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# Some Recent Developments on Complex Multivariate Distributions

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In this paper, the author gives a review of the literature on complex multivariate distributions. Some new results on these distributions are also given. Finally, the author discusses the applications of the complex multivariate distributions in the area of the inference on multiple time series.

## 1. INTRODUCTION

In recent years, there has been considerable interest in the area of complex multivariate distributions, since these distributions play an important role in various areas. In nuclear physics, these distributions are useful (e.g., see Porter [49], Carmeli [6]) in studying such problems as the distributions of the spacings between energy levels of nuclei in high excitation. In the area of the multiple time series, these distributions are useful in studying such problems as the structures of the spectral density matrix, since certain suitably defined estimates of the spectral density matrix of the stationary Gaussian multiple time series are approximately distributed as a complex Wishart matrix. The problems of studying the structures of the above spectral density matrix arise in the analysis of the data in numerous areas like the vibrations of the airframe structures, meteorological forecasts, and signal detection. For some discussion about the usefulness of the complex multivariate distributions in the area of multiple time series, the reader is referred to Hannan [13], Liggett [40, 41], Priestley, Subba Rao, and Tong [50], and Brillinger [5].

Wooding [61] and Goodman [9] studied the complex multivariate normal distribution. The joint distributions of the roots of some complex random matrices were derived by James [17], Wigner [59], and Khatri [20] by following

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similar lines as in the analogous real cases. Some work has been done in the past on the distribution problems associated with certain test statistics based on the eigenvalues of the complex Wishart, multivariate beta, multivariate $F$, and other random matrices. Some of these distribution problems may be solved by following similar lines as in the analogous real cases, whereas some problems need techniques different from the real cases. The object of this paper is to review some of the developments on complex multivariate distributions and present some new results. This review is by no means exhaustive. Some of the results that are new on the distributions were presented by the author at the ISI meeting held in 1973.

In Section 2 of this paper, we discuss the evaluation of certain integrals that are useful in the computation of the probability integrals of some complex multivariate distributions. Section 3 is devoted to a review of the literature on the distributions of some complex random matrices as well as the joint densities of the eigenvalues of these random matrices. These random matrices include complex Wishart, multivariate beta, multivariate $F$, and Gaussian matrices. The marginal distributions of few roots are discussed in Section 4. In Section 5, the distributions of the traces of some complex random matrices are considered. The distributions of various ratios of the roots of complex Wishart and multivariate beta matrices are reviewed in Section 6, whereas the results on the distributions of the likelihood ratio statistics for testing the hypotheses on the covariance structures and mean vectors of complex multivariate normal populations are discussed in Section 7. Finally, we discuss the applications of the complex multivariate distributions in inference on multiple time series.

## 2. Evaluation of Some Integrals

In this section, we discuss the evaluation of some integrals that are needed in the sequel.

Let $\phi(x_1, \ldots, x_p) = |(y_{ij})|$ and $\psi(x_1, \ldots, x_p) = |(z_{ij})|$, where $y_{ij} = \phi_i(x_j)$ and $z_{ij} = \psi_i(x_j)$. Also, let $\eta(x_1, \ldots, x_p)$ be a symmetric function of $x_1, \ldots, x_p$. Then, it is seen that

$$\begin{aligned} &\int \cdots \int_{a \leqslant x_1 \leqslant \cdots \leqslant x_p \leqslant b} \eta(x_1, \ldots, x_p)\, \phi(x_1, \ldots, x_p)\, \psi(x_1, \ldots, x_p)\, dx_1 \cdots dx_p \\ &\qquad = \int_a^b \cdots \int_a^b \eta(x_1, \ldots, x_p)\, |(a_{ij})|\, dx_1 \cdots dx_p, \end{aligned} \tag{2.1}$$

where $a_{ij} = \phi_i(x_j)\, \psi_j(x_j)$. Starting from Eq. (2.1), we obtain the following easily:

LEMMA 2.1. *Let the symmetric function* $\eta(x_1, ..., x_p)$ *be of the form*

$$\eta(x_1, ..., x_p) = \sum_{\mathbf{m}} c(m_1, ..., m_p)\, x_1^{m_1} \cdots x_p^{m_p}, \tag{2.2}$$

*where* $c(m_1, ..., m_p)$ *is a constant depending upon* $m_1, ..., m_p$, *and the summation is over the values of* $m_1, ..., m_p$. *Then*

$$\begin{aligned} &\int_{a\leqslant x_1\leqslant\cdots\leqslant x_p\leqslant b} \cdots \int \eta(x_1, ..., x_p)\, \phi(x_1, ..., x_p)\, \psi(x_1, ..., x_p)\, dx_1 \cdots dx_p \\ &\quad = \sum_{\mathbf{m}} c(m_1, ..., m_p)\, | B(m_1, ..., m_p)|, \end{aligned} \tag{2.3}$$

*where* $B(m_1, ..., m_p) = (b_{ij})$, *and*

$$b_{ij} = \int_{a<x<b} x^{m_j} \phi_i(x)\, \psi_j(x)\, dx.$$

When $\eta(x_1, ..., x_p) = 1$, the above lemma was proved in Andreief [2].

Next, let $\eta(x_1, ..., x_p)$ be any symmetric function of $x_1, ..., x_p$. Then, it is seen that

$$\begin{aligned} &\int_{D_1} \cdots \int \eta(x_1, ..., x_p)\, \phi(x_1, ..., x_p)\, \psi(x_1, ..., x_p)\, dx_1 \cdots dx_p \\ &= \frac{1}{r!\, p-r!} \sum\nolimits_1 \sum\nolimits_2 (-1)^{\Sigma\delta_i + \Sigma\alpha_i} \int_{D_2} \cdots \int \eta(x_1, ..., x_p)\, | B_1 |\, | B_2 |\, dx_1 \cdots dx_p, \end{aligned} \tag{2.4}$$

where the domains $D_1$, $D_2$ of integration are given by $D_1 : a \leqslant x_1 \leqslant \cdots \leqslant x_r \leqslant x \leqslant x_{r+1} \leqslant \cdots \leqslant x_p \leqslant b$ and $D_2 : a \leqslant x_i \leqslant x$ $(i = 1, ..., r)$, $x \leqslant x_j \leqslant b$ $(j = r + 1, ..., p)$. In Eq. (2.4), $\delta_1 < \cdots < \delta_r$ is a subset of the integers $1, 2, ..., p$ and $\nu_1 < \cdots < \nu_{p-r}$ is the subset complementary to $\delta_1, ..., \delta_r$ and $\sum_1$ denotes the summation over all $\binom{p}{r}$ possible choices of $\delta_1 < \cdots < \delta_r$. Similarly, $\alpha_1 < \cdots < \alpha_r$ is a subset of the integers $1, 2, ..., p$ and $\beta_1 < \cdots < \beta_{p-r}$ is the subset complementary to $\alpha_1, ..., \alpha_r$, and $\sum_2$ denotes the summation over $\alpha_1 < \cdots < \alpha_r$. In addition, $B_1 = (b_{1gh})$ and $B_2 = (b_{2gh})$, where

$$b_{1gh} = \sum_i \phi_{\delta_g}(x_i)\, \psi_{\alpha_h}(x_i), \; b_{2gh} = \sum_j \phi_{\nu_g}(x_j)\, \psi_{\beta_h}(x_j).$$

But

$$\begin{aligned} &\int_{D_2} \cdots \int \eta(x_1, ..., x_p)\, | B_1 |\, | B_2 |\, dx_1 \cdots dx_p \\ &\quad = \int_{D_2} \cdots \int \eta(x_1, ..., x_p)\, | B_1^* |\, | B_2^* |\, dx_1 \cdots dx_p, \end{aligned}$$

where $B_1^* = (b_{1gh}^*)$, $B_2^* = (b_{2gh}^*)$, $b_{1gh}^* = \phi_{\delta_g}(x_h)\,\psi_{\alpha_h}(x_h)$, and $b_{2gh}^* = \phi_{\nu_g}(x_{r+h}) \times \psi_{\beta_h}(x_{r+h})$. Thus, we have the following:

LEMMA 2.2. *Let* $\eta(x_1, \dots, x_p) = \sum_m c(m_1, \dots, m_p)\, x_1^{m_1}, \dots, x_p^{m_p}$ *be a symmetric function of* $x_1, \dots, x_p$. *Then*

$$\begin{aligned} \int \cdots \int_{D_1} \eta(x_1, \dots, x_p)\, \phi(x_1, \dots, x_p)\, \psi(x_1, \dots, x_p)\, dx_1 \cdots dx_p \\ = \textstyle\sum_1 \sum_2 \sum_{\mathbf{m}} (-1)^{\Sigma\delta_i + \Sigma\alpha_i}\, c(m_1, \dots, m_p)\, |B_3|\, |B_4|, \end{aligned} \tag{2.5}$$

*where* $B_3 = (b_{3gh})$, $B_4 = (b_{4gh})$, *and*

$$b_{3gh} = \int_a^x y^{m_h} \phi_{\delta_g}(y)\, \psi_{\alpha_h}(y)\, dy, \quad b_{4gh} = \int_x^b y^{m_{r+h}} \phi_{\nu_g}(y)\, \psi_{\beta_h}(y)\, dy.$$

When $\eta(x_1, \dots, x_p) = 1$, an alternative expression was given in Khatri [23]. But, the expression given on the right side of Eq. (2.5) is better than that given in Khatri [23] from a computational point of view.

Next, let us expand $\phi(x_1, \dots, x_p)$ as follows:

$$\begin{aligned} \phi(x_1, \dots, x_p) = \textstyle\sum_3 \sum_4 (-1)^{d(r,s)}\, & V(x_1, \dots, x_r\,;\, a_1, \dots, a_r) \\ & \times V(x_{r+1}, \dots, x_{r+s}\,;\, \alpha_1, \dots, \alpha_s) \\ & \times V(x_{r+s+1}, \dots, x_p\,;\, \beta_{r+s+1}, \dots, \beta_p), \end{aligned} \tag{2.6}$$

where $d(r, s) = (r(r+1)/2) + (s(s+1)/2) + \sum a_i + \sum \alpha_i$ and

$$V(x_1, \dots, x_n\,;\, b_1, \dots, b_n) = \begin{vmatrix} \phi_{b_1}(x_1) & \cdots & \phi_{b_1}(x_n) \\ \vdots & & \\ \phi_{b_n}(x_1) & \cdots & \phi_{b_n}(x_n) \end{vmatrix}.$$

Here, $(a_1 < \cdots < a_r)$ is a subset of $(1, \dots, p)$, and $(t_1, \dots, t_{p-r})$ is its complementary set. Similarly, $(\alpha_1 < \cdots < \alpha_s)$ is a subset of $(t_1, \dots, t_{p-r})$ and $(\beta_{r+s+1}, \dots, \beta_p)$ is its complementary set. In addition, $\sum_3$ denotes the summation over all possible $\binom{p}{r}$ choices of $(a_1, \dots, a_r)$, and $\sum_4$ denotes the summation over all $\binom{p-r}{s}$ possible choices of $(\alpha_1 < \cdots < \alpha_s)$. Similarly, we can express $\psi(x_1, \dots, x_p)$ as follows:

$$\begin{aligned} \psi(x_1, \dots, x_p) = \textstyle\sum_5 \sum_6 (-1)^{d^*(r,s)}\, & V^*(x_1, \dots, x_r\,;\, a_1^*, \dots, a_r^*) \\ & \times V^*(x_{r+1}, \dots, x_{r+s}\,;\, \alpha_1^*, \dots, \alpha_s^*) \\ & \times V^*(x_{r+s+1}, \dots, x_p\,;\, \beta_{r+s+1}^*, \dots, \beta_p^*), \end{aligned} \tag{2.7}$$

where $d^*(r, s) = (r(r+1)/2) + (s(s+1)/2) + \sum a_i^* + \sum \alpha_i^*$, and

$$V^*(x_1, ..., x_n ; b_1, ..., b_n) = \begin{vmatrix} \psi_{b_1}(x_1) & \cdots & \psi_{b_1}(x_n) \\ \vdots & \cdots & \vdots \\ \psi_{b_n}(x_1) & \cdots & \psi_{b_n}(x_n) \end{vmatrix}.$$

Also, $(a_1^* < \cdots < a_r^*)$ is a subset of $(1, ..., p)$ and $(t_1^*, ..., t_{p-r}^*)$ is its complementary set. Similarly, $(\alpha_1^* < \cdots < \alpha_s^*)$ is a subset of $(t_1^*, ..., t_{p-r}^*)$ and $(\beta_{r+s+1}^*, ..., \beta_p^*)$ is its complementary set. In addition, $\sum_5$ denotes the summation over all possible $\binom{p}{r}$ choices of $(a_1^*, ..., a_r^*)$ and $\sum_6$ denotes the summation over all $\binom{p-r}{s}$ possible choices of $(\alpha_1^* < \cdots < \alpha_s^*)$. Using Eqs. (2.6) and (2.7) and Lemma 2.1, we obtain the following:

LEMMA 2.3. *Let* $D_3 : a \leqslant x_1 \leqslant \cdots \leqslant x_r \leqslant x \leqslant x_{r+1} \leqslant \cdots \leqslant x_{r+s} \leqslant y \leqslant x_{r+s+1} \leqslant \cdots \leqslant x_p \leqslant b$. *Then*

$$\begin{aligned} &\int \cdots \int_{D_3} \phi(x_1, ..., x_p)\, \psi(x_1, ..., x_p)\, dx_1 \cdots dx_p \\ &\qquad = \textstyle\sum_3 \sum_4 \sum_5 \sum_6 (-1)^{\Sigma a_i + \Sigma a_i^* + \Sigma \alpha_i + \Sigma \alpha_i^*} \,|B_5|\,|B_6|\,|B_7|, \end{aligned} \tag{2.8}$$

*where* $B_5 = (b_{5gh})$, $B_6 = (b_{6gh})$, $B_7 = (b_{7gh})$, *and*

$$b_{5gh} = \int_a^x \phi_{a_g}(z)\, \psi_{a_h^*}(z)\, dz, \qquad g, h = 1, 2, ..., r,$$

$$b_{6gh} = \int_x^y \phi_{\alpha_g}(z)\, \psi_{\alpha_h^*}(z)\, dz, \qquad g, h = r+1, ..., r+s,$$

$$b_{7gh} = \int_y^b \phi_{\beta_g}(z)\, \psi_{\beta_h}(z)\, dz, \qquad g, h = r+s+1, ..., p.$$

LEMMA 2.4. *Let* $D_4 : a \leqslant x_1 \leqslant \cdots \leqslant x_r \leqslant x_{r+1} \leqslant x_{r+s} \leqslant x_{r+s+1} \leqslant \cdots \leqslant x_p \leqslant b$. *Then*

$$\begin{aligned} &\int \cdots \int_{D_4} \phi(x_1, ..., x_p)\, \psi(x_1, ..., x_p)\, dx_1 \cdots dx_r\, dx_{r+s+1} \cdots dx_p \\ &\quad = \textstyle\sum_3 \sum_4 \sum_5 \sum_6 (-1)^{\Sigma a_i + \Sigma \alpha_i + \Sigma a_i^* + \Sigma \alpha_i^*}\, V(x_{r+1}, ..., x_{r+s} ; \alpha_1, ..., \alpha_s) \\ &\qquad \times V^*(x_{r+1}, ..., x_{r+s} ; \alpha_1^*, ..., \alpha_s^*) \times |B_8|\,|B_9|, \end{aligned} \tag{2.9}$$

*where* $B_8 = (b_{8gh})$, $B_9 = (b_{9gh})$, *and*

$$b_{8gh} = \int_a^{x_{r+1}} \phi_{a_g}(z)\,\psi_{a_h^*}(z)\,dz, \qquad g, h = 1, 2,..., r$$

$$b_{9gh} = \int_{x_{r+s}}^{b} \phi_{\beta_g}(z)\,\psi_{\beta_h}(z)\,dz, \qquad g, h = r + s + 1,..., p.$$

## 3. Distributions of Some Complex Random Matrices and their Eigenvalues

We need the following definitions (see James [17]) in the sequel.

Let $\kappa = (k_1, ..., k_p)$ be a partition of $k$ such that $k_p \geqslant \cdots \geqslant k_1 \geqslant 0$ and $k_1 + \cdots + k_p = k$. Also, let $\tilde{C}_\kappa(A)$ denote the zonal polynomial of a Hermitian matrix $A$ of order $p \times p$. The hypergeometric functions with matrix arguments are as given below:

$$_r\tilde{F}_q(c_1, ..., c_r; d_1, ..., d_q; A) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(c_1)_\kappa \cdots (c_r)_\kappa \tilde{C}_\kappa(A)}{(d_1)_\kappa \cdots (d_q)_\kappa k!}, \tag{3.1}$$

$$_r\tilde{F}_q(c_1, ..., c_r; d_1, ..., d_q; G, H) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(c_1)_\kappa \cdots (c_r)_\kappa \tilde{C}_\kappa(G)\, \tilde{C}_\kappa(H)}{(d_1)_\kappa \cdots (d_q)_\kappa \tilde{C}_\kappa(I_p)\, k!}, \tag{3.2}$$

where $c_1, ..., c_r, d_1, ..., d_q$ are real or complex constants. Here, we note that $_0\tilde{F}_0(A) = \operatorname{etr} A$ and $_1\tilde{F}_0(a; A) = |I - A|^{-a}$. When $A = 0$, the right side of (3.1) is equal to 1. Also, the right side of (3.2) is equal to 1 when $G = 0$ or $H = 0$. Throughout this paper, etr $B$ denotes the exponential of the trace of $B$, and

$$(a)_\kappa = \prod_{i=1}^{p} (a - i + 1)_{\kappa_i}, \qquad (a)_\kappa = a(a + 1) \cdots (a + \kappa - 1)$$

$$\tilde{\Gamma}_p(a) = \pi^{p(p-1)/2} \prod_{i=1}^{p} (a - i + 1)_{\kappa_i}.$$

The zonal polynomials of the matrix variables were first considered by Hua [15] and later by James [16] independently. The general system of hypergeometric functions with matrix arguments is due to Herz [14] who defined them in integral forms. Since the computation of the expressions involving zonal polynomials is complicated, it would be of interest to try to get approximations to the above hypergeometric functions by starting with the expressions of Herz [14].

We will now review the results on the joint distributions of the roots of some random matrices.

Let $Z = X + iY$, where $X$ and $Y$ are random matrices of order $m \times p$. Also, let the rows of $(X, Y)$ be distributed independently as $2p$-variate normal with mean vector $(\mu_1', \mu_2')$ and covariance matrix $E$, where

$$E = \begin{pmatrix} E_1 & E_2 \\ -E_2 & E_1 \end{pmatrix}. \tag{3.3}$$

Then, the rows of $Z$ are known to be distributed as complex multivariate normal. Now, let $A_1 = Z\bar{Z}'$, where $\bar{Z}'$ is the transpose of the complex conjugate of $Z$. Then, the distribution of $A_1$ is known to be a central (noncentral) complex Wishart matrix with $m$ degrees of freedom when $M = 0$ $(M \neq 0)$, where $E(Z) = M$. Also, $E(A_1/m) = \Sigma_1 = 2(E_1 - iE_2)$. Next, let $A_2 : p \times p$ be a central complex Wishart matrix with $n$ degrees of freedom, and $E(A_2/n) = \Sigma_2$. Then, $F = A_1A_2^{-1}$ is known to be a central (noncentral) complex multivariate $F$ matrix when $M = 0$ $(M \neq 0)$. Similarly, $B = A_1(A_1 + A_2)^{-1}$ is said to be a central or noncentral complex multivariate beta matrix accordingly as $M = 0$ or $M \neq 0$.

Next, let $E_2 = 0$, $E_1 = \text{diag.}(\lambda_1, \ldots, \lambda_p)$, $B_0 = ((A - 2mE_1)(2m)^{1/2}) = B_1 + iB_2$ where $B_1 = (b_{1ij})$ and $B_2 = (b_{2ij})$. When $m \to \infty$, the random variables $b_{1ij}$ and $b_{2ij}$ by central limit theorem, are distributed independently and normally with zero means and variances given by $E(b_{1ij}^2) = \lambda_i\lambda_j$ $(i \neq j)$, $E(b_{1ii}^2) = 2\lambda_i^2$, and $E(b_{2ij}^2) = \lambda_i\lambda_j$ ; here, we note that $b_{2ii} = 0$.

Now, let $A = (a_{ij}) = R + iS$, where $A : p \times p$ is a Hermitian random matrix, $R = (r_{ij})$ and $S = (s_{ij})$. Then $s_{ii} = 0$. Now, let the elements of $R$ and the off-diagonal elements of $S$ be distributed independently. We assume that the variances of the off-diagonal elements of $R$ and $S$ are equal to 1 and the variances of the diagonal elements $R$ are equal to 2. Then, we refer to $A = R + iS$ as the central or noncentral complex Gaussian matrix accordingly as $E(A) = 0$ or $E(A) \neq 0$.

The complex multivariate normal distribution was derived by Wooding [61]. The density function of the complex multivariate normal is given by

$$f(Z) = \frac{1}{\pi^{pm} \mid \Sigma \mid^m} \operatorname{etr}[-\Sigma^{-1}(Z - M)(\bar{Z} - \bar{M})']. \tag{3.4}$$

The distribution of the complex Wishart matrix is known (see Goodman [9]) to be

$$f(A_1) = \frac{1}{\tilde{\Gamma}_p(m) \mid \Sigma \mid^m} \operatorname{etr}(-(1/2)\, \Sigma_1^{-1} A_1) \mid A_1 \mid^{m-p}, \tag{3.5}$$

where $M = 0$. When $M \neq 0$, the distribution of $A_1$ is known to be

$$f(A_1) = \operatorname{etr}(-\Sigma^{-1}M\bar{M}')\, {}_0\tilde{F}_1(m; \Sigma^{-1}M\bar{M}'\Sigma^{-1}A_1)$$
$$\times \frac{1}{\tilde{\Gamma}_p(m) \mid \Sigma \mid^m} \operatorname{etr}(-\Sigma^{-1}A_1) \mid A_1 \mid^{m-p}. \tag{3.6}$$

Srivastava [54] gave a simplified derivation of the central complex Wishart distribution.

Let $w_p \geqslant \cdots \geqslant w_1$ be the latent roots of the complex Wishart matrix $A_1\Sigma^{-1}$ with $m$ degrees of freedom. The joint density of the roots $w_1, \ldots, w_p$ is given by

$$h_1(w_1, \ldots, w_p) = \mathrm{etr}(-\Omega)\, {}_0\tilde{F}_1(m; \Omega; A_1) \times \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(m)\, \tilde{\Gamma}_p(p)} \mathrm{etr}(-A_1) \mid A_1 \mid^{m-p} \prod_{i>j}^{p} (w_i - w_j)^2, \tag{3.7}$$

where $\Omega$ is a diagonal matrix whose elements are the roots of $M\overline{M}'\Sigma^{-1}$. When $M = 0$, the joint density of the roots $(a_1 \leqslant \cdots \leqslant a_p)$ of $A_1$ is

$$h_1(a_1, \ldots, a_p) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(m)\, \tilde{\Gamma}_p(p)} \mid \Sigma \mid^{-m} \mathrm{etr}(-\Sigma^{-1}A_1) \times \mid A_1 \mid^{m-p} \prod_{i>j}^{p} (a_i - a_j)^2. \tag{3.8}$$

Next, let us assume that the rank of $A_1$ is $q$ and let the nonzero roots of $A_1A_2^{-1}$ be $f_q \geqslant \cdots \geqslant f_1$. When $m \leqslant p$, and $\Sigma = \Sigma_2$ the joint density of $f_1, \ldots, f_q$ is given by

$$h_2(f_1, \ldots, f_q) = \mathrm{etr}(-\Omega)\, {}_1\tilde{F}_1(m+n; p; \Omega, (I+F^{-1})^{-1}) \times \frac{\pi^{m(m-1)}\, \tilde{\Gamma}_m(m+n)}{\tilde{\Gamma}_m(p)\, \tilde{\Gamma}_m(m+n-p)\, \tilde{\Gamma}_m(m)} \frac{\mid F \mid^{p-m}}{\mid I+F \mid^{m+n}} \prod_{i>h} (f_i - f_j)^2, \tag{3.9}$$

where $F = A_1A_2^{-1}$ and $\Omega = \overline{M}'\Sigma^{-1}M$. When $m \geqslant p$, the joint density of $f_1, \ldots, f_q$ is given by Eq. (109) in James [17]. Also formulas (3.6), (3.7), and (3.9) were given in James [17].

Now, let $a_p \geqslant \cdots \geqslant a_1$ be the latent roots of the Gaussian matrix $A$. Then, the joint density of $a_1, \ldots, a_p$ is known (see Waikar, Chang, and Krishnaiah [58] to be

$$h_3(a_1, \ldots, a_p) = C\, \mathrm{etr}(-(1/2)M^2) \times \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa((1/2)M)}{k!\, \tilde{C}_\kappa(I_p)} \tilde{C}_\kappa(A) \prod_{i=1}^{p} \exp(-(1/4)a_i^2) \times \prod_{i>j=1}^{p} (a_i - a_j)^2, \qquad -\infty \leqslant A_i \leqslant \infty. \tag{3.10}$$

Also, $C = \pi^{p(p-1)}/\tilde{\Gamma}_p(p)\, 2^{(p^2+p)/2}\, \pi^{p^2/2}$. When $M = 0$, the joint density of $a_1, \ldots, a_p$ is known (see Wigner [59]) to be

$$h_3(a_1, \ldots, a_p) = C \prod_{i=1}^{p} \exp(-(1/4)a_i^2) \prod_{i>j=1}^{p} (a_i - a_j)^2. \tag{3.11}$$

Next, let the $m$ rows of $(Z_1 \vdots Z_2)$ be distributed independently as $(p+q)$ variate complex normal with zero means and covariance matrix

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

and let $p \leqslant q$. Also, let $r_p^2 \geqslant \cdots \geqslant r_1^2$ be the latent roots of $R^2 = (Z_1\bar{Z}_1')^{-1}(Z_1\bar{Z}_2')(Z_2\bar{Z}_2')^{-1} Z_2\bar{Z}_1'$. Then, the joint density of $r_1^2,\dots, r_p^2$ is known to be

$$\begin{aligned} h_4(r_1^2,\dots, r_p^2) &= | I - P^2 |^m \, {}_2\tilde{F}_1(m, m; q; P^2, R^2) \\ &\times \frac{\tilde{\Gamma}_p(m)\, \pi^{p(p-1)}}{\tilde{\Gamma}_p(m-q)\, \tilde{\Gamma}_p(q)\, \tilde{\Gamma}_p(p)} | R^2 |^{q-p} \, | I - R^2 |^{m-q-p} \prod_{i>j} (r_i^2 - r_j^2)^2, \end{aligned} \tag{3.12}$$

where $P^2 = \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. The above formula was given in James [17].

When $M = 0$, we will refer to the distribution of $A_1$ as the central complex Wishart matrix or the central complex Wishart matrix with $\Sigma_1 \neq I$ accordingly as $\Sigma = I$ or $\Sigma \neq I$. Similarly, when $M = 0$, we will refer to the distribution of $A_1(A_1 + A_2)^{-1}$ as the central multivariate beta matrix (central multivariate beta matrix with $\Sigma \neq \Sigma_2$) when $\Sigma = \Sigma_2 (\Sigma \neq \Sigma_2)$. Goodman [9] expressed the densities of multiple coherence and partial coherence as infinite series while Kabe [18] expressed them as finite series.

Now, let $Z'$ be partitioned as $Z' = (Z_1',\dots, Z_q')$, where $Z_j$ is of order $m \times p_j$ and the distribution of $Z$ is given by (3.4). In addition, let $S_j = Z_j\bar{Z}_j'A_j$, where the elements of $A_j$'s are constants. Then, the joint characteristic function of $S_1,\dots, S_q$ is given by

$$\begin{aligned} \phi(\theta_1,\dots, \theta_q) &= E\{\operatorname{etr} i(\theta_1 S_1 + \cdots + \theta_q S_q)\} \\ &= | I - A^*\Sigma |^{-m} \operatorname{etr}\{A^*(I - \Sigma A^*)^{-1} M\bar{M}'\}, \end{aligned} \tag{3.13}$$

where $A^* = \operatorname{diag.}(A_1^*,\dots, A_q^*)$, and $A_j^* = i\theta_j A_j$. If $p_1 = \cdots = p_q$, the characteristic function of $S_1 + \cdots + S_q$ is obtained from the above equation by choosing $\theta_1,\dots, \theta_q$ to be equal. The joint characteristic function of $y_1,\dots, y_q$, where $y_j = \operatorname{tr} S_j$ $(j = 1,\dots, q)$, is given by

$$| I - B^*\Sigma |^{-m} \operatorname{etr}\{B^*(I - \Sigma B^*)^{-1} M\bar{M}'\}, \tag{3.14}$$

where $B^* = \operatorname{diag.}(B_1^*,\dots, B_q^*)$, and $B_j^* = \mathrm{i}t_j A_j$. When $m = 1$ and $q = 1$, Turin [55] derived Eq. (3.14).

## 4. Marginal Distributions of Few Roots

Let $l_1, \ldots, l_p$ be the latent roots of a class of random matrices, and let the joint density of these roots be of the form

$$\begin{aligned} h(l_1, \ldots, l_p) = C \prod_{i=1}^{p} \psi(l_i) \sum_{k=0}^{\infty} \sum_{\kappa} c(\kappa, \Omega) \, |(l_j^{i-1})| \\ \times |(l_j^{i-1+k_{p-i+1}})|, \qquad a \leqslant l_1 \leqslant \cdots \leqslant l_p \leqslant b, \end{aligned} \tag{4.1}$$

where $C$ is a constant, $\kappa = (k_1, \ldots, k_p)$ is a partition of $k$ such that $k_p \geqslant \cdots \geqslant k_1$, $\psi(x)$ is a function of $x$, and $c(\kappa, \Omega)$ depends on $\kappa$ and the population parameter matrix $\Omega$. The joint densities of the eigenvalues of the random matrices considered in the preceding section are special cases of Eq. (4.1).

The probability integral of the joint distribution of the extreme roots $l_1$ and $l_p$ is given by

$$P[c \leqslant l_1 \leqslant l_p \leqslant d] = C \sum_{k=0}^{\infty} \sum_{\kappa} c(\kappa, \Omega) \, | A |, \tag{4.2}$$

where $A = (a_{ij})$ and

$$a_{ij} = \int_{c \leqslant x \leqslant d} \psi(y) y^{i+j+k_{p-i+1}-2} \, dy.$$

Equation (4.2) follows immediately by applying Lemma 2.1. The c.d.f. of the largest root $l_p$ is obtained by putting $c = a$ in Eq. (4.2). The c.d.f. of the smallest root $l_1$ is given by

$$P[l_1 \leqslant c] = 1 - P[c \leqslant l_1 \leqslant l_p \leqslant b], \tag{4.3}$$

where the right side of Eq. (4.3) can be evaluated by applying Eq. (4.2). The c.d.f. of the intermediate root $l_s$ $(2 \leqslant s \leqslant p-1)$ is given by

$$\begin{aligned} P[l_s \leqslant c] = P[l_{s+1} \leqslant c] + P[a \leqslant l_1 \leqslant \cdots \leqslant l_s \leqslant c \leqslant l_{s+1} \leqslant \cdots \leqslant l_p \leqslant b], \\ a \leqslant l_1 \leqslant \cdots \leqslant l_p \leqslant b. \end{aligned} \tag{4.4}$$

But, applying Lemma 2.2, we obtain

$$\begin{aligned} P[l_1 \leqslant \cdots \leqslant l_s \leqslant x \leqslant l_{s+1} \leqslant \cdots \leqslant l_p] \\ = C \sum_{k=0}^{\infty} \sum_{\kappa} \sum\nolimits_1 \sum\nolimits_2 (-1)^{\Sigma\alpha_i + \Sigma\delta_i} \, | B_3 | \, | B_4 |, \end{aligned} \tag{4.5}$$

where $\sum_1, \sum_2$, $\alpha_i$'s and $\delta_i$'s were defined in Eq. (2.4), $B_3 = (b_{3gh})$ and $B_4 = (b_{4gh})$.

Here

$$b_{3gh} = \int_a^x \phi_{\delta_g}(y)\, \psi_{\alpha_h}(y)\, dy,\ b_{4gh} = \int_x^b \phi_{\nu_g}(y)\, \psi_{\beta_h}(y)\, dy,$$

and

$$\phi_i(y) = y^{i-1}, \qquad \psi_i(y) = \psi(y) y^{k_{p-i+1}+i-1}.$$

The joint density of $l_{r+1}, \dots, l_{r+s}$ follows immediately by applying Lemma 2.4.

Similarly, one can derive the joint density of any few ordered roots that are not necessarily consecutive, but the resulting expressions are complicated. We will now discuss the joint probability integral associated with any pair of roots $l_r, l_s$ $(1 \leqslant r < s \leqslant p)$.

We know that

$$\begin{aligned} P[x_1 \leqslant l_r \leqslant l_s \leqslant x_2] = {} & P[l_{r-1} \leqslant x_1 \leqslant l_r \leqslant l_s \leqslant x_2 \leqslant l_{s+1}] \\ & + P[x_1 \leqslant l_{r-1} \leqslant l_r \leqslant l_s \leqslant x_2 \leqslant l_{s+1}] \\ & + P[l_{r-1} \leqslant x_1 \leqslant l_r \leqslant l_s \leqslant l_{s+1} \leqslant x_2] \\ & + P[x_1 \leqslant l_{r-1} \leqslant l_r \leqslant l_s \leqslant l_{s+1} \leqslant x_2]. \end{aligned} \tag{4.6}$$

Each quantity on the right side of Eq. (4.6) can be evaluated using Lemma 2.3.

Khatri [19] gave expressions for the extreme roots of the central complex Wishart and multivariate beta matrices whereas Al-Ani [1] derived the expressions for the intermediate roots of the above random matrices. Also, Khatri [22] derived the distributions of the individual roots of the central Wishart matrix $A_1$ with $\Sigma \neq I$, noncentral multivariate beta matrix, central multivariate beta matrix with $\Sigma \neq \Sigma_2$, and noncentral canonical correlation matrix in the complex cases. The method used in the above papers for obtaining the distributions is different from the method used in the present paper. The expressions given in this paper for the distributions of the intermediate roots are better, from computational point of view, than the corresponding expressions given in Khatri [19, 22] and Al-Ani [1]. Approximate percentage points of the largest root of the Wishart matrix and multivariate beta matrix were given by Pillai and Young [45] and Pillai and Jouris [48], respectively, in the central complex cases. Exact percentage points of the smallest root and intermediate roots of the central complex Wishart matrix were constructed by Schuurmann and Waikar [53] and Krishnaiah and Schuurmann [29]. Krishnaiah and Schuurmann [29] also constructed the exact percentage points of the distributions of the individual roots of the central complex multivariate beta matrix. Exact percentage points of the joint distribution of the extreme roots of the central complex Wishart matrix as well as that of the central complex multivariate beta matrix were given by Krishnaiah and Schuurmann [31].

The joint densities of any few unordered roots of the complex Wishart, complex multivariate beta matrix, and complex Gaussian matrix were known (see Wigner [59] and Mehta [42]) in the literature for the central cases. Waikar, Chang, and Krishnaiah [58] derived the joint densities of any few unordered roots of the complex Gaussian ensemble matrix, complex Wishart matrix $A_1$ with $\Sigma \neq I$, complex multivariate beta matrix, and complex canonical correlation matrix in the noncentral cases. We now discuss the joint distribution of any few unordered roots of a class of random matrices.

Let $l_1, \ldots, l_p$ be the unordered roots of a class of random matrices and let their joint density be given by

$$\begin{aligned} f_1(l_1, \ldots, l_p) &= (C/p!) \prod_{i=1}^{p} \psi(l_i) \sum_{k=0}^{\infty} \sum_{\kappa} c(\kappa, \Omega) \\ &\quad \times |(l_j^{i-1})| \, |(l_j^{i-1+k_{p-i+1}})|, \qquad a \leqslant l_i \leqslant b, \; i = 1, \ldots, p. \end{aligned} \tag{4.7}$$

Then, the joint density of $l_1, \ldots, l_r$ is given by

$$f_2(l_1, \ldots, l_r) = \int_a^b \cdots \int_a^b f(l_1, \ldots, l_p) \, dl_{r+1} \cdots dl_p \,. \tag{4.8}$$

The right side of Eq. (4.8) can be evaluated using the following lemma:

LEMMA 4.1. *Let $\phi(x_1, \ldots, x_p)$ and $\psi(x_1, \ldots, x_p)$ be as defined in Section 2. Then*

$$\int_a^b \cdots \int_a^b \phi(x_1, \ldots, x_p) \, \psi(x_1, \ldots, x_p) \, dx_{r+1} \cdots dx_p = \Sigma_1 \Sigma_2 (-1)^{\Sigma \delta_i + \Sigma \alpha_i} \, | B_1 | \, | B_5 |,$$

*where $B_1$, $\delta_i$'s, $\alpha_i$'s and the summations $\sum_1$ and $\sum_2$ are as defined in Eq. (2.4), and $B_5 = (b_{5gh})$, where*

$$b_{5gh} = (p - r)! \int_a^b \phi_{\nu_g}(y) \, \psi_{\beta_h}(y) \, dy.$$

## 5. Moments of the Elementary Symmetric Functions and the Distributions of the Traces

Let the joint density of the roots $l_1 < \cdots < l_p$ of a random matrix be given by Eq. (4.1). Also, let

$$\zeta_q(l_1, \ldots, l_p) = \sum l_{i_1} \cdots l_{i_q} \,, \qquad q \leqslant p,$$

where the summation is over all possible values of $i_1 < \cdots < i_q$. Then,

$$\{\zeta_q(l_1,..., l_p)\}^s = \sum_{\mathbf{s}} \frac{s!}{s_1! \cdots s_{p^*}!} l_1^{\eta_1} \cdots l_p^{\eta_p},$$

where $p^* = \binom{p}{q}$, $s = \sum_{i=1}^{p^*} s_i$, and $\eta_1,..., \eta_p$ depend upon $s_1,..., s_{p^*}$; the summation is over all possible values of $s_1,..., s_{p^*}$. Then, the $s$th moment of $\zeta_q(l_1,..., l_p)$ is given by

$$E\{\zeta_q(l_1,..., l_p)\}^s = \int_{a\leqslant l_1\leqslant\cdots\leqslant l_p\leqslant b} \cdots \int \{\zeta_q(l_1,..., l_p)\}^s f(l_1,..., l_p)\, dl_1 \cdots dl_p,$$

where $f(l_1,..., l_p)$ is given by Eq. (4.1). Now, applying Lemma 2.1, we get the following:

LEMMA 5.1. *The* $s$th *moment of* $q$th *order elementary symmetric function* $\zeta_q(l_1,..., l_p)$ *is given by*

$$E\{\zeta_q(l_1,..., l_p)\}^s = C \sum_k \sum_\kappa \sum_{\mathbf{s}} c(\kappa, \Omega) \frac{s!}{s_1! \cdots s_{p^*}!} \mid B \mid, \tag{5.1}$$

*where* $B = (b_{ij})$ *and*

$$b_{ij} = \int_a^b \psi(x) x^{i+j+\eta_j+k_{p-i+1}-2}\, dx. \tag{5.2}$$

When $a = 0$, $b = 1$, and $\psi(x) = x^n(1-x)^q$, $b_{ij}$ in Eq. (5.2) reduces to $b_{ij} = \beta(n+i+j+\eta_j+k_{p-i+1}-1, q+1)$. For $a = 0$, $b = \infty$, and $\psi(x) = \exp(-x)x^n$, we obtain $b_{ij} = \beta(n+i+j+\eta_j+k_{p-i+1}-1, q+1)$. If $a = -\infty$ $b = \infty$, and $\psi(x) = \exp(-x^2/4)$, then

$$\begin{aligned} b_{ij} &= 0, \qquad \text{if} \quad i+j+\eta_j+k_{p-i+1}-2 \text{ is odd}, \\ &= 4^{(i+j+\eta_j+k_{p-i+1}-1)/2}\, \Gamma((i+j+\eta_j+k_{p-i+1}-2)/2), \\ &\qquad \text{if} \quad i+j+\eta_j+k_{p-i+1}-2 \text{ is even}. \end{aligned}$$

The Laplace transformation of the statistic $T = \sum_{i=1}^p l_i$ is given by

$$\begin{aligned} \mathscr{L}(t; T) &= \int_{a\leqslant l_1\leqslant\cdots\leqslant l_p\leqslant b} \cdots \int \exp(-tT) f(l_1,..., l_p)\, dl_1 \cdots dl_p \\ &= C \sum_{k=0}^{\infty} \sum_\kappa c(\kappa, \Omega) \mid B^* \mid, \end{aligned} \tag{5.3}$$

where $B^* = (b_{ij}^*)$ and

$$b_{ij}^* = \int_a^b \exp(-tx) x^{i+j+k_{p-i+1}-2}\, \psi(x)\, dx. \tag{5.4}$$

We now derive the distribution of $T$ when $a = 0, b = 1$, and $\psi(x) = x^n(1-x)^q$. In this special case, we get

$$b_{ij}^* = \sum_{l=0}^{\infty} (-1)^l \binom{q}{l} \frac{(l+i+j+n-1+k_{p-i+1})!}{t^{l+i+j+n-1+k_{p-i+1}}} \left[1 - \exp(-t) \sum \frac{t^m}{m!}\right], \tag{5.5}$$

where the summation in the square bracket is over values of $m$ from 0 to $l + i + j + n - 1 + k_{p-i+1}$ ; here, we note that $\binom{q}{l} = 0$ when $q$ is integer and is less than $l$. The Laplace transformation is of the form

$$\mathscr{L}(t;\, T) = C \sum_{k=0}^{\infty} \sum_{\kappa} c(\kappa, \Omega) \left\{\sum_i d_i \frac{\exp(-t\alpha_i)}{t^{\beta_i}}\right\}, \tag{5.6}$$

where the coefficients $d_i$, $\alpha_i$, and $\beta_i$ depend on $n$, $q$, $p$, and the elements of the partition $\kappa$. Now, inverting the right side of Eq. (5.6), we obtain the following expression for the density of $T$:

$$g(T) = C \sum_{k=0}^{\infty} \sum_{\kappa} c(\kappa, \Omega) \left\{\sum_i d_i \frac{(T - \alpha_i)_+^{\beta_i - 1}}{(\beta_i - 1)!}\right\}, \qquad 0 \leqslant T \leqslant p, \tag{5.7}$$

where $(x)_+$ is equal to $x$ or 0 according as $x \geqslant 0$ or $x < 0$.

We will now consider the distribution of $T_1 = \sum_i (l_i/1 - l_i)$. The Laplace transformation of $T_1$ is given by

$$\mathscr{L}(t;\, T_1) = \int_0^{\infty} \exp(-tT_1) f(T_1)\, dT_1 = C \sum_{k=0}^{\infty} \sum_{\kappa} c(\kappa, \Omega) \mid D \mid, \tag{5.8}$$

where $D = (d_{ij})$ and

$$d_{ij} = \int_a^b \exp(-tx/1 - x) x^{i+j+k_{p-i+1}-2}\, \psi(x)\, dx.$$

Now, let $a = 0$, $b = 1$, and $\psi(x) = x^n(1 - x)^q$. Then,

$$d_{ij} = \int_0^{\infty} \exp(-tz) \frac{z^{n+i+j+k_{p-i+1}-2}}{(1 + z)^{n+q+i+j+k_{p-i+1}}}\, dz = \mathscr{L}(t;\, \eta(i, j;\, z)), \tag{5.9}$$

where

$$\eta(i, j;\, z) = z^{n+i+j+k_{p-i+1}-2}/(1 + z)^{n+q+i+j+k_{p-i+1}}. \tag{5.10}$$

Thus, in this special case, we obtain

$$\mathscr{L}(t; T_1) = C \sum_{k=0}^{\infty} \sum_{\kappa} c(\kappa, \Omega) \left\{ \sum \pm \mathscr{L}(t; \eta(i_1, i_2; z)) \cdots \mathscr{L}(t; \eta(i_{p-1}, i_p; z)) \right\}, \tag{5.11}$$

where the summation inside the curly bracket is taken over all permutations $(i_1, ..., i_p)$ of $(1, ..., p)$ with a plus sign if $(i_1, ..., i_p)$ is an even permutation and a minus sign if it is an odd permutation. Thus, the distribution of $T_1$ is given by

$$g_1(T_1) = C \sum_{k=0}^{\infty} \sum_{\kappa} c(\kappa, \Omega) \left\{ \sum \pm \eta(i_1, i_2; T_1) * \cdots * \eta(i_{p-1}, i_p; T_1) \right\}, \tag{5.12}$$

where * in Eq. (5.12) denotes convolution.

The distributions of the traces of the central complex bivariate beta matrix and the central complex bivariate $F$ matrix were derived by Pillai and Jouris [46]. Using Eq. (5.7), exact percentage points of the distribution of the trace of the central complex multivariate beta matrix were computed by Krishnaiah and Schuurmann [27]; these authors [32] have also extended these tables by approximating the above distribution with Pearson's Type I distribution. The accuracy of this approximation is satisfactory for several practical situations. Krishnaiah and Schuurmann [32] also constructed tables for the distribution of the trace of the central complex multivariate $F$ matrix by approximating this distribution with a suitable Pearson type distribution. Khatri [23] derived the moments of the traces of the complex multivariate beta matrix and the complex multivariate $F$ matrix for some nonnull cases.

## 6. Distributions of the Ratios of the Roots

Let the joint density of the roots be given by Eq. (4.1) and let $a \geqslant 0$. Making the transformations $l_p = l_p$ and $l_i = f_{ip} l_p$ for $i = 1, 2, ..., p-1$ in Eq. (4.1) and integrating out $l_p$, we obtain the following expression for the joint density of $f_{1p}, ..., f_{p-1,p}$

$$\begin{aligned} g_1(f_{1p}, ..., f_{p-1,p}) &= C \sum_{k} \sum_{\kappa} c(\kappa, \Omega) \, |(f_{jp}^{i-1})| \\ &\quad \times |(f_{jp}^{i-1+k_{p-i+1}})| \int_a^b \prod_{i=1}^{p} \psi(l_p f_{ip}) l_p^{k+p^2-1} \, dl_p . \end{aligned} \tag{6.1}$$

If we make the transformations $f_i = l_i / \sum l_j$ for $i = 1, 2, ..., p-1$ and $f_p = \sum l_i$

in Eq. (4.1) and integrate out $f_p$, we obtain the following expression for the joint density of $f_1, \ldots, f_{p-1}$

$$\begin{aligned} g_2(f_1, \ldots, f_{p-1}) = C \sum_k \sum_\kappa c(\kappa, \Omega) \, |(f_j^{*i-1})| \, |(f_j^{*i-1+k_{p-i+1}})| \\ \times \int_a^b f_p^{k+p^2-1} \prod_{i=1}^{p-1} \psi(f_i f_p) \, \psi\left(f_p\left(1 - \sum_{i=1}^{p-1} f_i\right)\right) df_p, \end{aligned} \tag{6.2}$$

where $f_p^* = 1 - \sum_{i=1}^{p-1} f_i$, and $f_i^* = f_i$ for $i = 1, 2, \ldots, p-1$. Next, let us make the transformations $f_{i,i+1} = l_{i+1}/l_i$ for $i = 1, 2, \ldots, p-1$ and $l_1 = l_1$ in Eq. (4.1) and integrate out $l_1$. Then, we obtain the following expression for the joint density of $f_{12}, \ldots, f_{p-1,p}$

$$\begin{aligned} g_3(f_{12}, \ldots, f_{p-1,p}) &= C \sum_k \sum_\kappa c(\kappa, \Omega) \prod_{i=1}^{p-2} f_{i,i+1}^{p-i-1} \\ &\times \left| \left(\prod_{i=0}^{j-1} f_{i,i+1}\right)^{i-1} \right| \left| \left(\prod_{i=0}^{j-1} f_{i,i+1}\right)^{i-1+k_{p-i+1}} \right| \\ &\times \int_a^b l_1^{p-1+p(p-1)k} \prod_{i=1}^{p} \psi \left\{ l_1 \left( \prod_{j=0}^{i-1} f_{j,j+1} \right) \right\} dl_1 . \end{aligned}$$

Similarly, we can obtain the joint distributions of other ratios like $f_{21}, \ldots, f_{p1}$ or $f_2, \ldots, f_p$.

When $l_p \geqslant \cdots \geqslant l_1$ are the roots of the central complex Wishart matrix, Krishnaiah and Schuurmann [28] derived the exact marginal distributions of the statistics $l_1/l_p$ and $l_i/\sum_{j=1}^{p} l_j$ for $i = 1, \ldots, p$, and computed percentage points of these statistics. Krishnaiah and Schuurmann [30] also derived the exact distribution of the ratio of the extreme roots of the central complex multivariate beta matrix and computed percentage points of this distribution.

## 7. Distributions of the Likelihood Ratio Test Statistics for Complex Multivariate Normal Populations

Following the same lines as in the real case, Goodman [10] showed that the distribution of the determinant of the central complex Wishart matrix is the product of the distributions of the central chi-square variates.

Giri [8] proved some optimum properties of the likelihood ratio tests for the hypothesis that the mean vector is equal to specified value and the hypothesis of independence of one variable with a set of variables when the underlying distribution is complex multivariate normal. Wahba [57] and Pillai and Nagarsenker [47] considered the distribution of the likelihood ratio test statistic for sphericity of the complex multivariate normal population. Gupta [12] computed exact percentage points of the distribution of the determinant of the central complex multivariate beta matrix for some special cases. We will now briefly review the recent work of Krishnaiah, Lee, and Chang on the distributions of the likelihood ratio statistics for testing certain hypotheses.

Let $\mathbf{Z}' = (\mathbf{Z}_1', \ldots, \mathbf{Z}_q')$ be distributed as a complex multivariate normal population with mean vector $\boldsymbol{\mu}' = (\boldsymbol{\mu}_1', \ldots, \boldsymbol{\mu}_q')$ and covariance matrix $\Sigma$, and let $\mathbf{Z}_i$ be of order $p_i \times 1$. Also, let $E\{(\mathbf{Z}_i - \boldsymbol{\mu}_i)\overline{(\mathbf{Z}_j - \boldsymbol{\mu}_j)'}\} = \Sigma_{ij}$. In addition, let $H_1$, $H_2$, $H_3$, and $H_4$ denote the following hypotheses:

$$\begin{aligned} H_1 &: \Sigma_{ij} = 0, \qquad (i \neq j = 1, \ldots, q), \\ H_2 &: \Sigma = \sigma^2 \Sigma_0, \\ H_3 &: \Sigma = \Sigma_0, \\ H_4 &: \Sigma = \Sigma_0, \qquad \boldsymbol{\mu} = \boldsymbol{\mu}_0, \end{aligned}$$

where $\sigma^2$ is unknown, and $\boldsymbol{\mu}_0$ and $\Sigma_0$ are known. If we denote the likelihood ratio test statistics for $H_1$, $H_2$, $H_3$, and $H_4$ by $\lambda_1$, $\lambda_2$, $\lambda_3$, and $\lambda_4$, respectively, then it is known that

$$\lambda_1 = \frac{|A|}{\prod_{i=1}^{q} |A_{ii}|} \tag{7.1}$$

$$\lambda_2 = \frac{|A\Sigma_0^{-1}|}{(\operatorname{tr} A\Sigma_0^{-1}/s)^s} \tag{7.2}$$

$$\lambda_3 = (e/n)^{sn} \, |A\Sigma_0^{-1}|^n \operatorname{etr}(-A\Sigma_0^{-1}) \tag{7.3}$$

$$\lambda_4 = (e/N)^{sN} \, |A\Sigma_0^{-1}|^N \operatorname{etr}[-\Sigma_0^{-1}\{A + N(\mathbf{Z}_. - \boldsymbol{\mu}_0)\overline{(\mathbf{Z}_. - \boldsymbol{\mu}_0)'}\}], \tag{7.4}$$

where $s = \sum_{i=1}^{q} p_i$, and $n = N - 1$. In the above equations,

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ \vdots & \vdots & & \vdots \\ A_{q1} & A_{q2} & \cdots & A_{qq} \end{bmatrix}$$

where $A_{lm} = \sum_{j=1}^{N} (\mathbf{Z}_{lj} - \mathbf{Z}_{l.})\overline{(\mathbf{Z}_{mj} - \mathbf{Z}_{m.})'}$, $NZ_{l.} = \sum_{j=1}^{N} \mathbf{Z}_{lj}$, and $\mathbf{Z}_{ij}$ denotes

$j$th independent observation on $\mathbf{Z}_i$. The moments of $\lambda_1$, $\lambda_2$, $\lambda_3$, and $\lambda_4$ are also known to be

$$E\{\lambda_1^h\} = \left\{\prod_{j=1}^{s} \frac{\Gamma(n+h-j+1)}{\Gamma(n-j+1)}\right\} \left\{\prod_{i=1}^{q} \prod_{j=1}^{p_i} \frac{\Gamma(n-j+1)}{\Gamma(n+h-j+1)}\right\}, \tag{7.5}$$

$$E\{\lambda_2^h\} = \frac{s^{hs}\,\Gamma(sn)}{\Gamma(sn+sh)} \prod_{j=1}^{s} \frac{\Gamma(n+h-j+1)}{\Gamma(n-j+1)}, \tag{7.6}$$

$$\begin{aligned} E\{\lambda_3^h\} &= (e/n)^{shn} \mid \Sigma_0 \mid^{nh} \mid I + h\Sigma_0 \mid^{-n(1+h)} \\ &\quad \times \prod_{i=1}^{s} \{\Gamma(n+nh+1-i)/\Gamma(n+1-i)\}, \end{aligned} \tag{7.7}$$

$$E\{\lambda_4^h\} = (e/N)^{shN} \frac{1}{(1+h)^{sN(1+h)}} \prod_{i=1}^{s} \frac{\Gamma(N-i+Nh)}{\Gamma(N-i)}. \tag{7.8}$$

Next, let us assume that $p_1 = \cdots = p_q = p$ and that $\Sigma_{ij} = 0$ $(i \neq j = 1,\ldots, q)$. Also, let $H_5$ denote the following hypothesis:

$$H_5 : \begin{cases} \Sigma_{11} & = \cdots = \Sigma_{q_1,q_1} \\ \Sigma_{q_1+1,q_1+1} & = \cdots = \Sigma_{q_2^*,q_2^*} \\ \quad\vdots & \\ \Sigma_{q_{k-1}^*+1,q_{k-1}^*+1} & = \cdots = \Sigma_{q,q}, \end{cases}$$

where $q_0^* = 0$, $q_1^* = q_1$, $q_j^* = \sum_{i=1}^{j} q_i$ and $q_k^* = q$. We assume that $N_i$ independent observations are available on $\mathbf{Z}_i$. In addition, let $N_i\mathbf{Z}_{i.} = \sum_{j=1}^{N_i} \mathbf{Z}_{ij}$ and

$$A_{ii} = \frac{1}{N_i} \sum_{j=1}^{N_i} (\mathbf{Z}_{ij} - \mathbf{Z}_{i.})\,\overline{(\mathbf{Z}_{ij} - \mathbf{Z}_{i.})'},$$

for $i = 1,\ldots, q$. The likelihood ratio test statistic $\lambda_5$ for $H_5$ and the moments of $\lambda_5$ are known to be as follows:

$$\lambda_5 = \frac{\prod_{i=1}^{q} \mid A_{ii}/n_i \mid^{n_i}}{\prod_{j=1}^{k} \mid \sum_{i=q_{j-1}^*+1}^{q_j^*} A_{ii}/n_i^* \mid^{n_j^*}}, \tag{7.9}$$

$$\begin{aligned} E\{\lambda_5^h\} &= \left[\frac{\prod_{i=1}^{k} n_i^{*phn_i^*}}{\prod_{i=1}^{q} n_i^{phn_i}}\right] \prod_{i=1}^{p} \prod_{j=1}^{k} \left[\left\{\prod_{g=q_{j-1}^*+1}^{q_j^*} \frac{\Gamma(n_g + hn_g + 1 - i)}{\Gamma(n_g + 1 - i)}\right\}\right. \\ &\quad \left. \times \frac{\Gamma(n_j^* + 1 - i)}{\Gamma(n_j^* + hn_j^* + 1 - i)}\right], \end{aligned} \tag{7.10}$$

where $n_i = N_i - 1$, and $n_j{}^* = \sum_{i=q_{j-1}^*+1}^{q_j^*} n_i$. Next, consider the hypothesis $H_6$, where

$$H_6: \begin{cases} \Sigma_{11} = \cdots = \Sigma_{qq} \\ \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_q \\ \text{(under the assumption that } p_1 = \cdots = p_q = p \text{ and } \Sigma_{ij} = 0 \text{ for } i \neq j). \end{cases}$$

The likelihood ratio test statistic $\lambda_6$ for $H_6$ and the moments of $\lambda_6$ are known to be

$$\lambda_6 = \frac{n_0^{pn_0} \prod_{i=1}^q | A_{ii} |^{n_i}}{\prod_{i=1}^q n_i^{pn_0} | \sum_{i=1}^q A_{ii} + \sum_{i=1}^q N_i(\mathbf{Z}_{i.} - \mathbf{Z}_{..})(\mathbf{Z}_{i.} - \mathbf{Z}_{..})' |^{n_0}},$$

$$E\{\lambda_6{}^h\} = \frac{n_0^{pn_0}}{\prod_{i=1}^q n_i^{phn_i}} \prod_{i=1}^p \prod_{j=1}^q \left\{\frac{\Gamma(n_j + hn_j + 1 - i)}{\Gamma(n_j + 1 - i)}\right\} \frac{\Gamma(n_0 + q - i)}{\Gamma(n_0 + hn_0 + q - i)}$$

(using the same notation as in $\lambda_5$, and $n_0 = \sum_{i=1}^q n_i$).

The derivation of the likelihood ratio test statistics $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$, $\lambda_5$, and $\lambda_6$, and their moments follow easily by following the same lines as in the corresponding real cases.

Box [4] derived an asymptotic expression for the distribution function of a class of statistics $W$ $(0 \leqslant W \leqslant 1)$ whose moments are of the form

$$E\{W^h\} = K \prod_{j=1}^c y_j^{y_j} \prod_{k=1}^a x_k^{x_k} \times \frac{\prod_{k=1}^a \Gamma(x_k(1+h) + \xi_k]}{\prod_{j=1}^c \Gamma[y_j(1+h) + \eta_j]}, \quad h = 0, 1, \ldots, \tag{7.11}$$

where $K$ is a normalizing constant such that $E\{W^0\} = 1$ and $\sum_{k=1}^a x_k = \sum_{j=1}^c y_j$.

Box gave the first few terms only in the asymptotic expression. In several situations, the first few terms do not give the desired degree of accuracy. Using Box's method, Lee, Chang, and Krishnaiah [39] gave the terms up to the order of $n^{-15}$; these terms are linear combinations of the distribution functions of the central chi-square variates. The moments of $\lambda_1$, $\lambda_2$, $\lambda_5$, and $\lambda_6$ are of the form (7.11). However, the asymptotic expression of Box is complicated if we have to take several terms in the series to get the desired degree of accuracy.

The distributions of certain powers of the statistics $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$, $\lambda_5$, and $\lambda_6$ are approximated in Krishnaiah, Lee, and Chang [37], Lee, Krishnaiah, and Chang [38], and Chang, Krishnaiah, and Lee [7] with Pearson's Type I distribution by using the first four moments of these distributions. Using these approximations, they have also computed percentage points of the distributions of $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{\lambda}_3$, $\tilde{\lambda}_4$, $\tilde{\lambda}_5$, and $\tilde{\lambda}_6$, where $\tilde{\lambda}_i = -2 \log \lambda_i$ for $i = 1, 2, \ldots, 6$. The accuracy of these approximations is sufficient for practical purposes. Nagarsenker and Das [43] have also computed the percentage points of the distribution of $\lambda_2$ for some values of the parameters by using a different method.

Khatri [21] derived the likelihood ratio test statistic[1] for the reality of the covariance matrix of the complex multivariate normal population. We also can use other functions (like elementary symmetric functions, ratios) of the roots of $AA_1^{-1}$ for testing the hypothesis of the reality of the covariance matrix; here, $A_1$ denotes the real part of the sample $SP$ matrix $A$. A certain power of the likelihood ratio statistic can be approximated with the Pearson's Type I distribution and the degree of accuracy of the approximation is sufficient for practical purposes. Khatri [20] also derived the moments of the likelihood ratio statistic for testing the hypothesis that the mean vector of a complex multivariate normal distribution is equal to a known vector. The distribution of this statistic is related to the distribution of $\lambda_1$ when $q = 2$.

## 8. Applications

Let $\mathbf{X}'(t) = (\mathbf{X}_1'(t),..., \mathbf{X}_q'(t))$ $(t = 1,..., T)$ be a $1 \times u$ random vector that is distributed as a stationary Gaussian multivariate time series with zero mean vector and covariance matrix $R(v) = E\{\mathbf{X}(t)\,\mathbf{X}'(t + v)\}$. Also, let the spectral density matrix $F(\omega) = (1/2\pi) \sum_{v=-\infty}^{\infty} \exp(-iv\omega)\, R(v)$ be partitioned as

$$F(\omega) = \begin{bmatrix} F_{11}(\omega) & F_{12}(\omega) & \cdots & F_{1q}(\omega) \\ F_{21}(\omega) & F_{22}(\omega) & \cdots & F_{2q}(\omega) \\ \vdots & \vdots & & \vdots \\ F_{q1}(\omega) & F_{q2}(\omega) & \cdots & F_{qq}(\omega) \end{bmatrix}, \tag{8.1}$$

where $F_{jk}(\omega)$ is of order $p_j \times p_k$, and $\mathbf{X}_j(t)$ is of order $p_j \times 1$. A well-known estimate (e.g., see Parzen [44], Brillinger [5]) of $F(\omega)$ is given by $\hat{F}(\omega) = (\hat{f}_{jk}(\omega))$, where

$$\begin{aligned} \hat{f}_{jk}(\omega) &= \sum_{a=-m}^{m} w_a I_{jk}(\omega + (2\pi a/T)), \\ I_{jk}(\lambda) &= Z_j(\lambda)\, \bar{Z}_k(\lambda), \\ Z_j(\lambda) &= (1/(2\pi T)^{1/2}) \sum_{t=1}^{T} X_j(t) \exp(-it\lambda). \end{aligned} \tag{8.2}$$

In the sequel, we assume that the weights $w_a$ are equal to $1/(2m + 1)$.

It is known (see Goodman [9], Wahba [56], and Brillinger [5]) that $(2m + 1)$ $\hat{F}(\omega)$ is approximately distributed as the central complex Wishart matrix with $(2m + 1)$ degrees of freedom.

[1] A. K. Gupta (*J. Statist. Comp. Simul.* 2 (1973), pp. 333–342) gave tables for the distribution of this statistic for a few special cases.

We will now discuss the problem of testing the hypothesis $H_1(\omega)$ where

$$H_1(\omega) : F_{jk}(\omega) = 0, \qquad (j \neq k = 1,\ldots, q).$$

Let $s_i = \min(p_i, p_1 + \cdots + p_{i-1})$, and let $c_{i1} \leqslant \cdots \leqslant c_{is_i}$ be the eigenvalues of $\hat{\beta}_{ii}$, where

$$\hat{\beta}_{ii}(\omega) = \hat{F}_{ii}^{-1}(\omega)(\hat{F}_{i1}(\omega),\ldots, \hat{F}_{i,i-1}(\omega)) \begin{bmatrix} \hat{F}_{11}(\omega) & \cdots & \hat{F}_{1,i-1}(\omega) \\ \hat{F}_{21}(\omega) & \cdots & \hat{F}_{2,i-1}(\omega) \\ \vdots & & \vdots \\ \hat{F}_{i-1,1}(\omega) & \cdots & \hat{F}_{i-1,i-1}(\omega) \end{bmatrix}^{-1} \begin{bmatrix} \hat{F}_{1i}(\omega) \\ \vdots \\ \hat{F}_{i-1,i}(\omega) \end{bmatrix}. \tag{8.3}$$

The hypothesis $H_1(\omega)$ can be expressed as $H_1(\omega) = \bigcap_{j=2}^{q} H_{1j}(\omega)$, where $H_{1j}(\omega)$: $(F_{j1}(\omega),\ldots, F_{j,j-1}(\omega)) = 0$. We can test the hypothesis $H_1(\omega)$ as follows using a conditional approach. We first test the hypothesis $H_{12}(\omega)$. If $H_{12}(\omega)$ is rejected, we conclude that $H_1(\omega)$ is rejected. If $H_{12}(\omega)$ is accepted, we test $H_{13}(\omega)$ given $H_{12}(\omega)$. If $H_{13}(\omega)$ is rejected, we conclude that $H_1(\omega)$ is rejected; otherwise, we test $H_{14}(\omega)$ given $H_{13}(\omega)$. This procedure is continued until a decision is made about the acceptance or rejection of $H_1(\omega)$. Now, let $T_i(c_{i1},\ldots, c_{is_i})$ denote a suitable function of $c_{i1},\ldots, c_{is_i}$. Then, the hypothesis $H_{1j}$ given $\bigcap_{k=1}^{j-1} H_{1k}$ is accepted or rejected according as

$$T(c_{j1},\ldots, c_{js_j}) \lesseqgtr d_j, \tag{8.4}$$

for $j = 2, 3,\ldots, q$, with the understanding that $H_{12}$ given $H_{11}$ is equivalent to $H_{12}$. In Eq. (8.4), the constants $d_j$ are chosen such that

$$\begin{aligned} &P[T(c_{j1},\ldots, c_{js_j}) \leqslant d_j\,; j = 2,\ldots, q \mid H_1] \\ &\qquad = \prod_{j=2}^{q} P[T(c_{j1},\ldots, c_{js_j}) \leqslant d_j \mid H_1] = (1-\alpha). \end{aligned} \tag{8.5}$$

It is known that $(2m + 1)\hat{F}(\omega)$ is approximately distributed as the complex Wishart matrix with $(2m + 1)$ degrees of freedom and $E(\hat{F}(\omega)) = (2m + 1)F(\omega)$. Thus, when $H_1$ is true, the joint density of $c_{j1},\ldots, c_{js_j}$ is approximately of the same form as Eq. (3.12) after replacing $p$, $m$, and $q$ with $s_j$, $(2m + 1)$, and $\max(p_j, p_1 + \cdots + p_{j-1})$, respectively. When $T(c_{j1},\ldots, c_{js_j}) = c_{js_j}$, the procedure discussed above is similar to the conditional approach used by Roy and Bargmann [52] for testing the multiple independence of several sets of variables when their joint distribution is real multivariate normal. The test statistics $T(c_{j1},\ldots, c_{js_j})$ can be also chosen to be equal to $\prod_{i=1}^{s_j} (1 - c_{ji})$, $c_{js_j}/c_{j1}$, $c_{js_j}/\sum_{i=1}^{s_j} c_{ji}$, elementary symmetric functions of the roots, or some other suitable functions When $q = 2$, analogous test statistics were used in the literature by various authors for testing the independence of two sets of variables when their joint distribution is real multivariate normal. For example, test statistics analogous to $\prod_{i=1}^{s_2}(1 - c_{2i})$ and $\sum_{i=1}^{s_2} c_{2i}$ were used by Wilks [60] and Bartlett [3], respectively, whereas test statistics analogous to $c_{2s_2}/c_{21}$ and $c_{2s_2}/\sum_{i=1}^{s_2} c_{2i}$ were considered by Krishnaiah and Waikar [25, 26]. In (8.4), one can of course use different types of statistics $T_j(c_{j1},..., c_{js_j})$ to test $H_{1j}$'s instead of using the same type of statistic $T(c_{j1},..., c_{js_j})$ at each stage of conditioning.

We will now consider the problem of testing the hypothesis $H_2(\omega): F_{11}(\omega) = \cdots = F_{qq}(\omega)$, when[2] $R_{ij}(v) = R_{12}(v)$ $(i \neq j = 1,..., q)$. Let $\mathbf{Y}_1(t) = \{\mathbf{X}_1(t) + \cdots + \mathbf{X}_q(t)\}/q$, and $\mathbf{Y}_i(t) = \mathbf{X}_i(t) - \mathbf{Y}_1(t)$ for $i = 2,..., q$. Then, $\mathbf{Y}'(t) = (Y_1'(t),..., Y_q'(t))$ is a Gaussian stationary multiple time series with covariance matrix $R^*(v)$. The problem of testing the hypothesis $H_2(\omega)$ is equivalent to testing the hypothesis of $F_{12}^*(\omega) = 0$, where

$$F^*(\omega) = \begin{bmatrix} F_{11}^*(\omega) & F_{12}^*(\omega) \\ F_{21}^*(\omega) & F_{22}^*(\omega) \end{bmatrix}$$

is the spectral density matrix of the time series $\{\mathbf{Y}(t)\}$, and $F_{11}^*(\omega)$ is the spectral density matrix of the time series $\{\mathbf{Y}_1(t)\}$. The hypothesis that $F_{12}^*(\omega) = 0$ can be tested by using the method described before. The method described above for testing $H_2(\omega)$ is analogous to the method used by Krishnaiah [36] for testing the equality of the diagonal blocks of the covariance matrix of the multivariate normal population when the off-diagonal blocks are equal.

Next, consider the problem of testing the hypothesis $H_3(\omega): F(\omega) = F_0(\omega)$, where the matrix $F_0(\omega)$ is completely known. Let $c_p(\omega) \geqslant \cdots \geqslant c_1(\omega)$ be the latent roots of $\hat{F}(\omega)F_0^{-1}(\omega)$, and let $T(c_1(\omega),..., c_p(\omega))$ be a suitable function of these roots. Also, let $\lambda_p(\omega) \geqslant \cdots \geqslant \lambda_1(\omega)$ be the latent roots of $\hat{F}(\omega)F_0^{-1}(\omega)$. Then the hypothesis $H_3(\omega)$ when tested against $T(\lambda_1(\omega),..., \lambda_p(\omega)) > T(1,..., 1)$ is accepted or rejected accordingly as

$$T(c_1(\omega),..., c_p(\omega)) \lesseqgtr d_{3\alpha},$$

where

$$P[T(c_1(\omega),..., c_p(\omega)) \leqslant d_{3\alpha} \mid H_3(\omega)] = (1 - \alpha). \tag{8.6}$$

We can similarly propose a test procedure against two sided alternatives. When $H_3(\omega)$ is true, $\hat{F}(\omega)F_0^{-1}(\omega)$ is approximately distributed as the central complex Wishart matrix with $(2m + 1)$ degrees of freedom. Hence, the distributions of some of the statistics $T(c_1(\omega),..., c_p(\omega))$ can be evaluated by using the results discussed in this paper. Some possible choices of $T(c_1(\omega), ..., c_p(\omega))$, are $c_p(\omega)$, $\sum_{j=1}^{p} c_j(\omega)$, $c_p(\omega) - c_1(\omega)$, $\max_i(c_{i+1} - c_i)$, $c_p - \sum_{i=1}^{p} c_i/p$, or a statistic analogous to $\lambda_3$ in the preceding section. Another procedure for testing $H_3(\omega)$ against the alternative that $F(\omega) \neq F_0(\omega)$ is to accept $H_3(\omega)$ if

$$d_{3\alpha} \leqslant c_1(\omega) \leqslant c_p(\omega) \leqslant d_{4\alpha}$$

[2] Here $R_{ij}(v) = E[\mathbf{X}_i(t)\mathbf{X}_j'(t + v)]$.

and reject it otherwise, where

$$P[d_{3\alpha} \leqslant c_1(\omega) \leqslant c_p(\omega) \leqslant d_{4\alpha} \mid H_3(\omega)] = (1 - \alpha). \tag{8.7}$$

The hypothesis $H_4(\omega) : F(\omega) = \sigma^2(\omega) I_p$ can be tested by using various ratios of the roots of $\hat{F}(\omega)$; for a review of these methods, the reader is referred to Krishnaiah and Schuurmann [28]. Of course, one can test $H_4(\omega)$ by using $|\hat{F}(\omega) F_0^{-1}(\omega)| / \{\operatorname{tr} \hat{F}(\omega) F_0^{-1}(\omega)/s\}^s$ as a test statistic. The distribution problems associated with this statistic were discussed in the preceding section.

Now, let $\{\mathbf{X}_1(t)\},\dots, \{\mathbf{X}_q(t)\}$ be $q$ independently distributed stationary, Gaussian $p$-variate time series with spectral density matrices $F_1(\omega),\dots, F_q(\omega)$. Also, let the record of $i$th time series be $T_i$. In addition, let the sample estimate $\hat{F}_i(\omega)$ of $F_i(\omega)$ be defined in the same way as Eq. (8.2) by taking the averages of $(2m_i + 1)$ periodograms. In addition, let $c_{ijp}(\omega) \geqslant \cdots \geqslant c_{ij1}(\omega)$ be the roots of $\hat{F}_i(\omega) \hat{F}_j^{-1}(\omega)$, and let $\lambda_{ijp} \geqslant \cdots \geqslant \lambda_{ij1}$ be the roots of $F_i(\omega) F_j^{-1}(\omega)$.

Also, let $\psi(\lambda_{ij1},\dots, \lambda_{ijp})$ be a suitable function of $\lambda_{ij1},\dots, \lambda_{ijp}$ and $\psi(1,\dots, 1) = d$. We will now discuss procedures for testing the hypothesis $H_5(\omega) : F_1(\omega) = \cdots = F_q(\omega)$. The hypothesis $H_5(\omega)$ when tested against $\bigcup_{i=1}^{q-1} [\psi(\lambda_{i,i+1,1}(\omega),\dots, \lambda_{i,i+1,p}(\omega)) \geqslant d]$ is accepted if

$$\psi(c_{i,i+1,1}(\omega),\dots, c_{i,i+1,s}(\omega)) \leqslant c_\alpha,$$

for $i = 1,\dots, q - 1$, and rejected otherwise, where $c_\alpha$ is chosen such that

$$P[\psi(c_{i,i+1,1}(\omega),\dots, c_{i,i+1,s}(\omega)) \leqslant c_\alpha ;\, i = 1,\dots, q - 1 \mid H_5(\omega)] = (1 - \alpha). \tag{8.8}$$

When $H_5(\omega)$ is true, $\hat{F}_i(\omega) \hat{F}_j^{-1}(\omega)$ is distributed as the central complex multivariate $F$ matrix with $E(\hat{F}_i) = E(\hat{F}_j(\omega))$. Thus, we can use Bonferroni's inequality to compute bounds on the values of $\alpha$ in Eq. (8.8). Similarly, we can propose procedures against the alternatives $\bigcup_{i=1}^{q-1} [\psi(\lambda_{i,i+1,1}(\omega),\dots, \lambda_{i,i+1,p}(\omega)) \leqslant d]$ and $\bigcup_{i=1}^{q-1} [\psi(\lambda_{i,i+1,1}(\omega),\dots, \lambda_{i,i+1,p}(\omega)) \neq d]$. We also can propose procedures for testing $H_5(\omega)$ against $\bigcup_{i=1}^{q-1} [\psi(\lambda_{iq1},\dots, \lambda_{iqp}) \neq d]$ (or one-sided alternatives) by using $\psi(c_{iq1},\dots, c_{iqp})$ $(i = 1,\dots, q - 1)$ as test statistics. If we test $H_5(\omega)$ against $\bigcup_{i<j}^{q} [\psi(\lambda_{ij1},\dots, \lambda_{ijp}) \neq d]$ (or one-sided alternatives), we use $\psi(c_{ij1},\dots, c_{ijp})$ $(i < j = 1,\dots, q)$ as test statistics. The hypothesis $H_5(\omega)$ can be tested against $\bigcup_{i=1}^{q-1} [F_i(\omega) \neq F_{i+1}(\omega)]$, $\bigcup_{i=1}^{q-1} [F_i(\omega) \neq F_q(\omega)]$, and $\bigcup_{i<j}^{q} [F_i(\omega) \neq F_j(\omega)]$ by using procedures analogous to those considered by Krishnaiah [34] and Krishnaiah and Pathak [24] for testing the equality of the covariance matrices of real multivariate normal populations. Procedures analogous to those considered for testing $H_5(\omega)$ can be also used for testing the hypothesis $F(\omega_1) = \cdots = F(\omega_k)$ against different alternatives when $\omega_1,\dots, \omega_k$ are widely separated, since, in this case, $\hat{F}(\omega_1),\dots, \hat{F}(\omega_k)$ are distributed independently.

Next, consider the problem of testing the hypothesis

$$H_6(\omega) : F(\omega) = G_1(\omega) \otimes \Omega_1(\omega) + \cdots + G_k(\omega) \otimes \Omega_k(\omega),$$

where $G_1(\omega),\ldots, G_k(\omega)$ are known matrices, $\Omega_1(\omega),\ldots, \Omega_k(\omega)$ are unknown matrices, and $\otimes$ denotes the Kronecker product. The hypothesis $H_6(\omega)$ can be tested by using the procedures analogous to those considered by Krishnaiah and Lee [33] for testing the linear structures of the covariance matrices of real multivariate normal populations.

The techniques used above in the area of the inference on multiple time series are also useful in the area of the inference on the spectral density matrices of multivariate point processes.

We will now discuss some procedures for testing the hypothesis on the adequacy of a given number of discriminators to discriminate between complex multivariate normal populations.

Let the rows of $Z_i = (z_{iju})$: $m_i \times p$ $(i = 1,\ldots, k)$ be distributed independently as complex multivariate normal with mean vector $\boldsymbol{\mu}_i'$ and covariance matrix $\Sigma$. Also, we assume that $Z_1,\ldots, Z_k$ are distributed independently. The between group sums of squares and cross-products ($SP$) matrix and the within group $SP$ matrix are respectively given by $S_1 = (s_{1uv})$ and $S_2 = (s_{2uv})$, where

$$s_{1uv} = \sum_{i=1}^{k} m_i(z_{i\cdot u} - z_{\cdot\cdot u})(\bar{z}_{i\cdot v} - \bar{z}_{\cdot\cdot v}),$$

$$s_{2uv} = \sum_{i=1}^{k} \sum_{j=1}^{m_i} (z_{iju} - z_{i\cdot u})(\bar{z}_{ijv} - \bar{z}_{i\cdot v}),$$

$m_i z_{i.u} = \sum_{j=1}^{m_i} z_{iju}$, $m z_{\cdot\cdot u} = \sum_{i=1}^{k} \sum_{j=1}^{m_i} z_{iju}$, and $m = \sum_{i=1}^{k} m_i$. We know that $S_1$ is distributed as the noncentral complex Wishart matrix with $k - 1$ d.f. and $E(S_1/k - 1) = \Sigma + (1/(k-1))\, \boldsymbol{\nu}\boldsymbol{\nu}'$, where $\boldsymbol{\nu} = (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_.,\ldots, \boldsymbol{\mu}_k - \boldsymbol{\mu}_.)$ and $k\boldsymbol{\mu} = \sum_{j=1}^{k} \boldsymbol{\mu}_j$. Also, $S_2$ is distributed as the central complex Wishart matrix with $m - k$ degrees of freedom. In the real case, likelihood ratio test for reducing the dimensionality was discussed in the literature (see Rao [51]). In the complex case, the analogous test was discussed in Young [62]. Alternative procedures for the reduction of dimensionality are discussed below.

Let $l_p \geqslant \cdots \geqslant l_1$ denote the eigenvalues of $S_1 S_2^{-1}$ and let $\lambda_p \geqslant \cdots \geqslant \lambda_1$ be the eigenvalues of $\Omega = (1/(k-1))\, \boldsymbol{\nu}\boldsymbol{\nu}'\Sigma^{-1}$. Let $H_i : \lambda_i = 0$, and $A_i : \lambda_i > 0$. Then, the nested hypotheses $H_1,\ldots, H_p$ can be tested simultaneously as follows. We accept or reject $H_i$ against $A_i$ accordingly as

$$l_i \lesseqgtr c_\alpha,$$

where

$$P[l_p \leqslant c_\alpha \mid H_p] = (1 - \alpha). \tag{8.9}$$

The evaluation of the distribution of $l_p$ in the central and noncentral cases was discussed in Section 4. Sometimes, the experimenter knows in advance that

$\lambda_j > 0$ for $j = i + 1,..., p$. Then, he has to test $H_1$ ,..., $H_i$ only. In this case, the critical value $c_\alpha$ is chosen such that

$$P[l_i \leqslant c_\alpha \mid H_i] = (1 - \alpha). \tag{8.10}$$

But the distribution of $l_i$ , under $H_i$ , involves $\lambda_{i+1}$ ,..., $\lambda_p$ as nuisance parameters. Thus, it would be of interest to obtain bounds (free from nuisance parameters) on the probability integral in Eq. (8.10).

Next, let $H_{ij} : \lambda_i = \lambda_j$ , $A_{ij} : \lambda_i > \lambda_j$ , and $f_{ij} = l_i/l_j$ for $i > j$. Then, the hypotheses $H_{ij}$ $(i > j)$ can be tested simultaneously against $A_{ij}$ as follows. We accept or reject $H_{ij}$ accordingly as

$$f_{ij} \lesseqgtr d_\alpha ,$$

where

$$P[f_{p1} \leqslant d_\alpha \mid H_{p1}] = (1 - \alpha). \tag{8.11}$$

The evaluation of the distribution of $f_{p1}$ was discussed in Section 6. When $H_{p1}$ is true, the probability integral in (8.11) involves $\lambda_1$ as a nuisance parameter. Thus, it would be of interest to obtain a bound (free from nuisance parameters) on the probability integral in Eq. (8.11). When $p > k - 1$, then $\lambda_1 = \cdots = \lambda_{p-k+1} = 0$, and so the probability integral in (8.11) does not involve nuisance parameters. Also, in some practical situations, $\lambda_1$ is not significantly different from zero and so it may be replaced with zero.

The procedures discussed above are proposed in the same spirit as the procedures discussed in Krishnaiah and Waikar [25, 26] for the analogous real cases.

Next, consider the problem of testing the hypotheses $H_{t0}$ ,..., $H_{q0}$ $(t \leqslant q \leqslant p)$ simultaneously against the alternatives $A_{t0}$ ,..., $A_{q0}$ , where $H_{j0} : \lambda_j \leqslant c \sum_{i=1}^{p} \lambda_i$ and $A_{j0} : \lambda_j > c \sum_{i=1}^{p} \lambda_i$ and $c$ is a known constant. In this case, we accept or reject $H_{j0}$ $(j = t,..., q)$ accordingly as

$$l_j/c \sum_{j=1}^{p} l_j \lesseqgtr d_\alpha ,$$

where

$$P\left[l_q/c \sum_{j=1}^{p} l_j \leqslant d_\alpha \mid H_{q0}\right] = (1 - \alpha). \tag{8.12}$$

Similarly, we can test the hypothesis that $\sum_{i=t}^{q} \lambda_i < c \sum_{i=1}^{p} \lambda_i$ by using $\sum_{i=t}^{q} l_i/c \sum_{i=1}^{p} l_i$ as a test statistic. The probability integrals, in both of the above cases, are not only complicated, but would involve nuisance parameters. Hence, it would be of interest to obtain bounds (free from nuisance parameters) for these probability integrals. Similar tests can be used for drawing the inference on the eigenvalues of the spectral density matrix.

When $\Sigma$ is known, we can use the above procedure after replacing $S_2/(m-k)$ with $\Sigma$.

Procedures similar to those discussed above for drawing inference on the eigenvalues of $\Omega$ may be used for drawing inference on the canonical correlations when the two sets of variables are jointly distributed as a complex multivariate normal.

In the methods discussed above for drawing inference on the spectral density matrices of the multiple time series, we may, of course, use alternative estimates $\hat{\hat{F}}(\omega)$ instead of using $\hat{F}(\omega)$ for estimating $F(\omega)$. If these estimates $\hat{\hat{F}}(\omega)$ are approximately distributed as complex Wishart matrices, the distributions discussed in this paper are useful in computing the critical values. For a discussion of the alternative estimates of $F(\omega)$, the reader is referred to Hannan [13] and Brillinger [5].

Next, let us consider a matrix $\Sigma^*$ and let $S^*$ be a suitable estimate of $\Sigma^*$. For example, in the area of principal component analysis, we may treat $\Sigma^*$ and $S^*$ as the population and sample covariance matrices, respectively. Similarly, in the area of canonical correlation analysis, we may treat $\Sigma^*$ and $S^*$ as the population canonical correlation matrix and sample canonical correlation matrix, respectively. Now, let $\theta_p \geqslant \cdots \geqslant \theta_1$ be the eigenvalues of $S^*$, whereas $\lambda_p \geqslant \cdots \geqslant \lambda_1$ denote the eigenvalues of $\Sigma^*$. In some situations, the experimenter knows in advance that the $\lambda_i$'s differ from each other. In these situations, he may be interested in simultaneous testing of the hypotheses $H_{ij}$ against $A_{ij}$ $(i > j)$, where $H_{ij} : \lambda_i < d\lambda_j$ $(d > 1)$ and $A_{ij} : \lambda_i > d\lambda_j$. In this case, we accept or reject $H_{ij}$ accordingly as

$$l_i/dl_j \lesseqgtr c_\alpha ,$$

where

$$P\,[l_p/l_1 \leqslant dc_\alpha \mid \lambda_p < d\lambda_1] = (1-\alpha). \tag{8.13}$$

A bound on the critical value $c_\alpha$ may be obtained by constructing a bound (free from nuisance parameters) on the left side of Eq. (8.13) and equating it to $(1-\alpha)$. One may similarly be interested in testing the hypotheses $H_{ij}$ : $\lambda_i - \lambda_j < d$, $(d > 0)$ against the alternatives $A_{ij} : \lambda_i - \lambda_j > d$. In this case, we accept or reject $H_{ij}$ against $A_{ij}$ accordingly as

$$(l_i - l_j - d) \lesseqgtr c_\alpha ,$$

where

$$P[l_p - l_1 \leqslant d + c_\alpha \mid \lambda_p - \lambda_1 < d] = (1-\alpha). \tag{8.14}$$

Here also, one may attempt to get a bound on $c_\alpha$ by constructing a lower bound (free from nuisance parameters )on the left side of Eq. (8.14).

We will now give the definitions of some complex multivariate processes. Let $\{X_j(t)\}$ $(j = 1,..., n)$ denote $n$ independent and identically distributed stationary $p$-variate continuous parameter $(-\infty < t < \infty)$ complex Gaussian stochastic process. Also, let $S(t) = X(t)\,\bar{X}'(t)$, where $X(t) = [\mathbf{X}_1(t),..., \mathbf{X}_u(t)]$. The Hermitian matrix valued stochastic process $S(t)$, $-\infty < t < \infty$, obtained by varying $t$ is known (see Goodmann and Dubman [11]) to be as complex Wishart process. Now, let $S_1(t) = X(t)\,A\bar{X}'(t)$ and $S_2(t) = X(t)\,\bar{X}'(t)B$, where $A : n \times n$ and $B : p \times p$ are symmetric matrices. The processes $\{S_1(t)\}$ and $\{S_2(t)\}$, $-\infty < t < \infty$, obtained by varying the time $t$ are generalizations of complex Wishart process. Next, let us consider a complex Wishart process $\{S_0(t)\}$ $(-\infty < t < \infty)$ obtained by varying the time $t$. Then, it would be of interest to investigate the properties of the processes $\{S_1(t)\,S_0^{-1}(t)\}$, $\{S_0^{-1/2}(t)S_1(t)\,S_0^{-1/2}(t)\}$, $\{(S_0(t) + S_1(t)^{-1/2}S_1(t)(S_0(t) + S_1(t))^{-1/2}\}$, $\{S_1(t)(S_0(t) + S_1(t))^{-1}\}$, $(-\infty < t < \infty)$ obtained by varying the time $t$. Similarly, we can define the processes obtained by replacing $S_1(t)$ with $S_2(t)$ in the above processes. It is also of interest to study the above processes for the discrete cases.

## 9. Comments on Real Multivariate Distributions

The joint density of the eigenvalues $\theta_p \geqslant \cdots \geqslant \theta_1$ of a wide class of real random matrices in certain noncentral cases are of the form

$$g(\theta_1,..., \theta_p) = C_1 \prod_{i=1}^{p} h(\theta_i) \prod_{i>j} (\theta_i - \theta_j) \times \sum_{k=0}^{\infty} \sum_{\kappa} b(\kappa)\, \eta_\kappa(\Phi), \tag{9.1}$$

where $a \leqslant \theta_1 \leqslant \cdots \leqslant \theta_p \leqslant b$, $\Phi = \text{diag.}(\theta_1,..., \theta_p)$, $\eta_\kappa(\Phi)$ is a symmetric function of $\theta_1,..., \theta_p$, and $a(\kappa)$ depends upon the population parameters and the partition $\kappa$. The joint density of the eigenvalues of the analogous class of complex random matrices is given by (4.1). This joint density is a symmetric function of the roots, whereas the corresponding density in the real case given by Eq. (9.1) is not a symmetric function of the roots. Thus, the distribution problems associated with the individual roots or certain functions of the roots in the real cases are more complicated than the corresponding problems in the complex cases. Krishnaiah and Chattopadhyay [35] derived the marginal distributions of the roots $\theta_r,..., \theta_s$ $(1 \leqslant r \leqslant s \leqslant p)$, moments of the elementary symmetric functions of the above roots, and the Laplace transformation of $\sum_{i=1}^{p} \theta_i$. They have also discussed the evaluation of the distribution of $\sum_{i=1}^{p} \theta_i$ when $h(\theta)$ is of the form $\theta^r(1 - \theta)^m$ $(0 \leqslant \theta \leqslant 1)$ or $\theta^m(1 + \theta)^{-(m+r+p+1)}$ $(0 \leqslant \theta \leqslant \infty)$. A review of the literature on real multivariate distributions will be given in a separate paper.

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