ON ANALYTIC FIBER BUNDLES—I

HOLOMORPHIC FIBER BUNDLES WITH INFINITE DIMENSIONAL FIBERS†

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(Received 8 November 1966)

§1. INTRODUCTION

Let $X$ be a complex analytic space and $G$ a complex Lie group. In a series of papers [7–9] Grauert proves that if $X$ is a Stein space then two complex analytic principal bundles with structure group $G$ over $X$ are analytically equivalent if and only if they are equivalent as topological fiber bundles with structure group $G$. A couple of years ago, F. Browder asked the question whether Grauert's theorem holds also with $G$ the general linear group $GL(H)$ of a complex Hilbert space $H$, which is an infinite dimensional complex Lie group. It turned out that the answer is affirmative. With the discovery by Kuiper [14] that $GL(H)$ is contractible, it yields the interesting result that every complex analytic vector bundle over a Stein analytic space with fiber an infinite dimensional Hilbert space, is trivial. So it seemed worthwhile to have the result finally written up. In fact we prove that Grauert's theorem holds for any complex analytic Lie group with a Banach space as space of parameters.

Actually, we need to know little about the theory of infinite dimensional complex analytic Lie groups; just their definition and the existence of the canonical holomorphic mapping $\exp$ from the tangent space $T_1$ into the Lie group, which is biholomorphic in a neighborhood of 0 in $T$. This theory can be developed as in Maissen [16], where the real analytic case is discussed. (See in particular §§7 and 9 of that paper. The results on analytic functions on Banach spaces which are used there, can be found, for instance, in §§41–44 of Nachbin [17].) In the proofs of our theorem we have heavily leant on Cartan’s presentation [4] of Grauert’s theorem. Cartan throws the “hard” part of the proof into two basic propositions (Propositions 1 and 2) which he puts off till the end; he deduces from them Grauert’s theorem by “soft” methods. It turns out that the “soft” part of Cartan’s presentation is valid for infinite dimensional holomorphic fiber bundles as it stands, so we do not reproduce it here. We are therefore left with proving the fundamental Propositions 1 and 2. Since the theory of vector valued holomorphic functions as developed in Bishop [1] and Bungart [2] is not powerful enough, we have to deduce from it some results on analytic sheaves over the sheaf of germs of holomorphic functions with values in a Banach algebra.

† Research supported in part by the National Science Foundation.
(which could well form a basis for a general theory). This is done in §§2 and 3. These results are used in §4 to obtain resolutions for holomorphic vector bundles with a Banach space as fiber. In §5 we state Propositions 1 and 2, which are then proved in §§6 and 7. Section 8 contains the precise statement of the final result and applications.

We use the terminology of Gunning and Rossi [11]. All function spaces (or spaces of sections) are given the compact open topology. As for holomorphic functions with values in a Frechet space we rely on the author's paper [2]. The results can also be found in Bishop [1] if one wants to assume that all analytic spaces and subvarieties which occur in the following, have only regular points. However, not all needed results are explicitly stated in that paper. By the time this article appears in print, Kripke's work [13] will probably be available.

§2. CARTAN'S LEMMA FOR BANACH ALGEBRAS

The basic tool in proving the theorems on the cohomology of coherent analytic sheaves is "Cartan's Lemma" [3, 5]. We need a corresponding theorem for holomorphic functions with values in (the multiplicative group of invertible elements in) a complex Banach algebra. Only minor modifications are necessary in the standard proof as presented for instance in Gunning and Rossi [11], pp. 199–200.

2.1 Theorem (Cartan’s Lemma). Suppose $D_1$ and $D_2$ are two compact rectangles in the complex plane intersecting in a rectangle $D_0$ such that $D_1 \cup D_2$ is also a rectangle. Let $f$ be a holomorphic function on (a neighborhood of) $D_0$ with values in the group $G$ of invertible elements of a (complex) Banach algebra $A$ (with unit). There are holomorphic functions $f'$ and $f''$ on $D_1$ and $D_2$, respectively, having values in $G$ such that

$$f(z) = f'(z)f''(z)$$

for $z \in D_0 = D_1 \cap D_2$.

Proof. We sketch the proof only. Suppose $f$ is defined in an $\varepsilon$-neighborhood of $D_0$. Let $V_1, V_2$ be $\varepsilon^{-1}$-neighborhoods of $D_1, D_2$ and define $V_j = V_{1j} \cap V_{2j}$. We can write $\partial V_j$ as the union of two curves $a_j$ and $b_j$ such that $a_j \cap V_{1j} = \emptyset = b_j \cap V_{2j}$. Let

$$\ell = \sup\{|a_j|, |b_j| : j \geq 1\}.$$

Now choose $\delta$ so small that $\exp$ maps the $\delta$-ball around 0 in $A$ homeomorphically onto a neighborhood of 1 in $G$ and such that there is a constant $M$ with

$$\|(1 + x)^{-1}\| \leq M$$

for $\|x\| \leq \delta$.

We may assume also

$$0 < \delta < \pi e/4M^2 \ell.$$

Then choose $\rho > 0$ with

$$4M^2 \delta^2 < \rho < \pi \delta e \ell.$$

Suppose $\|f - 1\| < \rho/4$ on $V_1$ and let $g_1 = f - 1$. Define by induction $g'_j, g''_j, g_{j+1}$ on $V_{1j}, V_{2j}$ and $V_{j+1}$, respectively, such that

1. $\|g'_j\| < \delta 2^{-j}$ on $V_{1j+1}, \|g''_j\| < \delta 2^{-j}$ on $V_{2j+1}$. 


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(2) \( g_j = g'_j + g''_j \) on \( V_j \).

(3) \( 1 + g_{j+1} = (1 + g'_j)^{-1}(1 + g_j)(1 + g''_j)^{-1} \) on \( V_{j+1} \).

(4) \( \|g_{j+1}\| < p/4^{j+1} \) on \( V_{j+1} \).

It suffices to let

\[
g'_j(z) = (2\pi i)^{-1} \int_{a_j} g_j(t)(t-z) \, dt,
\]

\[
g''_j(z) = (2\pi i)^{-1} \int_{b_j} g_j(t)(t-z) \, dt
\]

and define \( g_{j+1} \) by (3). We have

\[
f = f_1 \cdots f'_j f_{j+1} f''_j \cdots f''_{f+1} \text{ on } V_{j+1}
\]

where \( f'_j = 1 + g'_j \) and \( f''_j = 1 + g''_j \). Now

\[
f' = \lim f'_j, \quad f'' = \lim f''_j
\]

(limits in \( A \)) have the property \( f = f'f'' \) on \( D_0 \). Since we could have taken \( D_1 \) and \( D_2 \) slightly larger in the proof we may assume that \( f' \) and \( f'' \) are holomorphic on \( D_1 \) and \( D_2 \) respectively. The values of \( f' \) belong to \( G \) since \( \Pi_{j>k} f'_j \) is close to 1 for \( k \) large enough, and similarly for \( f'' \).

Above we have assumed that \( f \) is close to 1. If not, we proceed as follows. First we may assume that \( f \) has values in the connected component \( G_0 \) of \( G \) which contains 1 (multiplying \( f \) by a suitable element of \( G \) if necessary). Since the topological group of holomorphic functions on \( V_1 \) with values in \( G_0 \) is connected, we can write \( f = h_1 \cdots h_k \) where \( h_j \) have values close to 1. Each \( h_j \) can be approximated by \( \exp P_j \) where \( P_j \) is a polynomial with coefficients in \( A \) i.e., \( f \) can be approximated by a holomorphic function \( h \) defined on the whole plane and with values in \( G_0 \). Now we need only factor \( fh^{-1} = f'f'' \) as above and obtain \( f = f'(f''h) \).

2.2 Corollary. Suppose \( \Gamma_1 \) and \( \Gamma_2 \) are compact cubes in \( C^n \) intersecting in a cube \( \Gamma_0 \) such that \( \Gamma_1 \cup \Gamma_2 \) is again a cube. Let \( f \) be a holomorphic function on \( \Gamma_0 \) with values in the group \( G \) of invertible elements of a Banach algebra \( A \). There are holomorphic functions \( f' \) and \( f'' \) on \( \Gamma_1 \) and \( \Gamma_2 \), respectively, having values in \( G \) such that

\[
f(z) = f'(z)f''(z) \text{ for } z \in \Gamma_0 = \Gamma_1 \cap \Gamma_2 \.
\]

Proof. We may as well assume that \( \Gamma_i = D_i \times Q \), where the \( D_i \) are as in the theorem and \( Q \) is a cube in \( C^{n-1} \). Then we apply the theorem with the Banach algebra \( A \) replaced by the Banach algebra of bounded holomorphic \( A \)-valued functions on a (suitable) fixed neighborhood of \( Q \) in \( C^{n-1} \).

2.3 Definition. A compact subset \( K \) of an analytic space \( X \) is called special (see Cartan [4], Definition 3, p. 99) if there is a holomorphic map \( h : X \rightarrow C^n \) for some \( n \) such that

(1) \( h \) is biholomorphic in a neighborhood \( U \) of \( K \),

(2) there is a cube \( \Gamma \) in \( C^n \) such that \( K = U \cap h^{-1}(\Gamma) \).
A triplet \((K, K', K'')\) of special compact subsets is called a special configuration (see [4], p. 109) if

(3) there is a decomposition \(\Gamma = \Gamma_1 \cup \Gamma_2\) of \(\Gamma\) into a union of cubes \(\Gamma_1\) and \(\Gamma_2\) such that \(\Gamma_1 \cap \Gamma_2\) has lower (real) dimension than \(\Gamma\) and

\[ K' = U \cap h^{-1}(\Gamma_1), \quad K'' = U \cap h^{-1}(\Gamma_2). \]

2.4 COROLLARY. Suppose \((K, K', K'')\) is a special configuration of special analytic sets and \(f\) is a holomorphic function on (a neighborhood of) \(K' \cap K''\) with values in the group \(G\) of invertible elements of a Banach algebra \(A\). If \(f\) takes values on \(K' \cap K''\) which are sufficiently close to the identity, then there are holomorphic functions \(f'\) and \(f''\) on \(K'\) and \(K''\), respectively, having values in \(G\) such that

\[ f(z) = f'(z)f''(z) \text{ for } z \in K' \cap K''. \]

Proof. We consider \(K = K' \cup K''\) as embedded in \(C^n\) (via the holomorphic mapping of the above definition). Then a neighborhood \(U\) of \(K\) is realized as a closed subvariety of a domain of holomorphy containing \(\Gamma\) and \(K' = \Gamma_1 \cap U, K'' = \Gamma_2 \cap U\) (in the notation of the definition). Since \(f\) has values close to the identity, we can form \(g = \log f(= \exp^{-1}f)\). The function \(g\) can be extended to an \(A\)-valued holomorphic function \(\tilde{g}\) on a neighborhood of \(\Gamma_1 \cap \Gamma_2 = \Gamma_0\) (Bungart [2], Corollary 12.1). \(\tilde{f} = \exp \tilde{g}\) is a holomorphic function on \(\Gamma_0\) with values in \(G\) which extends \(f\). Now we can apply the previous corollary to \(\tilde{f}\).

2.5 COROLLARY. If the function \(f\) in the last corollary depends continuously on a parameter in a compact Hausdorff space \(H\), then the functions \(f'\) and \(f''\) can be chosen with the same property.

Proof. Replace \(A\) in the last corollary by the Banach algebra of continuous \(A\)-valued functions on \(H\).

§3. SYZYGETIC SHEAVES

In the following \(A\) is a complex Banach algebra. We denote the sheaf of germs of \(A\)-valued holomorphic functions on a complex analytic space \(X\) by \(A\). If \(X = C^n\) we denote the sheaf of germs of \(A\)-valued holomorphic functions on \(C^n\) (or any subdomain of \(C^n\)) by \(A\). Since \(A\) is a sheaf of rings, we can talk about sheaves of \(A\)-modules.

3.1 DEFINITION. A sheaf \(\mathcal{F}\) of right \(A\)-modules on an analytic space \(X\) is called syzygetic (the terminology is due to Gunning [10]) if every point of \(X\) has a neighborhood over which \(\mathcal{F}\) admits a finite free resolution

\[ 0 \to (A)^n \to \cdots \to (A)^1 \to \mathcal{F} \to 0. \]

of sheaves of right \(A\)-modules.

3.2 THEOREM. Suppose \(\mathcal{F}\) is a syzygetic sheaf of right \(A\)-modules on an open set \(U\) in \(C^n\). For any compact cube \(\Gamma\) in \(U\) there is a finite free resolution of \(\mathcal{F}\) over a neighborhood of \(\Gamma\).
Proof. The construction of such a resolution (or syzygy) is achieved by amalgamation of local syzygies using Cartan’s lemma (applied to the Banach algebra of $n_j \times n_j$ matrices with entries in $A$). For details see Gunning and Rossi [11, pp. 201–207].

3.3 Lemma. Let $V$ be a closed subvariety of an open set $U$ in $\mathbb{C}^n$ and suppose $\mathcal{F}$ is a locally free sheaf of right $\mathfrak{O}^A$-modules of finite rank on $V$. We extend $\mathcal{F}$ trivially to all of $U$. Then $\mathcal{F}$ is a syzygetic sheaf of $\mathfrak{O}^A$-modules on $U$.

Proof. Since being syzygetic is a local property, we may assume that $\mathcal{F} = \mathfrak{O}^A$ and that the sheaf $\mathfrak{O}$ of germs of (scalar valued) holomorphic functions on $V$ has a free resolution

$$0 \to \mathfrak{O}^n \to \cdots \to \mathfrak{O}^1 \to \mathfrak{O} \to 0,$$

where $\mathfrak{O}^n$ is a power of the sheaf of germs of holomorphic functions on $U \subset \mathbb{C}^n$. By vectorization of this exact sequence we obtain an exact sequence

$$0 \to (\mathfrak{O}^A)^n \to \cdots \to (\mathfrak{O}^A)^1 \to \mathfrak{O}^A \to 0$$

of sheaves of right $\mathfrak{O}^A$-modules (see Proposition 9.4 in Bungart [2]).

3.4 Corollary. Suppose $\mathcal{F}$ is a locally free sheaf of right $\mathfrak{O}^A$-modules (of finite rank) on an analytic space $X$. For any special compact subset $K$ of $X$ (see Definition 2.3) there is an exact sequence

$$0 \to (\mathfrak{O}^A)^n \to \cdots \to (\mathfrak{O}^A)^1 \to \mathcal{F} \to 0$$

of sheaves of right $\mathfrak{O}^A$-modules on a neighborhood of $K$.

3.5 Remarks. If $X$ is a Stein analytic space then $(\mathfrak{O}^A)^n$ has trivial cohomology (Theorem B, p. 331 in [2]) and hence the sheaf $\mathcal{F}$ in Corollary 3.4 has trivial cohomology, too. On the other hand, if $\mathcal{F}$ is any syzygetic sheaf of right $\mathfrak{O}^A$-modules on an analytic space $X$, one can prove Corollary 3.4 for $\mathcal{F}$ (one has to show that an $\mathfrak{O}^A$-syzygetic sheaf on a subvariety $V$ of an open set $U$ in $\mathbb{C}^n$ is also $\mathfrak{O}^A$-syzygetic by pasting together a lot of exact sequences). Thus such a sheaf $\mathcal{F}$ has also trivial cohomology if $X$ is a Stein space.

§4. Resolutions for Analytic $\mathfrak{B}$-Vector Bundles

We show now how the results of the previous section can be used to construct global cross sections in analytic vector bundles over a Stein space whose fibers are isomorphic to a (complex) Banach space.

4.1 Definition. A (complex) analytic $\mathfrak{B}$-vector bundle $V$ with fiber a Banach space $\mathfrak{B}$ on an analytic space is a topological vector bundle with fiber $\mathfrak{B}$ which can be defined by holomorphic transition functions with values in the general linear group $GL(\mathfrak{B})$ of $\mathfrak{B}$ (of invertible transformations on $\mathfrak{B}$).

We consider $GL(\mathfrak{B})$ always as embedded in the algebra $A = L(\mathfrak{B})$ of continuous linear endomorphisms of $\mathfrak{B}$. 
4.2 Theorem. Let $V$ be an analytic $B$-vector bundle on an analytic space $X$, and $\mathcal{Y}$ the sheaf of germs of holomorphic cross sections of $V$. For any special compact set $K$ in $X$ (see Definition 2.3) there is an exact sequence

$$0 \rightarrow (\mathcal{O}_0^B)^{n_0} \rightarrow \cdots \rightarrow (\mathcal{O}_0^B)^{n_1} \rightarrow \mathcal{Y} \rightarrow 0$$

over a neighborhood of $K$, where the components of the maps are holomorphic functions with values in $A = L(B)$ (locally in case of the last map).

Proof. Let $\mathcal{F}$ be the sheaf of germs of fiber preserving holomorphic mappings from $X \times B$ into $V$ which are linear on the fibers. Then $\mathcal{F}$ is a locally free sheaf of right $\mathcal{O}_0^B$-modules. So there is an exact sequence

$$0 \rightarrow (\mathcal{O}_0^A)^{n_0} \rightarrow \cdots \rightarrow (\mathcal{O}_0^A)^{n_1} \rightarrow \mathcal{F} \rightarrow 0$$

of sheaves of right $\mathcal{O}_0^A$-modules over a neighborhood of $K$ (by Corollary 3.4). The mappings in this sequence can locally be given by matrices whose entries are holomorphic functions with values in $A$. Thus they define a sequence of sheaves

$$(*) 0 \rightarrow (\mathcal{O}_0^B)^{n_0} \rightarrow \cdots \rightarrow (\mathcal{O}_0^B)^{n_1} \rightarrow \mathcal{Y} \rightarrow 0$$

in a neighborhood of $K$ which is to be shown to be exact. Since this is a local problem we may assume that $V$ is a product bundle so that $\mathcal{F} = (\mathcal{O}_0^A)^{n_0}$ and $\mathcal{Y} = (\mathcal{O}_0^B)^{n_0}$ (with $n_0 = 1$). Furthermore, we need consider the exact sequence $(*)$ only at each point $0$. The morphism

$$f_1 : (\mathcal{O}_0^A)^{n_1} \rightarrow (\mathcal{O}_0^A)^{n_0} \rightarrow 0$$

splits; so there is a morphism

$$g_1 : (\mathcal{O}_0^A)^{n_0} \rightarrow (\mathcal{O}_0^A)^{n_1}$$

with $f_1 \circ g_1 = \text{identity}$. Thus

$$f_1 : (\mathcal{O}_0^B)^{n_1} \rightarrow (\mathcal{O}_0^B)^{n_0}$$

splits and is therefore onto. The kernel of this map is the image of the projection $id - g_1 \circ f_1$. The morphism

$$f_2 : (\mathcal{O}_0^A)^{n_2} \rightarrow (id - g_1 \circ f_1)(\mathcal{O}_0^A)^{n_1}$$

splits since the image is a direct summand of $(\mathcal{O}_0^A)^{n_1}$, so there is a morphism

$$g_2 : (\mathcal{O}_0^A)^{n_1} \rightarrow (\mathcal{O}_0^A)^{n_2}$$

with $f_2 \circ g_2 = id - g_1 \circ f_1$ which implies that $(*)$ is exact at $(\mathcal{O}_0^B)^{n_0}$. Similarly we show exactness of $(*)$ at the other places.

4.3 Corollary. Let $V$ be an analytic $B$-vector bundle on a Stein analytic space $X$ and $Y$ a closed subvariety of $X$. Denote by $\mathcal{Y}^0$ the sheaf of germs of holomorphic cross sections of $V$ that vanish on $Y$. For any relatively compact holomorphically convex domain $U$ in $X$ there is an integer $n$ and an epimorphism

$$\Phi : (\mathcal{O}_0^B)^n \rightarrow \mathcal{Y}^0$$
where $\mathcal{F}^B$ is the sheaf of germs of holomorphic $B$-valued functions vanishing on $Y$ and where the components of $\Phi$ are locally given by holomorphic functions with values in $A = L(B)$. Furthermore, $\Phi$ induces an epimorphism

$$\Phi: H^0(U_0, \mathcal{F}^B) \to H^0(U_0, \mathcal{V}^0)$$

for any holomorphically convex domain $U_0 \subset U$.

**Proof.** There is a special compact set $K$ containing $U$ and hence, by the last theorem, an exact sequence

$$0 \to (\mathcal{O}_B)^n_{K} \to \cdots \to (\mathcal{X}_B)^n_{K} \to \mathcal{V}^0 \to 0$$

over $U$, where $\mathcal{V}^0$ is the sheaf of germs of cross sections of $V$. Since the sheaves in this exact sequence are locally free, we can tensor over $\mathcal{O}$ with the sheaf $\mathcal{X}_B$ of germs of holomorphic functions and retain exactness. Now we use that

$$\mathcal{X}_B \otimes \mathcal{O}_B = \mathcal{X}_B \quad (\mathcal{O} = \mathcal{O}_B)$$

(Bungart [2], Proposition 10.3 in conjunction with the remark preceding it; see also Bishop [1], Theorem 3). Thus we have an exact sequence

$$0 \to (\mathcal{X}_B)^n_{U_0} \to \cdots \to (\mathcal{X}_B)^n_{U_0} \to \mathcal{V}^0 \to 0$$

over $U$. Since the sheaves $\mathcal{X}_B$ have zero cohomology over any holomorphically convex subdomain $U_0$ of $U$ (Theorem B on page 331 in [2]), the kernel $\mathcal{K}$ of $\Phi$ also has zero cohomology over each such $U_0$. This in turn implies that $\Phi$ induces an epimorphism of cross sections over each $U_0$.

4.4 **Remarks.** The above theorem implies, of course, that $\mathcal{V}^0$ and $\mathcal{V}^0$ have zero cohomology if $X$ is Stein. The method used in proving the theorem can be used to develop a general theory of analytic $B$-sheaves satisfying suitable conditions. However, we do not have any use for such a theory in this paper.

§5. **HOLOMORPHIC FIBER BUNDLES WITH INFINITE DIMENSIONAL STRUCTURE GROUPS**

Let $G$ be a complex analytic Lie group with space of parameters $a$ (complex) Banach space $B$ (the definition is analogous to Definition 1, p. 235 of Maissen [16]). The complex tangent space $T$ to $G$ at 1 is isomorphic to $B$, and canonically isomorphic with the Lie algebra of $G$ ([16] Satz 3.1, p. 239). There is a holomorphic map

$$\exp: T \to G$$

canonically attached to the pair $(G, T)$, which is biholomorphic in a neighborhood of 0 ([16] Satz 6.1, p. 245 and Satz 7.1, p. 249).

Now let $E$ be an analytic fiber bundle with fiber $G$ on a complex analytic space $X$. That means, there is an open covering $\{U_i\}$ of $X$ such that $E$ is isomorphic (as a topological fiber bundle) to $U_i \times G$ on $U_i$ and the transition functions from $U_i \times G$ to $U_j \times G$ can
be given by holomorphic mappings

\[ f_{ij} : (U_i \cap U_j) \times G \to G \]

where \( f_{ij}(x, \cdot) \) is an automorphism of \( G \) for each \( x \in U_i \cap U_j \).

Suppose \( C \) is a compact space of parameters, and \( N, H \) are closed subsets of \( C \) with \( N \subset H \). Let \( Y \) be a closed subvariety of the complex analytic space \( X \) and denote by \( \mathcal{G}^r(U) \) the topological group of continuous cross sections of \( E \) over an open set \( U \) that equal 1 on \( Y \). Similarly, \( \mathcal{G}^u(U) \) is the subalgebra of \( \mathcal{G}^r(U) \) of holomorphic cross sections of \( E \) over \( U \) which equal 1 on \( Y \). A cross section \( \varphi \) of \( E \) of type \( (N, H, C) \) over \( U \) is a continuous mapping

\[ \varphi : C \to \mathcal{G}^r(U) \]

such that

1. \( \varphi(t) = 1 \) for \( t \in N \),
2. \( \varphi(t) \in \mathcal{G}^u(U) \) for \( t \in H \).

We remark here, that cross sections of type \( (N, H, C) \) are similarly defined for a holomorphic vector bundle, with 1 replaced by 0.

Proposition 1 and 2 of Cartan [4] read now as follows.

5.1 PROPOSITION. Suppose \( X \) is a Stein analytic space and \( K \) is a special compact subset of \( X \) (Definition 2.3). Let \( \mathcal{F} \) be the sheaf of germs of cross sections of type \( (N, H, C) \) of the analytic Lie group bundle \( E \). Then every cross section of \( \mathcal{F} \) over a neighborhood of \( K \) which is sufficiently close to the identity, can be uniformly approximated on \( K \) by cross sections of \( \mathcal{F} \) over \( X \) (i.e. the image of \( H^0(X, \mathcal{F}) \to H^0(K, \mathcal{F}) \) is dense in some neighborhood of 1 if we give \( H^0(K, \mathcal{F}) \) the inductive limit topology relative to the maps \( H^0(U, \mathcal{F}) \to H^0(K, \mathcal{F}), U \) an open neighborhood of \( K \)).

5.2 PROPOSITION. Suppose \( (K, K', K^*) \) is a special configuration of special compact sets in the analytic space \( X \) (Definition 2.3). If \( f \in H^0(K' \cap K^*, \mathcal{F}) \) is sufficiently close to the identity, then

\[ f = f' \cdot f^{-1}, \]

where \( f' \in H^0(K', \mathcal{F}) \) and \( f'' \in H^0(K^*, \mathcal{F}) \).

The proofs are very much like those in Cartan's paper [4]; only slight modifications are necessary in their formulation in order to make use of the results of the previous sections. We shall sketch the proofs.

§6. PROOF OF PROPOSITION 5.1

Before starting on the proof let us recall the following theorem on Frechet spaces which is proved in the appendix of Cartan [4].

LEMMA. Suppose \( \Phi : F \to F_1 \) is a continuous linear epimorphism of Frechet spaces. Let \( f : C \to F_1 \) be a continuous map from a compact Hausdorff space \( C \) and \( g : H \to F \) a
continuous map from a closed subset $H$ of $C$ such that $\Phi \circ g = f|H$. Then there is a continuous extension $\tilde{g}$ of $g$ to all of $C$ satisfying $\Phi \circ \tilde{g} = f$.

Associated with the analytic Lie group bundle $E$ is the complex analytic $T$-vector bundle $T(E)$ of complex tangent spaces to the fibers of $E$ at $1$. The fibers of $T(E)$ are isomorphic to the Lie algebra $T$ of $G$. The exponential

$$\exp: T(E) \to E$$

defines a holomorphic mapping of analytic fiber bundles, and $\exp$ is biholomorphic in a neighborhood of the zero cross section of $T(E)$. Thus it suffices to prove Proposition 5.1 for cross sections of type $(N, H, C)$ of the $T$-vector bundle $T(E)$. Furthermore, it suffices to show that for any relatively compact holomorphically convex domain $U$ containing $K$, the given section $f \in H^0(K, \mathcal{F})$ can be approximated on $K$ by sections of type $(N, H, C)$ over $U$. Namely, if $\{U_n\}$ is a sequence of relatively compact holomorphically convex domains with $K = K_0 \subset U_0 \subset K_n \subset U_n$, $n \geq 1$ and $\cup U_n = X$, then we approximate the given $f$ on $K_0$ by an element $f_1 \in H^0(U_1, \mathcal{F})$ and $f_1$ on $K_1$ by an element $f_2 \in H^0(U_2, \mathcal{F})$ etc., thus obtaining a sequence $\{f_n\}$ which can be made to converge uniformly on compact sets to a section $g \in H^0(X, \mathcal{F})$ which approximates $f$ on $K$.

By Corollary 4.3, there is a continuous linear epimorphism

$$\Phi^a : H^0(U, x^T)^n \to H^0(U, \mathcal{V}^a),$$

where $\mathcal{V}^a$ is the sheaf of germs of holomorphic cross sections of $T(E)$ vanishing on the subvariety $Y$ and where the components of $\Phi^a$ are locally given by holomorphic $L(T)$-valued functions. Moreover, $\Phi^a$ induces an epimorphism if $U$ is replaced by a holomorphically convex subdomain $U_0$ on which the given section $f$ of type $(N, H, C)$ of $T(E)$ is defined. $\Phi^a$ defines a continuous linear map

$$\Phi^c : H^0(U_0, x^T)^n \to H^0(U_0, \mathcal{V}^c)$$

where $\mathcal{V}^c$ is the sheaf of germs of continuous cross sections of $T(E)$ vanishing on $Y$ and $x^T$ is the sheaf of germs of continuous $T$-valued functions vanishing on $Y$. $\Phi^c$ is in fact an epimorphism. The easiest way to show this is to appeal to the proof of Corollary 4.3, where $\Phi^a$ was constructed from an epimorphism $\Phi^a : H^0(U, x^T)^n \to H^0(U, \mathcal{F}(E))$ (where $\mathcal{F}(E)$ is the sheaf of germs of holomorphic cross sections of $T(E)$) which, by the proof of Theorem 4.2, has locally a left inverse (splitting) $\Psi$. Thus, writing a germ $x \in \mathcal{V}^c$ as $x = \gamma \beta$ where $\beta \in \mathcal{V}^c$ and $\gamma$ is a germ of scalar valued continuous function vanishing on $Y$, we have $\gamma \Psi(\beta) \in x^T$ which maps by $\Phi^c$ onto $a$. Hence $\Phi^c : (x^T)^n \to \mathcal{V}^c$ is an epimorphism on $U$ whose kernel is a fine sheaf with zero cohomology. (However, one can also show directly that $\Phi^c : (x^T)^n \to \mathcal{V}^c$ is an epimorphism). Employing the theorem on Frechet spaces stated in the beginning of this section, we can extend the zero function $N \to H^0(U_0, x^T)^n$ to a continuous function

$$g_0 : H \to H^0(U_0, x^T)^n.$$
such that $\Phi^* \circ g_0$ coincides with the given $f$ on $H$. Considering $g_0$ with values in $H^0(U_0, X^T)^\pi$, we can extend $g_0$ to a continuous function
$$g : C \to H^0(U_0, X^T)^\pi$$
such that $\Phi^* \circ g = f$. It suffices now to approximate the section $g$ of type $(N, H, C)$ of $X \times T^\pi$ on $K$ by a section of type $(N, H, C)$ of $X \times T^\pi$ over $U$. Using partition of unity on $C$, this is reduced to approximating a holomorphic (continuous) $T$-valued function on $U_0 \supset K$, which vanishes on $Y$, by a $T$-valued function on $U$ with the same properties. This is possible since $U \supset U_0$ are holomorphically convex (see Theorem C, p. 332 in Bungart [2]).

§7. PROOF OF PROPOSITION 5.2

The proposition is first solved for $t \in H$ so that $f$ has the properties
(a) $f(x, t)$ is continuous in $(x, t)$,
(b) $f(\cdot, t)$ is holomorphic for $t \in H$,
(c) $f(x, t) = 1$ for $x \in Y$ or $t \in N$.

Since $f$ has values sufficiently close to 1, we can find a cross section $a(x, t)$ of the bundle $T(E)$ of Lie algebras associated with $E$ such that
$$\exp a(x, t) = f(x, t)$$
and $a(x, t)$ satisfies conditions similar to (a) to (c) (with 1 replaced by 0). Let
$$f(x, t, u) = \exp(u \cdot a(x, t)), \quad u \in [0, 1]$$
so that
$$\frac{\partial f}{\partial u} = a(x, t) \cdot f(x, t, u).$$
Finding $f'(x, t, u)$ and $f''(x, t, u)$ with $f = f'f''$ is equivalent (by differentiation) to solving the following equations in $T(E)$ with suitable sections $a'$ and $a''$ over $K'$ and $K''$, respectively:
$$a'(x, t, u) = a(x, t) + \lambda(x, t, u) \cdot a''(x, t, u)$$
$$\frac{\partial f'}{\partial u} = a', \quad \frac{\partial f''}{\partial u} = a'', \quad f'(0) = 1 = f''(0),$$
where $\lambda(x, t, u) = ad(f(x, t, u))$.

Let $\mathcal{H}$ be the sheaf of germs of fiber preserving holomorphic mappings from $X \times T$ into $T(E)$ which are linear on the fibers ($T$ is the typical fiber of $T(E)$). By Corollary 3.4 there is an epimorphism of sheaves of right $\mathcal{O}A$-modules
$$\Phi : (\mathcal{O}A)^\pi \to \mathcal{H},$$
where $A$ is the algebra of bounded linear endomorphisms of $T$, and this morphism induces an epimorphism
$$\Phi : H^0(U, \mathcal{O}A)^\pi \to H^0(U, \mathcal{H})$$
on any holomorphically convex domain $U \supset K' \cap K''$ in the domain of definition of $\lambda$ (or $f$). Note that $\lambda(\cdot, t, u)$ defines an $H^0(U, \mathcal{O}A)$-isomorphism $\Lambda(t, u)$ of $H^0(U, \mathcal{H})$ into
itself by acting on the left on $H^0(U, \mathcal{H})$. Now consider the Frechet space $F_1$ of $H^0(U, \mathcal{O})$-morphisms
\[ H^0(U, \mathcal{O})^n \to H^0(U, \mathcal{H}). \]
The Frechet space $F$ of $H^0(U, \mathcal{O})$-endomorphisms of $H^0(U, \mathcal{O})^n$ is mapped by
\[ \gamma \to \Phi \cdot \gamma \]
on to $F_1$ since $H^0(U, \mathcal{O})^n$ is free. Thus we can lift the continuous map
\[ \Lambda \circ \Phi : H \times [0, 1] \to F_1 \]
to a continuous map
\[ \varphi : H \times [0, 1] \to F. \]
Since $\Lambda$ is close to the identity, we can choose $\varphi$ close to the identity, too, by the open mapping theorem. Thus $\varphi(x, t, u)$ is an invertible matrix close to the identity. Furthermore, by Proposition 5.1 we can find a section $\alpha(x, t)$ of $X \times T^n$ over $U$ vanishing for $x \in Y$ and for $t \in N$ such that $\Phi(x) = a$. It suffices now to solve the equation
\[ \alpha'(x, t, u) = \alpha(x, t) + \varphi(x, t, u) \cdot \alpha(x, t, u). \]
By Corollaries 2.4 and 2.5, we can find invertible matrices $\varphi'$ and $\varphi''$ whose entries are holomorphic $A$-valued functions on a neighborhood of $K'$ and $K''$, respectively, and which depend continuously on $t$ and $u$, such that
\[ \varphi' \varphi = \varphi'' \]
in a neighborhood of $K' \cap K''$. Therefore we can write our equation as
\[ \varphi' \alpha' = \varphi' \alpha + \varphi'' \alpha'', \]
that is, we have to solve
\[ \varphi' \alpha = \beta = \beta' - \beta'' \]
for $\beta'$ and $\beta''$. For this we let $\Gamma, \Gamma_1, \Gamma_2$ be the cubes in euclidean space in which $K, K_1, K_2$ can be realized (see Definition 2.3). We extend $\beta$ into a neighborhood of $\Gamma_1 \cap \Gamma_2$ (Bungart [2] Corollary 12.1) and write
\[ \beta = \sum \beta_j g_j \]
where $g_j$ are scalar valued holomorphic functions in a neighborhood of $\Gamma = \Gamma_1 \cup \Gamma_2$ vanishing on $Y$ ([2] Proposition 12.2 and Proof). Then we solve
\[ \beta_j = \beta'_j - \beta''_j \]
with $\beta'_j$ and $\beta''_j$ holomorphic in a neighborhood of $\Gamma_1$ and $\Gamma_2$, respectively, by the Cauchy integral formula (see equation (*) in the proof of Theorem 1).
Now we have solved the equation
\[ a'(x, t, u) = a(x, t) + \lambda(x, t, u) \cdot a''(x, t, u) \]
for $t \in H$. We extend $a'$ and $a''$ by continuity for $t \in C$ such that they vanish for $x \in Y$. 
Then
\[ a' - \lambda \cdot a'' - a = \varphi \]
is holomorphic in \( x \) in a neighborhood of \( K' \cap K'' \) and vanishes for \( x \in Y \) or \( t \in H \). We extend \( \varphi \) for \( t \in C \) to a continuous section \( \Phi \) of \( T(E) \) in a neighborhood of \( K' \) such that \( \Phi \) vanishes for \( x \in Y \) and for \( t \in H \). Then
\[ b' = a' - \Phi, \quad b'' = a'' \]
satisfy the equation
\[ b'(x, t, u) = a(x, t) + \lambda(x, t, u) \cdot b''(x, t, u) \]
as desired. This completes the proof as we have remarked at the outset.

§8. ANALYTIC BUNDLES WITH INFINITE DIMENSIONAL FIBERS

Let \( E \) be a complex analytic fiber bundle of Lie groups over a complex analytic space \( X \) whose fibers are all isomorphic to a complex analytic Lie group \( G \) with space of parameters a complex Banach space \( B \) (see the beginning of §5).

8.1 Theorem. Let \( \delta^a \), respectively \( \delta^c \), denote the sheaf of germs of holomorphic (continuous) cross sections of \( E \). If \( X \) is a Stein space then the natural morphism of sheaves of groups \( \delta^a \to \delta^c \) induces an isomorphism
\[ H^1(X, \delta^a) \to H^1(X, \delta^c). \]

Proof: Now that Proposition 1 and 2 of Cartan [4] have been established for \( E \), the remainder of the proof as described by Cartan [4] (for a finite dimensional Lie group \( G \)) can be copied word by word.

Geometrically, the theorem says that two complex analytic \( E \)-principal fiber bundles are analytically isomorphic if and only if they are isomorphic as topological \( E \)-principal fiber bundles. In particular (taking \( E = X \times G \)), we obtain that a complex analytic principal \( G \)-bundle is analytically trivial if and only if it is trivial as a topological principal \( G \)-bundle. If \( G \) is the general linear group \( GL(H) \) of invertible operators on an infinite dimensional complex Hilbert space \( H \) then every principal \( G \)-bundle is trivial since \( G \) is contractible (Corollary 1, p. 29 in Kuiper [14]. An analytic space is a CW complex by the results of [6] or [15]. See also the exposition by Illusie in [12]). Hence

8.2 Theorem. Every principal complex analytic fiber bundle over a Stein space, with fiber the general linear group of an infinite dimensional complex Hilbert space, is analytically trivial.

An interesting question in this connection is whether \( GL(H) \) has a holomorphic contraction if \( H \) is infinite dimensional. Propositions 1 and 2 can be generalized in a suitable way for bundles \( E \) over an infinite dimensional analytic space without too much effort. However, Cartan's proof of Theorem 8.1 uses induction on the dimension of certain special compact sets. The author has not yet been able to substitute that argument by one which will work on infinite dimensional analytic spaces. If this should prove possible,
an application of Theorem 3, p. 103, in Cartan [4] (or its generalization to infinite dimensions in Theorem 8.3 below) will establish the existence of a holomorphic contraction for $GL(H)$.

In the course of the proof of Theorem 8.1 a la Cartan, the following result is obtained. Let $E$ be a complex analytic fiber bundle of Lie groups over a Stein analytic space $X$ whose fibers are all isomorphic to a complex analytic Banach Lie group $G$.

**8.3 Theorem.** Let $f$ and $f'$ be holomorphic sections of an analytic $E$-principal bundle $P$ such that $f$ and $f'$ are homotopic as continuous sections. Then there is a holomorphic homotopy $h : X \times [0, 1] \to P$ of sections between $f$ and $f'$ (where $[0, 1]$ is considered as a subset of $C$).

Similarly, Théorème 1 bis and Théorème 2 bis of [4] generalize to our setting with $G$ a Banach Lie group:

**8.4 Theorem.** Let $P$ be as above, $Y$ a closed subvariety of $X$ and $U$ a holomorphically convex domain in $X$. If a continuous section $f : X \to P$ is such that $f|Y - g$ is holomorphic, then $f$ is homotopic to a holomorphic section in the space of continuous sections that induce $g$ on $Y$. If $f : U \to P, g : Y \to P$ are holomorphic sections such that $f|Y \cap U = g|Y \cap U$, then $f$ can be approximated by holomorphic sections $h : X \to Y$ with $h|Y = g$ (uniformly on compacta in $U$) if and only if it can be approximated by continuous sections $h$ with $h|Y = g$.

Finally, let us mention that Propositions 1 and 2 of Cartan [4], and their generalization to Banach Lie groups in Propositions 5.1 and 5.2, can also be obtained for real (Banach) Lie group bundles over real analytic manifolds with a suitable concept of approximation. Whether this implies the main result (Theorem 8.1) is presently under study. To my knowledge, Theorem 8.1 is unknown even for the case of a Lie group bundle over a real analytic manifold with a finite dimensional real Lie group as fiber.

**Added in Proof.** In his paper *Une généralisation de la notion de diviseur* (Atti del Convegno Internazionale di geometria algebrica, Torino 1961) P. Dolbeault quotes on p. 145 an unpublished paper by H. Cartan according to which Theorem 8.1 and its consequences are true for finite dimensional real analytic Lie groups.

**REFERENCES**


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