Bernstein Algebras Given by Symmetric Bilinear Forms*

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ABSTRACT

Let (A, ω) be a finite dimensional Bernstein algebra and N the kernel of ω. We study the algebras where dim N^2 is 1. The algebras fall into two general classes. For the first of these classes we give the multiplication tables for the complete set of nonisomorphic algebras. For the second of these classes we give the multiplication tables for what we call "complete algebras." We show that any algebra of the second class can be embedded in a complete algebra. The multiplication in the complete algebras is easy to describe. The Bernstein algebras of the second class are then characterized as subalgebras of the complete algebras. For Jordan Bernstein algebras satisfying dim $N^2 = 1$ we give the complete classification.

1. INTRODUCTION

A Bernstein algebra is a pair (A, ω) consisting of a commutative nonassociative algebra A over a field K and a nonzero homomorphism of algebras

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$w: A \to K$ such that

$$x^2x^2 = \omega(x)^2x^2 \qquad (\text{for all} \quad x \in A). \tag{1}$$

If (A, ω_1) and (A, ω_2) are both Bernstein algebras, then $\omega_1 = \omega_2$ (see [3, Lemma 1]), so the homomorphism ω is uniquely determined. These algebras were introduced by Holgate [4] in connection with the problem, proposed by Bernstein, of classifying populations that achieve equilibrium at the second generation (see Lyubich [2]).

Let N denote the kernel of ω. In [7] Costa studied the problem of classifying finite dimensional Bernstein algebras such that N^2 is one dimensional. In this case, $N^2 = Kc$ and the multiplication in N is given by $xy = b(x, y)c$, where b is a symmetric bilinear form. When the Witt index of b is zero a complete classification has been obtained.

In this paper we show that the classification of these algebras splits into two parts according as $N^2 \subseteq Z$ or $N^2 \subseteq U$. The first is completely classified. The second is partially classified by giving the basis multiplication for complete algebras and proving every such algebra is a subalgebra of a complete algebra. Furthermore, we completely classify the algebras that have the additional property of being Jordan.

The following algebra with parameters s, d, k, t $(0 < s,\ 0 \leqslant d,\ 0 \leqslant k,\ 0 \leqslant t, k + t \leqslant s)$ and element c is called a complete algebra on the parameters s, d, k, t, and c.

Take a vector space V of dimension $2s + d + 1$ over a field K. Pick a basis of V: e, u_i, z_j $(1 \leqslant i \leqslant s + d, 1 \leqslant j \leqslant s)$. Pick c in the subspace spanned by the u_i. The nonzero products of V are

$$\begin{gathered} e^2 = e, \qquad eu_i = \tfrac{1}{2}u_i \quad (1 \leqslant i \leqslant s + d), \\ u_iz_i = c \qquad (1 \leqslant i \leqslant s), \\ z_i^2 = \begin{cases} c, & 1 \leqslant i \leqslant k, \\ -c, & k < i \leqslant k + t. \end{cases} \end{gathered} \tag{2}$$

The vector space V with this product becomes a Bernstein algebra of type $(s + d + 1, s)$. If U is the span of the u_i and Z is the span of the z_j, then $Ke \oplus U \oplus Z$ gives the idempotent decomposition of the algebra relative to the idempotent e. Notice that $N^2 = Kc \subseteq U$.

For the definition and elementary structure theorems, K may be any field of characteristic $\neq 2$. Because our proofs require orthogonally diagonalizing symmetric matrices over the field K, in our theorems we restrict K to be the real numbers and indicate this by calling the field $\mathbb{R}$. We will consider only finite dimensional Bernstein algebras. We will reserve the letter N to represent the kernel of the homomorphism ω. N is an ideal of A.

2. CHARACTERIZATION

Let (A, ω) be a Bernstein algebra. If $y \in A$ and $\omega(y) \neq 0$, let $x = [\omega(y)]^{-1} y$. Now $\omega(x) = 1$, so $x^2 \neq 0$, and by the identity (1) x^2 is an idempotent. We choose some idempotent e and write A as a vector space direct sum $A = Ke \oplus N$. If R_e denotes right multiplication by e acting on N, we have $2R_e^2 = R_e$. This gives the decomposition $A = Ke \oplus U \oplus Z$, where U is the kernel of $2R_e - I$ and Z the kernel of R_e. The subspaces U and Z satisfy

$$UZ \subset U, \qquad Z^2 \subset U, \qquad U^2 \subset Z. \tag{3}$$

All these facts are obtained from the linearized form of the identity (1). The above decomposition depends on the choice of the idempotent. However, it is known that the dimension of U and consequently the dimension of Z are invariants of A. The pair $(\dim U + 1, \dim Z)$ is called the type of A. See [5].

We now assume that $N^2 = Kc$. From (3) it follows that

$$UZ \subset U \cap Kc, \qquad Z^2 \subset U \cap Kc, \qquad U^2 \subset Z \cap Kc.$$

There are two possibilities. If $U^2 \neq 0$, then $UZ = Z^2 = 0$ and the multiplication table is known when the products $u_i u_j = b_{ij} c$ are specified for a basis $u_1, \ldots, u_r$ of U. In this case $c \in Z$. If $U^2 = 0$, then the multiplication table requires the products $z_i u_j = p_{ij} c$ and $z_i z_j = q_{ij} c$ for some basis $u_1, \ldots, u_r$ of U and $z_1, \ldots, z_s$ of Z. In this case $c \in U$ and $c^2 = 0$.

Theorem 1. *Let $A = \mathbb{R}e \oplus U \oplus Z$ be a real Bernstein algebra of type $(r+1, s)$ such that $N^2 = \mathbb{R}c$. Then one of the following assertions holds:*

(i) *A has a basis e, u_i, z_j $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s)$, and the nonzero products are*

$$e^2 = e, \qquad eu_i = \tfrac{1}{2} u_i \quad (1 \leqslant i \leqslant r),$$

$$u_i^2 = \begin{cases} c & (1 \leqslant i \leqslant k), \\ -c & (k < i \leqslant k+t) \end{cases} \qquad (\textit{for some} \quad k \geqslant 0, \quad t \geqslant 0, \quad k + t \leqslant r). \tag{4}$$

(ii) *A is a subalgebra of a complete (Bernstein) algebra with parameters s, d, k, t, and c.*

Proof. If $U^2 \neq 0$ then $UZ = Z^2 = 0$. Let $u_1, \ldots, u_r$ be a basis of U, and $u_i u_j = b_{ij} c$. The multiplication table is given by the matrix $B = (b_{ij})$. This matrix represents a symmetric bilinear form on U, and by a change of basis it can be reduced to

$$\begin{bmatrix} I_{k\times k} & & \\ & -I_{t\times t} & \\ & & O_{l\times l} \end{bmatrix},$$

where $k + t + l = r$. The integers k, t, l are unique by Theorem 5 of [1, p. 296]. This is the multiplication table given in (4).

Now assume $U^2 = 0$. Pick basis $\tilde{U} = \{u_1, \ldots, u_r\}$ of U and $\tilde{Z} = \{z_1, \ldots, z_s\}$ of Z. We can give the multiplication table for this algebra by giving the matrix $[P \mid Q]$ with submatrices P and Q. P is $s \times r$, Q is $s \times s$, $z_i u_j = p_{ij} c$, *and* $z_i z_j = q_{ij} c$. *If we change the basis of U by* $\tilde{U} = R\tilde{U}'$ *and change the basis of Z by* $\tilde{Z} = S\tilde{Z}'$, *the multiplication table changes to* $[S^tPR \mid S^tQS]$. Suppose that P is invertible. Choosing S in such a way that $S^tQS = J$, where

$$J = \begin{bmatrix} I_{k\times k} & & \\ & -I_{t\times t} & \\ & & O_{l\times l} \end{bmatrix} \qquad (k + t + l = s),$$

and setting $R = (S^tP)^{-1}$, we obtain the multiplication table $[I_{s\times s} \mid J]$. This is the multiplication table for a complete algebra with parameters s, d, k, t and element c. The parameter d is zero. If P is not invertible, then we proceed in two steps:

1. We augment P with enough columns until we obtain a matrix $P^{\#}$ with rank s. Now $[P^{\#} \mid Q]$ gives the multiplication of a complete algebra which has A as a subalgebra.

2. Since $P^{\#}$ has rank s, we can change the basis as in the previous case to obtain

$$\left[S^tP^{\#}R \mid S^tQS\right] = \left[I_{s\times s} \quad O_{s\times d} \quad \mid J\right].$$

This is the multiplication table for a complete algebra with parameters s, d, k, t and element c. ∎

The multiplication of the complete algebra is determined by the integers s, d, k, t and the element c. These parameters are determined by U and Z. The value of s is the dimension of Z; $k + t$ and $k - t$ are the rank and the signature of the symmetric bilinear form which gives the multiplication for Z. The subspace $\{u \in U : uZ = 0\}$ depends only on N, and its dimension is d. The equality $N^2 = \mathbb{R}c$ determines the element c up to a scalar multiple. Positive multiples leads to exactly the same values of k and t, while negative multiples lead to complete algebras with k and t interchanged. To specify one multiple over another, we could ask that c be chosen to maximize the signature.

The complete algebra is determined by $Ke \oplus U \oplus Z$. However, complete algebras defined by different multiplication tables can still be isomorphic. The algebras given by the following two tables are isomorphic:

$$A: \quad \begin{array}{c c} & \begin{array}{cccc} u_1 & u_2 & z_1 & z_2 \end{array} \\ \begin{array}{c} z_1 \\ z_2 \end{array} & \left[\begin{array}{cc|cc} c & 0 & c & 0 \\ 0 & c & 0 & c \end{array}\right] \end{array} \qquad c = u_1 + u_2,$$

$$A': \quad \begin{array}{c c} & \begin{array}{cccc} u_1' & u_2' & z_1' & z_2' \end{array} \\ \begin{array}{c} z_1' \\ z_2' \end{array} & \left[\begin{array}{cc|cc} c' & 0 & c' & 0 \\ 0 & c' & 0 & -c' \end{array}\right] \end{array} \qquad c' = \sqrt{2}\, u_2'.$$

The matrix for an isomorphism from N' to N relative to $\{u_1', u_2', z_1', z_2'\}$ and $\{u_1, u_2, z_1, z_2\}$ is

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & -1 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

The idempotent e' is mapped to the idempotent $e + \frac{1}{4}(u_1 + u_2)$.

Our proof shows how to embed the algebra into a complete algebra by simply enlarging the space U. If we start with a complete algebra $Ke \oplus U^* \oplus Z$ and let U be any subspace of U^* containing c, then the subspace $Ke \oplus U \oplus Z$ will also be a subalgebra. The set of all Bernstein algebras satisfying $N^2 = \mathbb{R}c$ with $c \in U$ can be generated in this way. The further question of which subspaces of U^* lead to isomorphic algebras seems to be a very difficult one and is related to the classical problems of bilinear and quadratic forms.

3. THE JORDAN PROPERTY

A Jordan algebra is a commutative nonassociative algebra that satisfies the identity

$$(x^2y)x = x^2(yx).$$

As shown in [8], a Bernstein algebra $A = Ke \oplus U \oplus Z$ is a Jordan algebra if and only if

$$Z^2 = 0, \qquad (uz)z = 0 \quad \text{for any } u \in U,\ z \in Z.$$

Now assume that $N^2 = Kc$. When $U^2 \neq 0$ the algebra is always Jordan. In the case when $U^2 = 0$, the algebra is Jordan if and only if $Z^2 = 0$ and $cN = 0$.

THEOREM 2. *Let A be a real Bernstein algebra of type $(r+1, s)$ such that $N^2 = \mathbb{R}c$. Then A is a Jordan algebra if and only if one of the following statements is true*:

(i) *A is the algebra given in part* (i) *of Theorem* 1.

(ii) *A has a basis e, u_i, z_j $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s$, with $u_r = c)$, and the nonzero products are*

$$e^2 = e, \qquad eu_i = \tfrac{1}{2}u_i \quad (1 \leqslant i \leqslant r),$$

$$u_i z_i = u_r = c \qquad (1 \leqslant i \leqslant k, \text{ for some } k < r \text{ and } k \leqslant s). \tag{5}$$

Proof. The algebras given in (i) and (ii) are clearly Bernstein Jordan algebras. Conversely, assume that A is Jordan. As in Theorem 1, we have two cases. If $U^2 \neq 0$, then A is necessarily the algebra given in part (i) of Theorem 1. Now assume that $U^2 = 0$. We know that $c \in U$, $cN = 0$, and $Z^2 = 0$. The only products left to specify are ZU. Let $\tilde{U} = \{u_1, \dots, u_r\}$ be a basis of U with $u_r = c$, and $\tilde{Z} = \{z_1, \dots, z_s\}$ be a basis of Z. The multiplication is given by $z_i u_j = p_{ij}c$. If we change the basis $\tilde{U} = R\tilde{U}'$ and $\tilde{Z} = S\tilde{Z}'$, the new table is given by the matrix $P' = S^t PR$ where $P = (p_{ij})$. Then rank P = rank $P' = k$, and we can choose S and R such that

$$P' = \begin{bmatrix} I_{k\times k} & O \\ O & O \end{bmatrix}.$$

This is the table for the algebra in (5). ∎

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