Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 208 (2007) 316–330
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

A posteriori error bound methods for the inclusion of polynomial zeros☆

Miodrag S. Petković[a,*], Snežana Ilić[b], Ivan Petković[a]

[a] *Faculty of Electronic Engineering, Department of Mathematics, University of Niš, P.O. Box 73, 18000 Niš, Serbia, Serbia and Montenegro*
[b] *Faculty of Science, Department of Mathematics, University of Niš, 18000 Niš, Serbia, Serbia and Montenegro*

Received 1 August 2006; received in revised form 28 September 2006

Abstract

Using Carstensen's results from 1991 we state a theorem concerning the localization of polynomial zeros and derive two a posteriori error bound methods with the convergence order 3 and 4. These methods possess useful property of inclusion methods to produce disks containing all simple zeros of a polynomial. We establish computationally verifiable initial conditions that guarantee the convergence of these methods. Some computational aspects and the possibility of implementation on parallel computers are considered, including two numerical examples. A comparison of a posteriori error bound methods with the corresponding circular interval methods, regarding the computational costs and sizes of produced inclusion disks, were given.

MSC: 65H05; 65G20; 30C15

Keywords: Polynomial zeros; Localization of zeros; A posteriori error bounds; Inclusion methods; Parallel implementation

1. Localization of zeros and a posteriori error bound methods

Before running any locally convergent iterative method for the simultaneous determination of polynomial zeros, it is necessary to apply a multi-stage globally convergent composite algorithm that can provide sufficiently close initial approximations (see, e.g., [3,20,27]). The localization of zeros is an important part of this composite algorithm; a numerous references have been devoted to this subject, including famous books [16,13, Chapter 6]. One of the most beautiful results in this topic, connected with Gerschgorin's theorem and localization of zeros, is due to Carstensen [5] (Section 1). Adapting this result we can state computationally verifiable initial conditions for a number of simultaneous methods for finding polynomial zeros (see [24,25]) and construct iterative methods that produce disks in the complex plane containing the sought zeros (Section 2). The centers of these disks are calculated by some suitable iterative method with fast convergence and they present approximations to the zeros. The radii are evaluated using a corollary of Carstensen's result and give a posteriori error bounds related to these approximations.

☆ This research was supported by the Serbian Ministry of Science under Grant number 144024.
* Corresponding author.
E-mail addresses: msp@eunet.yu, msp@junis.ni.ac.yu (M.S. Petković).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2006.09.014

In this paper we combine good properties of iterative methods with fast convergence and a posteriori error bounds to construct efficient inclusion methods for polynomial complex zeros. Simultaneous determination of both centers and radii leads to iterative error bound methods which have very convenient inclusion property at each iteration. This class of methods possesses a high computational efficiency since it requires less numerical operations compared to usual interval methods realized in interval arithmetic. Numerical experiments demonstrate equal or even better convergence behavior of these methods than the corresponding interval methods realized in circular complex arithmetic (Section 3). In this paper the main attention is devoted to the construction of inclusion error bound methods together with its efficient implementation and initial conditions for the guaranteed convergence, and to the study of the convergence rate of a posteriori error bounds.

Let us return to Carstensen's result. Let $P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ $(a_i \in \mathbb{C})$ be a monic polynomial and let

$$W(z_i) = \frac{P(z_i)}{\prod_{\substack{j=1\\ j\neq i}}^{n} (z_i - z_j)} \quad (i \in I_n := \{1, \ldots, n\})$$

be Weierstrass' correction, where $z_1, \ldots, z_n$ are distinct approximations to the simple zeros $\zeta_1, \ldots, \zeta_n$ of P. Sometimes, we will write $W(z_i) = W_i$. Starting from Carstensen's result [6] which are concerned with the best Gerschgorin disks in a class of problems dealing with Weierstrass' corrections, we obtain the sharpest inclusion disks in the mentioned class given in the following theorem (see [25, Section 1.2, 29] for more details):

Theorem 1 (*Carstensen [5]*). *Let $\eta_i := z_i - W_i \in \mathbb{C}\backslash\{z_1, \ldots, z_n\}$ and set*

$$\delta_i := |W_i| \cdot \max_{j=1,\ldots,n,\, j\neq i} |z_j - \eta_i|^{-1}, \quad \sigma_i := \sum_{j=1,\, j\neq i}^{n} \frac{|W_j|}{|z_j - \eta_i|} \ \ (i \in I_n).$$

If

$$\sqrt{1+\delta_i} > \sqrt{\delta_i} + \sqrt{\sigma_i} \quad \text{and} \quad \delta_i + 2\sigma_i < 1, \tag{1}$$

then there is exactly one zero of P in the disk with center η_i and radius

$$r_i^* = |W_i| \frac{\delta_i + \sigma_i}{1 - \sigma_i}. \tag{2}$$

Remark 1. The quantity W_i is often called *Weierstrass' correction* since it appears in the very familiar Weierstrass' iterative method for the simultaneous determination of all simple zeros of a polynomial

$$\hat{z}_i = z_i - W_i \quad (i \in I_n), \tag{3}$$

also called the Durand–Kerner method [10,15]. Let us note that η_i in Theorem 1 coincides with $\hat{z}_i$.

Studying the problem of calculation of zeros, it is of interest to consider simultaneously the problem of localization of zeros together with other important topics: distribution of initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$, their closeness and the convergence of a posteriori error bounds (shorter PEB) given by the size of inclusion regions containing zeros. An extensive research performed during the last two decades (see, e.g., [22,24,25,31]) showed that the mentioned study can be realized by using Theorem 1 and an initial condition of the form

$$w^{(0)} \leqslant c_n d^{(0)}, \tag{4}$$

where

$$w^{(m)} = \max_{1 \leqslant i \leqslant n} |W(z_i^{(m)})|, \quad d^{(m)} = \min_{\substack{1 \leqslant i,j \leqslant n\\ j\neq i}} |z_i^{(m)} - z_j^{(m)}| \ \ (m = 0, 1, \ldots),$$

and $m=0,1,2,\ldots$ is the iteration index. When we omit the iteration index, then we write simply w and d. The quantity c_n depends only on the polynomial degree n. This convenient property explains the importance of results given in Theorem 1. Apart from the localization of polynomial zeros into separate disks, Theorem 1 points to the quadratic convergence of PEB corresponding to Weierstrass' method (3).

Corollary 1. *If c_n appearing in* (4) *is not greater than* $1/2n$, *then both inequalities* (1) *hold and the radii of the inclusion disks given in Theorem* 1 *are not greater than* $n|W_i|w/(d(1-c_n)-(n-1)w)$.

Proof. According to the definition of the minimal distance d, we have $|z_j-\eta_i|=|z_j-z_i+W_i|\geqslant|z_j-z_i|-|W_i|\geqslant d-w$ so that, by (4), we estimate

$$\delta_i\leqslant\frac{w}{d-w}\leqslant\frac{w}{d(1-c_n)},\quad \sigma_i\leqslant\frac{(n-1)w}{d-w}\leqslant\frac{(n-1)w}{d(1-c_n)}.$$

Hence, taking $w/d\leqslant c_n\leqslant 1/(2n)$, we prove the validity of (1). The radius r_i^* given by (2) becomes

$$r_i^*=|W_i|\frac{\delta_i+\sigma_i}{1-\sigma_i}\leqslant|W_i|\left(\frac{w}{d(1-c_n)}+\frac{(n-1)w}{d(1-c_n)}\right)\cdot\frac{1}{1-\dfrac{(n-1)w}{d(1-c_n)}},$$

wherefrom

$$r_i^*\leqslant\frac{n|W_i|w}{d(1-c_n)-(n-1)w}=:r_i. \qquad \square \tag{5}$$

Let $\varepsilon_i=z_i-\zeta_i$, $\varepsilon=\max_{1\leqslant i\leqslant n}|\varepsilon_i|$. Having in mind that $|W_i|=\mathrm{O}(\varepsilon)$, from (5) and the fact that η_i from Theorem 1 coincides with $\hat{z}_i$ given by (3), we conclude that the bounds $r_i^{(m)}$ (given by (5) for the iteration index m) can be expressed as a square of $\varepsilon^{(m)}$. Since Weierstrass' method converges quadratically under the condition $w^{(0)}<c_nd^{(0)}\leqslant d^{(0)}/(2n)$ (see [2]), that is, $\varepsilon^{(m+1)}=\mathrm{O}((\varepsilon^{(m)})^2)$, it follows that the sequences $\{r_i^{(m)}\}$ $(i\in I_n)$ of PEB, corresponding to Weierstrass' method (3), also converge quadratically.

From Corollary 1 we can derive the following useful inclusion which has the main role in our consideration.

Corollary 2. *Under the condition* (4) *each of disks D_i defined by*

$$D_i=\left\{z_i;\frac{|W_i|}{1-nc_n}\right\}=\{z_i;\rho_i\}\quad(i\in I_n)$$

contains exactly one zero of P.

The proof follows from Theorem 1 and (5) taking into account the inequality $w/d\leqslant c_n$. Indeed,

$$\left\{z_i-W_i;\frac{n|W_i|w}{d(1-c_n)-(n-1)w}\right\}\subseteq\left\{z_i-W_i;\frac{n|W_i|c_n}{1-nc_n}\right\}\subseteq\left\{z_i;\frac{n|W_i|c_n}{1-nc_n}+|W_i|\right\}=\left\{z_i;\frac{|W_i|}{1-nc_n}\right\}.$$

If the centers z_i of disks D_i are calculated by an iterative method, then we can generate the sequences of disks $D_i^{(m)}$ $(m=0,1,\ldots)$ whose radii $\rho_i^{(m)}$ converge to 0 under some suitable conditions. It should be noted that only those methods which use quantities already calculated in the previous iterative step (in our case, the corrections W_i) provide a high computational efficiency. For this reason, we restrict our choice to the class of derivative free methods which deal with Weierstrass' corrections, the so-called *W-class*. In what follows, aside from Weierstrass' method (3), we will consider the following two efficient simultaneous methods:

Börsch-Supan's method [4], shorter BS method, the convergence order 3:

$$z_i^{(m+1)}=z_i^{(m)}-\frac{W_i^{(m)}}{1+\sum_{j\neq i}\frac{W_j^{(m)}}{z_i^{(m)}-z_j^{(m)}}}\quad(i\in I_n;\ m=0,1,\ldots). \tag{6}$$

Börsch-Supan's method with Weierstrass' correction [18], shorter BSW method, the convergence order 4:

$$z_i^{(m+1)} = z_i^{(m)} - \frac{W_i^{(m)}}{1 + \sum_{j \neq i} \frac{W_j^{(m)}}{z_i^{(m)} - W_i^{(m)} - z_j^{(m)}}} \quad (i \in I_n;\; m = 0, 1, \ldots). \tag{7}$$

Let us note that $W_i^{(m)} = W(z_i^{(m)})$.

Let $z_1^{(0)}, \ldots, z_n^{(0)}$ be given initial approximations and let

$$z_i^{(m)} = \Phi_W(z_i^{(m-1)}) \quad (i \in I_n;\; m = 1, 2, \ldots) \tag{8}$$

be one of the derivative free iterative methods based on Weierstrass' corrections (belonging to W-class), which is indicated by the subscript index "W". For example, the methods (3), (6) and (7) belong to the W-class. Another iterative methods of Weierstrass' class are given in [11,26,33].

Combining the results of Corollary 2 and (8), we can state the following inclusion method in a general form:

A posteriori error bound method: A posteriori error bound method (shorter PEB method) is defined by the sequences of disks $\{D_i^{(m)}\}$ $(i \in I_n)$,

$$\begin{aligned} D_i^{(0)} &= \left\{ z_i^{(0)}; \frac{|W(z_i^{(0)})|}{1 - nc_n} \right\}, \\ D_i^{(m)} &= \{z_i^{(m)}; \rho_i^{(m)}\} \quad (i \in I_n;\; m = 1, 2, \ldots), \\ z_i^{(m)} &= \Phi_W(z_i^{(m-1)}), \quad \rho_i^{(m)} = \frac{|W(z_i^{(m)})|}{1 - nc_n}, \end{aligned} \tag{9}$$

assuming that the initial condition (4) (with $c_n \leqslant 1/(2n)$) holds.

Remark 2. The sequences of disks given by (9) can be regarded as a quasi-interval method, which differs structurally from typical interval methods that deal with disks as arguments; for instance, let us present the following circular interval methods which do not use the polynomial derivatives:

Weierstrass-like interval method [32], the order 2:

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{P(z_i^{(m)})}{\prod_{j=1, j \neq i}^{n} (z_i^{(m)} - Z_j^{(m)})} \quad (i \in I_n;\; m = 0, 1, \ldots). \tag{10}$$

Börsch-Supan-like interval method [19], the order 3:

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{W_i^{(m)}}{1 + \sum_{j=1, j \neq i}^{n} \frac{W_j^{(m)}}{Z_i^{(m)} - z_j^{(m)}}} \quad (i \in I_n;\; m = 0, 1, \ldots). \tag{11}$$

Börsch-Supan-like interval method with Weierstrass' correction [21], the order 4 (centered inversion):

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{W_i^{(m)}}{1 + \sum_{j=1, j \neq i}^{n} \frac{W_j^{(m)}}{Z_i^{(m)} - W_i^{(m)} - z_j^{(m)}}} \quad (i \in I_n;\; m = 0, 1, \ldots). \tag{12}$$

All methods (9)–(12) possess the crucial inclusion property: each of the produced disks contains exactly one zero in each iteration. More about interval methods for solving polynomial equations can be found in [20,28].

Studying the convergence of error bounds in the case of Weierstrass' method and having in mind aforementioned remarks, the following important tasks arise:

(1) Determine the convergence order of a posteriori error bound method when the centers $z_i^{(m)}$ of disks

$$D_i^{(m)} = \left\{ z_i^{(m)}; \frac{|W(z_i^{(m)})|}{1 - nc_n} \right\} \quad (i \in I_n; \ m = 0, 1, \ldots) \tag{13}$$

are calculated by an iterative method of order k $(\geqslant 2)$.

(2) State computationally verifiable initial condition that guarantees the convergence of the sequences of radii $\{\text{rad}\, D_i^{(m)}\}$. We note that this problem, very important in the theory and practice of iterative processes in general, is a part of Smale's "point estimation theory" [30] which has attracted a great attention during the last two decades (see [25] for details). In the case of algebraic polynomials, initial conditions should depend only on attainable data—initial approximations, polynomial degree and polynomial coefficients.

(3) Compare the computational efficiencies of the PEB methods and the existing circular interval methods (given, for instance, by (10)–(12)). Which of these two classes of methods is more efficient?

(4) Using numerical experiments, compare the size of inclusion disks produced by the PEB methods and the corresponding interval methods (10)–(12). Whether the construction of PEB methods is justified?

The study of these subjects is the main goal of this paper.

2. Convergence of PEB methods

Starting from the PEB method (9), where the centers are calculated by the BS method (6) or the BSW method (7), in this section we study initial conditions for the guaranteed convergence of the sequences of PEB $\{\rho_i^{(m)}\}$. We will present the convergence analysis of the PEB method based on the BS method (6) in details, while the convergence theorem concerned with the BSW method (7) will be given without a proof since it uses a similar technique. For simplicity, we will omit sometimes the iteration index m and denote quantities in the latter $(m+1)$st iteration by $\hat{}$ ("hat").

First, we give some necessary estimates.

Lemma 1. *If the inequality*

$$w < \frac{d}{2n} \tag{14}$$

holds, then for the iterative method (6) *and* $i \in I_n$ *we have*

(i) $\left| 1 + \sum_{j \neq i} \frac{W_j}{z_i - z_j} \right| > \frac{n+1}{2n}$;

(ii) $|\hat{z}_i - z_i| < \frac{2n}{n+1} |W_i| < \frac{d}{n+1}$;

(iii) $|\hat{z}_i - z_j| > \frac{n}{n+1} d$;

(iv) $|\hat{z}_i - \hat{z}_j| > \frac{n-1}{n+1} d$;

(v) $\left|\sum_{j=1}^{n}\frac{W_j}{\hat{z}_i - z_j}+1\right| < \frac{n-1}{2n^2}$;

(vi) $\prod_{j\neq i}\left|\frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}\right| < \left(1+\frac{1}{n-1}\right)^{n-1}$.

Proof. Of (i): Using (14) and the definition of d one obtains

$$\left|1+\sum_{j\neq i}\frac{W_j}{z_i - z_j}\right| \geqslant 1-\sum_{j\neq i}\frac{|W_j|}{|z_i - z_j|} \geqslant 1-\frac{(n-1)w}{d} > 1-\frac{n-1}{2n} = \frac{n+1}{2n}.$$

Of (ii): By (i) and (14) we get from (6)

$$|\hat{z}_i - z_i| = \left|\frac{W_i}{1+\sum_{j\neq i}\frac{W_j}{z_i - z_j}}\right| < \frac{|W_i|}{\frac{n+1}{2n}} = \frac{2n}{n+1}|W_i| < \frac{d}{n+1}.$$

Of (iii): Using (ii) we find

$$|\hat{z}_i - z_j| \geqslant |z_i - z_j| - |\hat{z}_i - z_i| > d - \frac{d}{n+1} = \frac{n}{n+1}d.$$

Of (iv): By (ii) one gets

$$|\hat{z}_i - \hat{z}_j| \geqslant |z_i - z_j| - |\hat{z}_i - z_i| - |\hat{z}_j - z_j| > d - 2\cdot\frac{d}{n+1} = \frac{n-1}{n+1}d.$$

Of (v): From the iterative formula (6) we obtain

$$\frac{W_i}{\hat{z}_i - z_i} = -1-\sum_{j\neq i}\frac{W_j}{z_i - z_j}$$

so that

$$\left|\sum_{j=1}^{n}\frac{W_j}{\hat{z}_i - z_j}+1\right| = \left|\frac{W_i}{\hat{z}_i - z_i}+\sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j}+1\right| = \left|\sum_{j\neq i}\frac{W_j(z_i-\hat{z}_i)}{(\hat{z}_i - z_j)(z_i - z_j)}\right|,$$

whence, by (ii), (iii) and (14),

$$\begin{aligned}\left|\sum_{j=1}^{n}\frac{W_j}{\hat{z}_i - z_j}+1\right| &\leqslant |\hat{z}_i - z_i|\sum_{j\neq i}\frac{|W_j|}{|\hat{z}_i - z_j||z_i - z_j|}\\ &< \frac{d}{n+1}\cdot\frac{(n-1)w}{\left(\frac{n}{n+1}d\right)\cdot d} < \frac{n-1}{2n^2}.\end{aligned}$$

Of (vi): By (ii) and (iv) we estimate

$$\prod_{j\neq i}\left|\frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}\right| \leqslant \prod_{j\neq i}\left(1+\frac{|\hat{z}_j - z_j|}{|\hat{z}_i - \hat{z}_j|}\right) < \prod_{j\neq i}\left(1+\frac{\dfrac{d}{n+1}}{\dfrac{(n-1)d}{n+1}}\right) = \left(1+\frac{1}{n-1}\right)^{n-1}. \qquad \square$$

Lemma 2. *Let us consider the PEB method* (9) *based on the Börsch-Supan's method* (6). *If the inequality* (14) *holds, then for* $i \in I_n$ *we have*

(i) $|\hat{W}_i| < \beta_n |W_i|, \quad \beta_n = \dfrac{n-1}{n(n+1)}\left(1+\dfrac{1}{n-1}\right)^{n-1}$;

(ii) $\hat{w} < \dfrac{\hat{d}}{2n}$;

(iii) $\hat{\rho}_i < \dfrac{\gamma_n}{d^2}\rho_i^2 \sum_{j\neq i}\rho_j, \quad \gamma_n = \begin{cases} \frac{27}{16} & \text{if } n=3,\\ e & \text{if } n\geqslant 4.\end{cases}$

Proof. From (6) we find

$$\frac{W_i}{\hat{z}_i - z_i} = -1 - \sum_{j\neq i}\frac{W_j}{z_i - z_j},$$

so that

$$\sum_{j=1}^{n}\frac{W_j}{\hat{z}_i - z_j} + 1 = \frac{W_i}{\hat{z}_i - z_i} + \sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j} + 1 = -(\hat{z}_i - z_i)\sum_{j\neq i}\frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)}. \tag{15}$$

Now we use the well-known result from the interpolation theory: if $z_1, \ldots, z_n$ are distinct complex numbers, then the polynomial P can be expressed by the Lagrange interpolation formula

$$P(z) = \left(\sum_{j=1}^{n}\frac{W_j}{z - z_j} + 1\right)\prod_{j=1}^{n}(z - z_j). \tag{16}$$

Putting $z = \hat{z}_i$ in (16) one gets

$$P(\hat{z}_i) = (\hat{z}_i - z_i)\left(\sum_{j=1}^{n}\frac{W_j}{\hat{z}_i - z_j} + 1\right)\prod_{j\neq i}(\hat{z}_i - z_j).$$

After dividing $P(\hat{z}_i)$ by $\prod_{j\neq i}(\hat{z}_i - \hat{z}_j)$, we find

$$\hat{W}_i = (\hat{z}_i - z_i)\left(\sum_{j=1}^{n}\frac{W_j}{\hat{z}_i - z_j} + 1\right)\prod_{j\neq i}\frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}. \tag{17}$$

Using (ii), (v) and (vi) of Lemma 1, we start from (17) and find

$$
\begin{aligned}
|\hat{W}_i| &= |\hat{z}_i - z_i| \left| \sum_{j=1}^{n} \frac{W_j}{|\hat{z}_i - z_j} + 1 \right| \prod_{j \neq i} \frac{|\hat{z}_i - z_j|}{|\hat{z}_i - \hat{z}_j|} \\
&< \frac{2n}{n+1} |W_i| \frac{n-1}{2n^2} \left(1 + \frac{1}{n-1}\right)^{n-1} \\
&= \frac{n-1}{n(n+1)} \left(1 + \frac{1}{n-1}\right)^{n-1} |W_i| = \beta_n |W_i|,
\end{aligned}
$$

which proves the assertion (i).

From Lemma 1 (assertion (iv)) we observe that $\hat{d} > [(n-1)/(n+1)]d$. According to this and (i) of Lemma 2, taking into account that $\beta_n \leqslant \frac{3}{8}$ we find

$$
|\hat{W}_i| < \beta_n |W_i| < \beta_n \cdot \frac{d}{2n} < \frac{(n+1)\beta_n}{2n(n-1)} \cdot \hat{d} \leqslant \frac{3(n+1)}{16n(n-1)} \cdot \hat{d} < \frac{\hat{d}}{2n}.
$$

This proves the implication

$$
w < \frac{d}{2n+2} \quad \Rightarrow \quad \hat{w} < \frac{\hat{d}}{2n+2}.
$$

To prove (iii) we use (15) and from (17) we find

$$
\hat{W}_i = -(\hat{z}_i - z_i)^2 \sum_{j \neq i} \frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)} \prod_{j \neq i} \frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}.
$$

According to the estimates (ii), (iii) and (vi) of Lemma 1, from the last relation we obtain

$$
\begin{aligned}
|\hat{W}_i| &= |\hat{z}_i - z_i|^2 \sum_{j \neq i} \frac{|W_j|}{|\hat{z}_i - z_j||z_i - z_j|} \prod_{j \neq i} \left| \frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j} \right| \\
&< \left(\frac{2n}{n+1}\right)^2 |W_i|^2 \sum_{j \neq i} \frac{|W_j|}{\dfrac{n}{n+1} d \cdot d} \left(1 + \frac{1}{n-1}\right)^{n-1} \\
&< \frac{4n}{n+1} \left(1 + \frac{1}{n-1}\right)^{n-1} \frac{1}{d^2} |W_i|^2 \sum_{j \neq i} |W_j| \\
&\leqslant \frac{4\gamma_n}{d^2} |W_i|^2 \sum_{j \neq i} |W_j|.
\end{aligned}
$$

Multiplying both sides of the last inequality with $1/(1 - nc_n) = 2$, we get

$$
\hat{\rho}_i < \frac{\gamma_n}{d^2} \rho_i^2 \sum_{j \neq i} \rho_j \tag{18}
$$

for every $n \geqslant 3$. □

The initial disks $D_i^{(0)}$ for $c_n = 1/(2n)$ are given by

$$
D_i^{(0)} = \{z_i^{(0)}; 2|W(z_i^{(0)})|\} \quad (i \in I_n).
$$

By (9) we define the sequences of inclusion disks

$$D_i^{(m)} = \{z_i^{(m)}; 2|W(z_i^{(m)})|\} = \{z_i^{(m)}; \rho_i^{(m)}\} \quad (i \in I_n;\ m = 1, 2, \ldots), \tag{19}$$

where $z_i^{(m)}$ is calculated by the Börsch-Supan's iterative formula (6) and $\rho_i^{(m)} = 2|W(z_i^{(m)})|$.

Theorem 2. *The PEB method* (19), *based on Börsch-Supan's method* (6), *converges cubically if the initial condition*

$$w^{(0)} < \frac{d^{(0)}}{2n} \tag{20}$$

holds.

Proof. As usually in the convergence analysis of interval methods, we have to prove that the sequences of a posteriori error bounds $\{\rho_i^{(m)}\}$ $(i \in I_n)$ converge cubically. The proof is by induction with the argumentation used in the proofs of Lemmas 1 and 2. The initial condition (20) coincides with (14), which implies that all assertions of Lemmas 1 and 2 hold for the index $m = 1$. In fact, this is the part of the proof with respect to $m = 1$. The inequality (ii) of Lemma 2 again reduces to the condition of the form (14) and, therefore, the assertions of Lemmas 1 and 2 hold for the next index, and so on. Actually, the implication

$$w^{(m)} < \frac{d^{(m)}}{2n} \quad \Rightarrow \quad w^{(m+1)} < \frac{d^{(m+1)}}{2n}$$

plays a key role because it involves the initial condition (20) which leads to the validity of all inequalities given in Lemmas 1 and 2 for each $m = 0, 1, \ldots$. In particular, we have for every $i \in I_n$

$$\rho_i^{(m+1)} < \frac{\gamma_n}{(d^{(m)})^2}(\rho_i^{(m)})^2 \sum_{j \neq i} \rho_j^{(m)}, \tag{21}$$

$$\frac{d^{(m)}}{d^{(m+1)}} < \frac{n+1}{n-1}, \tag{22}$$

$$|W_i^{(m+1)}| < \beta_n |W_i^{(m)}|, \tag{23}$$

and

$$|z_i^{(m+1)} - z_i^{(m)}| < \frac{2n}{n+1}|W_i^{(m)}|. \tag{24}$$

Substituting

$$h_i^{(m)} = \frac{\rho_i^{(m)}}{d^{(m)}}\sqrt{(n+1)\gamma_n} \quad (i \in I_n), \tag{25}$$

the inequalities (21) become

$$h_i^{(m+1)} < \frac{1}{n+1} \cdot \frac{d^{(m)}}{d^{(m+1)}}(h_i^{(m)})^2 \sum_{j \neq i} h_j^{(m)} \quad (i \in I_n). \tag{26}$$

Applying (22), from (26) one obtains

$$h_i^{(m+1)} < \frac{1}{n-1}(h_i^{(m)})^2 \sum_{j \neq i} h_j^{(m)} \quad (i \in I_n). \tag{27}$$

Using (20) we find

$$h_i^{(0)} < \frac{\rho_i^{(0)}}{d^{(0)}}\sqrt{(n+1)\gamma_n} = \frac{2|W_i^{(0)}|}{d^{(0)}}\sqrt{(n+1)\gamma_n} < \frac{1}{n}\sqrt{(n+1)\gamma_n} < 1.$$

Starting from the inequality $h_i^{(0)} < 1$ $(i \in I_n)$, by successive application of (27) we conclude that the sequences $\{h_i^{(m)}\}$ $(i \in I_n)$ monotonically converge to 0. Since $d^{(m)}$ is bounded and tends to $\min_{j\neq i}|\zeta_i - \zeta_j|$, in regard to the substitution (25) we conclude that the sequences $\{\rho_i^{(m)}\}$ $(i \in I_n)$ also converge to 0.

By successive application of (23), (24) and the condition (20), we estimate

$$\begin{aligned} d^{(m)} &\geqslant |z_i^{(m)} - z_j^{(m)}| \geqslant |z_i^{(m-1)} - z_j^{(m-1)}| - |z_i^{(m)} - z_i^{(m-1)}| - |z_j^{(m)} - z_j^{(m-1)}| \\ &> d^{(m-1)} - 2\cdot\frac{2n}{n+1}w^{(m-1)} > d^{(m-2)} - 2\cdot\frac{2n}{n+1}w^{(m-2)} - 2\cdot\frac{2n}{n+1}w^{(m-1)} \\ &\ \ \vdots \\ &> d^{(0)} - \frac{4n}{n+1}\left(w^{(0)} + w^{(1)} + \cdots + w^{(m-1)}\right) \\ &> d^{(0)} - \frac{4n}{n+1}w^{(0)}\left(1 + \frac{3}{8} + \left(\frac{3}{8}\right)^2 + \cdots + \left(\frac{3}{8}\right)^{m-1}\right) \\ &> d^{(0)} - \frac{32n}{5(n+1)}w^{(0)} > d^{(0)} - \frac{32n}{5(n+1)}\cdot\frac{d^{(0)}}{2n} = \frac{5n-11}{5n+5}d^{(0)}. \end{aligned}$$

Setting the inequality $d^{(m)} > [(5n-11)/(5n+5)]d^{(0)}$ in (21) we obtain

$$\rho_i^{(m+1)} < \left(\frac{5n+5}{5n-11}\right)^2 \frac{e}{(d^{(0)})^2}(\rho_i^{(m)})^2 \sum_{j\neq i}\rho_j^{(m)} \quad (i \in I_n).$$

Let $\rho^{(m)} = \max_{1\leqslant i\leqslant n}\rho_i^{(m)}$. From the last inequality we obtain

$$\rho_i^{(m+1)} < \frac{(n-1)e}{(d^{(0)})^2}\left(\frac{5n+5}{5n-11}\right)^2(\rho^{(m)})^3 \quad (i \in I_n),$$

which means that the sequences of PEB $\{\rho_i^{(m)}\}$ converge cubically. □

An extensive but elementary analysis, similar to that given in the proofs of Lemmas 1, 2 and Theorem 2, allows us to state computationally verifiable condition for the convergence of the PEB method based on BSW formula (7) in the form

$$w^{(0)} < \frac{d^{(0)}}{2n+1}. \tag{28}$$

In this case PEB is given by

$$\rho_i^{(m)} = \frac{|W(z_i^{(m)})|}{1 - nc_n} = \frac{2n+1}{n+1}|W(z_i^{(m)})|,$$

while the centers of disks (9) are now calculated by the Börsch-Supan's method with Weierstrass' corrections (7).

Theorem 3. *The PEB method*

$$D_i^{(m)} = \left\{z_i^{(m)}; \frac{2n+1}{n+1}|W(z_i^{(m)})|\right\} \quad (i \in I_n;\ m = 0, 1, \ldots),$$

based on the Börsch-Supan method with Weierstrass correction (7), *converges with the order* 4 *if the initial condition* (28) *is valid*.

Remark 3. It is not difficult to prove that the initial conditions (20) and (28) are sufficient to ensure the convergence of the iterative methods (6) and (7), respectively. Moreover, they improve the initial conditions given in [23].

We conclude this section emphasizing that the initial conditions (20) and (28) that guarantee the convergence of the PEB methods (9)–(6) and (9)–(7) (respectively) depend only of attainable data, which is of great practical importance.

3. Computational points

In this section we give practical aspects of the presented theoretical results and some practical procedures in the implementation of the proposed methods. As mentioned above, the computational cost significantly decreases if the quantities $W_i^{(0)}, W_i^{(1)}, \ldots$ $(i \in I_n)$, necessary in the calculation of PEB $\rho_i^{(m)} = |W_i^{(m)}|/(1 - nc_n)$, are applied in the calculation of the centers $z_i^{(m+1)}$ defined by the employed iterative formula from the *W*-class. Regarding the iterative formulas (3), (6) and (7) we observe that this requirement is satisfied. A general calculating procedure can be described by the following algorithm:

Calculating procedure (I). *Given* $z_1^{(0)}, \ldots, z_n^{(0)}$ *and the tolerance parameter* τ;
Set $m = 0$;

(1) *Calculate Weierstrass' corrections* $W_1^{(m)}, \ldots, W_n^{(m)}$ *at the points* $z_1^{(m)}, \ldots, z_n^{(m)}$;
(2) *Calculate the radii* $\rho_i^{(m)} = |W_i^{(m)}|/(1 - nc_n)$ $(i = 1, \ldots, n)$;
(3) *If* $\max_{1 \leqslant i \leqslant n} \rho_i^{(m)} < \tau$, *then STOP*
(3) *otherwise*, *GO TO* (4);
(4) *Calculate the new approximations* $z_1^{(m+1)}, \ldots, z_n^{(m+1)}$ *by a suitable iterative formula from the W-class* (*for instance*, *by* (3), (6) *or* (7));
(5) *Set* $m := m + 1$ *and GO TO the step* (1).

Following the procedure (I) we realized many numerical examples and, for demonstration, we select the following one.

Example 1. We considered the polynomial

$$\begin{aligned} P(z) &= z^{12} - (2+5i)z^{11} - (1-10i)z^{10} + (12-25i)z^9 - 30z^8 \\ &\quad - z^4 + (2+5i)z^3 + (1-10i)z^2 - (12-25i)z + 30 \\ &= (z^8 - 1)(z^2 - 2z + 5)(z - 2i)(z - 3i). \end{aligned}$$

Starting from sufficiently close initial approximations $z_1^{(0)}, \ldots, z_{12}^{(0)}$ we applied a posteriori error bound method (9) and obtained the inclusion disks $D_i^{(m)} = \{z_i^{(m)}; \rho_i^{(m)}\}$ $(i \in I_{12})$. The approximations $z_i^{(m)}$ $(m \geqslant 1)$ were calculated by the iterative formulas (3), (6) and (7) and the corresponding inclusion methods are referred to as (I-W), (I-BS) and (I-BSW), respectively. For comparison purpose, we also tested the interval methods (10)–(12). The largest radii of the disks obtained in the first four iterations are presented in Table 1, where $A(-q)$ means $A \times 10^{-q}$.

The size of disks in the first iteration are of the same order for all three PEB methods since we use the formula $\rho^{(0)} = |W(z_i^{(0)})|/(1 - nc_n)$, where we used $c_n = 1/(2n)$ for the methods (3) and (6) and $c_n = 1/(2n+1)$ for the method

Table 1
The largest radii of disks obtained by procedure (I) and by interval methods (10)–(12)

Methods	$\max \rho_i^{(0)}$	$\max \rho_i^{(1)}$	$\max \rho_i^{(2)}$	$\max \rho_i^{(3)}$
(I-W) (9)–(3)	1.82(−1)	5.20(−2)	3.28(−3)	5.70(−6)
Interval W (10)	1.70(−1)	No inclusions	No inclusions	No inclusions
(I-BS) (9)–(6)	1.82(−1)	4.25(−3)	1.13(−8)	3.29(−25)
Interval BS (11)	1.70(−1)	8.44(−3)	2.29(−7)	6.60(−22)
(I-BSW) (9)–(7)	1.76(−1)	8.54(−4)	1.14(−13)	2.68(−53)
Interval BSW (12)	1.70(−1)	1.01(−2)	4.74(−9)	2.60(−35)

Table 2
The largest radii of disks obtained by (I-W), (I-BS), (I-BSW)—Procedure (II)

Methods	$\max \rho_i^{(0)}$	$\max \rho_i^{(1)}$	$\max \rho_i^{(2)}$
(I-W) (9)–(3)	1.51(−3)	3.79(−6)	2.27(−11)
(I-BS) (9)–(6)	1.51(−3)	4.10(−9)	8.31(−26)
(I-BSW) (9)–(7)	1.46(−3)	9.64(−12)	1.60(−44)

(7). The initial radii of the applied interval methods (10)–(12) were rounded to 0.17. In our calculation we employed multi-precision arithmetic since the tested methods converge very fast producing very small disks. From Table 1 we observe that the PEB methods are equal or better than the corresponding methods (of the same order) (10)–(12) realized in complex interval arithmetic. A number of numerical experiments showed similar convergence behavior of the tested methods. The Weierstrass interval method (10) shows poor results since it uses the product of disks which is not an exact operation in circular arithmetic and gives too large disks.

Calculation procedure (I) assumes the knowledge of initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ in advance. The determination of these approximations is usually realized by a slowly convergent multi-stage composite algorithm. Sometimes, the following simple approach gives good results in practice.

Calculating procedure (II).

(1) *Find the disk centered in the origin with the radius*

$$R = 2 \max_{1 \leqslant k \leqslant n} \left| \frac{a_k}{a_0} \right|^{1/k} \quad (\textit{see [13, Corollary 6.4k]}),$$

which contains all zeros of the polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$.

(2) *Calculate Aberth's initial approximations* [1]

$$z_\nu^{(0)} = -\frac{a_1}{n} + r_0 \exp(\mathrm{i}\theta_\nu), \quad \mathrm{i} = \sqrt{-1}, \quad \theta_\nu = \frac{\pi}{n}\left(2\nu - \frac{3}{2}\right) \quad (\nu = 1, \ldots, n),$$

equidistantly distributed along the circle $|z + a_1/n| = r_0, \ r_0 \leqslant R$.

(3) *Apply the simultaneous method* (3) *or* (6) *starting with Aberth's approximations*; *stop the iterative process when the condition*

$$\max_{1 \leqslant i \leqslant n} |W(z_i^{(m)})| < c_n \min_{i \neq j} |z_i^{(m)} - z_j^{(m)}| \tag{29}$$

is satisfied.

(4)–(8) *The same as the steps* (1)–(5) *of procedure* (I).

We applied procedure (II) on the following example.

Example 2. To find approximations to the zeros of the polynomial

$$z^{15} + z^{14} + 1 = 0$$

satisfying the condition (29) (with $c_n = 1/(2n)$), we applied Börsch-Supan's method (6) with Aberth's initial approximations taking $a_1 = 1, \ n = 15, \ r_0 = 2$. The condition (29) was satisfied after seven iterative steps. The obtained approximations were used to start the inclusion methods (I-W), (I-BS) and (I-BSW). After three iterations we obtained disks whose largest radii are given in Table 2.

From Tables 1 and 2 we observe that the results obtained by the methods (I-W), (I-BS) and (I-BSW) coincide with the theoretical results given in Corollary 1 and Theorems 2 and 3; in other words, the order of convergence in practice matches very well the order derived in the presented theoretical analysis.

Table 3
The number of operations

	AS(n)	M(n)	D(n)
(I-W) (9)–(3)	$8n^2+n$	$8n^2+2n$	2n
Interval W (10)	$22n^2-6n$	$25n^2-6n$	$8n^2-n$
(I-BS) (9)–(6)	$15n^2-6n$	$14n^2+2n$	$2n^2+2n$
Interval BS (11)	$23n^2-4n$	$23n^2+2n$	$7n^2+2n$
(I-BSW) (9)–(7)	$15n^2-4n$	$14n^2+2n$	$2n^2+2n$
Interval BSW (12)	$23n^2-2n$	$23n^2+2n$	$7n^2+2n$

At the beginning of the paper we have mentioned that the PEB methods requires less numerical operations compared to their counterparts in complex interval arithmetic. In Table 3 we give the total number of numerical operations per one iteration, reduced to real arithmetic operations. We have used the following abbreviations:

$AS(n)$ (total number of additions and subtractions)

$M(n)$ (multiplications)

$D(n)$ (divisions)

From Table 3 we observe that the PEB methods needs significantly less numerical operations in reference to the corresponding interval methods. One of the reasons for this advantage is the use of the already calculated Weierstrass corrections W_i in the evaluation of the radii ρ_i.

Parallel implementation: It is worth noting that the error bound method (9) for the simultaneous determination of all zeros of a polynomial is very suitable for the implementation on parallel computers since it runs in several identical versions. In this manner, a great deal of computation can be executed simultaneously. An analysis of total running time of a parallel iteration and the determination of the optimal number of processors points to some undoubted advantages of the implementation of simultaneous methods on parallel processing computers, see, e.g., [7–9,12,23]. The parallel processing becomes of significantly great interest to speed up the determination of zeros when one should treat polynomials with degree 100 and higher, appearing in mathematical models in scientific engineering, including digital signal processing or automatic control [14,17].

The model of parallel implementation is as follows: it is assumed that the number of processors $k\ (\leqslant n)$ is given in advance. Let $\boldsymbol{W}^{(m)}=(W_1^{(m)},\ldots,W_n^{(m)})$, $\boldsymbol{\rho}^{(m)}=(\rho_1^{(m)},\ldots,\rho_n^{(m)})$, $\boldsymbol{z}^{(m)}=(z_1^{(m)},\ldots,z_n^{(m)})$ denote vectors in the mth iterative step, where $\rho_i^{(m)}=|W(z_i^{(m)})|/(1-nc_n)$, and $z_i^{(m)}$ is obtained by the iterative formula $z_i^{(m)}=\Phi_W(z_i^{(m-1)})$ $(i\in I_n)$. The starting vector $\boldsymbol{z}^{(0)}$ is computed by all processors $C_1,\ldots,C_k$ using some suitable globally convergent method based on a subdivided procedure and the inclusion annulus $\{z: r\leqslant|z|\leqslant R\}$ which contains all zeros, where

$$r=\frac{1}{2}\min_{1\leqslant k\leqslant n}\left|\frac{a_n}{a_{n-k}}\right|^{1/k},\qquad R=2\max_{1\leqslant k\leqslant n}\left|\frac{a_k}{a_0}\right|^{1/k}$$

(see [13, Theorem 6.4b, Corollary 6.4k]).

In the next stage, each step of the algorithm consists in sharing the calculation of $W_i^{(m)}$, $\rho_i^{(m)}$, $z_i^{(m+1)}$ among the processors and in updating their data through a broadcast procedures (shorter BCAST($\boldsymbol{W}^{(m)}$, $\boldsymbol{\rho}^{(m)}$), BCAST($\boldsymbol{z}^{(m+1)}$)). As in [8], let $I_1,\ldots,I_k$ be disjunctive partitions of the set $\{1,\ldots,n\}$ where $\cup I_j=\{1,\ldots,n\}$. To provide good load balancing between the processors, the index sets $I_1,\ldots,I_k$ are chosen so that the number of their components $\mathcal{N}(I_j)$ $(j=1,\ldots,k)$ is determined as $\mathcal{N}(I_j)\leqslant[n/k]$. At the mth iterative step the processor $C_j\ (j=1,\ldots,k)$ computes $W_i^{(m)}$, $\rho_i^{(m)}$ and, if necessary, $z_i^{(m+1)}$ for all $i\in I_j$ and then it transmits these values to all other processors using a broadcast procedure. The program terminates when some stopping criterion is satisfied, say, if for a given tolerance τ

the inequality

$$\max_{1\leqslant i\leqslant n} |\rho_i^{(m)}| < \tau$$

holds. A program written in pseudocode (following [8]) for a parallel implementation of the error bound method (9) is given below:

Program A Posteriori Error Bound Method

```
begin
    for all j = 1, ..., k do determination of the approximations z^(0);
    m := 0
    C := false
    do
        for all j = 1, ..., k do in parallel
        begin
            Compute W_i^(m),  i ∈ I_j;
            Compute ρ_i^(m),  i ∈ I_j;
            Communication: BCAST(W^(m), ρ^(m));
        end
        if max_{1≤i≤n} ρ_i^(m) < τ; C :=true
        else
            m := m + 1
            for all j = 1, ..., k do in parallel
            begin
                Compute z_i^(m),  i ∈ I_j, by (8);
                Communication: BCAST(z^(m));
            end
        endif
    until C
    OUTPUT z^(m), ρ^(m)
end
```

4. Conclusions

Using Carstensen's results [5] concerning Gerschgorin's theorem and localization of polynomial zeros, we construct an array of nonintersecting inclusion disks, each of them containing one and only one simple zero of a given polynomial. The centers of these disks are calculated by derivative free iterative methods for the simultaneous determination of polynomial zeros, based on Weierstrass' corrections W_i. These corrections are also used for the calculation of the radii $|W_i|/(1 - nc_n)$ of the produced disks, which is efficient from the practical point of view.

The stated methods can be regarded as a posteriori error bound methods that possess very important property—inclusion of zeros, giving automatically the upper error bounds of the calculated approximations, actually, the radii of disks. We have shown that these methods require significantly less numerical operations compared to the corresponding interval methods realized in circular interval arithmetic and even better convergence behavior in most of the tested polynomial equations.

We consider two efficient simultaneous derivative free methods of the third and fourth order and establish the convergence theorems under computationally verifiable conditions of the form

$$\max_{1\leqslant i\leqslant n} |W_i^{(0)}| < c_n \min_{\substack{1\leqslant j\leqslant n \\ j\neq i}} |z_i^{(0)} - z_j^{(0)}|,$$

depending only on attainable data. These conditions guarantee the convergence of the stated a posteriori error bound methods.

Two calculating procedures of practical importance are given and illustrated on numerical examples. A possibility of implementation on parallel computers is also considered.

References

[1] O. Aberth, Iteration methods for finding all zeros of a polynomial simultaneously, Math. Comp. 27 (1973) 339–344.
[2] P. Batra, Improvement of a convergence condition for the Durand–Kerner iteration, J. Comput. Appl. Math. 96 (1998) 117–125.
[3] D.A. Bini, G. Fiorentino, Design, analysis and implementation of a multiprecision polynomial rootfinder, Numer. Algorithms 23 (2000) 127–173.
[4] W. Börsch-Supan, Residuenabschätzung für Polynom-Nullstellen mittels Lagrange-Interpolation, Numer. Math. 14 (1970) 287–296.
[5] C. Carstensen, Anwendungen von Begleitmatrizen, Z. Angew. Math. Mech. 71 (1991) 809–812.
[6] C. Carstensen, Inclusion of the roots of a polynomial based on Gerschgorin's theorem, Numer. Math. 59 (1991) 349–360.
[7] M. Cosnard, P. Fraigniaud, Asynchronous Durand–Kerner and Aberth polynomial root finding methods on a distributed memory multicomputer, Parallel Comput. 9 (1989) 79–84.
[8] M. Cosnard, P. Fraigniaud, Finding the roots of a polynomial on an MIMD multicomputer, Parallel Comput. 15 (1990) 75–85.
[9] M. Cosnard, P. Fraigniaud, Analysis of asynchronous polynomial root finding methods on a distributed memory multicomputer, IEEE Trans. Parallel Distributed Systems 5 (1994) 639–648.
[10] E. Durand, Solution numériques des álgébraiques, Tom. I: Équations du Type $F(x) = 0$; Racines d'un Polynôme, Masson, Paris, 1960.
[11] G.H. Ellis, L.T. Watson, A parallel algorithm for simple roots of polynomials, Comput. Math. Appl. 10 (1984) 107–121.
[12] T.L. Freeman, Calculating polynomial zeros on a local memory parallel computer, Parallel Comput. 12 (1989) 351–358.
[13] P. Henrici, Applied and Computational Complex Analysis, vol. I, Wiley, New York, 1974.
[14] L.H. Jamieson, T.A. Rice, A highly parallel algorithms for root extraction, IEEE Trans. Comput. 28 (1989) 443–449.
[15] I.O. Kerner, Ein Gesamtschrittverfahren zur Berechnung der Nullstellen von Polynomen, Numer. Math. 8 (1996) 290–294.
[16] M. Marden, The Geometry of Polynomials. Mathematical Surveys, American Mathematical Society, Providence, RI, 1966.
[17] J.L. Nicolas, A. Schinzel, Localisation des zéros de polynomes intervenant end théorie du signal, Research Report, University of Lyon 1, Lyon, 1988.
[18] A.W.M. Nourein, An iteration formula for the simultaneous determination of the zeroes of a polynomial, J. Comput. Appl. Math. 4 (1975) 251–254.
[19] M.S. Petković, On an iterative method for simultaneous inclusion of polynomial complex zeros, J. Comput. Appl. Math. 1 (1982) 51–56.
[20] M.S. Petković, Iterative Methods for Simultaneous Inclusion of Polynomial Zeros, Springer, Berlin, Heidelberg, New York, 1989.
[21] M.S. Petković, C. Carstensen, On some improved inclusion methods for polynomial roots with Weierstrass' corrections, Comput. Math. Appl. 25 (1993) 59–67.
[22] M.S. Petković, C. Carstensen, M. Trajković, Weierstrass' formula and zero-finding methods, Numer. Math. 69 (1995) 353–372.
[23] M.S. Petković, D. Herceg, Börsch-Supan-like methods: point estimation and parallel implementation, Internat. J. Comput. Math. 64 (1997) 327–341.
[24] M.S. Petković, D. Herceg, Point estimation of simultaneous methods for solving polynomial equations: a survey, J. Comput. Appl. Math. 136 (2001) 183–207.
[25] M.S. Petković, D. Herceg, S.M. Ilić, Point Estimation Theory and its Applications, Institute of Mathematics, Novi Sad, 1997.
[26] M.S. Petković, S. Ilić, S. Tričković, A family of simultaneous zero-finding methods, Comput. Math. Appl. 34 (1997) 49–59.
[27] M.S. Petković, D. Milošević, A higher order family for the simultaneous inclusion of multiple zeros of polynomials, Numer. Algorithms 39 (2005) 415–435.
[28] M.S. Petković, L.D. Petković, Complex Interval Arithmetic and its Applications, Wiley-VCH, Berlin, Weinhein, New York, 1998.
[29] M.S. Petković, L.D. Petković, On the convergence of the sequences of Gerschgorin-like disks, Numerical Algorithms, to appear.
[30] S. Smale, The fundamental theorem of algebra and complexity theory, Bull. Amer. Math. Soc. 4 (1981) 1–35.
[31] D. Wang, F. Zhao, The theory of Smale's point estimation and its application, J. Comput. Appl. Math. 60 (1995) 253–269.
[32] X. Wang, S. Zheng, The quasi-Newton method in parallel circular iteration, J. Comput. Math. 4 (1984) 305–309.
[33] S. Zheng, F. Sun, Some simultaneous iterations for finding all zeros of a polynomial with high order of convergence, Appl. Math. Comput. 99 (1999) 233–240.