Classical adjoint-commuting mappings on Hermitian and symmetric matrices

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ABSTRACT

Let m and n be integers such that m, n ⩾ 3, and let $\mathbb{F}$ and $\mathbb{K}$ be fields which possess involutions $\neg$ of $\mathbb{F}$ and $\wedge$ of $\mathbb{K}$, respectively. Let $\mathcal{H}_n(\mathbb{F})$ be the $\mathbb{F}$-linear space of $n \times n$ Hermitian matrices over $\mathbb{F}$. In this note, we address the general description of mappings $\psi$ satisfying one of the following conditions:

1. $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$, with either $|\mathbb{K}^\wedge| = 2$ or $|\mathbb{F}^\neg|, |\mathbb{K}^\wedge| > 3$ and $\mathbb{F}$ and $\mathbb{K}$ of characteristic $\neq 2$ if $\neg$ and $\wedge$ are the identity maps, and $\psi$ is surjective satisfying $\psi(\text{adj}(A - B)) = \text{adj}(\psi(A) - \psi(B))$ for every $A, B \in \mathcal{H}_n(\mathbb{F})$.

2. $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_n(\mathbb{F})$, with either $|\mathbb{F}^\neg| = 2$ or $|\mathbb{F}^\neg| > n + 1$, and $\psi(\text{adj}(A + \alpha B)) = \text{adj}(\psi(A) + \alpha\psi(B))$ for every $A, B \in \mathcal{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^\neg$.

3. $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ is additive with $\psi(\text{adj}(A)) = \psi(\text{adj}A)$ for every $A \in \mathcal{H}_n(\mathbb{F})$.

Here, $\mathbb{F}^\neg := \{a \in \mathbb{F} : \alpha = a\}$ and $\mathbb{K}^\wedge := \{a \in \mathbb{K} : \hat{a} = a\}$ are the fixed fields with respect to the involutions $\neg$ of $\mathbb{F}$ and $\wedge$ of $\mathbb{K}$, respectively, and $\text{adj}A$ denotes the classical adjoint of the matrix $A$.

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1. Introduction

Let \( \mathbb{F} \) be a field. A mapping \( \tau : \mathbb{F} \rightarrow \mathbb{F} \) is called an involution of \( \mathbb{F} \) if it satisfies \( \overline{ab} = \overline{b} \overline{a} = \overline{a} \overline{b} \), \( \overline{a} = a \) for all \( a, b \in \mathbb{F} \). We say that \( \mathbb{F} \) possesses an involution if it has an involution. Let \( \mathbb{F}^- := \{ a \in \mathbb{F} : \overline{a} = a \} \) denote the set of all symmetric elements of \( \mathbb{F} \) with respect to the involution \( \tau \) of \( \mathbb{F} \). Since \( \tau \) is an anti-automorphism with \( \overline{a} = a \) for every \( a \in \mathbb{F} \), one can easily check that \( \mathbb{F}^- \) is a subfield of \( \mathbb{F} \). We call \( \mathbb{F}^- \) the fixed field with respect to the involution \( \tau \) of \( \mathbb{F} \). Let \( m \) and \( n \) be integers with \( m, n \geq 2 \). Let \( \mathcal{M}_{m,n}(\mathbb{F}) \) denote the linear space of \( m \times n \) matrices over \( \mathbb{F} \) ( \( \mathcal{M}_n(\mathbb{F}) = \mathcal{M}_{n,n}(\mathbb{F}) \) for short). A matrix \( A \in \mathcal{M}_{n}(\mathbb{F}) \) is called Hermitian with respect to the involution \( \tau \) of \( \mathbb{F} \), or simply Hermitian if \( \overline{A} = A \), and \( A \) is symmetric if \( A^t = A \). Here, \( A^t \) stands for the transpose of \( A \), and \( \overline{A} \) is the matrix obtained from \( A \) by applying \( \tau \) entrywise. We denote by \( \mathcal{H}_{n}(\mathbb{F}) \) the \( \mathbb{F}^- \)-linear space of \( n \times n \) Hermitian matrices over \( \mathbb{F} \), and use \( S_n(\mathbb{F}) \) to denote the linear space of \( n \times n \) symmetric matrices over \( \mathbb{F} \). Clearly, \( \mathcal{H}_{n}(\mathbb{F}) = S_n(\mathbb{F}) \) if \( \mathbb{F}^- = \mathbb{F} \). Otherwise, the involution \( \tau \) is proper, and thus, there exists an element \( i \in \mathbb{F} \), with \( i^2 = -1 \) when \( \mathbb{F} \) has characteristic \( \neq 2 \), and \( i = 1 + i \) when \( \mathbb{F} \) has characteristic \( 2 \), such that \( \mathbb{F} = \mathbb{F}^- \oplus i \mathbb{F}^- \) as an \( \mathbb{F}^- \)-linear space, see [13].

Given a matrix \( A \in \mathcal{M}_{n}(\mathbb{F}) \), the classical adjoint of \( A \), written as \( \text{adj} A \), is defined by the transposed matrix of cofactors of \( A \). More precisely, \( \text{adj} A \) is the \( n \times n \) matrix whose \((i, j)\)th entry is

\[
(\text{adj} A)_{ij} = (-1)^{i+j} \det(A[j \mid i])
\]

where \( \det(A[j \mid i]) \) denotes the determinant of the \((n - 1) \times (n - 1)\) submatrix \( A[j \mid i] \) of \( A \) obtained by excluding its \( j \)th row and \( i \)th column. Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be matrix spaces, and let \( \text{adj} A \in \mathcal{M}_1 \) whenever \( A \in \mathcal{M}_1 \) for \( i = 1, 2 \). A mapping \( \psi : \mathcal{M}_1 \rightarrow \mathcal{M}_2 \) is said to be classical adjoint-commuting if

\[\psi(\text{adj} A) = \text{adj}(\psi(A))\]  

(11)

for all \( A \in \mathcal{M}_1 \). Classical adjoint-commuting linear mappings on \( \mathcal{M}_n(\mathbb{F}) \) were first studied by Sinkhorn in [14] over the complex field by using the classical result of Frobenius [7] concerning determinant preservers. Later on, similar problems on various matrix spaces were studied in [1–3, 15–18]. Recently, inspired by the works [5,8,10], the present authors studied classical adjoint-commuting mappings \( \psi \) between matrix algebras over an arbitrary field in [4] by dropping the linearity and the additivity of \( \psi \). They studied mappings \( \psi : \mathcal{M}_n(\mathbb{F}) \rightarrow \mathcal{M}_m(\mathbb{F}) \), with \( m, n \geq 3 \), satisfying one of the following two conditions: For any matrices \( A, B \in \mathcal{M}_n(\mathbb{F}) \) and scalar \( \alpha \in \mathbb{F} \),

\[
\begin{align*}
(\text{AH-1}) & \quad \psi(\text{adj}(A + \alpha B)) = \text{adj}(\psi(A) + \alpha \psi(B)). \\
(\text{AH-2}) & \quad \psi(\text{adj}(A - B)) = \text{adj}(\psi(A) - \psi(B)).
\end{align*}
\]

Notice that if \( \psi \) satisfies condition (\text{AH-1}) or (\text{AH-2}), then \( \psi(0) = 0 \). Consequently, condition (1.1) holds, and hence, \( \psi \) is a classical adjoint-commuting mapping.

Let \( m \) and \( n \) be integers with \( m, n \geq 3 \), and let \( \mathbb{F} \) and \( \mathbb{K} \) be fields which possess involutions \( \tau \) of \( \mathbb{F} \) and \( \hat{\tau} \) of \( \mathbb{K} \), respectively. Let \( \mathbb{F}^- \) and \( \mathbb{K}^- \) be the fixed fields on the involutions \( \tau \) of \( \mathbb{F} \) and \( \hat{\tau} \) of \( \mathbb{K} \), respectively. In this note, we continue the study of classical adjoint-commuting mappings on Hermitian and symmetric matrices. We investigate the structure of \( \psi : \mathcal{H}_{m}(\mathbb{F}) \rightarrow \mathcal{H}_{m}(\mathbb{K}) \) satisfying either condition (\text{AH-1}) for every \( A, B \in \mathcal{H}_{n}(\mathbb{F}) \) and \( \alpha \in \mathbb{F}^- \) with \( (\mathbb{K}^-, \hat{\tau}) = (\mathbb{F}^-, \tau) \), or condition (\text{AH-2}) for every \( A, B \in \mathcal{H}_{n}(\mathbb{F}) \). In the same note, a complete characterization of classical adjoint-commuting additive mappings from \( \mathcal{H}_{n}(\mathbb{F}) \) into \( \mathcal{H}_{m}(\mathbb{K}) \), with no condition imposed on the underlying fields \( \mathbb{F} \) and \( \mathbb{K} \), is also obtained (see Theorems 2.10 and 2.11), which generalizes some results in [16] and [17].

We should point out that, in order to obtain a nice structural result of \( \psi \) which satisfies (\text{AH-1}) or (\text{AH-2}), the condition of \( \psi(I_n) \neq 0 \) is an indispensable assumption, where \( I_n \) is the identity matrix. Firstly, if \( \psi \) satisfies (\text{AH-1}) or (\text{AH-2}) with \( \psi(I_n) \neq 0 \), then it can be proved that \( \psi \) is injective and satisfies \( \text{rank}(A - B) = n \) if and only if \( \text{rank}(\psi(A) - \psi(B)) = m \) (see Lemma 2.8(b)). Here, \( \text{rank} A \) denotes the rank of the matrix \( A \). Further, if \( \psi \) satisfies (\text{AH-1}) with \( \psi(I_n) \neq 0 \), then it turns out that \( \psi \) is additive (see Lemma 2.9). By using the structural results obtained in Theorems 2.10 and 2.11, the structure of \( \psi \) is classified (see Theorems 2.12 and 2.13). Secondly, if \( \psi \) satisfies (\text{AH-1}) or (\text{AH-2}) with \( \psi(I_n) = 0 \), then \( \psi(A) = 0 \) for every rank one matrix \( A \in \mathcal{H}_{n}(\mathbb{F}) \) (see, Lemma 2.8(a)). If \( \psi \) is
additive, then it is easily proved that $\psi = 0$. Nevertheless, under the condition of (AH-1) or (AH-2), beside the zero mapping, there are nonzero classical adjoint-commuting mappings sending rank one matrices to zero. Indeed, in Theorems 2.12 and 2.13, a necessary and sufficient condition of classical adjoint-commuting mappings $\psi$ satisfying (AH-1) is obtained.

Before starting the proofs, we give four examples of nonzero classical adjoint-commuting mappings on Hermitian and symmetric matrices that send rank one matrices to zero. We note that $\psi_2$ (for $m \geq 5$) and $\psi_3$ are not rank nonincreasing mappings, which means, they do not satisfy condition rank $\psi_i(A) \leq \operatorname{rank} A$ for $i = 2, 3$.

Example 1.1. Let $m$ and $n$ be integers with $m, n \geq 3$, and let $F$ and $K$ be fields which possess involutions $-\cdot$ of $F$ and $^\wedge$ of $K$, respectively. Let $F^-$ and $K^\wedge$ be the fixed fields with respect to the involutions $-\cdot$ of $F$ and $^\wedge$ of $K$, respectively.

(i) Let $\sigma : F^- \to K^\wedge$ be a nonzero function. Let $\psi_1 : \mathcal{H}_n(F) \to \mathcal{H}_m(K)$ be the mapping defined by

$$
\psi_1(A) = \begin{cases} 
\sigma(a_{11})E_{11} & \text{if } A = (a_{ij}) \in \mathcal{H}_n(F) \text{ is of rank } k \text{ with } 1 < k < n, \\
0 & \text{otherwise}.
\end{cases}
$$

(ii) Let $m, n \geq 4$. Let $f : \mathcal{H}_n(F) \to K^\wedge$ be a nonzero function and let $\tau : (F, -\cdot) \to (K, ^\wedge)$ be a nonzero field homomorphism such that $\tau(a) = \tau(a)$ for all $a \in F$. Define the mapping $\psi_2 : \mathcal{H}_n(F) \to \mathcal{H}_m(K)$ by

$$
\psi_2(A) = \begin{cases} 
\sum_{i=1}^{m-2} f(A)E_{ii} & \text{if } \operatorname{rank} A = 2, \\
\tau(a_{12})E_{12} + \tau(a_{21})E_{21} & \text{if } A = (a_{ij}) \in \mathcal{H}_n(F) \text{ is of rank } k, 2 < k < n, \\
0 & \text{otherwise}.
\end{cases}
$$

(iii) Let $m, n \geq 5$. Let $\sigma : F^- \to K^\wedge$ be a nonzero function, and let $\tau : (F, -\cdot) \to (K, ^\wedge)$ be a nonzero field homomorphism such that $\tau(a) = \tau(a)$ for all $a \in F$. Let $\psi_3 : \mathcal{H}_n(F) \to \mathcal{H}_m(K)$ be the mapping, for every $A = (a_{ij}) \in \mathcal{H}_n(F)$, defined by

$$
\psi_3(A) = \begin{cases} 
\sum_{i=1}^{m-2} \sigma(a_{ii})E_{ii} & \text{if } \operatorname{rank} A = k, 1 < k < n, k \text{ odd,} \\
\sigma(a_{11})E_{11} + \tau(a_{23})E_{23} + \tau(a_{32})E_{32} & \text{if } \operatorname{rank} A = k, 1 < k < n, k \text{ even,} \\
0 & \text{otherwise}.
\end{cases}
$$

(iv) Let $m \geq n + 2$ and let $\mathcal{H} := \{ \operatorname{adj} H : H \in \mathcal{H}_n(F) \text{ is invertible} \}$. Let $\phi : \mathcal{H}_n(F) \to \mathcal{H}_n(K)$ be a nonzero mapping. We define the mapping $\psi_4 : \mathcal{H}_n(F) \to \mathcal{H}_m(K)$ by

$$
\psi_4(A) = \begin{cases} 
0 & \text{if } A \in \mathcal{H}_n(F) \text{ is of rank } 0 \text{ or } 1, \text{ or } A \in \mathcal{H}, \\
\phi(A) \oplus 0_{m-n} & \text{otherwise}.
\end{cases}
$$

Here, $E_{ij}$ stands for the square matrix unit whose $(i,j)$th entry is one and the others are zero. It is not difficult to see that each mapping $\psi_i$ satisfies the conditions: $\operatorname{adj} (\psi_i(A) + \alpha \psi_i(B)) = 0$ and $\psi_i(\operatorname{adj} A) = 0$ for every matrices $A, B \in \mathcal{H}_n(F)$ and scalar $\alpha \in F^-$. Therefore, we have

$$
\psi_i(\operatorname{adj} (A + \alpha B)) = 0 = \operatorname{adj} (\psi_i(A) + \alpha \psi_i(B))
$$
for every matrices $A, B \in \mathcal{H}_n(F)$ and scalar $\alpha \in \mathbb{F}^-$. Hence, $\psi_i$ satisfies (AH-1) and (AH-2) with $\psi_i(I_n) = 0$. We note that these classical adjoint-commuting mappings are neither injective nor surjective, and also, the integers $m$ and $n$ are not necessarily the same.

2. Proof

Throughout this section, unless otherwise stated, let $F$ and $K$ be fields which possess involutions $- \in F$ and $^\wedge \in K$, respectively, and let $F^- := \{a \in F : \bar{a} = a\}$ and $K^- := \{a \in K : \bar{a} = a\}$ denote the fixed fields with respect to the involutions $- \in F$ and $^\wedge \in K$, respectively.

Recall that a matrix $A \in \mathcal{M}_n(F)$ is_alternate if $uAu^t = 0$ for every row vector $u \in \mathbb{F}^n$, or equivalently, if $A^t = -A$ with zero diagonal entries. We start with the following result proved in [19, Propositions 1.32 and 1.34] and [9, Theorems 2.5.1 and 2.5.3].

**Lemma 2.1.** Let $n \geq 2$, and let $A \in \mathcal{M}_n(F)$ be a nonzero matrix. Then $A$ is Hermitian if and only if there exist a positive integer $k \leq n$ and an invertible matrix $P \in \mathcal{M}_n(F)$ such that either

$$A = P \left( \sum_{i=1}^{k} \alpha_i E_{ii} \right) P^t$$

for some nonzero scalars $\alpha_1, \ldots, \alpha_k \in \mathbb{F}^-\wedge$; or

$$A = P(J_1 \oplus \cdots \oplus J_{k/2} \oplus 0_{n-k})P^t$$

when $A$ is alternate and the involution $-$ is identity. Here

$$J_1 = \cdots = J_{k/2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(F).$$

Moreover, if $A$ is of Form $(2.2)$, then $k$ is necessarily even and $F$ is of characteristic 2.

**Lemma 2.2.** Let $n \geq 2$. If $A \in \mathcal{H}_n(F)$ is of rank one, then there is a rank $n - 1$ matrix $B \in \mathcal{H}_n(F)$ such that $A = \text{adj} B$.

**Proof.** If $A$ is of rank one, then, in view of Lemma 2.1, we see that there exist an invertible matrix $P \in \mathcal{M}_n(F)$ and a nonzero scalar $\alpha \in \mathbb{F}^-\wedge$ such that $A = P(\alpha E_{11})P^t$. Let $H = \text{adj} P$ and $\theta = (\det P P^t)^{n-2}$. Clearly, $H$ is an invertible matrix in $\mathcal{M}_n(F)$ and $\theta$ is a nonzero scalar in $\mathbb{F}^-\wedge$. Since $\text{adj}(I_n - E_{11} + (\theta^{-1} \alpha - 1)E_{22}) = \theta^{-1} \alpha E_{11}$, it follows that

$$A = \theta^{-1} \alpha E_{11} = \text{adj} P \theta (\theta^{-1} \alpha E_{11}) \text{adj} P^t$$

$$= (\det P)^{n-2} P \text{adj}(I_n - E_{11} + (\theta^{-1} \alpha - 1)E_{22}) (\det P)^{n-2} P^t$$

$$= \text{adj} \text{adj} P \theta (\theta^{-1} \alpha - 1) \text{adj} P = \text{adj} B$$

where $B = \text{adj} P (I_n - E_{11} + (\theta^{-1} \alpha - 1)E_{22}) \in \mathcal{H}_n(F)$ is of rank $n - 1$. We are done. □

**Lemma 2.3.** Let $n \geq 2$ and let $F$ be a field which possesses an involution $-$ of $F$. If $A \in \mathcal{H}_n(F)$ is a nonzero rank $r$ matrix, then $A = A_1 + \cdots + A_k$ for some rank one matrices $A_1, \ldots, A_k \in \mathcal{H}_n(F)$ with

$$k = \begin{cases} r + 1 & \text{if } A \text{ is alternate and } - \text{ is identity}, \\ r & \text{otherwise}. \end{cases}$$
Proof. We divide our proof into two cases. If $A$ is of Form (2.1) in Lemma 2.1, i.e., $A = P(\alpha_1 E_{11} + \cdots + \alpha_r E_{rr})P^t$ for some invertible matrix $P \in \mathcal{M}_n(\mathbb{F})$ and some nonzero scalars $\alpha_1, \ldots, \alpha_r \in \mathbb{F}^-$, then we set $A_i = P(\alpha_i E_{ii})P^t$ for $i = 1, \ldots, r$. Clearly, each $A_i \in \mathcal{H}_n(\mathbb{F})$ is of rank one, and $A = A_1 + \cdots + A_r$, as desired. If $A$ is alternate and the involution $\bar{\cdot}$ of $\mathbb{F}$ is identity, then $A$ is of Form (2.2), i.e., $A = Q(I_1 \oplus \cdots \oplus I_{r/2} \oplus 0_{n-r})Q^t$ for some invertible matrix $Q \in \mathcal{M}_n(\mathbb{F})$, and thus, $r$ is even and $\mathbb{F}$ has characteristic 2. Let $B = Q(E_{11} + E_{22})Q^t$. Then $B \in \mathcal{H}_n(\mathbb{F})$ is of rank 2, and $A + B \in \mathcal{H}_n(\mathbb{F})$ is of odd rank $r - 1$. By Lemma 2.1, we see that $A + B$ is of Form (2.1). So, there exists an invertible matrix $R \in \mathcal{M}_n(\mathbb{F})$ such that

$$A + B = R(\beta_1 E_{11} + \cdots + \beta_{r-1} E_{r-1,r-1})R^t$$

for some nonzero scalars $\beta_1, \ldots, \beta_{r-1} \in \mathbb{F}^- = \mathbb{F}$. Let $A_i = R(\beta_i E_{ii})R^t$ for $i = 1, \ldots, r - 1$, and $A_r = Q(-E_{11})Q^t$ and $A_{r+1} = Q(-E_{22})Q^t$. Evidently, $A_i \in \mathcal{H}_n(\mathbb{F})$ is of rank one for $i = 1, \ldots, r - 1$, and $A = A_1 + \cdots + A_r + A_{r+1}$. We are done. □

Lemma 2.4. Let $n \geq 3$ and let $A, B \in \mathcal{H}_n(\mathbb{F})$. Then the following assertions hold.

(a) If $A$ is of rank $r$, then there is a rank $n - r$ matrix $C_1 \in \mathcal{H}_n(\mathbb{F})$ such that $rank(A + C_1) = n$.
(b) There exists a matrix $C_2 \in \mathcal{H}_n(\mathbb{F})$ such that $rank(A + C_2) = rank(B + C_2) = n$.
(c) There exists a nonzero matrix $C_3 \in \mathcal{H}_n(\mathbb{F})$ such that either $A$ or $C_3$ is of rank $n$ but not both with $rank(A + C_3) = n$.
(d) If $A$ is a nonzero matrix, then there exists a matrix $C_4 \in \mathcal{H}_n(\mathbb{F})$ with $rank(C_4) \leq n - 2$ such that $rank(A + C_4) = n - 1$.
(e) If $|\mathbb{F}^-| > n + 1$ and rank $(A + B) = n$, then there exists a scalar $\lambda_0 \in \mathbb{F}^-$ with $\lambda_0 \neq 1$ such that $rank(A + \lambda_0 B) = n$.

Proof. Firstly, we see that if $A \in \mathcal{H}_n(\mathbb{F})$ is a nonzero rank $k$ matrix, then, by Lemma 2.1, there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{F})$ such that either $A$ is of Form (2.1), i.e.,

$$A = P(\alpha_1 E_{11} + \cdots + \alpha_k E_{kk})P^t$$

for some nonzero scalars $\alpha_1, \ldots, \alpha_k \in \mathbb{F}^-; or if A is alternat and the involution $\bar{\cdot}$ of $\mathbb{F}$ is identity, then $A$ can be written as Form (2.2), i.e.,

$$A = P(j_1 \oplus \cdots \oplus j_{k/2} \oplus 0_{n-k})P^t$$

such that $k$ is even, $\mathbb{F}$ has characteristic 2, and

$$j_1 = \cdots = j_{k/2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{F}).$$

With this observation, we now proceed to prove the lemma.

(a) The result is clear when $r = 0$ (set $C_1 = I_n$) or $r = n$ (take $C_1 = 0$). Suppose $1 < r < n$. Then, we set

$$C_1 = \begin{cases} P(E_{r+1,r+1} + \cdots + E_{mm})P^t & \text{if } A \text{ is of Form (2.1)}, \\
P(E_{r+1,r+1} + \cdots + E_{mm})P^t & \text{if } A \text{ is of Form (2.2)}. \end{cases}$$

In all cases, we see that $C_1 \in \mathcal{H}_n(\mathbb{F})$ is of rank $n - r$ and rank $(A + C_1) = n$, as required.

(b) If $A = B$, then we select $C_2 = I_n - A$, as required. Suppose $A \neq B$. Let $H = A - B$. Then $H \in \mathcal{H}_n(\mathbb{F})$ and $0 < rank H = k \leq n$. We consider the following two cases. Case I: If $H$ is of Form
Case II: If $\alpha Z_{ij} := E_{ij} + E_{ji} - \alpha E_{ii} \in \mathcal{H}_n(\mathbb{F})$ for $1 \leq i < j \leq n$ and $\alpha \in \mathbb{F}^-$. Case II: If $H$ is alternate and the involution $- \cdot$ of $\mathbb{F}$ is identity, then $H$ is of Form (2.2). Let $p$ be the greatest integer less than or equal to $n/2$, and let $q$ be the smallest integer greater than or equal to $n/2$. Let $h$ be an odd integer satisfying $p - 1 \leq h \leq p$. Set

$$D = \begin{cases} P_{j1n} P^t & \text{if } k < q + 1, \\ P(j_n - S_h) P^t & \text{if } k \geq q + 1, \text{ and } h \neq p \text{ or } h \neq q, \\ P(j_{n-1} - S_{h-2} + E_{nn}) P^t & \text{if } k \geq q + 1 \text{ and } h = p = q, \end{cases}$$

where $j_{1r} := E_{1r} + E_{2r-1} + \cdots + E_{r1} \text{ for } 1 \leq r \leq n$, and $S_r := (E_{12} + E_{21}) + (E_{34} + E_{43}) + \cdots + (E_{r,r+1} + E_{r+1,r}) \text{ for } 1 \leq r < n \text{ with } r \text{ odd. By a direct verification, in both cases of } H, \text{ it can be checked that } D \in \mathcal{H}_n(\mathbb{F}), \text{ and } A + C_2 = H + D \text{ and } B + C_2 = D, \text{ as desired.}

(c) If $A$ is of rank $n$, then we take

$$C_3 = \begin{cases} P(-\alpha_1 E_{11} + E_{12} + E_{21}) P^t & \text{if } A \text{ is of Form (2.1),} \\ PE_{11} P^t & \text{if } A \text{ is of Form (2.2).} \end{cases}$$

In both cases of $A$, we see that $C_3 \in \mathcal{H}_n(\mathbb{F})$ with rank $C_3 < n$, and rank $(A + C_3) = n$. We now consider rank $A = k < n$. If $A = 0$, then we take $C_3 = I_n$. Suppose $A \neq 0$. Then we consider the following two cases. Case I: If $A$ is of Form (2.1), then, by using the definition of $\alpha Z_{ij}$ as given in Case I of (b), we let

$$C_3 = \begin{cases} P(\alpha_1 Z_{12} + \alpha_3 Z_{34} + \cdots + \alpha_{k-1} Z_{k-1,k} + E_{k+1,k+1} + \cdots + E_{nn}) P^t & \text{if } k \text{ is even,} \\ P(\alpha_1 Z_{12} + \alpha_3 Z_{34} + \cdots + \alpha_k Z_{k,k+1} + E_{k+2,k+2} + \cdots + E_{nn}) P^t & \text{if } k \text{ is odd.} \end{cases}$$

Case II: If $A$ is of Form (2.2), then, by using the definitions of $p$, $q$ and $h$ as given in Case II of (b), we let

$$C_3 = \begin{cases} P_{j1n} P^t & \text{if } k < q + 1 \\ P(j_n - S_h) P^t & \text{if } k \geq q + 1, \text{ and } h \neq p \text{ or } h \neq q \\ P(j_{n-1} - S_{h-2} + E_{nn}) P^t & \text{if } k \geq q + 1 \text{ and } h = p = q. \end{cases}$$

By a direct verification, in both cases of $A$, we can check that $C_3 \in \mathcal{H}_n(\mathbb{F})$ is of rank $n$ and rank $(A + C_3) = n$. 

(d) Suppose rank $A = k$. If $k = n - 1$, then the result holds when we select $C_4 = 0$. We now consider $1 \leq k < n - 1$. We set $C_4 = P(E_{k+1,k+1} + \cdots + E_{n-1,n-1}) P^t$. Clearly, $C_4 \in \mathcal{H}_n(\mathbb{F})$ and rank $C_4 \leq n - 2$. Clearly, in both Forms (2.1) and (2.2) of $A$, we have rank $(A + C_4) = n - 1$. If $k = n,
then we let

\[
C_4 = \begin{cases} 
  P(-\alpha_n E_{nn}) P^T & \text{if } A \text{ is of Form (2.1)}, \\
  P(E_{11} + E_{22}) P^T & \text{if } A \text{ is of Form (2.2)}. 
\end{cases}
\]

We note that if \( A \) is of Form (2.2) with \( \text{rank} \ A = n \), then \( n \geq 4 \). So, in both cases, \( C_4 \in \mathcal{H}_n(\mathbb{F}) \) with rank \( C_4 \leq n - 2 \), and rank \( A + C_4 = n - 1 \). We are done.

(e) For each element \( x \in \mathbb{F}^- \), we denote \( g(x) = \det(A + xB) \). Since \( g(1) \neq 0 \) and \( g(x) = \det(A + xB) = g(x) \) as \( A + xB \in \mathcal{H}_n(\mathbb{F}) \), it follows that \( g \) is a nonzero polynomial over \( \mathbb{F}^- \). If \( B = 0 \), then rank \( A = n \), and so, the result follows by choosing \( x = 0 \). We now consider \( B \neq 0 \). Let rank \( B = k \) with \( 1 \leq k \leq n \). Then

\[
g(x) = \begin{cases} 
  \alpha \det(E + x(\alpha_1 E_{11} + \cdots + \alpha_k E_{kk})) & \text{if } B \text{ is of Form (2.1)}, \\
  \beta \det(F + x(E_{12} + E_{21} + E_{34} + E_{43} + \cdots + E_{k-1,k} + E_{kk})) & \text{if } B \text{ is of Form (2.2)}
\end{cases}
\]

where \( E = P^{-1} A \mathbb{F}^{-1} \in \mathcal{H}_n(\mathbb{F}) \) and \( 0 \neq \alpha = \det(P \mathbb{F}^{-1}) \in \mathbb{F}^- \) when \( B \) is of Form (2.1), and \( F = P^{-1} A \mathbb{F}^{-1} \in \mathcal{H}_n(\mathbb{F}) \) and \( 0 \neq \beta = \det(P \mathbb{F}^{-1}) \in \mathbb{F}^- = \mathbb{F} \) when \( B \) is of Form (2.2). It is clear that \( g \) is a nonzero polynomial of degree at most \( k \leq n \). Since \( |\mathbb{F}^-| > n + 1 \), there exists a scalar \( \lambda_0 \in \mathbb{F}^- \) with \( \lambda_0 \neq 1 \) such that \( g(\lambda_0) \neq 0 \). Consequently, we have rank \( (A + \lambda_0 B) = n \). The proof is complete. \( \square \)

**Lemma 2.5.** Let \( m \) and \( n \) be integers with \( m, n \geq 3 \). Let \( \psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K}) \) be a mapping satisfying (AH-2) and let \( A \in \mathcal{H}_n(\mathbb{F}) \). Then the following statements hold.

(a) \( \text{rank} \ \psi(A) \leq 1 \) if \( \text{rank} \ A = 1 \).

(b) \( \text{rank} \ \psi(A) \leq m - 1 \) if \( \text{rank} \ A = n - 1 \).

(c) \( \text{rank} \ \psi(A) \leq m - 2 \) if \( \text{rank} \ A \leq n - 2 \).

**Proof.** (a) If \( A \) is of rank one, then \( \psi(A) = \psi(\text{adj} A) = \psi(0) = 0 \), and so, \( \text{rank} \ \psi(A) \neq m \). By Lemma 2.2, there exists a rank \( n - 1 \) matrix \( B \in \mathcal{H}_n(\mathbb{F}) \) such that \( A = \text{adj} B \). Thus, \( \psi(B) = \psi(A) \), and so rank \( \psi(B) < m \) since rank \( \psi(A) \neq m \). Consequently, by the facts \( \psi(A) = \psi(\text{adj} B) \) and rank \( \psi(B) < m \), we conclude \( \text{rank} \ \psi(A) \leq 1 \), as desired.

(b) If rank \( A = n - 1 \), then \( \text{adj} \ (\text{adj} \ \psi(A)) = \psi(\text{adj} \ (\text{adj} A)) = 0 \), and so, \( \text{rank} \ \psi(A) \leq m - 1 \).

(c) If rank \( A \leq n - 2 \), then \( \text{adj} \ \psi(A) = \psi(\text{adj} A) = \psi(0) = 0 \). So, \( \text{rank} \ \psi(A) \leq m - 2 \). \( \square \)

**Lemma 2.6.** Let \( m \) and \( n \) be integers with \( m, n \geq 3 \). If \( \psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K}) \) is a mapping satisfying (AH-2), then the following statements are equivalent.

(a) \( \psi \) is injective.

(b) \( \ker \ \psi = \{0\} \).

(c) \( \text{rank} \ A = n \) if and only if \( \text{rank} \ \psi(A) = m \).

**Proof.** (a) \( \Rightarrow \) (b): Trivial. (b) \( \Rightarrow \) (c): Let \( A \in \mathcal{H}_n(\mathbb{F}) \). If rank \( \psi(A) = m \), then rank \( A = n \) by Lemma 2.5 (b) and (c). Conversely, let rank \( A = n \). Suppose rank \( \psi(A) < m \). Then \( \psi(\text{adj} \ (\text{adj} A)) = \text{adj} \ (\text{adj} \ \psi(A)) = 0 \) as \( m \geq 3 \). Since \( \ker \ \psi = \{0\} \), it follows that \( \text{adj} \ (\text{adj} A) = 0 \), which contradicts to the invertibility of \( A \). Thus, rank \( \psi(A) = m \). (c) \( \Rightarrow \) (a): Let \( \psi(A) = \psi(B) \) for some \( A, B \in \mathcal{H}_n(\mathbb{F}) \). We claim that \( A = B \). Suppose \( A - B = C \) is of rank \( k \). Then, by Lemma 2.4 (a), there exists a rank \( n - k \) matrix \( C \in \mathcal{H}_n(\mathbb{F}) \) such that \( A - B + C \) is of rank \( n \). Then \( \text{adj} \ (A - B + C) \) is of rank \( n \). So, by (c), we have
rank \((\text{adj } \psi(A - B + C)) = \text{rank } (\psi(\text{adj } (A - B + C))) = m\). Then

\[
\text{adj } \psi(C) = \text{adj } \psi(B - (B - C)) = \text{adj } (\psi(B) - \psi(B - C)) = \text{adj } (\psi(A) - \psi(B - C)) = \text{adj } (A - B + C)
\]

is of rank \(m\). Therefore, rank \(\psi(C) = m\) implies rank \(C = n\) by (c). Thus, \(k = 0\), and hence, \(A = B\) as claimed. This proves that \(\psi\) is injective. We are done. \(\square\)

**Lemma 2.7.** Let \(m\) and \(n\) be integers with \(m, n \geq 3\). Let \(\psi : \mathcal{H}_n(F) \to \mathcal{H}_m(K)\) be a mapping satisfying (AH-2). Let \(P \in \mathcal{M}_n(F)\) be a fixed invertible matrix, and let \(T_P : \mathcal{H}_n(F) \to \mathcal{H}_m(K)\) be the mapping defined by

\[
T_P(A) = \psi(PAP^t) \quad \text{for every } A \in \mathcal{H}_n(F).
\]

If \(\text{rank } T_P(I_n) \neq m\), then \(T_P(A) = 0\) for all rank one \(A \in \mathcal{H}_n(F)\), and rank \(T_P(A) \leq m - 2\) for all \(A \in \mathcal{H}_n(F)\).

**Proof.** Firstly, we note, by the definition of \(T_P\), that results (a), (b) and (c) of Lemma 2.5 hold true for \(T_P\) as well. Denote \(\theta := \det (PAP^t)^{-2}\). Clearly, \(\theta\) is a nonzero scalar in \(F^\times\). We first show that

\[
T_P(\theta I_n) = 0.
\]  
(2.3)

Since \(\text{adj } (\text{adj } (PAP^t)) = \theta PAP^t\), we obtain \(T_P(\theta I_n) = \psi(\text{adj } (\text{adj } (PAP^t))) = \text{adj } (\text{adj } T_P(I_n)) = 0\) since rank \(T_P(I_n) \neq m\). We show that

\[
\text{adj } T_P(A - B) = \text{adj } (T_P(A) - T_P(B)) \quad \text{for every } A, B \in \mathcal{H}_n(F).
\]  
(2.4)

Let \(A, B \in \mathcal{H}_n(F)\). We see that \(\text{adj } T_P(A - B) = \psi(\text{adj } (P(A - B)P^t)) = \psi(\text{adj } (PAP^t - PBP^t)) = \psi(\text{adj } (PAP^t - \psi(PBP^t))) = \text{adj } (T_P(A) - T_P(B))\), as desired. Denote \(H := \text{adj } P\) and \(\vartheta := \theta^{-1}\). Evidently, \(H\) is an invertible matrix in \(\mathcal{M}_n(F)\) and \(\vartheta\) is a nonzero scalar in \(F^\times\). We now claim that

\[
\psi(\vartheta H^t H) = 0.
\]  
(2.5)

By (2.3), we have \(\psi(\vartheta H^t H) = \psi(\text{adj } (\psi(PH^t))) = \psi(\text{adj } \psi(PH^t)) = \text{adj } T_P(\theta I_n) = 0\), as claimed. Next, we prove that

\[
\psi(\vartheta H^t \partial E_{ii} H) = 0 \quad \text{for every } i = 1, \ldots, n.
\]  
(2.6)

Let \(1 \leq i \leq n\). Since \(\partial E_{ii} = \text{adj } (\partial (I_n - E_{ii}))\), by (2.3) and (2.4), we see that \(\psi(\vartheta H^t \partial E_{ii} H) = \psi(\text{adj } (\partial (I_n - E_{ii})) H^t)) = \text{adj } \psi(\text{adj } (\partial (I_n - E_{ii})) H^t)) = \text{adj } \psi(\partial (I_n - E_{ii}) H^t)) = \text{adj } (T_P(\partial E_{ii}) = 0\) because rank \(T_P(\partial E_{ii}) \leq 1\) and \(m \geq 3\). We claim that for each \(1 \leq i \leq n\),

\[
T_P(\alpha E_{ii}) = 0 \quad \text{for every } \alpha \in F^\times.
\]  
(2.7)

The result is clear when \(\alpha = 0\). Suppose \(\alpha \neq 0\). Since

\[
\text{adj } (\partial I_n - \partial E_{ii} - \partial E_{jj} + \vartheta^{-1} \vartheta^{2^{-n}} \alpha E_{jj}) = \vartheta^{-1} \alpha E_{ii}
\]
with \( j \neq i \), and \( \text{adj} \, H = (\det P)^{n-2} P \), it follows from (2.5) and (2.6) that

\[
T_P(\alpha E_{ii}) = \psi(P(\alpha E_{ii})P^T) = \psi((\det P)^{n-2} P(\partial I_n - \partial E_{ii} - \partial E_{jj} + (\theta^{-1}\partial^{2-n}\alpha)E_{jj}))(\det P)^{n-2} P^T
\]

\[
= \psi((\det P)^{n-2} P(\partial I_n - \partial E_{ii} - \partial E_{jj} + (\theta^{-1}\partial^{2-n}\alpha)E_{jj})\text{adj} P)
\]

\[
= \psi((\det P)^{n-2} P(\partial I_n - \partial E_{ii} - \partial E_{jj} + (\theta^{-1}\partial^{2-n}\alpha)E_{jj}))(\det P)^{n-2} P^T
\]

because \( \text{rank} \, \psi(\theta^{(1}\partial^{2-n}\alpha)E_{jj}H) \leq 1 \) and \( m \geq 3 \). We now prove, for each \( 1 \leq i \leq n \), that

\[
\psi(\theta^{2-n}\alpha)E_{jj}H = 0 \quad \text{for every} \quad \alpha \in \mathbb{F}^-.
\]

Let \( 1 \leq i \leq n \) and let \( \alpha \in \mathbb{F}^- \). Then, since \( \text{adj} \, (I_n - E_{ii} - E_{jj} + \alpha E_{jj}) = \alpha E_{ii} \), by the obtained results (2.4) and (2.7), we have

\[
\psi(\theta^{2-n}\alpha)E_{jj}H = \psi((\det P)^{n-2} P(\partial I_n - \partial E_{ii} - \partial E_{jj} + (\theta^{-1}\partial^{2-n}\alpha)E_{jj}))(\det P)^{n-2} P^T
\]

\[
= \psi((\det P)^{n-2} P(\partial I_n - \partial E_{ii} - \partial E_{jj} + (\theta^{-1}\partial^{2-n}\alpha)E_{jj}))(\det P)^{n-2} P^T
\]

\[
= \psi((\det P)^{n-2} P(\partial I_n - \partial E_{ii} - \partial E_{jj} + (\theta^{-1}\partial^{2-n}\alpha)E_{jj}))(\det P)^{n-2} P^T
\]

By the result of (2.8), it is easy to see that

\[
\text{adj} \, \psi(A) - \psi(\theta^{2-n}\alpha)E_{jj}H) = \text{adj} \, \psi(A)
\]

for every \( A \in \mathcal{H}_n(\mathbb{F}) \) and \( \alpha_1, \ldots, \alpha_n \in \mathbb{F}^- \). Let \( 1 \leq i < j \leq n \). We denote

\[
X_{ijk} := I_n - E_{ii} - E_{jj} - 2E_{kk}
\]

with \( k \neq i, j \). We claim that

\[
\psi(\theta^{2-n}\alpha)E_{jj} + \alpha X_{ijk})H = 0
\]

for every \( \alpha \in \mathbb{F} \) and \( 1 \leq i \neq j \leq n \) with \( k \neq i, j \). The result clearly holds when \( \alpha = 0 \). We now consider \( \alpha \neq 0 \). Since \( \alpha X_{ijk} \in \mathbb{F}^- \) and

\[
\text{adj} \, (\alpha E_{ij} + \alpha X_{ijk}) = (\alpha E_{ij} + \alpha X_{ijk}).
\]
it follows from (2.4), (2.7)–(2.9) that

\[
\psi(H^t (\alpha E_{ij} + \overline{\alpha} E_{ji} + \overline{\alpha} \alpha X_{ijk})H)
\]

\[
= \psi((\text{adj} P^t) \text{adj} (\alpha E_{ij} + \overline{\alpha} E_{ji} + X_{ijk}) \text{adj} P)
\]

\[
= \text{adj} \psi(P(\alpha E_{ij} + \overline{\alpha} E_{ji} + X_{ijk})P^t)
\]

\[
= \text{adj} \psi(T_P(\alpha E_{ij} + \overline{\alpha} E_{ji} + X_{ijk}))
\]

\[
= \text{adj} (T_P(\alpha E_{ij} + \overline{\alpha} E_{ji} + X_{ijk} - E_{ss}) - T_P(-E_{ss})) \text{ for } s \neq i, j
\]

\[
= \text{adj} T_P(\alpha E_{ij} + \overline{\alpha} E_{ji} + (X_{ijk} - E_{ss})) = \cdots
\]

\[
= \text{adj} T_P(\alpha E_{ij} + \overline{\alpha} E_{ji} - E_{kk})
\]

\[
= \text{adj} (T_P(\alpha E_{ij} + \overline{\alpha} E_{ji}) - T_P(E_{kk}))
\]

\[
= \text{adj} (T_P(\alpha E_{ij} + \overline{\alpha} E_{ji}))
\]

\[
= \psi((\text{adj} P^t) \text{adj} (\alpha E_{ij} + \overline{\alpha} E_{ji}) \text{adj} P)
\]

\[
= \psi(H^t EH) = 0
\]

where \(E = -\alpha \overline{\alpha} E_{kk}\) when \(n = 3\), or \(E = 0\) when \(n > 3\).

We now claim that \(T_P\) sends all rank one matrices into zero. Let \(A \in \mathcal{H}_n(\mathbb{F})\) be a rank one matrix. By Lemma 2.2, there exists a rank \(n - 1\) matrix \(B = (b_{ij}) \in \mathcal{H}_n(\mathbb{F})\) such that \(\theta^{-1}A = \text{adj} B\). Since \(B \in \mathcal{H}_n(\mathbb{F})\), we have \(b_{ij} = \overline{b}_{ji}\) for all \(1 \leq i < j \leq n\), and \(b_{ii} \in \mathbb{F}^*\) for all \(1 \leq i \leq n\). By the results of (2.8)–(2.10), we see that

\[
T_P(A) = \psi(\theta P(\theta^{-1}A)P^t)
\]

\[
= \text{adj} \psi(H^t BH)
\]

\[
= \text{adj} \psi \left( \sum_{1 \leq i < j \leq n} H^t (b_{ji}E_{ji} + \overline{b}_{ji}E_{ij})H + \sum_{i=1}^{n} H^t (b_{ii}E_{ii})H \right)
\]

\[
= \text{adj} \psi \left( \sum_{1 \leq i < j \leq n} H^t (b_{ji}E_{ji} + \overline{b}_{ji}E_{ij})H \right) \text{ by (2.9)}
\]

\[
= \text{adj} \psi \left( \sum_{1 \leq i < j \leq n, i \neq 1 \text{ and } j \neq 2} H^t [(b_{ji}E_{ji} + \overline{b}_{ji}E_{ij}) + \overline{a}aX_{12k} - (\alpha E_{21} + \overline{\alpha} E_{12} + \overline{\alpha}aX_{12k})]H \right)
\]

\[
= \text{adj} \psi \left( \sum_{1 \leq i < j \leq n, i \neq 1 \text{ and } j \neq 2} H^t (b_{ji}E_{ji} + \overline{b}_{ji}E_{ij})H + H^t (\overline{b}_{21}b_{21}X_{12k})H \right), \text{ where } a = -b_{21}.
\]

Again, by (2.9), we obtain

\[
T_P(A) = \text{adj} \psi \left( \sum_{1 \leq i < j \leq n, i \neq 1 \text{ and } j \neq 2} H^t (b_{ji}E_{ji} + \overline{b}_{ji}E_{ij})H \right).
\]
Continuing in this way, we have

\[ T_P(A) = \text{adj} \ \psi \left( \sum_{1 \leq i < j \leq n, \ i \neq 1 \ and \ j \neq 2, 3} H^t (b_{ji}E_{ji} + b_{j1}E_{j1}) \right) = \cdots \]

\[ = \text{adj} \ \psi \left( H^t (b_{n,n-1}E_{n,n-1} + b_{n,n-1}E_{n-1,n}) \right) = \cdots \]

\[ = \text{adj} \ \psi \left( H^t (\overline{b}X_{n-1,n,n-2} - (\overline{b})E_{n,n-1} + (b)E_{n-1,n} + (b)(b)X_{n-1,n,n-2}) \right) \]

\[ = \text{adj} \ \psi \left( H^t (\overline{b}X_{n-1,n,n-2}) \right) = \text{adj} \ \psi \left( 0 - (H^t (\overline{b}X_{n-1,n,n-2}) ) \right) \]

\[ = \text{adj} \ \psi \left( 0 \right) \text{ by (2.9)} \]

\[ = 0 \]

where \( b = b_{n,n-1} \). Therefore, \( T_P(A) = 0 \) for every rank one matrix \( A \in \mathcal{H}_n(F) \).

We now prove that \( T_P(A) = 0 \) for all \( A \in \mathcal{H}_n(F) \). Let \( A \in \mathcal{H}_n(F) \). The result is clear when \( A = 0 \) or rank \( A = 1 \). Suppose rank \( A = k \) with \( 1 < k \leq n \). Then, by Lemma 2.3, there exist rank one matrices \( A_1, \ldots, A_h \in \mathcal{H}_n(F) \) with \( k \leq h \leq k + 1 \) such that \( A = A_1 + \cdots + A_h \). By (2.4), we get

\[ \text{adj} \ T_P(A) = \text{adj} \ T_P((A_1 + \cdots + A_h - (-A_h))) \]

\[ = \text{adj} \ (T_P(A_1 + \cdots + A_{h-1}) - T_P(-A_h)) \]

\[ = \text{adj} \ T_P(A_1 + \cdots + A_{h-1}) = \cdots \]

\[ = \text{adj} \ T_P(A_1) = 0. \]

Hence, rank \( T_P(A) \leq m - 2 \) for every \( A \in \mathcal{H}_n(F) \). The proof is complete. □

**Lemma 2.8.** Let \( m \) and \( n \) be integers with \( m, n \geq 3 \). Let \( \psi : \mathcal{H}_n(F) \to \mathcal{H}_m(K) \) be a mapping satisfying (AH-2). Then:

(a) The following statements are equivalent.

(i) \( \psi(I_n) = 0 \).

(ii) \( \psi(A) = 0 \) for all rank one matrices \( A \in \mathcal{H}_n(F) \).

(iii) \( \text{rank} \ \psi(A) \leq m - 2 \) for all \( A \in \mathcal{H}_n(F) \).

(iv) \( \text{rank} \ \psi(A) = m - 2 \).

(b) Let \( A, B \in \mathcal{H}_n(F) \). If \( \psi(I_n) \neq 0 \), then

(i) \( \psi \) is injective,

(ii) \( \text{rank} \ (\psi(A) - \psi(B)) = m \) if and only if \( \text{rank} \ (A - B) = n \).

**Proof.** (a) As an immediate consequence of Lemma 2.7 with \( \psi = T_P \) by the setting \( P = I_n \), we have (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii). Also, (iv) \( \Rightarrow \) (i) is clear since \( \text{adj} \ I_n = I_n \). Let us show (iii) \( \Rightarrow \) (iv). Let \( A \in \mathcal{H}_n(F) \). Note that \( \psi(\text{adj} \ A) = \psi(A) = 0 \) because, by our hypothesis, rank \( \psi(A) \leq m - 2 \). We are done.

(b) (i) Let \( P \in \mathcal{H}_n(F) \) be an arbitrarily fixed invertible matrix. We define the mapping \( T_P : \mathcal{H}_n(F) \to \mathcal{H}_m(K) \) by

\[ T_P(A) = \psi(PA\overline{P}) \quad \text{for every} \ A \in \mathcal{H}_n(F). \quad (2.11) \]

Since \( \psi(I_n) \neq 0 \), it follows from the fact that \( \text{adj} \ \psi(I_n) = \psi(I_n) \), we have rank \( \psi(I_n) = m \), and so, \( T_P((\overline{P}P)^{-1}) \) is of rank \( m \). In view of Lemma 2.7, we conclude that rank \( T_P(I_n) = m \); otherwise, rank \( T_P(A) \leq m - 2 \) for all \( A \in \mathcal{H}_n(F) \), which contradicts to the fact that rank \( T_P((\overline{P}P)^{-1}) = m \).
Consequently, $\text{adj } T_P(I_n)$ is of rank $m$. We denote $H := \text{adj } P$. Then
\[
\psi(\overline{F}^t H) = \psi(\text{adj } \overline{F} \text{ adj } P) = \psi(\text{adj } (P \overline{F}^t)) = \text{adj } \psi(\overline{F}^t) = \text{adj } T_P(I_n).
\]
Thus, rank $\psi(\overline{F}^t H) = m$, and so
\[
\text{rank } \text{adj } \psi(\overline{F}^t H) = m.
\]
(2.12)
We first show that
\[
\text{rank } T_P(E_{ii}) = 1 \text{ for every } i = 1, \ldots, n.
\]
Since rank $T_P(I_n) = m$, and $I_n = E_{ii} - \left(\sum_{j=1, j \neq i}^n -E_{jj}\right)$ for all $i = 1, \ldots, n$, and by the fact (2.4), we have
\[
\text{rank } \text{adj } \left(T_P(E_{ii}) - T_P\left(\sum_{j=1, j \neq i}^n -E_{jj}\right)\right) = \text{rank } (T_P(I_n)) = m.
\]
Therefore, rank\(\left(T_P(E_{ii}) - T_P\left(\sum_{j=1, j \neq i}^n -E_{jj}\right)\right) = m\), and thus
\[
\text{rank } (T_P(E_{ii})) + \text{rank } \left(T_P\left(\sum_{j=1, j \neq i}^n -E_{jj}\right)\right) \geq m.
\]
Further, by the definition of $T_P$ in (2.11), we note that results (a), (b) and (c) of Lemma 2.5 hold true for $T_P$ as well. Therefore, we have rank $T_P(E_{ii}) \leq 1$ and rank $T_P\left(\sum_{j=1, j \neq i}^n -E_{jj}\right) \leq m - 1$. Consequently, we conclude that rank $T_P(E_{ii}) = 1$. We next show that
\[
\text{rank } T_P(aE_{ii}) = 1
\]
(2.13)
for all $1 \leq i \leq n$ and nonzero scalar $a \in \mathbb{F}^\times$. In view of (2.11), we see that rank $T_P(aE_{ii}) \leq 1$ for all scalars $a \in \mathbb{F}^\times$ and $1 \leq i \leq n$ by Lemma 2.5 (a). Suppose to the contrary that $T_P(a_0E_{ij_0}) = 0$ for some $1 \leq i_0 \leq n$ and some nonzero scalar $a_0 \in \mathbb{F}^\times$. Since $n \geq 3$, we can find two distinct integers $1 \leq s, t \leq n$ with $s, t \neq i_0$ such that
\[-E_{ss} = \text{adj } (I_n - E_{ss} - (1 + a_0)E_{ij_0} - (1 - a_0^{-1})E_{tt}).
\]
By (2.4) and (2.11), we see that
\[
\psi(\overline{F}^t (-E_{ss}) H) = \psi(\text{adj } (I_n - E_{ss} - (1 + a_0)E_{ij_0} - (1 - a_0^{-1})E_{tt}) \overline{F}^t))
\]
\[
= \text{adj } T_P(I_n - E_{ss} - (1 + a_0)E_{ij_0} - (1 - a_0^{-1})E_{tt})
\]
\[
= \text{adj } (T_P(I_n - E_{ss} - E_{ij_0} - (1 - a_0^{-1})E_{tt}) - T_P(a_0E_{ij_0}))
\]
\[
= \text{adj } (T_P(I_n - E_{ii} - E_{ij_0} - (1 - a_0^{-1})E_{tt}))
\]
\[
= \psi(\overline{F}^t \text{ adj } (I_n - E_{ss} - E_{ij_0} - (1 - a_0^{-1})E_{tt}) H)
\]
\[
= \psi(0) = 0
\]
since $\text{adj } (I_n - E_{ss} - E_{ij_0} - (1 - a_0^{-1})E_{tt}) = 0$. Likewise, we have $\psi(\overline{F}^t (-E_{tt}) H) = 0$. Next, we observe that
adj $\psi(\bar{H}^T H) = \text{adj } \psi(\bar{H}^T ((I_n - E_{ss} - E_{tt}) + E_{ss} + E_{tt})H) = \text{adj } \psi(\bar{H}^T ((I_n - E_{ss} - E_{tt}) + E_{ss})H - \bar{H}^T (-E_{tt})H) = \text{adj } (\psi(\bar{H}^T ((I_n - E_{ss} - E_{tt}) + E_{ss})H) - \psi(\bar{H}^T (-E_{tt})H)) = \text{adj } \psi(\bar{H}^T (I_n - E_{ss} - E_{tt})H - \bar{H}^T (-E_{ss})H) = \psi((\text{adj } H) (I_n - E_{ss} - E_{tt}) \text{adj } \bar{H}^T) = \psi(0) = 0$

since $\text{adj } (I_n - E_{ss} - E_{tt}) = 0$. This contradicts to (2.12). Hence, (2.13) is proved.

We are ready to show that $\psi$ preserves rank one matrices. Let $H \in \mathcal{H}_n(\mathbb{F})$ be a rank one matrix. By Lemma 2.1, there exist an invertible matrix $Q \in \mathcal{H}_n(\mathbb{F})$ and a nonzero scalar $\alpha \in \mathbb{F}^-$ such that $H = Q(\alpha E_{11}) \bar{Q}$. Therefore, $\psi(H) = \psi(Q(\alpha E_{11}) \bar{Q}) = T_Q(\alpha E_{11})$, where $T_Q : \mathcal{H}_n(\mathbb{F}) \rightarrow \mathcal{H}_m(\mathbb{K})$ is the mapping defined by $T_Q(A) = QA\bar{Q}^{-1}$ for all $A \in \mathcal{H}_n(\mathbb{F})$. Hence, by result (2.14) applied on the mapping $T_Q$, we conclude that $\psi(H)$ is of rank one, as desired.

We now prove that $\psi$ is injective. Let $A \in \mathcal{H}_n(\mathbb{F})$ be a matrix such that $\psi(A) = 0$. Suppose that $A \neq 0$. By Lemma 2.4(d), there exists a matrix $B \in \mathcal{H}_n(\mathbb{F})$ with rank $B \leq n - 2$ such that rank $(B - A) = n - 1$. So, rank $\text{adj } (B - A) = 1$, and thus, rank $\psi(\text{adj } (B - A)) = 1$ since $\psi$ preserves rank one matrices. On the other hand, we see that $\psi(\text{adj } (B - A)) = \text{adj } (\psi(B) - \psi(A)) = \text{adj } \psi(B) = \psi(\text{adj } B) = \psi(0) = 0$, a contradiction. Thus, we conclude that $A = 0$, and so, ker $\psi = \{0\}$. By Lemma 2.6, we prove that $\psi$ is injective.

(b)(ii) Let $A, B \in \mathcal{H}_n(\mathbb{F})$. Since $\psi(I_n) \neq 0$, we conclude that $\psi$ is injective by (b)(i). In view of Lemma 2.6(c), we have

$$\text{rank } (A - B) = n \Leftrightarrow \text{rank } \text{adj } (A - B) = n \Leftrightarrow \text{rank } \psi(\text{adj } (A - B)) = m \Leftrightarrow \text{rank } \text{adj } (\psi(A) - \psi(B)) = m \Leftrightarrow \text{rank } (\psi(A) - \psi(B)) = m.$$ 

We are done. □

Let $m$ and $n$ be integers $\geq 3$. Let $\mathbb{F}$ be a field which possesses an involution $\tau$ of $\mathbb{F}$. We note that if a mapping $\psi : \mathcal{H}_n(\mathbb{F}) \rightarrow \mathcal{H}_m(\mathbb{F})$ satisfies condition $\text{(AH-1)}$, then it satisfies condition $\text{(AH-2)}$ immediately. Furthermore, if $\psi(I_n) \neq 0$, then $\psi$ is an injection by Lemma 2.8(b)(i). By a similar argument as in the proof of Lemma 2.8(b)(ii), it can be verified that

$$\text{rank } (A + \alpha B) = n \Leftrightarrow \text{rank } (\psi(A) + \alpha \psi(B)) = m \quad (2.14)$$

for $A, B \in \mathcal{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$. 

**Lemma 2.9.** Let $m$ and $n$ be integers with $m, n \geq 3$. Let $\mathbb{F}$ be a field which possesses an involution $\tau$ of $\mathbb{F}$, and $\mathbb{F}^-$ be the fixed field of the involution $\tau$ of $\mathbb{F}$. If $|\mathbb{F}^-| = 2$ or $|\mathbb{F}^-| > n + 1$, and $\psi : \mathcal{H}_n(\mathbb{F}) \rightarrow \mathcal{H}_m(\mathbb{F})$ is a mapping satisfying $\text{(AH-1)}$ with $\psi(I_n) \neq 0$, then $\psi$ is additive.

**Proof.** Let $A$ and $B$ be matrices in $\mathcal{H}_n(\mathbb{F})$ and $\alpha$ be a scalar in $\mathbb{F}^-$ such that rank $(A + \alpha B) = n$. Since $\psi(I_n) \neq 0$, it follows from Lemma 2.8(b) and (2.14) that $\psi$ is injective and rank $\psi(A + \alpha B) = \text{rank } (\psi(A) + \alpha \psi(B)) = m$. Then

$$\psi(A + \alpha B) \text{adj } \psi(A + \alpha B) = (\det \psi(A + \alpha B)) I_m.$$
\[(\psi(A) + \alpha \psi(B)) \text{adj} (\psi(A) + \alpha \psi(B)) = (\det(\psi(A) + \alpha \psi(B)))I_m.\]

Further, since \(\text{adj} (\psi(A) + \alpha \psi(B)) = \text{adj} \psi(A + \alpha B),\) we have
\[
\frac{\psi(A + \alpha B)}{\det(\psi(A + \alpha B))} \text{adj} (\psi(A + \alpha B)) = I_m = \frac{\psi(A) + \alpha \psi(B)}{\det(\psi(A) + \alpha \psi(B))} \text{adj} \psi(A + \alpha B).
\]

By the uniqueness of the inverse of \(\text{adj} \psi(A + \alpha B),\) we conclude that
\[
\psi(A + \alpha B) = \left(\frac{\det(\psi(A) + \alpha B)}{\det(\psi(A) + \alpha \psi(B))}\right) (\psi(A) + \alpha \psi(B)). \quad (2.15)
\]

Repeating a similar argument as in (2.15), we obtain
\[
\psi(A + \alpha B) = \left(\frac{\det(\psi(A) + \alpha B)}{\det(\psi(A) + \alpha \psi(B))}\right) (\psi(A) + \alpha \psi(B)) + \psi(\alpha \psi(B)). \quad (2.16)
\]

If we select \(A = 0\) and \(\alpha B\) is of rank \(n\) in (2.15), then
\[
\psi(\alpha B) = \left(\frac{\det(\psi(\alpha B))}{\det(\alpha \psi(B))}\right) (\alpha \psi(B)). \quad (2.17)
\]

We first claim that
\[
\psi(\alpha A) = \alpha \psi(A) \quad (2.18)
\]

for every rank \(n\) matrix \(A \in \mathcal{H}_n(F)\) and \(\alpha \in F^{-}.\) The claim clearly holds when \(\alpha = 0.\) Let \(\alpha \in F^{-}\) be nonzero and \(A \in \mathcal{H}_n(F)\) with rank \(A = n.\) By Lemma 2.4(c), there exists a nonzero matrix \(C \in \mathcal{H}_n(F)\) with rank \(C < n\) such that rank \((C + \alpha A) = n.\) By Lemma 2.6(c) and (2.14), we conclude that \(\psi(C + \alpha A), \psi(C) + \alpha \psi(A)\) and \(\psi(C) + \psi(\alpha A)\) are of rank \(m.\) By (2.15) and (2.16), we obtain
\[
\frac{\psi(C) + \alpha \psi(A)}{\det(\psi(C) + \alpha \psi(A))} = \frac{\psi(C) + \psi(\alpha A)}{\det(\psi(C) + \psi(\alpha A))}. \quad (2.19)
\]

Then
\[
\lambda_1 \psi(\alpha A) - \lambda_2 \psi(A) = (\lambda_2 - \lambda_1) \psi(C) \quad (2.20)
\]

where \(\lambda_1 = \det(\psi(C) + \alpha \psi(A))\) and \(\lambda_2 = \det(\psi(C) + \psi(\alpha A))\) are nonzero scalars in \(F^{-}.\) Suppose \(\lambda_1 \neq \lambda_2.\) Since rank \(A = n,\) it follows from (2.17) that \(\psi(\alpha A)\) and \(\psi(A)\) are linearly dependent. So, \(\psi(\alpha A) = \lambda \psi(A)\) for some \(\lambda \in F^{-}\) since \(\psi(\alpha A), \psi(A) \in \mathcal{H}_m(F).\) Substituting into (2.20), we have
\[
(\lambda_1 \lambda - \lambda_2 \alpha) \psi(A) = (\lambda_2 - \lambda_1) \psi(C). \quad (2.21)
\]

Therefore, \(\psi(A)\) and \(\psi(C)\) are linearly dependent. Further, since \(\psi(A)\) and \(\psi(C)\) are nonzero, we conclude that rank \(\psi(A) = \text{rank} \psi(C).\) This leads to a contradiction since rank \(\psi(A) = m\) but rank \(\psi(C) < m\) by Lemma 2.6(c). Hence, \(\det(\psi(C) + \psi(\alpha A)) = \lambda_1 = \lambda_2 = \det(\psi(C) + \psi(\alpha A)),\) and thus, the desired conclusion follows immediately from (2.19).

We next claim that if \(A, B \in \mathcal{H}_n(F)\) with rank \((A + B) = n,\) then
\[
A, B \text{ are linearly independent} \quad \Rightarrow \quad \psi(A), \psi(B) \text{ are linearly independent.} \quad (2.21)
\]

Suppose to the contrary that \(\psi(A), \psi(B)\) are linearly dependent. Since \(A, B\) are linearly independent, by the injectivity of \(\psi,\) we conclude that \(\psi(A)\) and \(\psi(B)\) are nonzero Hermitian matrices. Then there
exists a scalar \( \gamma \in \mathbb{F}^- \) such that \( \psi(B) = \gamma \psi(A) \). Since \( \text{rank}(A + B) = n \), it follows from (2.14) that \( \text{rank}((1 + \gamma)\psi(A)) = \text{rank}(\psi(A) + \psi(B)) = m \), and so, \( \text{rank}\psi(A) = m \). Thus \( \text{rank}A = n \) by Lemma 2.6(c). Hence \( \psi(B) = \gamma \psi(A) = \psi(\gamma A) \) by (2.18). By the injectivity of \( \psi \), we get \( B = \gamma A \), which means \( A, B \) are linearly dependent, a contradiction. So, claim (2.21) is proved.

We now claim that if \( A, B \in \mathcal{H}_n(\mathbb{F}) \) are matrices such that \( \text{rank}(A + B) = n \) with \( 0 < \text{rank}A < n \) and \( \text{rank}B = n \), then

\[
\psi(A + B) = \psi(A) + \psi(B).
\]  

(2.22)

In view of (2.15), by taking \( \alpha = 1 \), we have

\[
\frac{\psi(A + B)}{\det \psi(A + B)} = \frac{\psi(A) + \psi(B)}{\det(\psi(A) + \psi(B))}.
\]  

(2.23)

Note that \( \psi(A + B) \) and \( \psi(A) + \psi(B) \) are Hermitian, so \( \psi(A + B) \), \( \det(\psi(A) + \psi(B)) \in \mathbb{F}^- \). If \( |\mathbb{F}^-| = 2 \), then \( \det(\psi(A + B)) = 1 = \det(\psi(A) + \psi(B)) \), so claim (2.22) follows immediately from (2.23). We now consider \( |\mathbb{F}^-| > n + 1 \). By Lemma 2.4(e), there exists a nonzero scalar \( \alpha_0 \in \mathbb{F}^- \) such that \( \text{rank}(A + (1 + \alpha_0)B) = n \). By (2.23), we have

\[
\frac{\psi(A + B) + \psi(\alpha_0B)}{\det(\psi(A + B) + \psi(\alpha_0B))} = \frac{\psi(A + B + \alpha_0B)}{\det(\psi(A + B + \alpha_0B))} = \frac{\psi(A) + \psi(B + \alpha_0B)}{\det(\psi(A) + \psi(B + \alpha_0B))}.
\]  

(2.24)

Since \( \text{rank}A < n \), we have \( 1 + \alpha_0 \neq 0 \), and so, \( \text{rank}((1 + \alpha_0)B) = n \). By (2.18), we get \( \psi(B + \alpha_0B) = (1 + \alpha_0)\psi(B) = \psi(B) + \psi(\alpha_0B) \). So

\[
\frac{\psi(A + B) + \psi(\alpha_0B)}{\det(\psi(A + B) + \psi(\alpha_0B))} = \frac{\psi(A) + \psi(B) + \psi(\alpha_0B)}{\det(\psi(A) + \psi(B + \alpha_0B))}.
\]  

By (2.24), we obtain

\[
(\mu_1 \gamma - \mu_2)\psi(A + B) + (\mu_2 - \mu_1)\psi(\alpha_0B) = 0.
\]  

(2.25)

Since \( A \) and \( B \) are linearly independent, it follows that \( A + B \) and \( \alpha_0B \) are linearly independent. Further, since \( \text{rank}(A + B + \alpha_0B) = n \), we have \( \psi(A + B) \) and \( \psi(\alpha_0B) \) are linearly independent by (2.21).

In view of (2.25), we get \( \mu_1 = \mu_2 \), and hence, \( \psi(A + B) = \psi(A) + \psi(B) \) by (2.24).

We next show that \( \psi \) is \( \mathbb{F}^- \)-homogeneous, that is

\[
\psi(\alpha A) = \alpha \psi(A) \quad \text{for every } A \in \mathcal{H}_n(\mathbb{F}) \text{ and } \alpha \in \mathbb{F}^-.
\]  

(2.26)

It is clear that (2.26) is true when \( \alpha = 0, A = 0 \), or \( \text{rank}A = n \). Let \( \alpha \) be a nonzero scalar in \( \mathbb{F}^- \) and \( A \) be a nonzero Hermitian matrix with \( \text{rank}A < n \). By Lemma 2.4(c), there exists a rank \( n \) matrix \( C \in \mathcal{H}_n(\mathbb{F}) \) such that \( \text{rank}(\alpha A + C) = n \), and so, \( \text{rank}(A + \alpha^{-1}C) = n \). It follows from (2.18) and (2.22) that \( \psi(\alpha A) + \psi(C) = \psi(\alpha A + C) = \psi(\alpha(A + \alpha^{-1}C)) = \alpha\psi(A + \alpha^{-1}C) = \alpha(\psi(A) + \psi(\alpha^{-1}C)) = \alpha\psi(A) + \alpha\psi(\alpha^{-1}C) = \alpha\psi(A) + \psi(C) \). Consequently, we have \( \psi(\alpha A) = \alpha \psi(A) \), as claimed.

We now claim that

\[
\psi(A + B) = \psi(A) + \psi(B)
\]  

(2.27)

for every \( A, B \in \mathcal{H}_n(\mathbb{F}) \) with \( \text{rank}(A + B) = n \). The claim clearly holds when \( |\mathbb{F}^-| = 2 \) by (2.15). We now consider \( |\mathbb{F}^-| > n + 1 \). Let \( A, B \in \mathcal{H}_n(\mathbb{F}) \) be nonzero matrices. If \( A \) and \( B \) are linearly dependent, then we have \( B = \gamma A \) for some scalar \( \gamma \in \mathbb{F}^- \). By the \( \mathbb{F}^- \)-homogeneity of \( \psi \), we see that
\( \psi(A + B) = \psi((1 + \gamma)A) = (1 + \gamma)\psi(A) = \psi(A) + \gamma \psi(A) = \psi(A) + \psi(\gamma A) = \psi(A) + \psi(\gamma B) \). We now consider \( A \) and \( B \) are linearly independent. Since rank \((A + B) = n\), by Lemma 2.4(e), there exists a nonzero scalar \( \gamma \in \mathbb{F}^- \) such that rank \((A + (1 + \gamma)B) = n\). By (2.23) and the \( \mathbb{F}^- \)-homogeneity of \( \psi \), we obtain

\[
\frac{\psi(A + B) + \psi(\gamma B)}{\det(\psi(A + B) + \psi(\gamma B))} = \frac{\psi(A) + \psi(\gamma B)}{\det(\psi(A) + \psi(\gamma B))}.
\]

(2.28)

Since \( A \) and \( B \) are linearly independent, we have \( A + B \) and \( \gamma B \) are linearly independent, and so, \( \psi(A + B) \) and \( \psi(\gamma B) \) are linearly independent by (2.21). By a similar argument in the proof of (2.25), it can be shown that \( \det(\psi(A + B) + \psi(\gamma B)) = \det(\psi(A) + \psi(B + \gamma B)) \), and hence, claim (2.27) follows from (2.28) immediately.

Finally, we show \( \psi \) is additive. Let \( A \) and \( B \) be any matrices in \( \mathcal{H}_n(\mathbb{F}) \). By Lemma 2.4(b), there exists a matrix \( C \in \mathcal{H}_n(\mathbb{F}) \) such that rank \((A + C) = \text{rank}(A + B + C) = n\). By (2.27), we have \( \psi(A + B) + \psi(C) = \psi(A + B + C) = \psi(A + C) + \psi(B) \). Since rank \((A + C) = n\), again by (2.27), we get \( \psi(A + C) = \psi(A) + \psi(C) \). Consequently, \( \psi(A + B) + \psi(C) = \psi(A) + \psi(B) + \psi(C) \), and so, \( \psi(A + B) = \psi(A) + \psi(B) \) for every matrices \( A, B \in \mathcal{H}_n(\mathbb{F}) \), as desired. \( \square \)

We are now ready to prove our main theorems.

**Theorem 2.10.** Let \( m \) and \( n \) be integers with \( m, n \geq 3 \). Let \( \mathbb{F} \) and \( \mathbb{K} \) be fields which possess involutions \( ^- \) of \( \mathbb{F} \) and \( ^^\wedge \) of \( \mathbb{K} \), respectively, and \( ^- \) is proper. Then \( \psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K}) \) is an adjoint-commuting additive mapping if and only if either \( \psi = 0 \), or \( m = n \) and

\[
\psi(A) = \zeta PA^\sigma \tilde{P}^t \quad \text{for all } A \in \mathcal{H}_n(\mathbb{F}).
\]

Here, \( \sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^^\wedge) \) is a nonzero field homomorphism satisfying \( \sigma(a) = \sigma(a) \) for all \( a \in \mathbb{F} \), \( A^\sigma \) is the matrix obtained from \( A \) by applying \( \sigma \) entrywise, \( P \in \mathcal{M}_n(\mathbb{K}) \) is invertible with \( \tilde{P}^t P = \lambda I_n \) and \( \zeta, \lambda \in \mathbb{K}^\wedge \) are scalars with \( (\zeta \lambda)^{n-2} = 1 \).

**Proof.** The sufficiency part is trivial. We now prove the necessity part. By the additivity of \( \psi \), it is clear that \( \psi \) satisfies (AH-2). Since \( \text{adj} \psi(I_n) = \psi(I_n) \), it follows that either \( \psi(I_n) = 0 \) or rank \( \psi(I_n) = m \). We divide our proof into the following two cases.

Case I: \( \psi(I_n) = 0 \). By Lemma 2.8(a), we have \( \psi(A) = 0 \) for all rank one matrices \( A \in \mathcal{H}_n(\mathbb{F}) \). By Lemma 2.3 and the additivity of \( \psi \), we obtain \( \psi = 0 \), as desired.

Case II: rank \( \psi(I_n) = m \). Then \( \psi \) is injective by Lemma 2.8(b)(i), and so, \( \psi \) preserves rank one matrices by Lemma 2.5(a). Furthermore, by the additivity of \( \psi \), we see that \( m = \text{rank} \psi(I_n) \leq \sum_{i=1}^n \text{rank} \psi(E_{ii}) = n \). If \( n > m \), then, by [6, Theorem 2.1], there exist integers \( 1 \leq s_1 < \cdots < s_k \leq n \) with \( m \leq k < n \) such that \( \text{rank} \psi(E_{s_1s_1} + \cdots + E_{s_ks_k}) = m \). Therefore, \( m = \text{rank} \text{adj} \psi(E_{s_1s_1} + \cdots + E_{s_ks_k}) \leq 1 \) since \( k < n \). This leads to a contradiction since \( m \geq 3 \). So, we obtain \( m = n \). By [13, Main Theorem, p. 603] and [12, Theorem 2.1 and Remark 2.4], we have \( \psi(A) = \zeta PA^\sigma \tilde{P}^t \) for every \( A \in \mathcal{H}_n(\mathbb{F}) \), where \( \sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^^\wedge) \) is a nonzero field homomorphism satisfying \( \sigma(a) = \sigma(a) \) for every \( a \in \mathbb{F} \), \( P \in \mathcal{M}_n(\mathbb{K}) \) is an invertible matrix and \( \zeta \in \mathbb{K}^\wedge \) is a nonzero scalar. We now claim that

\[
\tilde{P}^t P = \lambda I_n
\]

for some nonzero scalar \( \lambda \in \mathbb{K}^\wedge \). Since \( \text{adj} \psi(I_n) = \psi(I_n) \), we have \( \zeta^{n-2} (\text{adj} \tilde{P}^t)(\text{adj} P) = P \tilde{P}^t \), and so,

\[
(\tilde{P}^t P)^2 = \zeta^{n-2} \det(\tilde{P}^t P) I_n.
\]

(2.30)

Let \( 1 \leq i < j \leq n \). Since \( \text{adj} \psi(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) = -\psi(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) \),

\[
\tilde{P}^t P(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) \tilde{P}^t P = \zeta^{n-2} \det(\tilde{P}^t P)(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}).
\]
Together with (2.30), we get \( \tilde{P}^i P (I_n - E_{ii} - E_{jj} + E_{jj} + E_{ii}) = (I_n - E_{ii} - E_{jj} + E_{jj} + E_{ii}) \tilde{P}^i P \) for every \( 1 \leq i < j \leq n \). Hence \( \tilde{P}^i P = \lambda I_n \) for some nonzero scalar \( \lambda \in K \) since \( \tilde{P}^i P \in \mathcal{H}_n(K) \), as claimed. So, \( P \tilde{P}^i = \lambda I_n \). Since \( \text{adj}(\tilde{P}^i P) = \tilde{P}^i P \), it follows that \( \text{adj}(\tilde{P}^i I) = \tilde{P}^i I \), and so, \( (\tilde{P}^i \lambda)^{n-2} = 1 \). This completes the proof. \( \square \)

**Theorem 2.11.** Let \( m \) and \( n \) be integers with \( m \geq n \), and let \( F \) and \( K \) be arbitrary fields. Then \( \psi : S_n(F) \to S_m(K) \) is an adjoint-commuting additive mapping if and only if either \( \psi = 0 \), or \( m = n \) and \( \psi(A) = \varsigma PA^\sigma P^\tau \) for all \( A \in S_n(F) \).

Here, \( \sigma : F \to K \) is a nonzero field homomorphism, \( A^\sigma \) is the matrix obtained from \( A \) by applying \( \sigma \) entrywise, \( P \in \mathcal{M}_n(K) \) is invertible with \( P^i P = \lambda I_n \) and \( \varsigma, \lambda \in K \) are scalars with \( (\varsigma \lambda)^{n-2} = 1 \).

**Proof.** The sufficiency part can be easily checked. We now consider the necessity part. By a similar argument as in the proof of Theorem 2.10, we can prove that either \( \psi = 0 \), or \( m = n \), \( \psi \) is injective and preserves rank one matrices, and rank \( \text{adj} \psi(I_n) = n \). By [11, Theorem 2.1], we see that \( \psi \) is of the following forms:

(a) \( \psi(A) = \varsigma PA^\sigma P^\tau \) for every \( A \in S_n(F) \), or

(b) \( \psi(A) = Q\tau(A)Q^\tau \) for every \( A \in S_n(F) \), only when \( n = 3 \) and \( F = \mathbb{Z}_2 := \{0, 1\} \).

Here, \( \varsigma \in K \) is a nonzero scalar, \( \sigma : F \to K \) is a nonzero field homomorphism, \( P \in \mathcal{M}_n(K) \) and \( Q \in \mathcal{M}_3(K) \) are invertible matrices and \( \tau : S_3(\mathbb{Z}_2) \to S_3(K) \) is an additive mapping preserving rank one matrices with rank \( \tau(1) = 3 \).

- Case I: \( \psi \) is of form (a). By a similar argument as given in (2.29), we show that \( P^i P = \lambda I_n = \lambda P^\tau \) for some nonzero scalar \( \lambda \in K \). By the fact \( \text{adj} \psi(I_n) = \psi(I_n) \), we obtain \( \text{adj}(\varsigma \lambda I_n) = \varsigma \lambda I_n \), and so, \( (\varsigma \lambda)^{n-2} = 1 \). We are done.

- Case II: \( \psi \) is of form (b). We first note that since \( \sigma \neq 0 \), by the additivity of \( \tau \), it follows that \( K \) is of characteristic \( 2 \). By the fact \( \text{adj} \psi(A) = \psi(\text{adj} A) \), we obtain \( \tau(\text{adj} A) = \text{adj}(\tau(A))U^\tau \) for all \( A \in S_3(\mathbb{Z}_2) \), where \( U = (\text{adj}Q)(Q^{-1})^\tau \in \mathcal{M}_3(K) \). It is easily seen that \( \tau(A) \) is a singular matrix whenever \( A \in S_3(\mathbb{Z}_2) \) is singular. So, rank \( \tau(E_{ii} + E_{jj}) = 2 \) for all \( 1 \leq i \neq j \leq 3 \) because \( \tau \) preserves rank one matrices and rank \( \tau(1) = 3 \). Since rank \( \tau(E_{11}) = 1 \), it follows from Lemma 2.1 that there exists an invertible matrix \( V_1 \in \mathcal{M}_3(K) \) and a nonzero scalar \( \alpha_1 \in K \) such that \( \tau(E_{11}) = \alpha_1 V_1 E_{11} V_1^\dagger \).

Let

\[
\tau(E_{22}) = V_1 \begin{pmatrix} x_1 & y_1 \\ y_1 & x_1 \end{pmatrix} V_1^\dagger
\]

with \( x_1 \in K \), \( y_1 \in \mathcal{M}_{1,2}(K) \) and \( x_1 \in S_2(K) \). Note that \( x_1 \neq 0 \); otherwise, \( y_1 = 0 \) because rank \( \tau(E_{22}) = 1 \), and so, \( \tau(E_{11} + E_{22}) < 2 \), an impossibility. Since rank \( \tau(E_{22}) = 1 \), we have rank \( x_1 = 1 \). So, there exists an invertible matrix \( V_2 \in \mathcal{M}_2(K) \) and a nonzero scalar \( \alpha_2 \in K \) such that

\[
\tau(E_{22}) = V_1 \begin{pmatrix} x_1 & Y_1 \\ Y_1 & 0 \end{pmatrix} V_1^\dagger = V_1 \begin{pmatrix} 1 & 0 \\ 0 & V_2 \end{pmatrix} \begin{pmatrix} x_1 & Y_1 & Y_{12} \\ Y_1 & x_1 & 0 \\ Y_{12} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & V_2 \end{pmatrix} V_1^\dagger
\]

with \( Y_{12} \in K \). Since rank \( \tau(E_{22}) = 1 \), it follows that \( y_{12} = 0 \) and \( x_1 = y_{11}^{-1} \alpha_2^{-1} \). Let

\[
V_3 = V_1 \begin{pmatrix} 1 & 0 \\ 0 & V_2 \end{pmatrix} \begin{pmatrix} 1 & y_{11} \alpha_2^{-1} \\ y_{11} \alpha_2^{-1} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_3(K).
\]
Clearly, $V_3$ is invertible and $\tau(E_{ii}) = \alpha_i V_3 E_{ii} V_3^t$ for $i = 1, 2$. Let

$$\tau(E_{33}) = V_3 \begin{pmatrix} \alpha_1 & 0 \\ 0 & 1 \end{pmatrix} V_3^t$$

with $\alpha_3 \in \mathbb{K}$, $Y_2 \in M_{2,1}(\mathbb{K})$ and $X_2 \in S_2(\mathbb{K})$. Since rank $\tau(E_{33}) = 1$ and rank $\tau(I_3) = 3$, we get $\alpha_3 \neq 0$, and so, $X_2 = -\alpha_3^{-1} Y_2 Y_3^t$. Let

$$V_4 = V_3 \begin{pmatrix} \alpha_3^{-1} Y_2 \\ 0 \\ 1 \end{pmatrix} \in M_3(\mathbb{K}).$$

Evidently, $V_4$ is invertible and $\tau(E_{ii}) = \alpha_i V_4 E_{ii} V_4^t$ for $i = 1, 2, 3$. Hence, we obtain

$$\psi(E_{ii}) = \alpha_i P E_{ii} P^t$$

for $i = 1, 2, 3$ where $P = QV_4 \in M_3(\mathbb{K})$ is invertible.

Let $i, j, k$ denote three distinct integers such that $1 \leq i, j, k \leq 3$. Since adj $(E_{jj} + E_{kk}) = E_{ii}$, it follows that $P(\alpha_i E_{ii}) P^t = \text{adj}(P(\alpha_j E_{jj} + \alpha_k E_{kk})) P^t)$, and so, $P P(\alpha_i E_{ii}) = P^t \text{adj}(P(\alpha_j E_{jj} + \alpha_k E_{kk})) P^t) (P^t)^{-1} = (\alpha_i \alpha_j E_{jj}) \text{adj}(P P^t)$ for $i = 1, 2, 3$. Hence, $P P^t$ is diagonal $(\lambda_1, \lambda_2, \lambda_3)$ with nonzero scalars $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{K}$. So, we have

$$\psi(E_{ii}) = \alpha_i P E_{ii} P^t = \alpha_i P E_{ii} (P^t P)^{-1} = \beta_i P E_{ii} (P P)^{-1} \text{ for } i = 1, 2, 3$$

where $\beta_i = \alpha_i \lambda_i \in \mathbb{K}$ is nonzero. Denote $H_{ij} := E_{ij} + E_{ji} \in S_3(\mathbb{Z}_2)$. Let $\psi(H_{ij}) = P A_{ij} P^{-1}$ with $A_{ij} = (a_{ij}) \in S_3(\mathbb{K})$. We claim that $\beta_i = \beta_j = \beta_k = 1$ and $A_{ij} = H_{ij}$. Since adj $H_{ij} = E_{kk}$, we get $\psi(H_{ij}) = \psi(E_{kk})$ is of rank one, and so, rank $\psi(H_{ij}) = 2$. Therefore, $\psi(H_{ij}) = \psi(H_{ij})^t = \psi(H_{ij})^t = \det \psi(H_{ij}) = 2 = 0$. So, $A_{ij} E_{kk} = E_{kk} A_{ij} = 0$, and thus, $a_{ik} = a_{ki} = 0$ for $s = 1, 2, 3$.

Since rank $(H_{ij} + E_{ii} + E_{jj}) = 1$, we get rank $(A_{ij} + \beta_i E_{ii} + \beta_j E_{jj}) = 1$, and so

$$(a_{ii} + \beta_i)(a_{jj} + \beta_j) = a_{ij}^2. \quad (2.31)$$

Suppose $a_{ij} = 0$. Since $A_{ij} \neq 0$, by (2.31), we see that $a_{ii} = -\beta_i$ or $a_{jj} = -\beta_j$ but not both. If $a_{ii} = -\beta_i$, then rank $\psi(H_{ij} + E_{ii} + E_{kk}) = \text{rank}(a_{ij} E_{jj} + \beta_k E_{kk}) < 3$, which contradicts to the injectivity of $\psi$, by Lemma 2.6. Similarly, if $a_{jj} = -\beta_j$, then rank $\psi(H_{ij} + E_{jj} + E_{kk}) < 3$, a contradiction. Therefore, $a_{ij} \neq 0$. Next, in view of adj $(H_{ij} + E_{kk}) = H_{ij} + E_{kk}$, we see that $\text{adj}(A_{ij} + \beta_k E_{kk}) = A_{ij} + \beta_k E_{kk}$, and so, $-\beta_k a_{ij} = a_{ij} = a_{ij} \beta_k = a_{ij}$ and

$$a_{ii} a_{ij} - a_{ij}^2 = \beta_k \quad (2.32)$$

Since $\mathbb{K}$ has characteristic 2 and $a_{ij} \neq 0$, we conclude that $\beta_k = 1$ and $a_{ii} = a_{ij}$. By using the facts adj $(E_{ii} + E_{kk}) = E_{ij}$ and adj $(E_{ii} + E_{jj}) = E_{kk}$, we prove that $\beta_i = \beta_j = 1$ as well. Further, since adj $(H_{ij} + E_{jj} = E_{kk}$, we obtain adj $(A_{ij} + E_{jj}) = E_{kk}$, and hence

$$a_{ii} (a_{ij} + 1) - a_{ij}^2 = 1. \quad (2.33)$$

By (2.32) and (2.33), we obtain $a_{ii} = 0$, and so, $a_{ij} = 0$. Since $\mathbb{K}$ has characteristic 2, from (2.33), we get $a_{ij} = 1$, and so, $A_{ij} = H_{ij}$, as claimed. Therefore $\psi(A) = P A P^{-1}$ for all $A \in S_3(\mathbb{Z}_2)$. Since $P A P^{-1} = A \psi^t$ and $P^t P = \zeta^{-1} A_3$ for some nonzero scalar $\zeta \in \mathbb{K}$. Therefore, we achieve

$$\psi(A) = \zeta P A P^t$$

for every $A \in S_3(\mathbb{Z}_2)$.

The proof is complete. \(\square\)
As an immediate consequence of Lemmas 2.8 (a) and 2.9, and Theorem 2.10, we obtain a complete classification of mappings \( \psi : \mathcal{H}_n(\mathbb{F}) \rightarrow \mathcal{H}_m(\mathbb{F}) \) satisfying condition (AH-1).

**Theorem 2.12.** Let \( m \) and \( n \) be integers with \( m, n \geq 3 \). Let \( \mathbb{F} \) be a field which possesses a proper involution \( - \) of \( \mathbb{F} \) with \( |\mathbb{F}| = 2 \) or \( |\mathbb{F}| > n + 1 \). Then \( \psi : \mathcal{H}_n(\mathbb{F}) \rightarrow \mathcal{H}_m(\mathbb{F}) \) is a mapping satisfying (AH-1) if and only if either \( \psi(A) = 0 \) for all rank one matrices \( A \in \mathcal{H}_n(\mathbb{F}) \) and

\[
\text{rank}(\psi(A) + \alpha \psi(B)) \leq m - 2
\]

for all \( A, B \in \mathcal{H}_n(\mathbb{F}) \) and \( \alpha \in \mathbb{F}^- \); or \( m = n \) and

\[
\psi(A) = \xi \mathbb{P} \mathbb{P}^t \quad \text{for all } A \in \mathcal{H}_n(\mathbb{F})
\]

where \( \mathbb{P} \in \mathcal{M}_n(\mathbb{F}) \) is invertible with \( \mathbb{P}^t \mathbb{P} = \lambda I_n \) and \( \xi, \lambda \in \mathbb{F}^- \) are scalars with \((\xi \lambda)^{n-2} = 1\).

**Proof.** The sufficiency part is clear. We now consider the necessity part. If \( \psi(I_n) \neq 0 \), then \( \psi \) is an additive mapping by Lemma 2.9. As a consequence of Theorem 2.10, the result follows immediately. We now consider \( \psi(I_n) = 0 \). By Lemma 2.8 (a), we have \( \psi(A) = 0 \) for all rank one matrices \( A \in \mathcal{H}_n(\mathbb{F}) \), and so \( \psi(\text{adj}(A)) = 0 \) for every \( A \in \mathcal{H}_n(\mathbb{F}) \). By condition (AH-1), we get \( \text{adj}(\psi(A) + \alpha \psi(B)) = \psi(\text{adj}(A + \alpha B)) = 0 \) for every \( A, B \in \mathcal{H}_n(\mathbb{F}) \) and \( \alpha \in \mathbb{F}^- \). Thus, \( \text{rank}(\psi(A) + \alpha \psi(B)) \leq m - 2 \) for every \( A, B \in \mathcal{H}_n(\mathbb{F}) \) and \( \alpha \in \mathbb{F}^- \). We are done. □

The following result follows immediately from Theorem 2.12.

**Theorem 2.13.** Let \( m \) and \( n \) be integers with \( m, n \geq 3 \). Let \( \mathbb{F} \) be a field such that \( |\mathbb{F}| = 2 \) or \( |\mathbb{F}| > n + 1 \). Then \( \psi : \mathcal{S}_n(\mathbb{F}) \rightarrow \mathcal{S}_m(\mathbb{F}) \) is a mapping satisfying (AH-1) if and only if either \( \psi(A) = 0 \) for all rank one matrices \( A \in \mathcal{S}_n(\mathbb{F}) \) and

\[
\text{rank}(\psi(A) + \alpha \psi(B)) \leq m - 2
\]

for all \( A, B \in \mathcal{S}_n(\mathbb{F}) \) and \( \alpha \in \mathbb{F}^- \); or \( m = n \) and

\[
\psi(A) = \xi \mathbb{P} \mathbb{P}^t \quad \text{for all } A \in \mathcal{S}_n(\mathbb{F})
\]

Here, \( \mathbb{P} \in \mathcal{M}_n(\mathbb{F}) \) is invertible with \( \mathbb{P}^t \mathbb{P} = \lambda I_n \) and \( \xi, \lambda \in \mathbb{F}^- \) are scalars with \((\xi \lambda)^{n-2} = 1\).

If we impose the surjectivity condition on \( \psi \), then we only require a weaker condition (AH-2). We have the following results.

**Theorem 2.14.** Let \( m \) and \( n \) be integers with \( m, n \geq 3 \). Let \( \mathbb{F} \) and \( \mathbb{K} \) be fields which possess involutions \( - \) of \( \mathbb{F} \) and \( \overset{\wedge}{\cdot} \) of \( \mathbb{K} \), respectively, such that either \( |\mathbb{F}| = 2 \), or \( |\mathbb{F}| > 3 \) and \( \text{char} \mathbb{F} \neq 2 \) and \( \text{char} \mathbb{K} \neq 2 \) if \( - \) and \( \overset{\wedge}{\cdot} \) are the identity maps. Then \( \psi : \mathcal{H}_n(\mathbb{F}) \rightarrow \mathcal{H}_m(\mathbb{K}) \) is a surjective mapping satisfying (AH-2) if and only if \( m = n \), \( \mathbb{F} \) and \( \mathbb{K} \) are isomorphic, and

\[
\psi(A) = \xi \mathbb{P} \sigma \overset{\wedge}{\mathbb{P}}^t \quad \text{for all } A \in \mathcal{H}_n(\mathbb{F})
\]

Here, \( \sigma : (\mathbb{F}, -) \rightarrow (\mathbb{K}, \overset{\wedge}{\cdot}) \) is a field isomorphism satisfying \( \sigma(a) = \overset{\wedge}{a} \) for every \( a \in \mathbb{F} \), \( A^\sigma \) is the matrix obtained from \( A \) by applying \( \sigma \) entrywise, \( P \in \mathcal{M}_n(\mathbb{K}) \) is invertible with \( \mathbb{P}^t \mathbb{P} = \lambda I_n \) and \( \xi, \lambda \in \mathbb{K}^\overset{\wedge}{\cdot} \) are scalars with \((\xi \lambda)^{n-2} = 1\).

**Proof.** The sufficiency part is trivial. We now prove the necessity part. Firstly, we note that \( \psi(I_n) \neq 0 \); otherwise, \( \psi(I_n) = 0 \) implies \( \text{rank} \psi(A) < m \) for every matrix \( A \in \mathcal{H}_n(\mathbb{F}) \) by Lemma 2.8 (a)(ii), which contradicts to the surjectivity of \( \psi \). In view of Lemma 2.8 (b), we see that \( \psi \) is injective, and hence, \( \psi \)
is bijective with rank \( (A - B) = n \) if and only if rank \( (\psi(A) - \psi(B)) = m \) for \( A, B \in \mathcal{H}_n(F) \). We divide our proof into the following two cases.

Case I: If \( |F^*|, |\mathbb{K}^\wedge| > 3 \), and \( F \) and \( \mathbb{K} \) do not have characteristic 2 when \(-\) and \(^\wedge\) are the identity maps, then, by combining [10, Theorem 3.6] and the fundamental theorem of the geometry of Hermitian matrices [9, Theorem 5.6.3], we conclude that \( m = n, F \) and \( \mathbb{K} \) are isomorphic, and

\[
\psi(A) = \varsigma PA^T \hat{P}^T + H_0 \quad \text{for every } A \in \mathcal{H}_n(F).
\]

Here, \( \sigma : (\mathbb{F}, -) \rightarrow (\mathbb{K}, \wedge) \) is a field isomorphism satisfying \( \sigma(a) = \hat{\sigma}(a) \) for all \( a \in F \), \( A^\sigma \) is the matrix obtained from \( A \) by applying \( \sigma \) entrywise, \( P \in \mathcal{M}_n(\mathbb{K}) \) is an invertible matrix, \( H_0 \in \mathcal{H}_n(\mathbb{K}) \) and \( \varsigma \in \mathbb{K}^\wedge \) is a nonzero scalar. Since \( \psi(0) = 0 \), we have \( H_0 = 0 \). By using a similar argument as given in (2.29), it can be shown that \( \hat{P}^T P = \lambda I_n \) for some nonzero scalar \( \lambda \in \mathbb{K}^\wedge \). Since \( \text{adj} \psi(I_n) = \psi(I_n) \), we obtain \( (\varsigma \lambda)^{n-2} = 1 \).

Case II: If \( |\mathbb{K}^\wedge| = 2 \), then we have rank \( (A - B) = n \) if and only if rank \( (\psi(A) + \psi(B)) = m \) for \( A, B \in \mathcal{H}_n(F) \). Let \( X, Y \in \mathcal{H}_n(F) \) with rank \( (X - Y) = n \). Then rank \( \psi(X - Y) = rank (\psi(X) + \psi(Y)) = m \). By a similar argument as in (2.15), we obtain

\[
\frac{\psi(X - Y)}{\det(\psi(X - Y))} = \frac{\psi(X) + \psi(Y)}{\det(\psi(X) + \psi(Y))}.
\]

Since \( \det(\psi(X - Y)), \det(\psi(X) + \psi(Y)) \in \mathbb{K}^\wedge \), it follows that \( \det(\psi(X - Y)) = 1 = \det(\psi(X) + \psi(Y)) \). Hence, we have \( \psi(A - B) = \psi(A) + \psi(B) \) for every \( A, B \in \mathcal{H}_n(F) \) with rank \( (A - B) = n \). By the injectivity of \( \psi \), we see that \( \psi(-I_n) = \psi(0 - I_n) = \psi(0) + \psi(I_n) = \psi(I_n) \) implies \( I_n = -I_n \). So, \( F \) is of characteristic 2. Then

\[
\psi(A + B) = \psi(A) + \psi(B) \quad \text{for every } A, B \in \mathcal{H}_n(F)
\]

with rank \( (A + B) = n \). By repeating a similar argument as in the last paragraph of Lemma 2.9, it can be shown that \( \psi \) is additive, and hence, by Theorem 2.10 and the bijectivity of \( \psi \), the result is proved.

**Theorem 2.15.** Let \( m \) and \( n \) be integers with \( m, n \geq 3 \). Let \( F \) and \( \mathbb{K} \) be fields such that either \( |\mathbb{K}| = 2 \), or \( |F|, |\mathbb{K}| > 3 \) with \( \text{char} \mathbb{F} \neq 2 \) and \( \text{char} \mathbb{K} \neq 2 \). Then \( \psi : \mathcal{S}_n(F) \rightarrow \mathcal{S}_n(\mathbb{K}) \) is a surjective mapping satisfying (AH-2) if and only if \( m = n, F \) and \( \mathbb{K} \) are isomorphic, and

\[
\psi(A) = \varsigma PA^T P^T \quad \text{for all } A \in \mathcal{S}_n(F).
\]

Here, \( \sigma : F \rightarrow \mathbb{K} \) is a field isomorphism, \( P \in \mathcal{M}_n(\mathbb{K}) \) is invertible with \( P^T P = \lambda I_n \) and \( \varsigma, \lambda \in \mathbb{K} \) are scalars with \( (\varsigma \lambda)^{n-2} = 1 \).

Let \( F \) and \( \mathbb{K} \) be fields which possess involutions \(-\) of \( F \) and \(^\wedge\) of \( \mathbb{K} \), respectively. Let \( \mathcal{U}_1, \ldots, \mathcal{U}_n \) be \( F^- \)-vector spaces over \( F \), and let \( \mathcal{V} \) be a \( \mathbb{K}^\wedge \)-vector space over \( \mathbb{K} \). Recall that if \( T : \mathcal{U}_1 \times \cdots \times \mathcal{U}_n \rightarrow \mathcal{V} \) is an additive mapping, then

\[
T(u_1, \ldots, u_n) = f_1(u_1) + \cdots + f_n(u_n) \quad \text{for every } (u_1, \ldots, u_n) \in \mathcal{U}_1 \times \cdots \times \mathcal{U}_n
\]

where \( f_i : \mathcal{U}_i \rightarrow \mathcal{V} \) is an additive mapping with \( f_i(u_i) = T(0, \ldots, 0, u_i, 0, \ldots, 0) \) for every \( u_i \in \mathcal{U}_i \) and \( i = 1, \ldots, n \). Moreover, if \( (\mathbb{K}, \wedge) = (F, -) \) and \( T \) is a \( F^- \)-linear mapping (i.e., \( T \) is additive and \( F^- \)-homogeneous), then each \( f_i \) is \( F^- \)-linear. Further, if \( \mathcal{U}_1 = \cdots = \mathcal{U}_n = \mathcal{V} = F^- \), then each \( f_i : F^- \rightarrow F^- \) is a linear mapping. So, for each \( 1 \leq i \leq n \), there exists a scalar \( \alpha_i \in F^- \) such that \( f_i(a_i) = \alpha_i a_i \) for every \( a_i \in F^- \), and so, we have

\[
T(a_1, \ldots, a_n) = \alpha_1 a_1 + \cdots + \alpha_n a_n \quad \text{for every } (a_1, \ldots, a_n) \in \mathcal{M}_{1,n}(F^-).
\]

With this observation, we have the following results.
Remark 2.16. Let $\mathbb{F}$ and $\mathbb{K}$ be fields which possess involutions $-$ and $^\wedge$, respectively. We recall that if $-$ and $^\wedge$ are proper, then there exists $i \in \mathbb{F}$ (with $i = -i$ when $\mathbb{F}$ has characteristic not 2, and $i = 1 + i$ when $\mathbb{F}$ has characteristic 2) such that $\mathbb{F} = \mathbb{F}^- \oplus i \mathbb{F}^-$, and respectively, there exists $j \in \mathbb{K}$ such that $\mathbb{K} = \mathbb{K}^\wedge \oplus j \mathbb{K}^\wedge$. We now give the general description of mappings $\mathcal{H}_2(\mathbb{F}) \to \mathcal{H}_2(\mathbb{K})$ satisfying either condition (AH-2), or condition (AH-1) with $(\mathbb{F}, \mathbb{K}, -) = (\mathbb{K}, -^\wedge)$.

(i) Let $\psi : \mathcal{H}_2(\mathbb{F}) \to \mathcal{H}_2(\mathbb{K})$ be a mapping satisfying condition (AH-2). Let $A, B \in \mathcal{H}_2(\mathbb{F})$. Then $\psi(A - B) = \psi(\text{adj} \, \text{adj}(A - B)) = \text{adj} \, \psi(A - B) = \psi(A) - \psi(B)$, and so, $\psi(-B) = -\psi(B)$. It follows that $\psi(A + B) = \psi(A) + \psi(B)$ for every $A, B \in \mathcal{H}_2(\mathbb{F})$. So, $\psi$ is a classical adjoint-commuting additive mapping.

- If the involutions $-$ and $^\wedge$ are the identity mappings on $\mathbb{F}$ and $\mathbb{K}$, respectively, then $\mathcal{H}_2(\mathbb{F}) = S_2(\mathbb{F})$ and $\mathcal{H}_2(\mathbb{K}) = S_2(\mathbb{K})$.

$$\psi \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} f_1(a) + f_2(b) + f_3(c) & g_1(a) + g_2(b) + g_3(c) \\ g_1(a) + g_2(b) + g_3(c) & h_1(a) + h_2(b) + h_3(c) \end{pmatrix} \in S_2(\mathbb{K})$$

for every $a, b, c \in \mathbb{F}$. Here, $f_1, f_2, f_3, g_1, g_2, g_3, h_1, h_2$ and $h_3$ are additive mappings from $\mathbb{F}$ into $\mathbb{K}$. Since $\text{adj}$ is linear and $\psi \circ \text{adj} = \text{adj} \circ \psi$, it follows that $h_1 = f_3, h_2 = -f_2, h_3 = f_1$ and $g_3 = -g_1$. So, we have

$$\psi \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} f_1(a) + f_2(b) + f_3(c) & g_1(a - c) + g_2(b) \\ g_1(a - c) + g_2(b) & f_3(a) - f_2(b) + f_1(c) \end{pmatrix}$$

for every $a, b, c \in \mathbb{F}$.

- If the involutions $-$ and $^\wedge$ are proper, then we have

$$\psi \begin{pmatrix} a & z \\ z & d \end{pmatrix} = \begin{pmatrix} f_1(a) + f_2(b) + f_3(c) + f_4(d) & g(d) + \overline{g(b, c)} + \overline{g_1(d)} \\ g(a) + G(b, c) + g_1(d) & h_1(a) + h_2(b) + h_3(c) + h_4(d) \end{pmatrix}$$

for every $a, b, c, d \in \mathbb{F}^-$ with $z = b + ic$. Here, $f_1, f_2, f_3, f_4, g_1, h_1, h_2, h_3, h_4 : \mathbb{F}^- \to \mathbb{K}^\wedge$ and $g, g_1 : \mathbb{F}^- \to \mathbb{K}^\wedge \oplus j \mathbb{K}^\wedge$ and $G : \mathbb{F}^- \times \mathbb{F}^- \to \mathbb{K}^\wedge \oplus j \mathbb{K}^\wedge$ are additive mappings. By $\psi \circ \text{adj} = \text{adj} \circ \psi$, we obtain $h_1 = f_4, h_2 = -f_2, h_3 = -f_3, h_4 = f_1$, and $g_1 = -g$. So, we have

$$\psi \begin{pmatrix} a & z \\ z & d \end{pmatrix} = \begin{pmatrix} f_1(a) + f_2(b) + f_3(c) + f_4(d) & g(a - d) + G(b, c) \\ g(a - d) + G(b, c) & f_4(a) - f_2(b) - f_3(c) + f_1(d) \end{pmatrix}$$

for every $a, b, c, d \in \mathbb{F}^-$ with $z = b + ic$. Again, by the additivity of $g$ and $G$, we have

$$g(a) = f_5(a) + j f_6(a) \quad \text{for all } a \in \mathbb{F}^-$$
$$G(b, c) = f_7(b) + f_8(c) + j (f_9(b) + f_{10}(c)) \quad \text{for all } b, c \in \mathbb{F}^-,$$

where $f_5, f_6, f_7, f_8, f_9, f_{10} : \mathbb{F}^- \to \mathbb{K}^\wedge$ are additive mappings.

(ii) Let $\psi : \mathcal{H}_2(\mathbb{F}) \to \mathcal{H}_2(\mathbb{F})$ be a mapping satisfying condition (AH-1). Then $\psi(A + \alpha B) = \psi(\text{adj} \, \text{adj}(A + \alpha B)) = \text{adj} \, \psi(A + \alpha B) = \psi(A) + \alpha \psi(B)$ for every $A, B \in \mathcal{H}_2(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$. So, $\psi$ is a classical adjoint-commuting $\mathbb{F}^-$-linear mapping.
If the involution $-$ is identity, then $\mathcal{H}_2(\mathbb{F}) = S_2(\mathbb{F})$, and thus, $\psi$ is a linear mapping. Therefore

$$
\psi\left( \begin{array}{cc} a & b \\ b & c \end{array} \right) = \left( \begin{array}{cc} \alpha_1 a + \alpha_2 b + \alpha_3 c & \alpha_4 (a - c) + \alpha_5 b \\ \alpha_4 (a - c) + \alpha_5 b & \alpha_3 a - \alpha_2 b + \alpha_1 c \end{array} \right)
$$

for every $a, b, c \in \mathbb{F}$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$ are some fixed scalars in $\mathbb{F}$. We remark that if $\mathbb{F}$ has characteristic not 2, then, by a similar argument as in [1, Theorem 3], we see that $\psi(A)$, with $A \in S_2(\mathbb{F})$, can also be represented as a linear combination of linear mappings of the form $P \ Adj P$ where $P \in M_2(\mathbb{F})$ is invertible with $\ Adj P = \pm P^t$. If the involution $-$ is proper, then, by a similar argument as given in (i), we obtain

$$
\psi\left( \begin{array}{cc} a & b + ic \\ b + ic & d \end{array} \right) = \left( \begin{array}{cc} \alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d & g(a - d) + G(b, c) \\ g(a - d) + G(b, c) & \alpha_4 a - \alpha_2 b - \alpha_3 c + \alpha_1 d \end{array} \right)
$$

for every $a, b, c, d \in \mathbb{F}^-$, where $g : \mathbb{F}^- \to \mathbb{F}^- \oplus i \mathbb{F}^-$ and $G : \mathbb{F}^- \times \mathbb{F}^- \to \mathbb{F}^- \oplus i \mathbb{F}^-$ are linear mappings given by

$$
g_1(a) = (\alpha_5 + i \alpha_6) a \quad \text{for all } a \in \mathbb{F}^-$$

$$G(b, c) = (\alpha_7 + i \alpha_8) b + (\alpha_9 + i \alpha_{10}) c \quad \text{for all } b, c \in \mathbb{F}^-,$$

and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9$ and $\alpha_{10}$ are some fixed scalars in $\mathbb{F}^-$.  

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**References**