Since $\mathcal{B}$ is a reducing band the measure $v_\nu$ lies in $\mathcal{B} \cap X^\perp$. We now have $\mu \ll v_\nu + v_\perp$, and $\mu \perp v_\perp$. Therefore $\mu \ll v_\nu$. Write $dv = Fd\nu$ for some $F \in L^1(v_\nu)$ and let $\{ g_n \}$ be a sequence in $C(K)$ so that $g_n \to F$ in norm. Evidently $\mu$ is in $E$ which implies $\mathcal{B} = E$.

**Proposition 3.2.** If $K$ is a metric space and $X$ is a tight subspace of $C(K)$ then $B_X/X^\perp$ is separable.

**Proof.** Since $K$ is metrizable we may find a dense sequence $\{ g_n \}$ in $C(K)$. Lemma 3.1 now implies that

$$
\mathcal{B}_{B_X}/X^\perp = \bigcup_{n=1} \mathcal{S}^*_n(X^\perp).
$$

Since weakly compact sets in the dual of a separable Banach space are norm separable and $\mathcal{S}^*_n$ is weakly compact for all $g$, the result follows.

Recall that if $A$ is a uniform algebra and $\{ \mathcal{B}_x \}$ is the collection of all nontrivial minimal reducing bands then $\bigoplus \mathcal{B}_x$ is isometric to a closed
subspace of $A^*$ and every nontrivial Gleason part corresponds to a distinct $\mathcal{B}_e$. We therefore have the following result which is not new but was proved in [6]. However, the present proof is more elementary.

**Corollary 3.3.** If $A$ is a tight uniform algebra on a metric space $K$ then $A$ has at most countably many nontrivial minimal reducing bands and at most countably many nontrivial Gleason parts.

The claim about the Gleason parts follows from the more basic fact that $\|\varphi - \psi\| = 2$ for points $\varphi$ and $\psi$ in distinct parts.

For example if $A = A(A \times A)$ is the bi-disk algebra, then $\{z \times A\}$ is a non-trivial Gleason part for every $z$ on the unit circle. In particular, the bi-disk algebra is not tight. For $R(K)$ where $K$ is a compact planar set, the fact about Gleason parts is well known. Any part of $R(K)$ containing a non-peak point has positive area (see [12]).

The only ingredient needed in the corollary is the separability of $\mathcal{B}_e/A^+$. We would like to mention that this is a special case of a more general phenomenon. We say a Banach space $Y$ is a separable distortion of an $L^1$-space if $Y = M \oplus L$ where $M$ is separable and $L = L^1(\mu)$. Since every band is isomorphic to $L^1(\mu)$ for some $\mu$, $A^*$ will be isomorphic to a separable distortion of an $L^1$-space whenever $\mathcal{B}_e/A^+$ is separable. The following theorem now extends the concept in the corollary.

**Theorem 3.4.** Let $A$ be a uniform algebra and suppose $A^*$ is isomorphic to a closed subspace of a separable distortion of an $L^1$-space. Then $A$ has at most countably many non-trivial minimal reducing bands and therefore at most countably many non-trivial minimal Gleason parts.

This type of phenomenon has its origins in the paper [13] of G. M. Henkin where it is shown that the ball-algebras $A(\mathbb{B}_n)$ are not isomorphic to the polydisk algebras $A(\mathbb{D}^m)$ when $m$ is greater than one (also, see [21]). Our result is a direct extension of Henkin’s work. We begin with some lemmas.

If $\mathcal{B}$ is any band then $H^\infty(\mathcal{B})$ is uniform algebra on its maximal ideal space (see Section 1). It is easy to see that $H^\infty(\mathcal{B})$ will be a proper uniform algebra on its maximal ideal space if and only if it fails to be self-adjoint. Note that if $\mathcal{B}$ is a minimal reducing band then $\mathcal{B}$ is nontrivial if and only if $\mathcal{B} \cap A^+ \neq \emptyset$. The following lemma shows that when $\mathcal{B}$ is a minimal reducing band then $H^\infty(\mathcal{B})$ is a proper uniform algebra on its maximal ideal space if and only if $\mathcal{B}$ is nontrivial.

**Lemma 3.5.** Let $A$ be a uniform algebra and let $\mathcal{B}$ be a reducing band. Then the following are equivalent:

1. $\mathcal{B}$ is nontrivial.
2. $H^\infty(\mathcal{B})$ is a proper uniform algebra on its maximal ideal space.
(a) $H^\infty(\mathcal{B}) = H^\infty(\mathcal{B})$.
(b) $H^\infty(\mathcal{B}) = L^\infty(\mathcal{B})$.
(c) $\mathcal{B} \cap A^\bot = 0$.

**Proof.** (a $\iff$ b) Assume $H^\infty(\mathcal{B}) = H^\infty(\mathcal{B})$. Note that

$$H^\infty(\mathcal{B}) = \left\{ F \in L^\infty(\mathcal{B}) \mid \int F_\mu \, d\mu = 0 \text{ for } \mu \in \mathcal{B} \cap A^\bot \right\}.$$  (3.2)

Therefore, if $f \in A$ then $\int f \, d\mu = 0$ for $\mu \in \mathcal{B} \cap A^\bot$. Now, if $g \in A$ and $\mu \in \mathcal{B} \cap A^\bot$ then $g \, d\mu \in \mathcal{B} \cap A^\bot$. It then follows that if $\mu \in \mathcal{B} \cap A^\bot$ then $\int fg \, d\mu = 0$ for all $f, g \in A$. Therefore $\mu = 0$ by the Stone–Weierstrass Theorem so $\mathcal{B} \cap A^\bot = 0$. It now follows from (3.2) that $H^\infty(\mathcal{B}) = L^\infty(\mathcal{B})$.

(b $\iff$ c) This follows immediately from (3.2).

The next result is a generalization of the fact that the space $L_1^1/H_0^1$ is not isomorphic to a subspace of an $L_1^1$-space. The main ingredient is the theorem of Kisliakov from [16] which states that no proper uniform algebra is isomorphic to a quotient of a $C(K)$ space.

**Proposition 3.6.** Suppose $\mathcal{B}$ is a nontrivial minimal reducing band for some uniform algebra $A$. Then $\mathcal{B}/\mathcal{B} \cap A^\bot$ is not isomorphic to a subspace of an $L_1^1$-space.

**Proof.** Suppose $T: \mathcal{B}/\mathcal{B} \cap A^\bot \to L_1^1(\mu)$ is an isomorphic embedding. Let $E$ be a compact space such that $C(E)$ is the dual of $L_1^1(\mu)$. Then $T^*: C(E) \to H^\infty(\mathcal{B})$ is surjective. Since $\mathcal{B}$ is nontrivial we have $\mathcal{B} \cap A^\bot \neq 0$. It then follows from Lemma 3.5 that $H^\infty(\mathcal{B})$ is a proper uniform algebra on its maximal ideal space which contradicts Kisliakov’s theorem.

If $X$ and $Y$ are Banach spaces we say $X$ is $C$-finitely representable in $Y$ if for every finite-dimensional subspace $F \subseteq X$ there exists a finite dimensional subspace $G \subseteq Y$ such that $d(F, G) \leq C$ where $d$ is the Banach–Mazur distance. If there exists such a $C$ we will simply say $X$ is finitely representable in $Y$. We are motivated by the following.

**Theorem 3.7 (Lindenstrauss-Pełczyński [17]).** Suppose $X$ is a Banach space which is finitely representable in $L_1^1(\mu)$ for some $\mu$. Then $X$ is isomorphic to a subspace of $L_1^1(\mu)$ for some $\mu$.

We now study products which embed into separable distortions.

**Proposition 3.8.** Let $\{E_n\}_{n \in \mathbb{N}}$ be a collection of Banach spaces and let $X = \bigoplus_{n \in \mathbb{N}} E_n$. Suppose $M$ is a separable Banach space and let $L = L_1^1(\mu)$ be
some $L^1$-space. Assume there exists an isomorphic embedding $T : X \rightarrow M \oplus L$ and let $C = \|T\| \cdot \|T^{-1}\|$. If $I_0$ is the set of $x$ in $I$ such that $E_x$ fails to be $2C$-finitely representable in $L$ then $I_0$ is countable.

**Proof.** Assume $I_0$ is uncountable. We may then assume that $I$ is uncountable and we have an isomorphic embedding $T : X \rightarrow M \oplus L$ where $E_x$ fails to be $2C$-finitely representable in $L$ for every $x \in I$ where $C = \|T\| \cdot \|T^{-1}\|$. Therefore, for every $x \in I$ we may find a finite dimensional subspace $F_x \subseteq E_x$ such that

$$d(F_x, G) > 2C$$

for every subspace $G$ of $L$ such that $\dim G = \dim F_x$.

Since $I$ is uncountable we may assume without loss of generality that there exists a fixed integer $n$ independent of $x$ such that $\dim F_x = n$ for all $x \in I$. Choose $\varepsilon > 0$ so that

$$\frac{2}{2-\varepsilon} (1 + 2\varepsilon) = \frac{3}{2}.$$  \hfill (3.4)

It is well known that the Banach-Mazur distance on the space of $n$-dimensional Banach spaces is a separable metric. We may therefore assume without loss of generality that

$$d(F_x, F_x') \leq 1 + \varepsilon$$

for all $x$ and $x'$ in $I$.

Let $x_0$ be any element of $I$. For every $x \in I$ let $U_x : F_{x_0} \rightarrow F_x$ be an isomorphism with

$$\|U_x\| \cdot \|U_x^{-1}\| \leq 1 + 2\varepsilon,$$  \hfill (3.6)

which can be done by (3.5). Furthermore, after multiplying by a constant we may assume

$$\|U_x^{-1}\| = \varepsilon$$

for all $x$.

Let $q_M$ and $q_L$ be the natural projections:

$$\begin{align*}
M & \rightarrow M \oplus L
\downarrow q_M \quad \downarrow q_L \\
L & \rightarrow M \oplus L
\end{align*}$$
For every $i \in I$ let $i : F \hookrightarrow X$ be the natural injection and define $S_{i} : F_{i} \to M$ by $S_{i} = q_{M} \circ T \circ i_{i} \circ U_{i}$. Note that the space of bounded linear operators $L(F_{i}, M)$ is separable. Since $\{S_{i}\}_{i \in I}$ is an uncountable collection in $L(F_{i}, M)$ we may find distinct elements $i_{1}$ and $i_{2}$ in $I$ so that

$$\|S_{i_{1}} - S_{i_{2}}\| < \frac{1}{|T^{-1}|}. \quad (3.8)$$

Define $W : F \to X$ by $W = (i_{1} \circ U_{i_{1}} - i_{2} \circ U_{i_{2}})$.

**Claim 1.** We have

$$\|W\| \|W^{-1}\| \leq 1 + 2\varepsilon. \quad (3.9)$$

If $x \in F$ then

$$\|Wx\| = \|U_{i_{1}}x\| + \|U_{i_{2}}x\| \geq \|x\| \left(\frac{1}{\|U_{i_{1}}^{-1}\|} + \frac{1}{\|U_{i_{2}}^{-1}\|}\right)$$

so

$$\|W^{-1}\| \leq \frac{\|U_{i_{1}}^{-1}\| \|U_{i_{2}}^{-1}\|}{\|U_{i_{1}}^{-1}\| + \|U_{i_{2}}^{-1}\|} = \frac{\varepsilon}{2} \quad (3.10)$$

by (3.7). Since $\|W\| \leq \|U_{i_{1}}\| + \|U_{i_{2}}\|$ we have

$$\|W\| \|W^{-1}\| \leq \|U_{i_{1}}\| \|U_{i_{2}}^{-1}\| \left(\frac{\|U_{i_{1}}^{-1}\|}{\|U_{i_{1}}^{-1}\| + \|U_{i_{2}}^{-1}\|}\right)$$

$$+ \|U_{i_{1}}\| \|U_{i_{2}}^{-1}\| \left(\frac{\|U_{i_{2}}^{-1}\|}{\|U_{i_{1}}^{-1}\| + \|U_{i_{2}}^{-1}\|}\right)$$

$$\leq 1 + 2\varepsilon$$

by (3.6).

Define $Q : F \to L$ by $Q = q_{L} \circ T \circ W$.

**Claim 2.** We have

$$\|Q\| \|Q^{-1}\| \leq \frac{\varepsilon}{2}C. \quad (3.11)$$
If \( x \in F_{s_0} \) then

\[
\| Qx \| = \| q_1(TWx) \|
\]

\[
= \| (TW)(x) \| - \| (q_0 TW)(x) \|
\]

\[
= \| (TW)(x) \| - \| (S_{s_0} - S_{s_0})(x) \|
\]

\[
\geq \frac{1}{1 - \| T^{-1} \| \| W^{-1} \| \| x \| - \| T^{-1} \| \| x \|}
\]

\[
= \frac{1}{1 - \| T^{-1} \| \| W^{-1} \|} \left( \frac{1}{\| W^{-1} \|} - 1 \right)
\]

so

\[
\| Q^{-1} \| \leq \frac{\| T^{-1} \| \| W^{-1} \|}{1 - \| T^{-1} \| \| W^{-1} \|}
\]

Furthermore, since \( \| Q \| \leq \| T \| \| W \| \) we have

\[
\| Q \| \| Q^{-1} \| \leq C \frac{\| W \| \| W^{-1} \|}{1 - \| W^{-1} \|}
\]

\[
\leq \frac{2}{2 - \varepsilon} C \| W \| \| W^{-1} \|
\]

\[
\leq \frac{2}{2 - \varepsilon} (1 + 2\varepsilon) C
\]

\[
= \frac{3}{2} C
\]

by (3.4).

If we let \( G = Q(F_{s_0}) \) then (3.11) implies \( d(F_{s_0}, G) \leq \frac{3}{2} C \). This contradicts (3.3). Hence, Proposition 3.8 is proved.

By Theorem 3.7 we have the following corollary.

**Corollary 3.9.** If \( \{ E_x \} \) is a collection of Banach spaces such that the product \( \bigoplus_j E_x \) embeds isomorphically into a separable distortion of an \( L^1 \)-space then all but a countable number of the \( E_x \) embed isomorphically into some \( L^1 \)-space (where the \( L^1 \)-space depends on \( x \)).

We now summarize our results in the following proof.

**Proof of Theorem 3.4.** Assume \( A \) is a uniform algebra and \( A^* \) is isomorphic to a subspace of a separable distortion of an \( L^1 \)-space. If \( \mathcal{B}_x \)
is the collection of all nontrivial minimal reducing bands then the sum \( \bigoplus_{\mathcal{B}} \mathcal{B}_\mathcal{B} \cap A^+ \) is isometric to a subspace of \( A^* \). By Corollary 3.9 this implies \( \mathcal{B}_\mathcal{B} \cap A^+ \) embeds in some \( L^1 \)-space for all but a countable number of \( x \). However, Proposition 3.6 states that \( \mathcal{B}_\mathcal{B} \cap A^+ \) fails to embed in an \( L^1 \)-space for every \( x \in I \). Therefore, the set \( I \) must be countable. Furthermore, every nontrivial Gleason part corresponds to a distinct nontrivial minimal reducing band, which finishes the proof of the proposition.

At the present time it is not known if there exists a uniform algebra \( A \) with the property that \( \mathcal{B}_\mathcal{B} \mid A^+ \) is nonseparable and the dual of \( A \) embeds into a separable distortion of an \( L^1 \)-space. One problem is that it is not clear when \( \mathcal{B} \mathcal{B} \cap A^+ \) is separable for an arbitrary nontrivial minimal reducing band \( \mathcal{B} \). Even if this problem is solved, the nontrivial minimal reducing bands do not exhaust \( \mathcal{B}_\mathcal{B} \mid A^+ \). A discussion of the complete decomposition of \( \mathcal{B}_\mathcal{B} \) can be found in [6].

The separability of \( \mathcal{B}_\mathcal{B} \mid A^+ \) can be applied to some ideas in pointwise approximation. To illustrate, let \( K \) be a compact planar set and let \( R(K) \) be the space of functions in \( C(K) \) which can be uniformly approximated by rational functions with poles off \( K \). Define \( Q \subset K \) to be the collection of non-peak points for \( R(K) \) and let \( \lambda_Q \) be the restriction of planar Lebesgue measure to \( Q \). Define \( H^\ast(\lambda_Q) \) be the weak-star closure of \( R(K) \) in \( L^\ast(\lambda_Q) \).

It is a theorem of A. M. Davie in [9] that if \( f \in H^\ast(\lambda_Q) \) then there exists a sequence of functions \( \{f_n\} \) in \( R(K) \) such that \( f_n \rightarrow f \) pointwise a.e. \( \lambda_Q \) and \( \|f_n\| \leq \|f\| \). This conclusion is sometimes referred to as pointwise bounded approximation with a reduction in norm. It is known that Davie's theorem implies, without much difficulty, that every \( z \in Q \) has a representing measure absolutely continuous with respect to \( \lambda_Q \).

We take the following approach to this problem (also, see [6] or [8]). Let \( A \) be an arbitrary uniform algebra. Given \( m \in \mathcal{B}_\mathcal{B} \), we have the natural projection

\[
H^\ast(\mathcal{B}_\mathcal{B}) \xrightarrow{\pi} H^\ast(m),
\]

which is the dual of the injection

\[
\frac{L^1(m)}{L^1(m) \cap A^+} \rightarrow \mathcal{B}_\mathcal{B} \rightarrow A^+.
\]

(see Section 1). Because the space \( H^\ast(\mathcal{B}_\mathcal{B}) \) is identified isometrically with a closed subspace of \( A^{**} \) it follows from Goldstine's Theorem that if \( F \in H^\ast(\mathcal{B}_\mathcal{B}) \) with \( \|F\| \leq 1 \) then there exists a net \( \{f_n\} \) in \( A \) with \( \|f_n\| \leq 1 \) such that \( f_n \rightharpoonup F \). Let \( m \) be any measure in \( \mathcal{B}_\mathcal{B} \). Recall that \( \mathcal{B}_\mathcal{B} \mid A^+ \) contains all the point evaluations for the points off the Choquet boundary.
Therefore this net also has the property that $f_n(z) \to f(z)$ for every $z \in Q$ where $Q$ is the complement of the Choquet boundary. Since the natural projection of $H^\infty(\mathscr{B}A)$ into $H^\infty(m)$ is weak-star continuous, we have $f_n \stackrel{w^*}{\to} F$ in $H^\infty(m)$ (where we are using $F$ as the symbol for an element of $H^\infty(\mathscr{B}A)$ as well as its projection $F_m$ into $H^\infty(m)$). It is now easy to see that there exists a sequence $\{f_n\}$ bounded by one such that $f_n(z) \to F(z)$ for all $z \in Q$ and $f_n \to F$ pointwise a.e. $[m]$. We therefore have the following.

**Lemma 3.10.** If $A$ is a uniform algebra and $m \in \mathscr{B}A$, then for any $F$ in the unit ball of $H^\infty(\mathscr{B}A)$ there exists a sequence $\{f_n\}$ from the unit ball of $A$ such that $f_n \to F$ weakly and $f_n \to F$ pointwise a.e. $[m]$.

Recall that a linear operator $T$ between the Banach spaces $X$ and $Y$ is a quotient map if the induced injection $S: X/Z \to Y$, where $Z = \ker T$, is a surjection. The following proposition relates the Davie phenomenon directly to the projection $\tau$.

**Proposition 3.11.** Let $A$ be a uniform algebra on an arbitrary compact space $K$ and let $m \in \mathscr{B}A$. Then the following are equivalent:

(a) For every $f \in H^\infty(m)$ there exists a sequence $\{f_n\}$ in $A$ with $\|f_n\| \leq \|f\|$ such that $f_n \to f$ pointwise a.e. $[m]$.

(b) The natural projection $H^\infty(\mathscr{B}A) \to H^\infty(m)$ is a quotient map.

**Proof.** (a $\Rightarrow$ b) Assume that (a) holds and let $I$ be the kernel of $\tau$ and $S: H^\infty(\mathscr{B}A)/I \to H^\infty(m)$ be the induced injection. Given $f \in H^\infty(m)$ let $\{f_n\} \subset A$ be the sequence mentioned in the statement of (a). Let $F$ be a weak-star accumulation point of $\{f_n\}$ in $H^\infty(\mathscr{B}A)$ so $\|F\| \leq \|f\|$. Since the map $\tau$ is a dual map it is continuous from the weak-star topology to the weak-star topology. Therefore, $\tau(f_n)$ accumulates weak-star at $\tau(F)$ in $H^\infty(m)$ and $\{f_n\}$ converges weak-star to $f$ so $\tau(F) = f$. Hence the map $S$ is onto. Furthermore, we have $\|\tau(F)\| \leq \|F\|$ by definition so $\|f\| = \|F\|$. Hence $S$ is an isometry and $\tau$ is a quotient map and therefore (b) holds.

(b $\Rightarrow$ a) Assume $\tau$ is a quotient map and let $f \in H^\infty(m)$. We may then find an $\hat{F} \in H^\infty(\mathscr{B}A)$ such that $\tau(\hat{F}) = f$ and $\|\hat{F} + I\| = \|f\|$ where $I = \ker \tau$. Since $I$ is a weak-star closed subspace of $H^\infty(\mathscr{B}A)$ we may find an $F \in H^\infty(\mathscr{B}A)$ such that $\tau(F) = \tau(\hat{F}) = f$ and $\|\hat{F} + I\| = \|F\|$. Hence, $\|F\| = \|f\|$. By Lemma 3.10 we may find a sequence $\{f_n\}$ in $A$ such that $\|f_n\| \leq \|F\|$ and $f_n \to F$ pointwise a.e. $[m]$ which is the desired conclusion.
It is possible that $\tau$ is a quotient map if and only if it is onto, but this is currently not known to be true. If $\tau$ were onto then note the kernel of $\tau$, call it $I$, is an ideal and the induced map $H^\infty(\mathcal{B}_A)/I \hookrightarrow H^\infty(m)$ is an algebra isomorphism from a Banach algebra to a uniform algebra. If $H^\infty(\mathcal{B}_A)/I$ is a uniform algebra then $\tau$ would be an isometry and so $\tau$ would be a quotient map. However, it is not clear when $H^\infty(\mathcal{B}_A)/I$ is a uniform algebra.

Nevertheless, the conclusion of Davie's theorem can now be stated in terms of specific properties of the natural projection. If $A$ is a uniform algebra on a compact space $K$ and $m \in \mathcal{B}_A$, we say $m$ is an ordinary Davie measure if $\tau$ is a quotient map and $m$ is a strong Davie measure if $\tau$ is an isometry. In general, a linear operator between Banach spaces is an isometric embedding if and only if its dual is a quotient map. Therefore, $m$ is an ordinary Davie measure if and only if $\sigma$ is an isometric embedding and is strong Davie measure if and only if $\sigma$ is an isometry. Since $\tau$ is an algebra homomorphism between uniform algebras, $\tau$ will be an isometry as soon as it is an isomorphism. Since $\sigma$ is always an injection, we see that $m$ is a strong Davie measure as soon as $\sigma$ is onto. Since the evaluations for the points off the Choquet boundary lie in $\mathcal{B}_A/A^+$, it follows easily that when $m$ is a strong Davie measure then every point of the Choquet boundary has a representing measure absolutely continuous with respect to $m$.

For the sake of completeness we will briefly discuss the injectivity of $\tau$. We say $m \in M(K)$ is a weakly rich (resp. strongly rich) measure for $A$ if when $\{f_n\}$ is a bounded sequence in $A$ such that $\int |f_n| \, dm \to 0$ then $f_n g + A \to 0$ (resp. $\|f_n g + A\| \to 0$) for every $g \in C(K)$. The concept of richness was introduced in [4] where it was shown that a uniform algebra $A$ (or even an arbitrary subspace of $C(K)$) has the Pečzynski property if there exists a strongly rich measure for $A$. Note that weakly rich measures on strongly tight spaces (where the operators $S_g$ are compact) are strongly rich.

The following result can be found in [22]. If $m \in M(K)$ let $m = m_a + m_s$ be the Lebesgue decomposition of $m$ with respect to $\mathcal{B}_A$.

**Proposition 3.12.** Let $A$ be a uniform algebra on a compact space $K$ and let $m$ be an element of $M(K)$. Then the following are equivalent:

(a) The natural projection $H^\infty(\mathcal{B}_A) \to H^\infty(m)$ is one-to-one.

(b) If $\{f_n\}$ is a bounded sequence in $A$ such that $\int |f_n| \, dm \to 0$ then $f_n \to 0$ in $L^\infty(\mu)$ for every $\mu \in A^+$.

(c) $m$ is a weakly rich measure for $A$.

Furthermore, if the above hold, and $K$ is metrizable, then $\mathcal{B}_A/A^+$ is separable.
We will now show that \( A \) possesses a strong Davie measure whenever \( A^\perp \) is separable. We approach this problem from a general point of view. Let \( X \) be a Banach space and let \( E \subseteq X \) be a closed subspace. If \( Y \) is a subspace of \( X \) we say \( Y \) is a full subspace with respect to \( E \) if the induced map

\[
\frac{Y}{Y \cap E} \xrightarrow{\sigma} \frac{X}{E}
\]

is an isometric embedding; that is, for every \( y \in Y \) we have

\[
\text{dist}(y, Y \cap E) = \text{dist}(y, E).
\]

Note that we do not assume \( Y \) to be a closed subspace of \( X \) which means that \( Y/Y \cap E \) may only have a semi-norm. Therefore, when we say isometric embedding in the definition, what we really mean is that \( \sigma \) preserves the semi-norm.

The case we should be thinking about is \( X = A^\perp \) the space \( E = A = m \) for some measure \( m \notin B^\perp \). When \( L^1(m) \) is full with respect to \( A^\perp \), then \( m \) is an ordinary Davie measure. If the map \( \sigma \) is onto, \( m \) will be a strong Davie measure. When the subspace \( E \) is clear we will simply refer to \( Y \) as being a full subspace. Furthermore, we will identify \( YY \cap E \) with its image \( \sigma(YY \cap E) \) in \( X/E \). Note that all of our Banach spaces are complex and \( \mathcal{F} \) refers to the closed complex linear span of \( F \).

**Proposition 3.13.** Let \( X \) be a Banach space, and let \( E \) be a closed subspace of \( X \). Suppose \( S \) is a separable subset of \( X \) (respectively, of \( X/E \)). Then there exists a closed, full, separable subspace \( Y \subseteq X \) such that \( S \subseteq Y \) (respectively, \( S \subseteq Y/Y \cap E \)).

**Proof.** Let \( \{x_n\} \) (respectively \( \{x_n+E\} \)) be a dense sequence in \( S \) and let

\[
\{x_n\}_{n=1}^{\infty} = \left\{ \sum_{k=1}^{N} a_k s_k : a_k = p + iq, \ p, q \in \mathbb{Q}, \ N \in \mathbb{N} \right\}.
\]

Choose \( x_n + E \) so that \( \lim_{N \to \infty} x_n + x_n + E = \|x_n + E\|. \) Let \( Y = \mathbb{P} \{ x_n \cup \{x_n + E\} \} \) so that \( S \subseteq Y \) (respectively, \( S \subseteq Y/Y \cap E \)). Now, since \( x_n + E \) is dense in \( Y/Y \cap E \),

\[
\|x_n + Y \cap E\| \leq \lim_{k \to \infty} \|x_n + x_n + E\| = \|x_n + E\| \leq \|x_n + Y \cap E\|
\]

so \( \|x_n + Y \cap E\| = \|x_n + E\| \) for all \( n \). Hence, the map \( \sigma \) in Eq. (3.14) is an isometry on \( \{x_n + Y \cap E\} \). By definition, the sequence \( \{x_n\} \) is dense in \( \mathbb{P} \{ x_n \} \) and therefore \( \{x_n + Y \cap E\} \) is dense in \( Y/Y \cap E \). Hence \( \sigma \) is an isometric embedding since it preserves the norm on a dense set. \( \blacksquare \)
If \( X \) is a band of measures and \( m \in X \) we identify \( L^1(m) \) with the subband of \( X \) consisting of all measures absolutely continuous with respect to \( m \).

**Proposition 3.14.** Let \( X \) be a band of measures on some compact metric space \( K \) and let \( E \subseteq X \) be a closed subspace. Suppose \( S \subseteq X/E \) is separable. Then there exists an \( m \in X \) such that \( L^1(m) \) is full and \( S \subseteq L^1(m)/L^1(m) \cap E \).

**Proof.** By Proposition 3.13 we can find a closed, full, separable sub-space \( Y_1 \subseteq X \) with \( S \subseteq Y_1/Y_1 \cap E \). Let \( \mathcal{B}_1 \) be the band generated by \( Y_1 \) so that \( \mathcal{B}_1 \) is separable (here we use the metrizability of \( K \)). Using Proposition 3.13 again we may find a closed, full, separable subspace \( Y_2 \) with \( \mathcal{B}_1 \subseteq Y_2 \). Let \( \mathcal{B}_2 \) be the band generated by \( Y_2 \) and repeat, so we have

\[
Y_1 \subseteq \mathcal{B}_1 \subseteq Y_2 \subseteq \mathcal{B}_2 \subseteq \cdots ,
\]

where \( Y_n \) is a closed, full, separable subspace and \( \mathcal{B}_n \) is a separable band. Let

\[
\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n
\]

so \( \mathcal{B} \) is separable. It is easy to see \( \mathcal{B} \) is a band.

**Claim.** \( \mathcal{B} \) is full. Note that \( \mathcal{B} = \bigcup_{n=1}^{\infty} Y_n \) Let \( Y = \bigcup_{n=1}^{\infty} Y_n \). If \( y \in Y \) then \( y \in Y_n \) for some \( n \) and

\[
\text{dist}(y, Y \cap E) \leq \text{dist}(y, Y_n \cap E) = \text{dist}(y, E) \leq \text{dist}(y, Y \cap E)
\]

since \( Y_n \) is full. This shows \( Y \) is full which clearly implies \( \mathcal{B} \) is full. Hence, \( \mathcal{B} \) is full.

Since \( \mathcal{B} \) is separable it follows that \( \mathcal{B} = L^1(m) \) for some \( m \in X \) and the proposition is proved. \( \square \)

**Corollary 3.15.** Suppose \( A \) is a uniform algebra on a compact metric space \( K \) and \( G \) is any subset of \( K \) which does not meet the Choquet boundary. Then the following are equivalent:

(a) \( G \) is separable in the Gleason metric.

(b) There exists an ordinary Davie measure \( m \) such that every point in \( G \) has a representing measure absolutely continuous with respect to \( m \).

**Proof.** Let \( G' \) be the subset of \( A^* \) consisting of point evaluations at the points in \( G \) so that \( G' \subseteq \mathcal{B}_{A^*}/A^* \).

(a \( \Rightarrow \) b) Assume (a) holds. Since the Gleason metric on \( G \) corresponds to the norm on \( \mathcal{B}_{A^*}/A^* \) it follows that \( G' \) is a separable subset
of $B/A\Rightarrow$. Using the above proposition with $X = B/A\Rightarrow$ and $E = A\Rightarrow$ we may find a measure $m$ in $B/A\Rightarrow$ such that $L^1(m)$ is a full subspace and

$$G' \subseteq \frac{L^1(m)}{L^1(m) \cap A\Rightarrow}.$$ 

Since $L^1(m)$ is full $m$ is an ordinary Davie measure. Furthermore, if $z \in G$ then there exists a representing measure $\mu$ for $z$ such that $\mu + A\Rightarrow \in G'$. We may then find a $g \in L^1(m)$ such that $\mu + A\Rightarrow = g \, dm + A\Rightarrow$ and so $g \, dm$ is a complex representing measure for $z$. We can then find a representing measure for $z$ absolutely continuous with respect to $m$. This proves (b).

(b $\Rightarrow$ a) follows from the fact then $L^1(m)$ is separable when $K$ is metrizable (we don't need $m$ to be an ordinary Davie measure here).

The next corollary is immediate.

**Corollary 3.16.** If $A$ is a uniform algebra on a compact metric space and $B/A\Rightarrow$ is separable then $A$ admits a strong Davie measure $m$.

Applying Proposition 3.12 we have another corollary.

**Corollary 3.17.** Let $A$ be a uniform algebra on a compact metric space $K$. Then the following are equivalent:

(a) $A$ admits a weakly rich measure.

(b) $A$ admits a strong Davie measure.

(c) $B/A\Rightarrow$ is separable.

From Proposition 3.2 we deduce the following.

**Theorem 3.18.** If $A$ is a tight uniform algebra on compact metric space $K$ then $A$ admits a strong Davie measure $m$. In particular, every point off the Choquet boundary has a representing measure absolutely continuous with respect to $m$.

Obviously $\lambda_Q$ is a strong Davie measure for $R(K)$. It follows from Theorem 5.2 that if $D$ is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ then the surface-area measure on $\partial D$ is also a strong Davie measure for $A(\partial D)$. It is known that $R(K)$ has the Pełczyński property for any compact planar set $K$ and $A(\partial D)$ has the Pełczyński property for the domains $D$ just mentioned (see the comments in Sections 1 and 5). It follows that these uniform algebras have weakly sequentially complete duals. If $A^*$ is weakly sequentially complete then $B/A\Rightarrow$ is weakly sequentially complete. If $m$ is an ordinary Davie measure then $L^1(m)/L^1(m) \cap A\Rightarrow$ is isomorphic to a subspace of $B/A\Rightarrow$ and is therefore weakly sequentially complete and
has $H^\infty(m)$ as its dual. This proves the following theorem, which has an analogous version for $R(K)$ and is a direct generalization of a theorem in [19] which deals with the unit disk.

**Theorem 3.19.** Suppose $D \subset \mathbb{C}^n$ is a bounded strictly pseudoconvex domain with $C^2$ boundary. Let $m$ be surface-area measure on $\partial D$. Suppose \{f_n\} is a sequence of functions in $L^1(m)$ such that $\lim \int f_n h \, dm$ exists for every $h \in H^\infty(m)$. Then there exists an $f \in L^1(m)$ such that $\lim \int f_n h \, dm = \int fh \, dm$ for every $h \in H^\infty(m)$.

4. UNIFORMLY CONVERGENT FOURIER SERIES

Let $\Gamma$ be the unit circle in $\mathbb{C}$ and let $U$ be the space of continuous functions $F$ on $\Gamma$ which extend to be analytic in the unit disk such that the series $\sum_{n=0}^{\infty} \hat{F}(n) z^n$ converges uniformly to $F$ where $\hat{F}(n) = \frac{1}{2\pi} \int_{\Gamma} F(e^{it}) e^{-int} dt$. For $g \in L^1(\partial \Gamma)$ let $P_n(g) = \sum_{k=0}^{n} \hat{g}(k) z^k$. If we define

$$\|F\|_U = \sup_{n \geq 0} \|P_n(F)\|_{L^1},$$

then $U$ becomes a Banach space with the norm $\|\cdot\|_U$. In this section we will prove the following.

**Theorem 4.1.** Let $U$ be the space of analytic uniformly convergent Fourier series on the unit circle with the above norm. Then $U$ has the following properties. Parts (a)–(c) originally appeared in [2] (see Section 1).

(a) $U$ is not isomorphic to a quotient of $C(G)$ for any compact space $G$.

(b) $U$ has the Pełczyński property.

(c) $U^*$ is weakly sequentially complete.

(d) $U$ and $U^*$ have the Dunford–Pettis property.

(e) $U^*$ is isomorphic to a separable distortion of an $L^1$-space.

**Proof of Theorem 4.1 (a).** Suppose, on the contrary, there exists a compact space $G$ and a surjective continuous linear operator $T: C(G) \to U$. Let $A$ be the disk algebra on $\Gamma$ and let $i: U \to A$ be the natural inclusion. Let $P: A \to l^1$ be the Paley operator $P(f) = \{\hat{f}(n)\}_{n=0}^{\infty}$. Then $P$ is 1-summing (see [21]). However $PT$ is also 1-summing and therefore compact (again, see [21]), which implies by the surjectivity of $T$ that $PT$ is compact. By examining $P(\bar{z}^n)$, we see that this is a contradiction.
It is well known that part (c) of Theorem 4.1 follows from (b). To prove (b) we will show that \( U \) embeds isometrically into some \( C(K) \) space as a tight subspace and apply Theorem 2.1.

Let \( K' = \{1/n\}_{n=1}^{\infty} \cup \{0\} \) and let \( K = K' \times I \). Define a sequence of closed subspaces of \( K \) by \( \Gamma_n = \{1/(n+1)\} \times I \) for \( n \geq 0 \) and let \( \Gamma_\infty = \{0\} \times I \). If \( \Phi \in C(K) \) we can write \( \Phi = (\varphi_\infty, \varphi_{0}, \varphi_{1}, \varphi_{2}, ...) \) where \( \Phi|_{\Gamma_n} = \varphi_n \) for \( n \geq 0 \), \( \Phi|_{\Gamma_\infty} = \varphi_\infty \) and \( \varphi_n \to \varphi_\infty \) uniformly on \( \Gamma \). Define an isometry \( i: U \to C(K) \) by \( i(F) = \Phi = (F, P_0(F), P_1(F), P_2(F), ...) \) and let \( X = i(U) \).

To show \( X \) is a tight subspace of \( C(K) \) we must study the operators \( S_{g} \) defined in Section 1. Given a compact space \( G \) and closed subspace \( Y \subset C(G) \), we define \( Y_{cg} \) to be the set of \( g \in C(K) \) such that \( S_{g} \) is weakly compact (the “c” represents Brian Cole and the “g” represents Ted Gamelin). It was shown in [6] that when \( A \) is a uniform algebra on \( G \) then \( A_{cg} \) is a closed subalgebra of \( C(G) \). However, the proof does not use the algebraic structure of \( A \) and goes as follows. From general theory we know that \( S_{g}: Y \to C/G \) is weakly compact if and only if \( S_{g}^{**}(Y^{**}) \subseteq C/G \). Since \( S_{g}^{**}(F) = gF + Y^{**} \), \( S_{g} \) is weakly compact if and only if \( gY^{**} \subseteq Y^{**} + C \).

It is now clear that \( Y_{cg} \) is a closed subalgebra of \( C(G) \).

Let \( Y_{cg} \) be the set of those \( g \) such that \( S_{g} \) is compact. It is an even easier task to show that \( Y_{cg} \) is again a closed subalgebra of \( C(G) \). This result is proved in [22] for algebras, however the proof does not use the algebraic structure.

The fact that \( X_{cg} \) is a closed subalgebra of \( C(K) \) means that we need only verify \( S_{\Phi} \) is weakly compact on a set of continuous functions \( g \) that is self-adjoint and separates the points of \( K \).

**Proposition 4.2.** \( X \) is a tight subspace of \( C(K) \).

**Proof.** By the observation above, we need only verify that \( S_{\Phi} \) is weakly compact for \( \Phi = (\varphi_\infty, \varphi_{0}, \varphi_{1}, \varphi_{2}, ...) \) where \( \Phi \) has the following form. First, suppose \( \varphi_n = \delta_{nm} z^n \) for some integer \( m \) and \( n \geq 0 \) where \( \delta_{mn} \) is the Kronecker delta function. Then \( S_{\Phi} \) is easily seen to be of finite rank.

Secondly, suppose \( \varphi_n = z^n \) for \( n \geq 0 \) or \( \varphi_n = \bar{z}^n \) for \( n \geq 0 \). If we prove \( S_{\Phi} \) is weakly compact in this case then, since the functions in the first and second cases form a separating self-adjoint family, we will have shown \( X_{cg} = C(K) \) by the comments above. This, by definition, means \( X \) is a tight subspace of \( C(K) \).

Suppose \( \varphi_n = z^n \) for \( n \geq 0 \). Suppose \( f \in X \) and \( f = \widehat{F} \) where \( F \in U \). Then if \( a_n = \widehat{F}(n) \) we have

\[
\Phi f = (zf, a_0z, a_0z + a_1z^2, ...) = (zF, 0, a_0z, a_0z + a_1z^2, ...) + (0, a_0z, a_1z^2, a_2z^3, ...).
\]
Therefore, if we define $V: X \to C(K)$ by $Vf = (0, a_0 z, a_1 z^2, a_2 z^3, ...)$ then $V$ is easily seen to be continuous and $S_\phi = q V$ where $q: C(K) \to C(K)/X$ is the natural quotient map. Furthermore, if we let $j: X \to l^2$ map $f$ to its Fourier coefficients and let $\tilde{V}: l^2 \to C(K)$ by $\tilde{V}x = (0, x(0) z, x(1) z^2, ...)$, then $j$ and $\tilde{V}$ are continuous and $V = \tilde{V} j$. Hence $V$ is weakly compact which implies $S_\phi$ is weakly compact. The argument for $\phi_n = \tilde{\phi}$ is similar.

Proof of Theorem 4.1 (b), (c), and (e). In Theorem 2.1 we proved that tight subspaces have the Pelczynski property, and therefore so does $X$. Part (b) now follows from the fact that $U$ is isomorphic to $X$. It is well known that Banach spaces with the Pelczynski property have weakly sequentially complete dual spaces, which takes care of (c). Part (e) is an immediate consequence of Proposition 3.2.

If $G$ is any compact space and $Y \subseteq C(G)$ is a closed subspace, let $Y_b$ and $Y_B$ be the space of functions $g \in C(G)$ such that $S_g$ (respectfully, $S_g^{**}$) is completely continuous. These are called the Bourgain algebras of $Y$. These spaces were first defined in [5]. It is not difficult to see that $Y_b$ and $Y_B$ are closed subalgebras of $C(G)$, as was shown in [5]. The motivation to study these spaces was the work of Bourgain in [4]. It can be deduced from Bourgain’s work that if $Y_B = C(K)$ then $Y$ and $Y^*$ have the Dunford–Pettis property. This is how we plan to prove part (d) of Theorem 4.1.

The lemma below follows immediately from Lemma 2.3.

Lemma 4.3. Suppose $A$, $B$ and $C$ are Banach spaces with continuous linear operators $S: A \to B$ and $T: A \to C$. Assume $T$ is weakly compact and $T^{**}$ is completely continuous. Suppose $T$ has the additional property that whenever $x_n$ is a bounded sequence in $A$ such that $\|Tx_n\| \to 0$ then $\|Sx_n\| \to 0$. Then $S^{**}$ is completely continuous.

Proposition 4.4. The Bourgain algebra $X_B$ of $X$ equals $C(K)$.

Proof. Since $X_B$ is a closed subalgebra of $C(K)$, it suffices to show $S_B^{**}$ is completely continuous for the family of functions $\Phi$ studied in the proof of Proposition 4.2. We need only consider the functions $\Phi = (z, z, z, ...)$ and $\Phi = (z, \bar{z}, \bar{z}, ...)$.

Let $m = (d/2\pi)|f|$, that is, $m \in M(K)$ is normalized Lebesgue measure on the set $\Gamma_{\infty}$. Let $T^*: C(K) \to L^1(m)$ be the natural inclusion. Then $T^*$ is weakly compact. Since $C(K)^{**}$ has the Dunford–Pettis property, $T^{**}$ is completely continuous. Let $T'$ be the restriction of $T^*$ to $X$. Let $V'$ be the operator defined in the proof of Proposition 4.2. Then

$$\|Vf\| \leq \frac{1}{2\pi} \int |f| \, dm. \quad (4.1)$$
Since \( S_\phi = qV \), we see that \( S_\phi \) and \( T \) satisfy the hypothesis of Lemma 4.3. Hence, \( S_\phi^* \) is completely continuous. As in Proposition 4.2, the proof for \( \Phi = (z, z, z, ... \) ) is similar.

A comment is in order. The inequality (4.1) implies that the operator \( V \) is 1-summing (strictly 1-integral, in fact; see [24] for the definitions). Therefore \( S_\phi \) is 1-summing. This provides us with another way of deducing the above properties of \( S_\phi \). It is well known that the second adjoint of a 1-summing operator is 1-summing, and that 1-summing operators are weakly compact and completely continuous.

**Proof of Theorem 4.1 (d).** In Proposition 4.4 we showed that \( S_\phi^* \) is completely continuous for every \( \Phi \in C(K) \). It can now be deduced from the work in [4] that \( X \) and \( X^* \) have the Dunford–Pettis property. Since \( U \) is isomorphic to \( X \), the proof is finished.

The space \( X \) is tight but it is not strongly tight; that is, the operators \( S_\phi \) are not compact for every \( \Phi \in C(K) \). We will make this result precise in the proposition below. This is interesting because in every known example where \( A \) is tight uniform algebra on some compact space \( G \), \( A \) turns out to be strongly tight. It if not known if this is true in general.

**Proposition 4.5.** The operator \( S_\phi \) is compact if and only if \( \Phi|_{F_\phi} \) is constant. That is,

\[
X_{CG} = \{ \Phi \in C(K) \mid \Phi|_{F_\phi} \text{ is constant} \}.
\]

**Proof.** If \( Y \) is the right-hand side of the above, is not hard to see that \( Y \subseteq X_{CG} \). If \( \varphi_\alpha = \delta_{\alpha n}z^{\alpha n} \) as in the beginning of the proof of Proposition 4.2 then \( S_\phi \) is compact. Since this is true for a self-adjoint family of functions which separates the points of \( \{ F_\phi \}_{n=0}^\infty \) and \( X_{CG} \) is a closed subalgebra of \( C(K) \), \( X_{CG} \) contains all functions which are constant on \( F_\phi \).

Let \( \phi_\alpha = \delta_{\alpha n}z^{\alpha n} \) so that \( \{ \phi_\alpha \}_{n=0}^\infty \) is a sequence in the unit ball of \( X \). Let \( \Phi \in C(K) \) be arbitrary.

**Claim.** \( S_\phi(\phi_\alpha) \rightharpoonup 0 \). Since \( S_\phi \) is weakly compact it suffices to show that zero is the only weak accumulation point of \( \{ S_\phi(\phi_\alpha) \} \). So we let \( \Psi + X \) be such a weak accumulation point. If we write \( \Phi = (\varphi_{-1}, \varphi_0, \varphi_1, \varphi_2, ...) \) then \( \Phi\varphi_\alpha = (z^{\alpha n}\varphi_{-1}, 0, 0, 0, 0, 0, z^{\alpha n}\varphi_1, z^{\alpha n}\varphi_{n+1}, ...) \). Write \( \Psi = (\psi_{-1}, \psi_0, \psi_1, ...) \). If \( V_0 \) is any measure on \( \Gamma \) with \( \int dV_0 = 0 \) then \( V_0|_{F_\phi} \in X_+^* \) which implies \( \int \psi_{-1}dV_0 = 0 \) and hence \( \psi_{-1} = c_0 \) is constant. Similarly we can show \( \psi_1 = c_1, \psi_2 = c_2z \ldots \). Similarly, by considering the annihilating measure \( z(d\theta/2\pi)|_{F_\phi} \), we see that \( \psi_1 = c_1 + c_1z \) and \( \psi_2 = c_0 + c_1z + c_2z^2 \).

We repeat and find that \( \Psi \) is in \( X \), which proves the claim.
Now, suppose $\Phi \in X_{\mathcal{C}^N}$. Let $N$ be a positive integer. Let $x_n = (\pi + N/2\pi) d\theta_{x_n}$ so that $\|v_n\| = 1$ and $v_n \in X^k$. Then $\int \Phi \phi_n dv_n = \phi_d(N)$ which implies

$$|\phi_d(N)| \leq \|\Phi\phi_n + X\|.$$

Since $S_d(\phi_n)$ tends to zero weakly and $S_d$ is compact it follows that $\|S_d(\phi_n)\| \to 0$ and so $\phi_d(N) = 0$. If $N$ is a negative integer then we prove $\phi_d(N) = 0$ by using the annihilating measure $z^{d-N}(d\theta/2\pi)|_{\theta = 0} - z^{d-N}(d\theta/2\pi)|_{\theta = 1}$ for $n > 1$. Hence, $\phi_d$ is constant and we are done.

5. A NOTE ON INNER FUNCTIONS

We conclude with an application of tightness to inner functions on strictly pseudoconvex domains.

Let $D$ be a domain in $\mathbb{C}^n$ with smooth boundary and let $A = A(\partial D)$ be the uniform algebra of functions in $C(\partial D)$ which extend to be analytic in $D$. Let $m$ be the normalized surface-area measure on $\partial D$ and let $H^\infty(\partial D) = H^\infty(m)$ be the corresponding Hardy space. For short we will write $C = C(\partial D)$ and $H^\infty = H^\infty(\partial D)$. Recall that a function $f \in H^\infty(\partial D)$ is an inner function if $|f| = 1$ a.e. $[m]$.

**Theorem 5.1.** Let $D$ be a strictly pseudoconvex domain with $C^2$ boundary in $\mathbb{C}^n$. Suppose $f$ is an inner function in $H^\infty(\partial D)$. If $f(z_n) \to 0$ for some sequence $\{z_n\}$ tending to $\partial D$ then $f \not\in H^\infty + C$. In particular, if $n > 1$ then $f \in H^\infty + C$ if and only if $f$ is constant. If $D$ is the unit disk then $f \in H^\infty + C$ if and only if $f$ is a finite Blaschke product.

The proof is indirect and utilizes the Pełczyński property and the theory of tight uniform algebras. As discussed in Section 1, it was proven in [6] (also, see [22]) that if $D$ is a strictly pseudoconvex domain with $C^2$ boundary in $\mathbb{C}^n$ then $A(D)$ is a strongly tight uniform algebra. Actually all that is needed is the solvability of the $\partial$-problem with Hölder estimates on the solutions, and therefore this is all that is needed in Theorem 5.1. It is proven in [22] that whenever a uniform algebra is strongly tight on some compact space $K$, it is strongly tight as a uniform algebra on its Shilov boundary. Hence, $A(\partial D)$ is strongly tight on $\partial D$. Since strongly tight uniform algebras are tight, it follows from Theorem 2.1 that $A(\partial D)$ has the Pełczyński property (this result was also in [22]).

We will need to lift the properties of the operators $S_d$ on $A(\partial D)$ to the corresponding operators on the uniform algebra $H^\infty$. We accomplish this by the following result, which can be found in [6] (this can also be deduced from the results in [7]).
Theorem 5.2. The measure $m$ is in $B_{A^+}$ and the natural projection $H^\infty(B_{A^+}) \to H^\infty(m)$ is a surjective isometry.

Consider $H^\infty(m)$ as a uniform algebra on the maximal ideal space of $L^\infty(m)$. Given a function $g \in L^\infty(m)$ define

$$S_{g, H^\infty(m)} : H^\infty(m) \to \frac{L^\infty(m)}{H^\infty(m)}$$

by

$$f \mapsto fg + H^\infty(m).$$

We define $(H^\infty(m))_{CG}$ to be set of those $g$ such that $S_{g, H^\infty}$ is compact.

Lemma 5.3. $H^\infty(m) + C(\partial D) \subseteq (H^\infty(m))_{CG}$.

Proof. Let $g \in C(\partial D)$. Then $S_g : A \to C/A$ is compact. Now

$$S_g^* : A^+ \to \frac{B_{A^+}}{A^\perp}$$

satisfies $S_g^*(A^+) \subseteq B_{A^+}/A^\perp$ and is given by $v \mapsto g dv + A^\perp$. Let $p : A^* \to B_{A^+}/A^\perp$ be the natural map so that $p \circ S_g^*$ is compact. Let

$$T_g = (p \circ S_g^*)|_{L^1(m) \cap A^\perp}$$

so

$$T_g : L^1(m) \cap A^\perp \to \frac{B_{A^+}}{A^\perp}$$

is compact.

By Theorem 5.2 the natural projection $\tau$ is an isometry and therefore its predual

$$\frac{L^1(m)}{L^1(m) \cap A^\perp} \xrightarrow{\tau} \frac{B_{A^+}}{A^\perp}$$

is also an isometry. Let $U_g = \sigma^{-1} \circ T_g$ so $U_g$ is compact and

$$U_g : L^1(m) \cap A^\perp \to \frac{L^1(m)}{L^1(m) \cap A^\perp}$$

by

$$f dm \mapsto fg dm + L^1(m) \cap A^\perp.$$
The adjoint of $U_g$ is $S_{g,H^\infty}$ which is therefore compact. Hence, $g \in (H^\infty)_{CG}$ and we have shown that $C(\partial D) \subseteq (H^\infty)_{CG}$. Since $S_{g,H^\infty} = 0$ for $g \in H^\infty$ the proposition is proved.

**Proof of Theorem 5.1.** Assume $f$ is an inner function and $f(z_n) \to 0$ for some sequence $\{z_n\}$ in $D$ such that $z_n \to z$ where $z \in \partial D$. We will show $f \notin (H^\infty(m))_{CG}$; in other words, $S_{f,H^\infty}$ fails to be compact. It then follows from Lemma 5.3 that $f \notin H^\infty + C$. Our method will be to show that $S_{f,H^\infty}$ is an isomorphism on a copy of $c_0$ in $H^\infty$.

Let $\{\varphi_n\} \subset A^*$ be the evaluation functionals at $\{z_n\}$. Since it is well-known that every point on $\partial D$ is a peak point for $A$ it follows that $\{\varphi_n\}$ is not relatively weakly compact. Since $A(\partial D)$ has the Pelczyński property we may find, after passing to a subsequence of $\{z_n\}$ if necessary, a weakly unconditionally Cauchy series $\sum \varphi_n A$ in $A$ and an $\varepsilon > 0$ such that $|\varphi_n(z_n)| > \varepsilon$ for all $n$. We claim that $\|f\varphi_n + H^\infty\| \to 0$. Otherwise there exist bounded sequences $\{f_n\} \subset H^\infty$ and $\{k_n\} \subset L^\infty(m)$ with $\|k_n\| \to 0$ such that $f\varphi_n + f_n = k_n$. Therefore, since $f$ is inner, $\varphi_n + f_n$ tends to zero uniformly which contradicts the fact that $f(z_n) \to 0$.

It is now clear that $S_{f,H^\infty}$ fails to be compact. In fact, it follows from the remarks in Section 1 that $S_{f,H^\infty}$ is an isomorphism on a copy of $c_0$.

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**REFERENCES**


