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Scattering States and Wave Operators in the Abstract Theory of Scattering*

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The abstract theory of scattering deals with pairs of self-adjoint operators H_j acting on Hilbert spaces $\mathscr{H}_j$, $j = 1, 2$, and the corresponding unitary groups $U_j(t) = \exp(-itH_j)$, $t \in R$. Most of the existing theories are concerned with finding criteria for the existence and completeness of the generalized wave operators, defined by

$$W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} U_2(-t)\, KU_1(t)\, P_1^{ac}$$

where $K: \mathscr{H}_1 \to \mathscr{H}_2$ is a linear bijection and P_1^{ac} is the orthogonal projection of $\mathscr{H}_1$ onto the subspace $\mathscr{H}_1^{ac}$ of absolute continuity for H_1. The theory can be applied to physical problems only if $\mathscr{H}_1^{ac}$ can be shown to coincide with the subspace of scattering states for $U_1(t)$. This paper presents a new abstract definition of the scattering states, based directly on the physical meaning of scattering, and develops a corresponding abstract theory of wave operators. The applicability of the theory is demonstrated for a class of wave propagation problems of classical physics.

1. INTRODUCTION

An abstract model for many wave propagation phenomena of classical and quantum physics is provided by a Hilbert space $\mathscr{H}$, a self-adjoint operator H on $\mathscr{H}$, and the corresponding group of unitary operators $U(t) = \exp(-itH)$, $t \in R$. The vectors $f \in \mathscr{H}$ are interpreted as states of a physical system whose time evolution is given by $u(t) = U(t)f$. The unitarity of $U(t)$ is equivalent to the conservation law $\| u(t)\| = \| f \|$ where $\| \cdot \|$ denotes the norm in $\mathscr{H}$.

The abstract theory of scattering is concerned with the asymptotic equality for $t \to \infty$ of the states of two physical systems. To formulate this precisely let $\mathscr{H}_j$, H_j, and $U_j(t) = \exp(-itH_j)$ $(j = 1, 2)$ repre-

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sent the systems and let $J_{21}: \mathscr{H}_1 \to \mathscr{H}_2$ denote a linear bijection with inverse $J_{12} = J_{21}^{-1}$. Then $U_1(t)$ may be said to be asymptotically equal to $U_2(t)$ on a subspace $\mathscr{M}_1 \subset \mathscr{H}_1$ if for each $f_1 \in \mathscr{M}_1$ there exists an $f_2 \in \mathscr{H}_2$ such that

$$\lim_{t\to\infty} \{U_1(t) f_1 - J_{12} U_2(t) f_2\} = 0 \qquad \text{in } \mathscr{H}_1 . \tag{1.1}$$

It is easy to show that (1.1) holds if and only if the wave operator

$$W = W(H_2, H_1, J_{21}, \mathscr{M}_1) = \operatorname*{s-lim}_{t\to\infty} U_2(-t)\, J_{21} U_1(t)\, P_{\mathscr{M}_1} \tag{1.2}$$

exists, where s-lim denotes the strong limit as an operator from $\mathscr{H}_1$ to $\mathscr{H}_2$ and $P_{\mathscr{M}_1}$ denotes the orthogonal projection of $\mathscr{H}_1$ onto $\mathscr{M}_1$. From this point of view the basic problem of the abstract theory of scattering is to discover those quadruples $(H_2, H_1, J_{21}, \mathscr{M}_1)$ for which the wave operator W exists. The problem so formulated is extremely general and only special cases of it have been treated.

Following the pioneering work of Kato [2, 3] and Kuroda [7] most of the existing work on the abstract theory of wave operators treats the case where $\mathscr{M}_1 = \mathscr{H}_1^{ac}$, the subspace of absolute continuity for H_1 [4]. This choice appears to have been motivated by its mathematical convenience (see [6]) and the observation that in many applications to physical problems it can be shown that

$$\mathscr{H}_1 = \mathscr{H}_1^{ac} \oplus \mathscr{H}_1^{p}, \tag{1.3}$$

where $\mathscr{H}_1^p$ is spanned by the eigenstates of H_1 whose time evolution is known. The abstract theory based on this choice has the disadvantage that for each physical application it is necessary either to establish (1.3) or to investigate whether there are states in $\mathscr{H}_1 \ominus \mathscr{H}_1^{ac}$ for which the wave operator exists.

The purpose of this paper is to present an abstract definition of the subspace $\mathscr{H}^s$ of all scattering states for an operator H, and to develop the theory of wave operators with $\mathscr{M}_1 = \mathscr{H}_1^s$. The theory is motivated by, and is applicable to, a large class of scattering problems which includes both the quantum mechanical problem of scattering by a potential and scattering problems of classical physics.

The paper is organized as follows. In Section 2 it is shown that the scattering states for quantum mechanical and classical waves in R^n can be defined by means of a family $\{Q_q\}$ of "localizing" operators on the state space $\mathscr{H}$. Then an axiomatic definition is given for families $\{Q_q\}$ of localizing operators on an abstract Hilbert space $\mathscr{H}$, and it is shown how a "scattering" subspace $\mathscr{H}^s$ can be associated with each self-adjoint operator H and family $\{Q_q\}$. In Section 3 a theory of wave operators on $\mathscr{H}^s$ is developed. The theory is parallel to, but independent of, the well-known theory of wave operators on $\mathscr{H}^{ac}$ originated by Kato and Kuroda. In Section 4 the applicability of the abstract theory is demonstrated for a class of scattering problems of classical physics. This application is based on a method due to Lax and Phillips [10] and extended by La Vita, Schulenberger and the author [8]. The results developed below are applied in [8] which should be read together with this paper.

2. Localizing Operators and Scattering Subspaces

The definition of a scattering subspace given below is suggested by consideration of the propagation of quantum mechanical and classical waves in R^n. In the first case (see [1]), $\mathscr{H} = L_2(R^n)$ and H is a self-adjoint extension of the Schrödinger operator

$$H = -\sum_{j=1}^{n} D_j^2 + V(x) \tag{2.1}$$

where $x = (x_1, x_2, ..., x_n) \in R^n$, $D_j = \partial/\partial x_j$ and $V(x)$ is a real-valued potential. If $u(x, t) = U(t) f(x)$ is a corresponding wave with initial state $f \in \mathscr{H}$, $\|f\| = 1$, and $K \subset R^n$ is a Lebesgue measurable set then

$$I(K, t) = \int_K |u(x, t)|^2 \, dx \tag{2.2}$$

is interpreted as the probability that the system corresponding to H is in the set K at time t. In the second case (see [17]), H is generated by a matrix partial differential operator

$$H = -iE(x)^{-1} \sum_{j=1}^{n} A_j D_j \tag{2.3}$$

where $E(x)$, $A_1, A_2, ..., A_n$ are $m \times m$ Hermitian matrices and $E(x)$ is positive definite. $\mathscr{H}$ is the Hilbert space generated by the inner product

$$(u, v) = \int_{R^n} u(x)^* E(x) v(x) \, dx \tag{2.4}$$

where $u(x)$ is an $m \times 1$ matrix and $u(x)^*$ is its Hermitian adjoint.

In this case if $u(x, t) = U(t)f(x)$ with $f \in \mathscr{H}$ and $K \subset R^n$ is Lebesgue measurable then

$$I(K, t) = \int_K u(x, t)^* E(x)\, u(x, t)\, dx \tag{2.5}$$

is interpreted as the energy in the set K at time t.

In both of the examples sketched above the fundamental meaning of the statement "$u(x, t)$ is a scattering wave" is that the wave ultimately propagates out of any bounded set K; i.e.,

$$\lim_{t\to\infty} I(K, t) = 0 \qquad \text{for every bounded measurable} \quad K \subset R^n. \tag{2.6}$$

Thus it is natural to say that $f \in \mathscr{H}$ is a scattering state if and only if (2.6) holds.

Condition (2.6) can be generalized to the setting of the abstract theory of scattering by reformulating it in operator-theoretic terms. First, note that (2.6) holds if and only if it holds for all balls $B_q = \{x: |x| \leqslant q\}$. Define an operator $Q_q: \mathscr{H} \to \mathscr{H}$ by

$$Q_q u(x) = \chi_q(x)\, u(x) \qquad \text{for all} \quad x \in R^n \tag{2.7}$$

where χ_q is the characteristic function of B_q. Then $I(B_q, t) = \| Q_q U(t) f \|^2$ and (2.6) is equivalent to

$$\lim_{t\to\infty} \| Q_q U(t) f \| = 0 \qquad \text{for every} \quad q \geqslant 0. \tag{2.8}$$

It is easy to verify that the family of operators $\{Q_q: 0 \leqslant q < \infty\}$ has the properties

$$Q_q \text{ is an orthogonal projection on } \mathscr{H} \text{ for each } q \geqslant 0. \tag{2.9}$$

$$\operatorname*{s-lim}_{q\to\infty} Q_q = 1, \qquad \text{the identity operator on } \mathscr{H}. \tag{2.10}$$

It is shown next, in the context of the abstract theory, that any such family determines a subspace of scattering states. To this end let $\mathscr{H}$ denote an arbitrary separable Hilbert space and let H denote a self-adjoint operator on $\mathscr{H}$ with corresponding unitary group $U(t) = \exp(-itH)$.

Definition. A family $\{Q_q: 0 \leqslant q < \infty\}$ is called a family of localizing operators on $\mathscr{H}$ if $Q_q: \mathscr{H} \to \mathscr{H}$ satisfies (2.9) and (2.10).

DEFINITION. A vector $f \in \mathscr{H}$ is said to be a scattering state for H and $\{Q_q: 0 \leqslant q < \infty\}$ if and only if (2.8) holds. The set of all scattering states for H and $\{Q_q: 0 \leqslant q < \infty\}$ is denoted by $\mathscr{H}^s$.

The space $\mathscr{H}$ has the decomposition

$$\mathscr{H} = \mathscr{H}^c \oplus \mathscr{H}^p \tag{2.11}$$

into the subspaces of continuity and discontinuity for H, respectively, and (2.11) reduces H [4, p. 515]. A first result concerning the scattering states is

THEOREM 2.1. $\mathscr{H}^s \subset \mathscr{H}^c$.

Proof. Since $\mathscr{H}^c = (\mathscr{H}^p)^\perp$ it is enough to show that $\mathscr{H}^s \subset (\mathscr{H}^p)^\perp$. Moreover, since $\mathscr{H}^p$ is the closed subspace spanned by the eigenvectors of H it is enough to prove that if $f \in \mathscr{H}^s$ and ϕ is any eigenvector of H then $(f, \phi) = 0$. To prove this note that

$$\begin{aligned}(Q_q U(t)f, \phi) &= (U(t)f, Q_q\phi) \\ &= (U(t)f, \phi) + (U(t)f, Q_q\phi - \phi) \\ &= (f, U(-t)\phi) + (U(t)f, Q_q\phi - \phi) \\ &= e^{i\lambda t}(f, \phi) + (U(t)f, Q_q\phi - \phi)\end{aligned} \tag{2.12}$$

where λ is the eigenvalue of ϕ. Thus

$$(f, \phi) = e^{-i\lambda t}(Q_q U(t)f, \phi) - e^{-i\lambda t}(U(t)f, Q_q\phi - \phi) \tag{2.13}$$

for all $q \geqslant 0$ and $t \in R$, and hence

$$|(f, \phi)| \leqslant \| Q_q U(t) f \| \| \phi \| + \| f \| \| Q_q \phi - \phi \|. \tag{2.14}$$

If first $t \to \infty$ and then $q \to \infty$ the right-hand side of (2.14) tends to zero, by (2.8) and (2.10). It follows that $(f, \phi) = 0$, which completes the proof.

The next result shows that H defines an operator on $\mathscr{H}^s$.

THEOREM 2.2. *$\mathscr{H}^s$ is a closed subspace of $\mathscr{H}^c$ and reduces H.*

Proof. $\mathscr{H}^s$ is clearly a linear manifold in $\mathscr{H}^c$. Suppose that $f \in \overline{\mathscr{H}^s}$, the closure of $\mathscr{H}^s$ in $\mathscr{H}^c$, so that $f = \lim_{n\to\infty} f_n$ where $f_n \in \mathscr{H}^s$. Then

$$\begin{aligned}\| Q_q U(t) f \| &\leqslant \| Q_q U(t)(f - f_n)\| + \| Q_q U(t) f_n \| \\ &\leqslant \| f - f_n \| + \| Q_q U(t) f_n \|\end{aligned} \tag{2.15}$$

for all t and $n = 1, 2, 3, \ldots$, because $\|Q_q U(t)\| = 1$. The last term tends to zero when $t \to \infty$ because $f_n \in \mathscr{H}^s$. Thus

$$\overline{\lim_{t\to\infty}} \|Q_q U(t) f\| \leqslant \|f - f_n\| \qquad \text{for} \quad n = 1, 2, 3, \ldots \tag{2.16}$$

It follows that f satisfies (2.8) because $\lim_{n\to\infty} f_n = f$. Hence $\overline{\mathscr{H}^s} = \mathscr{H}^s$.

$\mathscr{H}^s$ reduces H if and only if it reduces $U(\tau) = \exp(-i\tau H)$ for each $\tau \in R$. Hence it is sufficient to prove that $U(\tau)\mathscr{H}^s \subset \mathscr{H}^s$ for each $\tau \in R$ (which implies that $U(\tau)(\mathscr{H}^s)^\perp \subset (\mathscr{H}^s)^\perp$ for each $\tau \in R$ because $U(\tau)$ is unitary). But this property is immediate, because if $f \in \mathscr{H}^s$ then

$$\|Q_q U(t)\, U(\tau) f\| = \|Q_q U(t+\tau) f\| \to 0 \tag{2.17}$$

when $t \to \infty$ by (2.8). This completes the proof.

A primary problem of scattering theory, as developed here, is to identify or characterize the subspace $\mathscr{H}^s$ of scattering states. In [8] it is shown for a class of scattering problems of classical physics that $\mathscr{H}^s = \mathscr{H}^c = \mathscr{H}^{ac}$. This result cannot hold in all cases. However, the following partial result holds whenever the localizing operators Q_q are H-compact [4, p. 194].

THEOREM 2.3. *If Q_q is H-compact for every $q \geqslant 0$ then $\mathscr{H}^{ac} \subset \mathscr{H}^s$.*

Proof. It must be shown that each $f \in \mathscr{H}^{ac}$ satisfies (2.8). Note that it is sufficient to prove this for a dense subset of $\mathscr{H}^{ac}$ because $Q_q U(t)$ is uniformly bounded for all $q \geqslant 0$ and $t \in R$. The dense set $D(H) \cap \mathscr{H}^{ac}$ will be used. Note that if $f \in \mathscr{H}^{ac}$, $g \in \mathscr{H}$, and $\{\Pi(\lambda) : \lambda \in R\}$ is the spectral family for H, then $(\Pi(\lambda) f, g)$ is absolutely continuous and hence

$$(U(t) f, g) = \int_R \exp(-it\lambda) \frac{d(\Pi(\lambda) f, g)}{d\lambda}\, d\lambda \tag{2.18}$$

where $d(\Pi(\lambda) f, g)/d\lambda \in L_1(R)$. It follows by the Riemann–Lebesgue theorem that $U(t) f \to 0$ weakly in $\mathscr{H}$ when $t \to \infty$. Next if $f \in D(H)$ then $\|U(t) f\| = \|f\|$ and

$$\|HU(t) f\| = \|U(t)\, Hf\| = \|Hf\| \tag{2.19}$$

for all $t \in R$. Hence the set of vectors $\{U(t) f : t \in R\}$ is bounded in the graph norm of H. Therefore, since Q_q is H-compact, the set of vectors $\{Q_q U(t) f : t \in R\}$ is precompact in $\mathscr{H}$. In particular, any sequence $\{Q_q U(t_n) f\}$ with $t_n \to \infty$ contains a subsequence $\{Q_q U(t_n') f\}$ which converges in $\mathscr{H}$:

$$\lim_{n\to\infty} Q_q U(t_n') f = g \qquad \text{in } \mathscr{H}. \tag{2.20}$$

But if $f \in \mathscr{H}^{ac}$ then $Q_q U(t_n') f \to 0$ weakly in $\mathscr{H}$. Thus $g = 0$ by the uniqueness of weak limits. Finally, the entire sequence $\{Q_q U(t_n) f\}$ must converge strongly to zero, because every sequence contains a subsequence which does so. Thus $D(H) \cap \mathscr{H}^{ac} \subset \mathscr{H}^s$ and the proof is complete.

It is known that $\mathscr{H}^c$ has the reducing decomposition

$$\mathscr{H}^c = \mathscr{H}^{ac} \oplus \mathscr{H}^{sc} \tag{2.21}$$

where $\mathscr{H}^{sc}$ is the singularly continuous subspace [4, p. 516]. Hence Theorem 2.3 implies

COROLLARY 2.4. *If Q_q is H-compact for every $q \geqslant 0$ then $\mathscr{H}^{sc} = \{0\} \Rightarrow \mathscr{H}^s = \mathscr{H}^{ac}$.*

Proof. This is immediate because Theorem 2.3 implies the inclusions

$$\mathscr{H}^{ac} \subset \mathscr{H}^s \subset \mathscr{H}^c = \mathscr{H}^{ac} \oplus \mathscr{H}^{sc}. \tag{2.22}$$

States which scatter when $t \to \infty$ were defined and studied above. Corresponding results for states which scatter when $t \to -\infty$ are needed for the theory of the scattering operator. However, these follow immediately from the results already given, since

$$U_H(-t) = \exp(itH) = U_{-H}(t). \tag{2.23}$$

Hence, each H and localizing family $\{Q_q : 0 \leqslant q < \infty\}$ define a pair of scattering subspaces $\mathscr{H}_{\pm}^s$, corresponding to $t \to \pm\infty$, and Theorems 2.2 and 2.3 hold for both of them.

3. PROPERTIES OF THE WAVE OPERATORS ON THE SCATTERING SUBSPACES

In the abstract theory of scattering based on the subspace of absolute continuity $\mathscr{H}^{ac}$, as developed by Kato [4] and Kuroda [7], it is emphasized that the wave operators $W(H_2, H_1, J_{21}, \mathscr{H}_1^{ac})$ have many properties that follow directly from their existence. Moreover, these properties play a major role in the existence and completeness theory for the wave operators. In this section it is shown that an analogous theory based on a scattering subspace $\mathscr{H}^s$ can be developed which is parallel to, but independent of, the theory based on $\mathscr{H}^{ac}$.

Consider a pair of systems defined by separable Hilbert spaces $\mathscr{H}_j$ and self-adjoint operators H_j ($j = 1, 2$), and write $U_j(t) = \exp(-itH_j)$. Let $\{Q_q^j: 0 \leqslant q < \infty\}$ denote a family of localizing operators on $\mathscr{H}_j$ and let $\mathscr{H}_j^s$ denote the scattering subspace for H_j and $\{Q_q^j: 0 \leqslant q < \infty\}$. Let P_j^s denote the orthogonal projection of $\mathscr{H}_j$ onto $\mathscr{H}_j^s$. The wave operators

$$W(H_2, H_1, J_{21}, \mathscr{H}_1^s) = \underset{t\to\infty}{\text{s-lim}}\, U_2(-t)\, J_{21} U_1(t)\, P_1^s \tag{3.1}$$

will be considered, where $J_{21}: \mathscr{H}_1 \to \mathscr{H}_2$ is a linear bijection with inverse $J_{12} = J_{21}^{-1}$.

Kato has observed that different operators J_{21} may produce the same wave operators and has formulated the following criterion [5].

Definition. J_{21} and J_{21}' are said to be equivalent with respect to H_1 and P_1^s (in symbols, $J_{21} \sim J_{21}'(H_1, P_1^s)$) if and only if

$$\underset{t\to\infty}{\text{s-lim}}\, (J_{21} - J_{21}')\, U_1(t)\, P_1^s = 0. \tag{3.2}$$

It is easy to verify that this is an equivalence relation. Its importance for scattering theory is due to

Theorem 3.1. *Let $J_{21} \sim J_{21}'(H_1, P_1^s)$. Then $W(H_2, H_1, J_{21}, \mathscr{H}_1^s)$ exists if and only if $W(H_2, H_1, J_{21}', \mathscr{H}_1^s)$ exists. Moreover, if these wave operators exist they are equal:*

$$W(H_2, H_1, J_{21}', \mathscr{H}_1^s) = W(H_2, H_1, J_{21}, \mathscr{H}_1^s). \tag{3.3}$$

The analogous result with $\mathscr{H}_1^s$ replaced by $\mathscr{H}_1^{ac}$ was proved by Kato [5] and the same proof is applicable to Theorem 3.1.

Some of the properties of wave operators that follow directly from Definition (3.1) are derived next. The following notation is used for brevity:

$$W^{21} = W(H_2, H_1, J_{21}, \mathscr{H}_1^s). \tag{3.4}$$

Theorem 3.2. *If W^{21} exists then it is an intertwining operator for the pair H_1, H_2; that is,*

$$W^{21} H_1 \subset H_2 W^{21}. \tag{3.5}$$

The proof depends only on the fact that $P_1{}^s$ commutes with H_1 and is the same as the analog for $\mathscr{H}_1^{ac}$ proved in [5].

Several properties of the wave operators follow directly from the intertwining relation (3.5). These results are exact analogs of results for $\mathscr{H}_1^{ac}$ and may be proved by the same arguments (see [5] for details).

THEOREM 3.3. *If W^{21} exists and if P_1 and P_2 denote the orthogonal projections on $\mathscr{M}_1 = N(W^{21})^\perp$ and $\mathscr{M}_2 = N(W^{21*})^\perp$, respectively, then $| W^{21} |$ and P_1 commute with H_1, $| W^{21*} |$ and P_2 commute with H_2, and the parts of H_1 and H_2 in $\mathscr{M}_1$ and $\mathscr{M}_2$ are unitarily equivalent.*

Partial information concerning the nullspace and range of W^{21} is given by (see [5, Theorem 3.2]).

THEOREM 3.4. *If W^{21} exists then*

$$W^{21} = W^{21}P_1{}^s \qquad \text{and} \qquad P_1 \leqslant P_1{}^s. \tag{3.6}$$

DEFINITION. The wave operator W^{21} is said to be semicomplete $\Leftrightarrow P_1 = P_1{}^s$. W^{21} is said to be complete $\Leftrightarrow P_1 = P_1{}^s$ and $P_2 = P_2{}^s$.

In the abstract theory of scattering the wave operator W^{21} may or may not be a partial isometry and it may or may not be semicomplete or complete. The existence of one or more of these properties depends on the choice of the H_j, $\{Q_q{}^j: 0 \leqslant q < \infty\}$ $(j = 1, 2)$ and J_{21}, as does the existence of W^{21}. However, criteria which guarantee these additional properties can be formulated within the abstract theory. Several such results which are useful for applications are given below.

THEOREM 3.5. *Let W^{21} exist and assume*

$$\textit{there exists a unitary } J_{21}' \sim J_{21}(H_1, P_1{}^s). \tag{3.7}$$

Then W^{21} is a partial isometry with initial set $\mathscr{H}_1{}^s$:

$$W^{21*}W^{21} = P_1{}^s, \tag{3.8}$$

and W^{21} is semicomplete.

The proof is essentially the same as in Kato [5, Theorem 6.2].

THEOREM 3.6. *Let W^{21} exist and assume that* (3.7) *holds and*

$$Q_q{}^2 J_{21} = J_{21} Q_q{}^1 \qquad \text{for} \quad 0 \leqslant q < \infty. \tag{3.9}$$

Then

$$W^{21} = P_2{}^s W^{21} \quad \text{and} \quad P_2 \leqslant P_2{}^s. \tag{3.10}$$

Proof. P_2 is the orthogonal projection onto $N(W^{21*})^\perp = \overline{R(W^{21})}$. Thus both parts of (3.10) are equivalent to the statement that $R(W^{21}) \subset \mathscr{H}_2{}^s$. To prove this note that $u \in R(W^{21})$ if and only if $u = W^{21}v$ for some $v \in \mathscr{H}_1{}^s$. For this pair of vectors Definition (3.1) implies

$$\lim_{t\to\infty} \{U_2(t)u - J_{21}U_1(t)v\} = 0 \quad \text{in } \mathscr{H}_2. \tag{3.11}$$

Moreover, for any $q \geqslant 0$,

$$\begin{aligned} \| Q_q{}^2 U_2(t)u \|_2 &\leqslant \| Q_q{}^2\{U_2(t)u - J_{21}U_1(t)v\}\|_2 + \| Q_q{}^2 J_{21} U_1(t)v \|_2 \\ &\leqslant \| U_2(t)u - J_{21}U_1(t)v \|_2 + \| J_{21}Q_q{}^1 U_1(t)v \|_2 \\ &\leqslant \| U_2(t)u - J_{21}U_1(t)v \|_2 + \| J_{21} \| \, \| Q_q{}^1 U_1(t)v \|_1 . \end{aligned} \tag{3.12}$$

Note that hypothesis (3.9) was used in the second inequality of (3.12). Both terms on the right in (3.12) tend to zero when $t \to \infty$, by (3.11) and because $v \in \mathscr{H}_1{}^s$. Thus every $u \in R(W^{21})$ is a scattering state for H_2; that is, $R(W^{21}) \subset \mathscr{H}_2{}^s$. This completes the proof of Theorem 3.6.

The final result from the abstract theory that will be presented here is a completeness criterion. It is based on a chain rule for wave operators which can be formulated as follows.

THEOREM 3.7. *Let $W(H_2, H_1, J_{21}, \mathscr{H}_1{}^s)$ and $W(H_3, H_2, J_{32}, \mathscr{H}_2{}^s)$ exist and let J_{32} satisfy* (3.7) *and $Q_q{}^2$, $Q_q{}^3$ satisfy* (3.9). *Moreover, let $J_{31} \sim J_{32}J_{21}(H_1, P_1{}^s)$. Then $W(H_3, H_1, J_{31}, \mathscr{H}_1{}^s)$ exists and*

$$W(H_3, H_2, J_{32}, \mathscr{H}_2{}^s)\, W(H_2, H_1, J_{21}, \mathscr{H}_1{}^s) = W(H_3, H_1, J_{31}, \mathscr{H}_1{}^s). \tag{3.13}$$

An analogous result with $P_j{}^s$ replaced by P_j^{ac} was proved by Kato [5] and the same method applies to Theorem 3.7.

The completeness criterion follows directly from the chain rule. It may be stated as follows.

THEOREM 3.8. *Let both W^{21} and W^{12} exist. Moreover, let both J_{21} and J_{12} satisfy* (3.7) *and let $Q_q{}^1$ and $Q_q{}^2$ satisfy* (3.9). *Then both W^{21} and W^{12} are complete and*

$$W^{21*} = W^{12}. \tag{3.14}$$

Proof. The semicompleteness of W^{21} and W^{12} follows from Theorem 3.5. Moreover, Theorem 3.7 is applicable with $H_3 = H_1$, $J_{32} = J_{12}$, $Q_q^3 = Q_q^1$, and $J_{31} = 1$. Since $W(H_1, H_1, 1, \mathscr{H}_1^s) = P_1^s$, (3.13) gives

$$W^{12}W^{21} = P_1^s. \tag{3.15}$$

Since the indices 1 and 2 enter symmetrically in the hypotheses it also follows that

$$W^{21}W^{12} = P_2^s. \tag{3.16}$$

Next, note that $W^{21} = P_2^s W^{21}$ by Theorem 3.6. Taking adjoints and using (3.16) and (3.8) gives

$$\begin{aligned} W^{21*} &= W^{21*}P_2^s = W^{21*}(W^{21}W^{12}) = (W^{21*}W^{21})W^{12} \\ &= P_1^s W^{12} = W^{12} \end{aligned} \tag{3.17}$$

because $R(W^{12}) \subset \mathscr{H}_1^s$ by Theorem 3.6 applied to W^{12}. Thus (3.14) is verified. Combining (3.14) and (3.16) gives

$$W^{21}W^{21*} = W^{21}W^{12} = P_2^s \tag{3.18}$$

which proves the completeness of W^{21}. The completeness of W^{12} follows by symmetry.

The wave operators defined by (3.1) relate the behavior of $U_1(t)$ and $U_2(t)$ for $t \to +\infty$. Analogous operators relate their behavior for $t \to -\infty$, and these are needed in the theory of the scattering operator. In this connection note that if the two scattering subspaces for $t \to \pm\infty$ are defined by

$$\mathscr{H}_\pm^s = \mathscr{H}_\pm^s(H) = \{f\colon \lim_{t\to\pm\infty} \|Q_q \exp(-itH)f\| = 0 \text{ for all } q \geqslant 0\} \tag{3.19}$$

then

$$\mathscr{H}_\pm^s(H) = \mathscr{H}_\mp^s(-H), \tag{3.20}$$

and the corresponding orthogonal projections satisfy

$$P_\pm^s(H) = P_\mp^s(-H). \tag{3.21}$$

Thus, if

$$W_\pm(H_0, H, J, \mathscr{H}_\pm^s(H)) = \operatorname*{s-lim}_{t\to\pm\infty} \exp(itH_0)J\exp(-itH)\,P_\pm^s(H), \tag{3.22}$$

then

$$W_-(H_0, H, J, \mathscr{H}_-^s(H)) = W_+(-H_0, -H, J, \mathscr{H}_+^s(-H)). \quad (3.23)$$

Hence, the results proved above for the wave operators W_+ also hold for W_-.

4. Applications to Scattering Problems of Classical Physics

In this section the applicability of the theory developed in Sections 2 and 3 to scattering problems of classical physics is demonstrated. In particular, it is shown that the existence and completeness of the wave operators on the scattering subspaces $\mathscr{H}_\pm^s$ can be proved directly, on the basis of the results in Sections 2 and 3, without reference to theories based on $\mathscr{H}^{ac}$, $\mathscr{H}^c$, or other subspaces.

The method used here is an adaptation of a method due to Lax and Phillips [9, 10]. Lax and Phillips were the first to develop an abstract scattering theory which did not depend on the theory of $\mathscr{H}^{ac}$ (see [9] for references). Moreover, in [9, Chap. VI; 10] they applied their theory to self-adjoint matrix operators

$$H = -i \sum_{j=1}^{n} A_j(x) D_j + B(x) \quad (4.1)$$

under the assumptions

$$H \text{ is elliptic.} \quad (4.2)$$

The coefficients $A_j(x)$ ($j = 1, 2,..., n$) and $B(x)$ are smooth functions. (4.3)

$$A_j(x) = A_j^0 \quad \text{and} \quad B(x) = 0 \quad \text{for} \quad |x| \geqslant \alpha. \quad (4.4)$$

$$n \text{ is odd.} \quad (4.5)$$

In their work H is compared with the operator

$$H_0 = -i \sum_{j=1}^{n} A_j^0 D_j, \quad (4.6)$$

both operating in $\mathscr{H} = L_2(R^n)$, and it is shown that the wave operators $W_\pm(H_0, H, 1, \mathscr{H}^c)$ and $W_\pm(H, H_0, 1, \mathscr{H}_0^c)$ exist and are complete. The scattering subspaces $\mathscr{H}_\pm^s$ and $\mathscr{H}_{0,\pm}^s$ were not introduced in their work but their results imply that in the notation of this paper, $\mathscr{H}_\pm^s = \mathscr{H}^c = \mathscr{H}^{ac}$ and $\mathscr{H}_{0,\pm}^s = \mathscr{H}_0^c = \mathscr{H}_0^{ac}$.

The abstract theory of Lax and Phillips was applied to scattering problems of classical physics by La Vita, Schulenberger and the author [8]. The Hilbert space $\mathscr{H}$ and self-adjoint operator H were defined by (2.4) and (2.3) (see [17] for details) and it was assumed that

$E(x)$ is Lebesgue-measurable, bounded, and uniformly positive definite on R^n. (4.7)

$$\text{rank} \sum_{j=1}^{n} A_j p_j = m - k \qquad \text{for all} \quad p \in R^n - \{0\}. \tag{4.8}$$

In (4.8), k is a fixed integer with $0 \leqslant k < m$. The significance of this condition is discussed in [12] and [13]. The application of the Lax–Phillips theory to this class of operators was made possible by a local compactness theorem for the operators (2.3) due to Schulenberger [11].

In [8] H was assumed to satisfy both (4.7) and (4.8) and the hypotheses

$$E(x) = E_0 \qquad \text{for} \quad |x| \geqslant \alpha \tag{4.9}$$

and

$$n \text{ is odd.} \tag{4.10}$$

H was compared with the operator

$$H_0 = -iE_0^{-1} \sum_{j=1}^{n} A_j D_j \tag{4.11}$$

operating on the Hilbert space $\mathscr{H}_0$ with inner product

$$(u, v)_0 = \int_{R^n} u(x)^* E_0 v(x)\, dx. \tag{4.12}$$

The identification operator $J_0 \colon \mathscr{H} \to \mathscr{H}_0$, defined by

$$J_0 u(x) = u(x) \qquad \text{for all} \quad x \in R^n, \tag{4.13}$$

and its inverse $J = J_0^{-1}$ were used ($\mathscr{H}$ and $\mathscr{H}_0$ are equivalent Hilbert spaces). $\{Q_q\}$ and $\{Q_q^0\}$ were the families of localizing operators defined by (2.7) on $\mathscr{H}$ and $\mathscr{H}_0$ respectively.

Abstract localizing operators and scattering subspaces were first introduced and applied in [8]. However, an exposition of the abstract theory was left to the present paper. In particular, Theorem 3.8 (the completeness criterion) and Theorem 4.2 are needed to complete the exposition in [8]. The principal result of [8] is

THEOREM 4.1. *If* $E(x)$, $A_1, A_2, \ldots, A_n$ *satisfy* (4.7), (4.8), (4.9), *and* (4.10), *then the wave operators* $W_\pm(H_0, H, J_0, \mathscr{H}_\pm{}^s)$ *and* $W_\pm(H, H_0, J, \mathscr{H}^s_{0,\pm})$ *exist and are complete.*

The proof of this result given in [8] consists of the following major steps. (a) Construct incoming and outgoing subspaces $\mathscr{D}_\pm{}^\tau$ for $U(t) = \exp(-itH)$ *in the reducing subspace* $\mathscr{H}^c$. (b) Use $\mathscr{D}_\pm{}^\tau$ to prove that $\mathscr{H}_\pm{}^s = \mathscr{H}^c$. (c) Use (a) and (b) to prove the existence and completeness of the wave operators on $\mathscr{H}_\pm{}^s = \mathscr{H}^c$. Here, in order to emphasize that a scattering theory based on $\mathscr{H}_\pm{}^s$ can be developed independently of theories based on $\mathscr{H}^c$ or other subspaces, a direct proof of Theorem 4.1 is given. The proof is a simple modification of the one given in [8] and should be read in conjunction with the proof in [8].

Proof of Theorem 4.1. The proof is based on the observation that the subspaces $\mathscr{D}_-{}^\tau$ and $\mathscr{D}_+{}^\tau$ constructed in [8] are actually incoming and outgoing subspaces of $\mathscr{H}_-{}^s$ and $\mathscr{H}_+{}^s$, respectively. The fact that $\mathscr{D}_\pm{}^\tau \subset \mathscr{H}_\pm{}^s$ follows immediately from [8, Eqs. (6.2), (6.4), and Theorem 6.5]. To prove that $\mathscr{D}_\pm{}^\tau$ are incoming and outgoing subspaces of $\mathscr{H}_\pm{}^s$, the three axioms of the Lax–Phillips theory [8, Eqs. (3.1)–(3.3)] must be verified with $\mathscr{D}_\pm{}^\tau$ for $\mathscr{D}_\pm$ and $\mathscr{H}_\pm{}^s$ for $\mathscr{H}$. Only (3.3) is difficult to verify. It states that $\Omega_\pm{}^\tau = \bigcup_{\pm t \leqslant 0} U(t)\, \mathscr{D}_\pm{}^\tau$ is dense in $\mathscr{H}_\pm{}^s$. The density of $\Omega_\pm{}^\tau$ in $\mathscr{H}^c$ was proved in [8, Theorem 6.8]. A direct proof that $\Omega_\pm{}^\tau$ is dense in $\mathscr{H}_\pm{}^s$ can be given by the same method and is even easier. In fact, it is shown in the proof given in [8] that if $f \in \mathscr{H}^c$ and $f \perp \Omega_+{}^\tau$ then there exist positive constants μ, q, and K such that

$$\| Q_{\mu(t-\tau)} f \| \leqslant K \| Q_q U(t) f \| \tag{4.14}$$

for all $t > 2\tau$. In particular, this holds for any $f \in \mathscr{H}_+{}^s$ such that $f \perp \Omega_+{}^\tau$. It follows from (4.14), (2.8), and (2.10) that $f = 0$. Hence $\Omega_+{}^\tau$ is dense in $\mathscr{H}_+{}^s$, and a similar proof holds for $\Omega_-{}^\tau$ and $\mathscr{H}_-{}^s$.

The existence of $W_\pm(H_0, H, J_0, \mathscr{H}_\pm{}^s)$ follows immediately from the fact that $\mathscr{D}_\pm{}^\tau$ are incoming and outgoing subspaces of $\mathscr{H}_\pm{}^s$. The simple proof, due to Lax and Phillips, is given in [8, Theorem 7.2]. The completeness of $W_\pm(H_0, H, J_0, \mathscr{H}_\pm{}^s)$ follows from [8, Theorem 2.4]. The proof of this last result is based on the abstract theory and was left to the present paper. In fact, it follows immediately from Theorem 3.8 above, as soon as the hypotheses of that theorem are verified. To see what is needed, make the identifications

$$H_1 = H, \qquad H_2 = H_0, \qquad J_{21} = J_0, \qquad \mathscr{H}_1^s = \mathscr{H}_\pm^s \tag{4.15}$$

in Theorem 3.8. Then all the hypotheses will be satisfied if

$$Q_q^0 J_0 = J_0 Q_q \tag{4.16}$$

and there exist unitary operators J' and J_0' such that

$$J' \sim J(H_0, P_{0+}^s) \qquad \text{and} \qquad J_0' \sim J_0(H, P^s). \tag{4.17}$$

Equation (4.16) follows immediately from the definitions of J_0, Q_q and Q_q^0 given above. The unitary operators

$$J' = E^{-1/2}E_0^{1/2} \qquad \text{and} \qquad J_0' = E_0^{-1/2}E^{1/2} \tag{4.18}$$

may be used to verify (4.17). The correctness of (4.17) follows from

THEOREM 4.2. *Let $K\colon \mathscr{H} \to \mathscr{H}_0$ be defined by $Ku(x) = K(x)\, u(x)$ where $K(x)$ is an $m \times m$ matrix over C for each $x \in R^n$ with the properties that $K(x)$ is bounded and Lebesgue measurable on R^n and* $\lim_{|x|\to\infty} K(x) = 1$. *Then $K \sim J_0(H, P^s)$.*

Proof. It must be shown that $\lim_{t\to\infty} \|(K - J_0)\, U(t)u\|_0 = 0$ for all $u \in \mathscr{H}^s$; see (3.2). Now if $u(x, t) = U(t)\, u(x)$ then

$$\|(K - J_0)\, U(t)u\|_0^2 = \int_{R^n} ((K(x) - 1)\, u(x, t))^*\, E_0(K(x) - 1)\, u(x, t)\, dx. \tag{4.19}$$

Notice that if M is an $m \times m$ matrix and u is an $m \times 1$ matrix then $(Mu)_j = \sum_{k=1}^m M_{jk}u_k$, whence

$$|(Mu)_j|^2 \leqslant \sum_{k=1}^m |M_{jk}|^2 \sum_{k=1}^m |u_k|^2 \leqslant m \max_{1\leqslant j,k\leqslant m} |M_{jk}|^2\, u^*u \tag{4.20}$$

for $j = 1, 2, \ldots, m$. Therefore, if the notation

$$\|M\|_{\max} = \max_{1\leqslant j,k\leqslant m} |M_{jk}| \tag{4.21}$$

is used, then

$$(Mu)^*\, Mu \leqslant m^2\, \|M\|_{\max}^2\, u^*u. \tag{4.22}$$

Now $E(x)$ and E_0 are bounded and uniformly positive definite; that is,

$$cu^*u \leqslant u^*E(x)u, \quad u^*E_0u \leqslant c'u^*u \quad \text{for all} \quad x \in R^n \quad \text{and} \quad u \in C^m, \tag{4.23}$$

where $0 < c \leqslant c'$. Combining (4.22) and (4.23) gives

$$(Mu)^* E_0 Mu \leqslant c'(Mu)^* Mu \leqslant c'm^2 \| M \|_{\max}^2 u^*u \leqslant \bar{c} \| M \|_{\max}^2 u^*E(x)u \tag{4.24}$$

where $\bar{c} = c'm^2c^{-1}$ is independent of M, x, and u. Returning to (4.19), (4.24) gives

$$\begin{aligned} \|(K - J_0) U(t)u \|_0^2 &\leqslant \bar{c} \int_{R^n} \| K(x) - 1 \|_{\max}^2 u(x,t)^* E(x) u(x,t)\, dx \\ &\leqslant \bar{c} \max_{|x| \geqslant q} \| K(x) - 1 \|_{\max}^2 \int_{|x| \geqslant q} u(x,t)^* E(x) u(x,t)\, dx \\ &\quad + \bar{c} \max_{|x| \leqslant q} \| K(x) - 1 \|_{\max}^2 \int_{|x| \leqslant q} u(x,t)^* E(x) u(x,t)\, dx \\ &\leqslant \bar{c} \max_{|x| \geqslant q} \| K(x) - 1 \|_{\max}^2 \| U(t)u \|^2 \qquad (4.25) \\ &\quad + \bar{c} \max_{x \in R^n} \| K(x) - 1 \|_{\max}^2 \| Q_q U(t) u \|^2 \\ &\leqslant \bar{c} \max_{|x| \geqslant q} \| K(x) - 1 \|_{\max}^2 \| u \|^2 + \overline{c_1} \| Q_q U(t) u \|^2 \end{aligned}$$

because $K(x)$ is bounded on R^n. The last term tends to zero when $t \to \infty$ because $u \in \mathscr{H}^s$. Thus

$$\overline{\lim_{t \to \infty}} \|(K - J_0) U(t)u \|_0^2 \leqslant \bar{c} \max_{|x| \geqslant q} \| K(x) - 1 \|_{\max}^2 \| u \|^2. \tag{4.26}$$

Moreover, the right-hand side of (4.26) tends to zero when $q \to \infty$ because $\lim_{|x| \to \infty} K(x) = 1$. This completes the proof.

5. Concluding Remarks

The results given above show that an abstract theory of scattering based on the concepts of families of localizing operators and their scattering subspaces can be developed, and applied to concrete problems, independently of other theories based on the subspace of absolute continuity. Hence, a choice among these theories becomes a matter of taste. From the viewpoint of pure mathematics, where internal consistency and richness of results are the principal criteria, the theories based on $\mathscr{H}^{ac}$ might be preferred because they are better developed and have more applications at present. However, it seems likely that the theory based on $\mathscr{H}^{s}$ will be extended and new applications found. (In this connection, recent unpublished work by Lax and Phillips, extending their methods to differential operators on even-dimensional spaces, can be used to delete hypothesis (4.9) in Theorem 4.1.) From the viewpoint of mathematical physics it seems to the author that a condition like (2.8) must be accepted as the defining condition for scattering states. From this viewpoint, other theories must be regarded as incomplete until the relationship of $\mathscr{H}_{\pm}^{s}$ and $\mathscr{H}^{ac}$ is clarified. At present, in all the cases that have been analyzed completely it has been found that $\mathscr{H}_{\pm}^{s} = \mathscr{H}^{ac} = \mathscr{H}^{c}$. However, it seems certain that ultimately cases will be found where $\mathscr{H}_{\pm}^{s}$ differs from $\mathscr{H}^{ac}$ and/or $\mathscr{H}^{c}$. In such cases the mathematical physicist must give preference to $\mathscr{H}_{\pm}^{s}$.

References

1. J. D. Dollard, Quantum-mechanical scattering theory for short-range and Coulomb interactions, *Rocky Mountain J. Math.* 1 (1971), 5–88.
2. T. Kato, On finite-dimensional perturbation of self-adjoint operators, *Japan J. Math. Soc.* 9 (1957), 239–249.
3. T. Kato, Perturbation of continuous spectra by trace class operators, *Proc. Japan Acad.* 33 (1957), 260–264.
4. T. Kato, "Perturbation Theory for Linear Operators," Springer-Verlag, New York, 1966.
5. T. Kato, Scattering theory with two Hilbert spaces, *J. Functional Analysis* 1 (1967), 342–368.
6. T. Kato, Scattering theory and perturbation of continuous spectra, *Actes, Congrès internat. Math.* 1 (1970), 135–140, Gauthier–Villars, Paris, 1971.
7. S. T. Kuroda, On the existence and the unitary property of the scattering operator, *Nuovo Cimento* 12 (1959), 431–454.
8. J. A. La Vita, J. R. Schulenberger, and C. H. Wilcox, The scattering theory of Lax and Phillips and wave propagation problems of classical physics, *Applicable Anal.* (to appear).
9. P. D. Lax and R. S. Phillips, "Scattering theory," Academic Press, New York, 1967.
10. P. D. Lax and R. S. Phillips, Scattering theory, *Rocky Mountain J. Math.* 1 (1971), 173–223.
11. J. R. Schulenberger, A local compactness theorem for wave propagation problems of classical physics, *Indiana Univ. Math. J.* 22 (1972), 429–433.
12. J. R. Schulenberger and C. H. Wilcox, Coerciveness inequalities for non-elliptic systems of partial differential equations, *Ann. Mat. Pura Appl.* 88 (1971), 229–306.

13. J. R. Schulenberger and C. H. Wilcox, The singularities of the Green's matrix in anisotropic wave motion, *Indiana Univ. Math. J.* **20** (1971), 1093–1117.
14. J. R. Schulenberger and C. H. Wilcox, The limiting absorption principle and spectral theory for steady-state wave propagation in inhomogeneous anisotropic media, *Arch. Rational Mech. Anal.* **41** (1971), 46–65.
15. J. R. Schulenberger and C. H. Wilcox, Completeness of the wave operators for perturbations of uniformly propagative systems, *J. Functional Analysis* **7** (1971), 447–474.
16. J. R. Schulenberger and C. H. Wilcox, Completeness of the wave operators for scattering problems of classical physics, *Bull. Amer. Math. Soc.* (1971), 777–782.
17. C. H. Wilcox, Wave operators and asymptotic solutions of wave propagation problems of classical physics, *Arch. Rational Mech. Anal.* **22** (1966), 37–78.