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The Cauchy–Poisson Waves in an Inviscid Rotating Stratified Liquid

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1. FORMULATION

We consider the axisymmetric Cauchy–Poisson waves in an inviscid incompressible rotating stratified liquid of infinite depth. We use the cylindrical polar coordinates (r, θ, z) and consider a semi-infinite body of liquid bounded by $0 \le r \le \infty$, $-\infty < z \le z_0(r)$. The liquid is subjected to a uniform rotation with angular velocity Ω about the vertical axis $r = 0$ so that the equation of the paraboloidal free surface with 2ℓ as latus rectum is given by $z = z_0(r) = r^2/2\ell$.

We assume that the disturbed free surface is given by

$$z = z_0(r) + \eta(r,t) \tag{1.1}$$

due to superimposed initial elevation

$$z = z_0(r) = a\eta_0(r) = a\frac{\delta(r)}{r} \text{ at } t = 0, \tag{1.2}$$

where $2\pi a$ is the displaced volume associated with the initial elevation and $\delta(r)$ is the Dirac delta function.

The problem will be studied under the Boussinesq approximation. We assume the density field varies exponentially with the depth of the liquid. The Brunt–Väisälä frequency N is given by

$$N = \left[-\frac{g}{\rho_0}\frac{d\rho_0}{dz}\right]^{\frac{1}{2}} \tag{1.3}$$

where g is the acceleration due to gravity, $p_0(z)$ and $\rho_0(z)$ are the pressure and density in a reference state of hydrostatic equilibrium. The pressure p and the density ρ are expanded about p_0 and ρ_0 so that $\nabla p_0 = g\rho_0$, $p = p_0 + p'$, and $\rho = \rho_0 + \rho'$ where p' and ρ' are the perturbed quantities. We further assume that the density field varies exponentially with the depth so that N is real and positive for stable mean density distribution $(d\rho_0/dz < 0)$ and it remains constant throughout the flow field.

In view of these assumptions together with the acceleration potential $\chi = p'/\rho_0 + g(z - z_0)$, the basic equations in a rotating frame of reference are

$$\frac{\partial}{\partial t}(u, v, w) + 2\Omega(-v, u, 0) = -(\frac{\partial}{\partial r}, 0, \frac{\partial}{\partial z}\chi - (0, 0, \frac{g\rho'}{\rho_0}) \tag{1.4}$$

$$\frac{\partial \rho'}{\partial t} + w\frac{d\rho_0}{dz} = 0 \tag{1.5}$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{1.6}$$

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where (u, v, w) is the velocity vector. The free surface conditions are

$$\chi = g\eta, \ w = \eta_t + uz_0'(r) \quad \text{on } z = z_0(r) \tag{1.7}$$

The bottom boundary condition is

$$\chi_z \to 0 \quad \text{as } z \to -\infty \tag{1.8}$$

The wave motion is generated by the action of the initial surface elevation at $t = 0$ so that the initial conditions are

$$u = v = w = \chi = 0, \ \eta = a\eta_0(r) \quad \text{at } t = 0 \tag{1.9}$$

2. Formal Solution of the Problem

We first transform the initial value problem (1.4–1.9) to a boundary value problem by using the Laplace transform with respect to t. We next introduce a change of variable

$$r = \xi\zeta, \ z = \frac{1}{2}[\frac{\xi^2}{\ell} + \lambda^2\ell(1-\zeta^2)] \tag{2.1ab}$$

with $0 \le \xi \le \infty$ and $1 \le \zeta < \infty$ so that the free surface $z = z_0(r)$ corresponds to $\xi = r$ and $\zeta = 1$. The Laplace transformed equation for $\overline{\chi}(\xi, \zeta, s)$ and the associated boundary conditions become

$$(\lambda\ell)^2\xi^{-1}(\xi\overline{\chi}_\xi)_\xi + \zeta^{-1}(\xi\overline{\chi}_\zeta)_\zeta = 0 \tag{2.2}$$

$$\frac{s^2+N^2}{g}\overline{\chi} - (\lambda^2\ell)^{-1}\overline{\chi}_\zeta = \frac{a(s^2+N^2)}{s}\eta_0(\zeta) \text{ on } \zeta = 1 \tag{2.3}$$

where $\lambda^2 = (s^2+4\Omega^2)/(s^2+N^2)$ and s is the Laplace transform variable.

The solutions of this system are given by

$$\chi(\xi, \zeta, t) = ag\mathcal{L}^{-1}\int_0^\infty J_0(k\xi)\overline{Z}(k, s)P(\lambda k\ell\zeta)kdk, \tag{2.4}$$

$$\eta(\xi, t) = a\lim_{\zeta\to 1+}\mathcal{L}^{-1}\int_0^\infty J_0(k\xi)\overline{Z}(k, s)P(\lambda k\ell\zeta)kdk, \tag{2.5}$$

where $\mathcal{L}^{-1}$ stands for the inverse Laplace transform, $\overline{Z}(k, s)$ is the joint Laplace–Hankel transform of the free surface elevation in the limit $\zeta \to 1+$ given by

$$\overline{Z}(k, s) = as[s^2 + (gk/\lambda)\psi(\kappa)]^{-1}, \quad \kappa = \lambda k\ell, \tag{2.6}$$

$$\psi(\kappa) = \frac{K_1(\kappa)}{K_0(\kappa)} \sim 1 + \frac{1}{2\kappa} \text{ as } \kappa \to \infty \text{ and } P(\kappa\zeta) = \frac{K_0(\kappa\zeta)}{K_0(\kappa)} \tag{2.7ab}$$

where $K_0(\kappa)$ and $K_1(\kappa)$ are modified Bessel functions. The free surface elevation is then

$$\eta(r, t) = \mathcal{L}^{-1}\int_0^\infty \overline{Z}(k, s)J_0(k\xi)Qkdk \tag{2.8}$$

where $Q = \lim_{\zeta\to 1+} P(\lambda k\ell\zeta)$

The implicit form of the dispersion relation is obtained by setting $s = \pm i\omega$ in the denominator of the function $\overline{Z}(k, s)$ and then equating the denominator to zero as

$$\omega^2 = N^2 + (gk/\lambda)\psi(\lambda k\ell) \text{ or } \omega^2 = 4\Omega^2 + (gk\lambda)\psi(\lambda k\ell), \tag{2.9ab}$$

with $\lambda^2 = (\omega^2 - 4\Omega^2)/(\omega^2 - N^2)$. In the planar approximation ($\alpha = \frac{\Omega^2 \ell}{g} \to \infty$), this dispersion relation has the explicit form

$$\omega^4 - \omega^2(4\Omega^2 + N^2) + 4\Omega^2 N^2 - g^2 k^2 = 0 \tag{2.10}$$

In the limit $2\Omega \to 0$ or $N \to 0$, the dispersion relation reduces to known results (see [3,2]).

3. Asymptotic Solutions and Conclusions

In order to determine the wave structure in a rotating stratified liquid, the asymptotic solution for sufficiently large time is of special interest. The nondimensional form of the solution $\eta^* = (\frac{r^2}{a})\eta(r,t)$ can be expressed as the sum of two terms η_1 and η_2. The first term η_1 is made up of contributions from the poles of $\overline{Z}(k,s)$ at $s = 0$ and $s = \pm i\omega$ combined with the contribution from the stationary point of the integral (2.5). The second term η_2 is made up of the branch point contributions and contributions from the stationary points of the associated integral. In terms of nondimensional parameters $\mu \equiv \frac{gt^2}{r}$, $\theta \equiv \frac{r}{\ell}$, $\alpha \equiv \Omega^2\ell/g$, and $\beta = N^2\ell/g$, we obtain the asymptotic solution by using the stationary phase method in the form

$$\eta_1(r,t) \sim 2^{-3/2}(\mu - 12\alpha\theta - 3\beta\theta - \theta + 0(\frac{1}{\mu}))\cos(\frac{1}{4}\mu + 2\alpha\theta + \frac{1}{2}\theta + \frac{1}{2}\beta\theta + 0(\frac{1}{\mu})), \text{ as } \mu \to \infty \tag{3.1}$$

Similarly, we write down the asymptotic solution for $\eta_2(r,t)$. For the case $2\Omega \gg N$, the solution is

$$\begin{aligned}\eta_2(r,t) \sim\ & \frac{2\alpha\theta}{\sqrt{\pi}}(1 - \frac{N^2}{4\Omega^2})^{\frac{1}{2}}(\Omega t)^{-3/2}\cos(2\Omega t - \frac{3\pi}{4}) \\ & + \frac{2^{3/2}N\alpha\theta}{\Omega\sqrt{\pi}}(1 - \frac{N^2}{4\Omega^2})^{\frac{1}{2}}(Nt)^{+1/2}\cos(Nt + \frac{\pi}{4}),\ \Omega t \to \infty.\end{aligned} \tag{3.2}$$

For the case $N \gg 2\Omega$, the asymptotic solution has the form

$$\begin{aligned}\eta_2(r,t) \sim\ & \sqrt{\frac{2}{\pi}}\beta\theta(1 - \frac{4\Omega^2}{N^2})^{\frac{1}{2}}(Nt)^{-3/2}\cos(Nt - \frac{3\pi}{4}) \\ & + 2^{3/2}\frac{\theta\Omega}{N\sqrt{\pi}}(1 - \frac{4\Omega^2}{N^2})^{\frac{1}{2}}.(2\Omega t)^{-3/2}\cos(2\Omega t + \frac{\pi}{4}) \text{ as } Nt \to \infty.\end{aligned} \tag{3.3}$$

For the case $2\Omega \gg N$, the branch-point contribution to η_2 is given by

$$\eta_2(r,t) = \frac{2\alpha\theta}{\Omega t}J_1(2\Omega t) \sim \frac{2\alpha\theta}{\Omega t}.\frac{1}{\sqrt{\pi\Omega t}}\sin(2\Omega t - \frac{\pi}{4}) \text{ as } \Omega t \to \infty \tag{3.4}$$

On the other hand, the branch-point contribution to η_2 for the case $N \gg 2\Omega$ is

$$\eta_2 = \frac{\beta\theta}{Nt}J_1(Nt) \sim \frac{\beta\theta}{Nt}\frac{1}{\sqrt{\pi Nt}}\sin(Nt - \frac{\pi}{4}) \text{ as } Nt \to \infty \tag{3.5}$$

The asymptotic solution for $\eta(r,t)$ consists of several distinct wave terms. The first term represented by (3.1) corresponds to surface waves which are qualitatively similar to those in the classical Cauchy–Poisson problem for an inviscid nonrotating nonstratified liquid. However, the amplitude, and phase of the waves are modified by rotation, stratification, and the curved surface of the liquid. For a rapidly rotating liquid ($2\Omega \gg N$), the second term (3.2) represents inertial waves of frequency 2Ω and internal waves of frequency N. The amplitude and phase of these waves are modified, and the former decays as $\Omega t \to \infty$.

Finally, for the case of strong stratification, the solution consists of internal waves and inertial waves of decaying amplitude as $Nt \to \infty$. Thus the second term represented by η_2 has its existence entirely due to rotation and density stratification. These waves have no antecedents in an inviscid nonrotating and nonstratified liquid.

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