Some subalgebras of indefinite type Kac–Moody Lie algebras

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Received 3 November 2006; received in revised form 11 August 2007; accepted 1 September 2007
Available online 31 October 2007

Communicated by R. Parimala

Abstract

We show that every Kac–Moody Lie algebra of indefinite type contains a subalgebra with a Dynkin diagram having two adjacent vertices whose edge labels multiply to a number greater than or equal to five. Consequently, every Kac–Moody algebra of indefinite type contains a subalgebra of strictly hyperbolic type, and a free Lie algebra of rank two.

MSC: 17B05; 17B65; 17B67

1. Introduction

Berman [1], while studying the fixed point algebras of certain involutions involving the Cartan involution, presented examples of sequences of containments of Kac–Moody algebras that occurred as a consequence of replacing some roots by others which arose from Weyl group reflections. This latter part of Berman’s work prompted us to ask the following question: given an arbitrary indecomposable indefinite type Lie algebra $L$, could we always find a finite sequence of Weyl group reflections such that two of the subsequent roots, $\alpha$ and $\beta$, would have the property that $\alpha$ and $\beta$ were connected in the resulting diagram with labels that multiplied to be $\geq 5$, i.e., the subalgebra generated by root spaces $L_\alpha$ and $L_\beta$ would be strictly hyperbolic?

Using the Weyl group and some well-known facts about Kac–Moody Lie algebras, we are able to answer this question in the affirmative. A consequence of showing that every indecomposable indefinite type Kac–Moody algebra contains a subalgebra of strictly hyperbolic type is that every such algebra has a free subalgebra of rank two. Finally, our work provides an alternative characterization of the classification of Kac–Moody Lie algebras, into the three major classes, based on the number of lines between two real roots of the algebra in its diagram.

In Section 2.1, we set up the diagrammatical notation which we will work with, and in Section 2.2 we introduce three short technical lemmas which allow us to reason through and work effectively within the large collection of indefinite type Kac–Moody algebras. In Section 3 we prove the above result, and we conclude the paper with some brief remarks in Section 4.
2. Preliminaries

2.1. Notations

All algebras under consideration in this paper are Kac–Moody Lie algebras. In drawing our Dynkin diagrams, we use the conventions given in [4], Sections 4.7 and 4.8. (See [3,5] for further background on Kac–Moody algebras.)

We will often say that the Dynkin diagram \( \mathcal{D} \) or \((\mathcal{D}, (a_{ij}))\) is of indefinite type if \( \mathcal{D} \) is a diagram corresponding to an algebra \( \mathcal{L} \) of indefinite type with Cartan matrix \((a_{ij})\). We also say that the diagram \( \mathcal{D} \) “contains” the diagram \( \mathcal{E} \), or that \( \mathcal{E} \) is a “subdiagram” of \( \mathcal{D} \), if the Lie algebra corresponding to \( \mathcal{D} \) contains a subalgebra whose Dynkin diagram is \( \mathcal{E} \). Note that the term “subdiagram” is not being used in the graph-theoretic sense of a subgraph.

Let \((\mathcal{D}, (a_{ij}))\) be a Dynkin diagram of indefinite type. If \(a_{ij}a_{ji} = k\), with \(1 \leq k\), we call the corresponding edge, connecting vertices \(\alpha_i\) and \(\alpha_j\), a \(k\)-edge.

Since we deal with indefinite type diagrams, \(\mathcal{D}\) may contain edges both within cycles and outside of cycles. If a \(k\)-edge occurs within a cycle in \(\mathcal{D}\), we call it a cyclic \(k\)-edge; otherwise we refer to it as a non-cyclic \(k\)-edge. When the analysis at hand is independent of the edge being in a cycle or not, we refer to such an edge simply as a \(k\)-edge. So, for example,

\[
\begin{array}{c}
\circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ
\end{array}
\]

has five non-cyclic 1-edges, one cyclic 1-edge, one non-cyclic 2-edge, one cyclic 2-edge, one non-cyclic 4-edge, and one cyclic 4-edge.

We will often be in the situation where one of the edges in the diagram is a \(k\)-edge where \(1 \leq k \leq 4\). Here, the analysis will rely on the fact that \(k\) is a number between 1 and 4, and not on the various ways that this \(k\)-edge can be drawn. So we draw such an edge as

\[
\begin{array}{c}
\circ \longrightarrow \circ \longrightarrow \circ
\end{array}
\]

Unless indicated otherwise, drawing an edge this way denotes that we are working with a \(k\)-edge, where \(k \in \{1, 2, 3, 4\}\). The edge labels, which are the corresponding entries \(a_{ij}, a_{ji}\) in the Cartan matrix, are not displayed unless we wish to see how the \(k\)-value of the edge has changed after a sequence of Weyl group reflections. For example, suppose we begin with the diagram

\[
\begin{array}{c}
\circ \longrightarrow \circ \longrightarrow \circ
\end{array}
\]

The corresponding Cartan matrix is of the form

\[
\begin{bmatrix}
2 & a_{12} & 0 \\
a_{21} & 2 & -2 \\
0 & -2 & 2
\end{bmatrix},
\]

where \(a_{12}, a_{21} \leq 0\) and \(1 \leq a_{12}a_{21} \leq 4\). After reflecting root \(\alpha_3\) in root \(\alpha_2\), the diagram for the subalgebra generated by \(\{\alpha_1, r_{\alpha_2}(\alpha_3), \alpha_3\}\) is

\[
\begin{array}{c}
\circ \longrightarrow \circ \longrightarrow \circ
\end{array}
\]

with Cartan matrix

\[
\begin{bmatrix}
2 & 2a_{12} & 0 \\
2a_{21} & 2 & -2 \\
0 & -2 & 2
\end{bmatrix}.
\]
We end this subsection by mentioning two conventions that are frequently employed in this paper. The first is our use of expressions of the form “the subalgebra generated by roots \( \{\alpha_1, \ldots, \alpha_n\}\)”:

By this we mean the subalgebra of the given algebra \( L \) generated by the corresponding root spaces \( L_{\alpha_1}, \ldots, L_{\alpha_n} \).

The second is our use of expressions of the form “roots \( \alpha \) and \( \beta \) are connected (by \( m \) lines)”: By this we mean that the vertices corresponding to \( \alpha \) and \( \beta \) in the associated Dynkin diagram are joined (by \( m \) lines).

### 2.2. Three helpful lemmas

Say \((\mathcal{D}, (a_{ij}))\) is a Dynkin diagram on the vertex set of roots \( \alpha_1, \ldots, \alpha_n \). Let us reflect root \( \alpha_j \) in root \( \alpha_i \) and consider the resulting Cartan matrix \((a'_{ij})\) of the subalgebra generated by \( \{\alpha_1, \ldots, \alpha_{i-1}, r_{\alpha_i}(\alpha_j), \alpha_{i+1}, \ldots, \alpha_j, \ldots, \alpha_n\} \). All rows and columns of \((a'_{ij})\) other than the \( i \)th row and the \( i \)th column are unchanged and identical to the corresponding rows and columns in \((a_{ij})\). Along the \( i \)th column, given any \( 1 \leq k \leq n, k \neq i \),

\[
a'_{ki} = \langle r_{\alpha_i}(\alpha_j), \alpha_k^{\vee} \rangle = \langle \alpha_j - \langle \alpha_j, \alpha_i^{\vee} \rangle \alpha_i, \alpha_k^{\vee} \rangle = a_{kj} - a_{ij}a_{ki},
\]

where \( \alpha_k^{\vee} \) denotes the \( k \)th coroot. Along the \( i \)th row of the matrix \((a'_{ij})\), for \( 1 \leq l \leq n, l \neq i \),

\[
a'_{li} = \langle \alpha_l, \alpha_j^{\vee} - \langle \alpha_l, \alpha_j^{\vee} \rangle \alpha_i \rangle = a_{lj} - a_{ji}a_{li}.
\]

Finally, the \( a'_{ij} \) entry of \((a'_{ij})\) is

\[
a'_{ii} = \langle \alpha_j - \langle \alpha_j, \alpha_i^{\vee} \rangle \alpha_i, \alpha_j \rangle - \langle \alpha_i, \alpha_j \rangle \alpha_i^{\vee} \rangle
= a_{jj} - 2a_{ij}a_{ji} + a_{ij}a_{ji}a_{ii}
= 2,
\]

since \( a_{jj} = a_{ii} = 2 \).

Now using \((a'_{ij})\) we can draw the Dynkin diagram \( \mathcal{D}' \) corresponding to the subalgebra generated by \( \{\alpha_1, \ldots, \alpha_{i-1}, r_{\alpha_i}(\alpha_j), \alpha_{i+1}, \ldots, \alpha_j, \ldots, \alpha_n\} \). Below, when we say, given a starting Dynkin diagram \((\mathcal{D}, (a_{ij}))\), reflect root \( \alpha_j \) in root \( \alpha_i \) and consider \((\mathcal{D}', (a'_{ij}))\), we mean consider the Cartan matrix \((a'_{ij})\) and the Dynkin diagram \( \mathcal{D}' \) of the subalgebra generated by \( \{\alpha_1, \ldots, \alpha_{i-1}, r_{\alpha_i}(\alpha_j), \alpha_{i+1}, \ldots, \alpha_j, \ldots, \alpha_n\} \).

The next lemma allows us to work more effectively with Dynkin diagrams having a large number of vertices and 1-edges. Roughly speaking, it says that we can “collapse” as many 1-edges as we like.

**Lemma 1** *(Collapsing Lemma)*. Let \( \mathcal{D} \) contain a subdiagram of the type

\[
\begin{array}{c}
\alpha_1, \alpha_2, \alpha_3, \ldots = \alpha_{l-1}, \alpha_{l}, \alpha_{l+1}, \alpha_{l+2}, \ldots
\end{array}
\]

where \( l \geq 2 \). Then \( \mathcal{D} \) contains a subdiagram of the type

\[
\begin{array}{c}
\alpha_1, \alpha_2, \alpha_3, \beta = \alpha_{l+1}, \alpha_{l+2}, \ldots
\end{array}
\]

**Proof.** If \( l = 2 \), then let \( \beta = \alpha_2 \). Otherwise, for \( l \geq 3 \), use induction on \( l \), letting

\[
\beta = r_{\alpha_2}r_{\alpha_3} \cdots r_{\alpha_{l-1}}(\alpha_l).
\]

The subalgebra generated by \( \{\alpha_1, \beta, \alpha_{l+1}\} \) yields the desired result. \( \square \)

**Lemma 2** provides us, in a very rough sense, a version of the comparison theorem for the convergence of series in elementary analysis in our setting. It helps us conclude that if a sequence of reflections leads to the desired result for diagram \( \mathcal{D} \), then if \( \mathcal{D} \) is a “subgraph” of \( \mathcal{E} \), the same sequence of reflections leads to the desired result for \( \mathcal{E} \). The
Suppose we reflect root \( \alpha_j \) in root \( \alpha_k \) for both diagrams \( \mathcal{D} \) and \( \mathcal{E} \) getting diagrams \( \mathcal{D}', (a_{ij}') \) and \( \mathcal{E}', (b_{ij}') \), respectively. Given any index \( p \in \{1, \ldots, n\} \), if

1. Suppose we reflect root \( \alpha_l \) in root \( \alpha_k \) for both diagrams \( \mathcal{D} \) and \( \mathcal{E} \) getting diagrams \( \mathcal{D}', (a_{ij}') \) and \( \mathcal{E}', (b_{ij}') \), respectively. Given any index \( p \in \{1, \ldots, n\} \), if

   (a) \( p \neq l \) and \( a_{pl} = a_{lp} = 0 \), or

   (b) \( p = l \),

   then

   \[ b_{pk}'b_{kp}' = a_{pk}'a_{kp}'. \]

2. If root \( \alpha_m \) is reflected in root \( \alpha_l \) where \( m \neq l \) and \( m \neq k \) for both diagrams \( \mathcal{D} \) and \( \mathcal{E} \), yielding diagrams \( \mathcal{D}', (a_{ij}') \) and \( \mathcal{E}', (b_{ij}') \), respectively, then given any index \( p \in \{1, \ldots, n\} \), if

   (a) \( p = k \) and \( a_{km} = a_{mk} = 0 \), or

   (b) \( p \neq k \),

   then

   \[ b_{pl}'b_{lp}' = a_{pl}'a_{lp}'. \]
Proof. 1. (a) Suppose $p \neq l$ and $a_{pl} = a_{lp} = 0$. Also suppose $p \neq k$. Since matrices $(b_{ij})$ and $(a_{ij})$ are equal everywhere except perhaps at $b_{kl}$ and $b_{lk}$, with $b_{kl} = a_{lk}$ and $b_{lk} = a_{lk}$,

$$b'_{pk}b'_{kp} = (b_{pl} - b_{kl}b_{pk}) (b_{lp} - b_{lk}b_{kp})$$

$$= (a_{pl} - a_{lk}a_{pk}) (a_{lp} - a_{lk}a_{kp})$$

$$= (0 - a_{lk}a_{pk}) (0 - a_{lk}a_{kp})$$

$$= a_{lk}a_{pk}a_{lk}a_{kp}$$

$$= (a_{pl} - a_{kl}a_{pk}) (a_{lp} - a_{kl}a_{kp})$$

$$= a'_{pk}a'_{kp}.$$  

If $p = k$, then

$$b'_{pk}b'_{kp} = b'_{kk}b'_{kk} = a'_{kk}a'_{kk} = a'_{pk}a'_{kp}.$$  

(b) Suppose $p = l$. Then

$$b'_{pk}b'_{kp} = (b_{pl} - b_{kl}b_{pk}) (b_{lp} - b_{lk}b_{kp})$$

$$= (a_{pl} - a_{lk}a_{pk}) (a_{lp} - a_{lk}a_{kp})$$

$$= a'_{pk}a'_{kp}.$$  

2. If $p = l$, then

$$b'_{pl}b'_{lp} = b'_{ll}b'_{ll} = a'_{ll}a'_{ll} = a'_{pl}a'_{lp}.$$  

On the other hand, suppose $p \neq l$, and we have

(a) $p = k$ with $a_{km} = a_{mk} = 0$. Then

$$b'_{pl}b'_{lp} = (b_{pm} - b_{lm}b_{pl}) (b_{mp} - b_{lp}b_{ml})$$

$$= (a_{km} - a_{lm}a_{lk}) (a_{mk} - a_{lk}a_{ml})$$

$$= (0 - a_{mk}a_{lk}) (0 - a_{lk}a_{ml})$$

$$= a_{lk}a_{mk}a_{lk}a_{ml}$$

$$= (0 - a_{mk}a_{lk}) (0 - a_{lk}a_{ml})$$

$$= (a_{km} - a_{lm}a_{lk}) (a_{mk} - a_{lk}a_{ml})$$

$$= (a_{pm} - a_{im}a_{pl}) (a_{mp} - a_{lp}a_{ml})$$

$$= a'_{pl}a'_{lp}.$$  

(b) $p \neq k$. Then

$$b'_{pl}b'_{lp} = (b_{pm} - b_{lm}b_{pl}) (b_{mp} - b_{lp}b_{ml})$$

$$= (a_{pm} - a_{im}a_{pl}) (a_{mp} - a_{lp}a_{ml})$$

$$= a'_{pl}a'_{lp}.$$  

3. Main theorem

Theorem 1. Every indecomposable Kac–Moody Lie algebra of indefinite type has a homogeneous Lie subalgebra whose Dynkin diagram contains two vertices joined by a 5-edge.
Proof. Let $\mathcal{D}$ be the Dynkin diagram of a given indecomposable Kac–Moody Lie algebra $L$ of indefinite type. If $\mathcal{D}$ contains two vertices joined by an edge whose labels multiply to be $\geq 5$, then we are done: $L$ itself is the desired subalgebra.

So suppose $\mathcal{D}$ has no $k$-edge with $k \geq 5$.

Case: $\mathcal{D}$ has a 4-edge.

If $\mathcal{D}$ has a non-cyclic 4-edge, then because it is of indefinite type, after applying the collapsing lemma if necessary, $\mathcal{D}$ must contain a subdiagram $\mathcal{E}$ of the form

![Diagram](image)

In the first case, i.e., for $A_1^{(1)} . x_1$, if we reflect $\alpha_3$ in root $\alpha_2$, and then $\alpha_3$ in the root $\beta := r_{\alpha_2}(\alpha_3)$, we get

![Diagram](image)

Since $a_{12} \cdot a_{21} \geq 1$, it follows that $4a_{12} \cdot 4a_{21} \geq 16 > 5$. For the second possibility, i.e., $A_2^{(2)} . x_1$, reflecting $\alpha_3$ in root $\alpha_2$ leads to a diagram of the $A_1^{(1)} . x_1$ type

![Diagram](image)

and reduces to the first case.

Observe that we could have set out

![Diagram](image)

as a separate case, but Lemma 3 tells us that reflecting root $\alpha_3$ in root $\alpha_2$ also yields a diagram of the $A_1^{(1)} . x_1$ type.

If $\mathcal{D}$ has a cyclic 4-edge, then either, after applying the collapsing lemma if necessary, $\mathcal{D}$ contains an induced subgraph $A_1^{(1)} . x_1$ or $A_2^{(2)} . x_1$ or a diagram of type

![Diagram](image)

In the latter case, we can use the result derived below for a cycle containing a single 2-edge together with the comparison lemma.

Before proceeding to look at diagrams with $k$-edges where $k \leq 3$, let us consider the case where $\mathcal{D}$, after applying the collapsing lemma if necessary, has the following subdiagram.

![Diagram](image)

\footnote{$A_1^{(1)} . x_1$ may be read as “extension one” of the diagram $A_1^{(1)}$. Also note that we are applying the label $A_1^{(1)} . x_1$ to a family of diagrams since there are various possibilities for the edge between $\alpha_1$ and $\alpha_2$.}
Reflecting $\alpha_4$ in $\alpha_2$, the Dynkin diagram of the subalgebra generated by \{\alpha_1, r_{\alpha_2}(\alpha_4), \alpha_3\}, is the following

So we are done by $A_1^{(1)} \cdot x1$.

**Case: $D$ has a 3-edge.**

If $D$ has a 3-edge and no $k$-edge where $k \geq 4$, then after applying the collapsing lemma and Lemma 3, if necessary, we list below all the possibilities, and the corresponding reflections that lead to a subalgebra having the desired property or to one which we have already shown to have the desired property.

In $D_4^{(3)} \cdot x1$, we could have begun with the more general figure

with $a_{24} \cdot a_{42} \geq 1$. If $a_{24} \cdot a_{42}$ is strictly bigger than 1, then by focusing on the subalgebra generated by $\{\alpha_2, \alpha_3, \alpha_4\}$, and by applying Lemma 3, we are in the same situation as for $G_2^{(1)} \cdot x1$. This is why we just considered $D_4^{(3)} \cdot x1$ with $a_{24} \cdot a_{42} = 1$. A similar consideration holds for the product $a_{34} \cdot a_{43}$ in the figure $G_2^{(1)} \cdot x2$.

For $G_2^{(1)} \cdot x3$, we could have started with the more general figure

where $a_{12} \cdot a_{21} \geq 1$. If $a_{12} \cdot a_{21}$ is strictly bigger than 1, after applying the collapsing lemma once, we could look at the subalgebra generated by $\{\alpha_1, r_{\alpha_2}(\alpha_3), \alpha_4\}$ having Dynkin diagram

We would now be in the same situation as with $G_2^{(1)} \cdot x1$. This is why it suffices to consider $G_2^{(1)} \cdot x3$ with $a_{12} \cdot a_{21} = 1$. 
When it comes to cyclic 3-edges, again as in the 4-edge case, looking at induced subgraphs, collapsing to an already examined case, or using the comparison lemma gives us the result.

Case: \( D \) has a 2-edge.

Let us next consider diagrams with one or more 2-edges and in which there are no \( k \)-edges with \( k \geq 3 \). Among these, let us first consider the case when \( D \) has two or more 2-edges. Applying the lemmas when necessary, we get the following possibilities.

Now suppose \( k \leq 2 \) but \( D \) has only one 2-edge. The possibilities are as follows.
Case: \( D \) has a 1-edge.

Finally we consider the case where \( D \) only has 1-edges.
4. Concluding remarks

Given an indecomposable indefinite type algebra $L$, we see from our above work that it has a homogeneous subalgebra whose Cartan matrix is of the form

$$
\begin{bmatrix}
2 & a \\
b & 2
\end{bmatrix},
$$

where $a$, $b$ are negative integers and $ab \geq 5$. Since

$$
\begin{bmatrix}
2 & a \\
b & 2
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{b}{a} \end{bmatrix} \begin{bmatrix} 2 & a \\ a & 2a/b \end{bmatrix}
$$

is symmetrizable, by [4], Ex. 9.11, p. 166, it follows that this subalgebra contains a free Lie algebra of rank 2.

Another observation is that the reflections given above are not unique. For example, for $C_2^{(2)} \cdot x 1$ we could have reflected root $\alpha_2$ in $\alpha_3$ instead. This would have led to a different diagram containing an $A_1^{(1)} \cdot x 1$ type diagram from which we could conclude the result.

Finally, our work provides another characterization of the classification of Kac–Moody Lie algebras. Since we start with simple roots and only use fundamental reflections or, subsequently, reflections in real roots, we are always working with real roots. This allows us to consider the subalgebras generated by the corresponding root spaces and to consider the associated diagrams. So we have that a Kac Moody Lie algebra is of

<table>
<thead>
<tr>
<th>Type</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>finite type</td>
<td>any two real roots have at most three lines joining them</td>
</tr>
<tr>
<td>affine type</td>
<td>$\exists$ two real roots with exactly four lines joining them, and no two real roots have $\geq 5$ lines joining them</td>
</tr>
<tr>
<td>indefinite type</td>
<td>$\exists$ two real roots with five or more lines connecting them</td>
</tr>
</tbody>
</table>

For the finite case, see [2]. If the algebra is of indefinite type we, constructively, showed the existence of two real roots joined by five or more lines. On the other hand, if an algebra has two real roots joined by five or more lines, then the subalgebra generated by these two roots is of indefinite type. Hence, the algebra containing these two real roots must also be of indefinite type. Finally, given an affine algebra, take any real root $\alpha$. Then $\alpha$ and $-\alpha + \delta$ have four lines between them, where $\delta$ is the base isotropic root. Conversely, if an algebra has two real roots connected by exactly four lines, then it cannot be finite. Furthermore, if it has no real roots joined by five or more lines, it cannot be indefinite. So it must be of affine type.
Acknowledgements

We would like to thank our supervisor Professor Yun Gao for bringing this problem to our attention, and for his valuable feedback and encouragement. We would also like to thank the referees for their helpful suggestions.

References