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A q-enumeration of alternating permutations ☆

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ABSTRACT

A classical result of Euler states that the tangent numbers are an alternating sum of Eulerian numbers. A dual result of Roselle states that the secant numbers can be obtained by a signed enumeration of derangements. We show that both identities can be refined with the following statistics: the number of crossings in permutations and derangements, and the number of patterns 31-2 in alternating permutations.

Using previous results of Corteel, Rubey, Prellberg, and the author, we derive closed formulas for both q-tangent and q-secant numbers. There are two different methods for obtaining these formulas: one with permutation tableaux and one with weighted Motzkin paths (Laguerre histories).

1. Introduction

The classical Euler numbers E_n are given by the Taylor expansion of the tangent and secant functions:

$$\tan(x) + \sec(x) = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.$$

Since the tangent is an odd function and the secant is an even function, the integers E_{2n+1} are called the *tangent numbers* and the integers E_{2n} are called the *secant numbers*. More than a century ago, Désiré André [1] showed that E_n is the number of *alternating permutations* in $\mathfrak{S}_n$.

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For any permutation $\sigma \in \mathfrak{S}_n$, let $\mathrm{exc}(\sigma)$ be the number of exceedances of σ, *i.e.* the number of integers $i \in \{1, \ldots, n-1\}$ such that $\sigma(i) > i$. In [13] Foata and Schützenberger showed that

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{exc}(\sigma)} \Bigg|_{x=-1} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-1}{2}} E_n & \text{if } n \text{ is odd.} \end{cases} \tag{1}$$

This is a combinatorial version of a result by Euler [9]. Let $\mathfrak{D}_n \subset \mathfrak{S}_n$ be the set of derangements; then another result is that

$$\sum_{\sigma \in \mathfrak{D}_n} x^{\mathrm{exc}(\sigma)} \Bigg|_{x=-1} = \begin{cases} (-1)^{\frac{n}{2}} E_n & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{2}$$

This was first obtained by Roselle [24] using a slightly different combinatorial interpretation.

Recently Foata and Han [12] have derived q-analogs of the identities (1) and (2). The statistic involved in permutations and derangements is the major index, and the statistic involved in alternating permutations is the number of inversions.

There exist several q-analogs of the Euler numbers E_n. Firstly, we can mention the coefficients of the x-expansions of $\sin_q(x)/\cos_q(x)$ and $1/\cos_q(x)$, where $\sin_q(x)$ and $\cos_q(x)$ are the q-sine and q-cosine introduced by Jackson [18]. These are the classical q-Euler numbers, used by Foata and Han [12]. See also [2,3,11]. Secondly, there are the q-analogs defined by continued fraction expansion. Besides the approach made by Prodinger [15,21,22], there is also the q-analog introduced by Han, Randrianarivony, Zeng [17,4]. The latter is the one considered in this article and is defined as follows.

Definition 1.1. The *q-tangent numbers* $E_{2n+1}(q)$ are defined by

$$\sum_{n=0}^{\infty} E_{2n+1}(q) x^n = \cfrac{1}{1 - \cfrac{[1]_q[2]_q x}{1 - \cfrac{[2]_q[3]_q x}{1 - \cfrac{[3]_q[4]_q x}{\ddots}}}}, \tag{3}$$

where $[n]_q = \frac{1-q^n}{1-q}$, and the *$q$-secant numbers* $E_{2n}(q)$ are defined by

$$\sum_{n=0}^{\infty} E_{2n}(q) x^n = \cfrac{1}{1 - \cfrac{[1]_q^2 x}{1 - \cfrac{[2]_q^2 x}{1 - \cfrac{[3]_q^2 x}{\ddots}}}}. \tag{4}$$

The first values are $E_0(q) = E_1(q) = E_2(q) = 1$, $E_3(q) = 1 + q$, $E_4(q) = 2 + 2q + q^2$, $E_5(q) = 2+5q+5q^2+3q^3+q^4$. To compute these polynomials, we can use the continued fractions, but also some combinatorial interpretations given in [17,4]. See Remarks 2.1 and 3.3 for more precision.

The first purpose of this article is to construct other q-analogs of the combinatorial identities (1) and (2) using the above analog $E_n(q)$. To this end we make use of the notion of crossings introduced in [5]. A crossing of $\sigma \in \mathfrak{S}_n$ is a pair (i, j) such that either $i < j \le \sigma(i) < \sigma(j)$ or $\sigma(i) < \sigma(j) < i < j$. For any permutation $\sigma \in \mathfrak{S}_n$, let $\mathrm{wex}(\sigma)$ be the number of weak exceedances of σ, *i.e.* the number of integers $i \in \{1, \ldots, n\}$ such that $\sigma(i) \ge i$.

Theorem 1.2.

$$\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\mathrm{wex}(\sigma)} q^{\mathrm{cr}(\sigma)} = \begin{cases} 0 & \textit{if } n \textit{ is even,} \\ (-1)^{\frac{n+1}{2}} E_n(q) & \textit{if } n \textit{ is odd.} \end{cases} \tag{5}$$

Theorem 1.3.

$$\sum_{\sigma\in\mathfrak{D}_n}\left(-\frac{1}{q}\right)^{\mathrm{wex}(\sigma)}q^{\mathrm{cr}(\sigma)}=\begin{cases}\left(-\frac{1}{q}\right)^{\frac{n}{2}}E_n(q) & \textit{if } n \textit{ is even},\\ 0 & \textit{if } n \textit{ is odd}.\end{cases}\tag{6}$$

With $A_n(y,q)=\sum_{\sigma\in\mathfrak{S}_n}y^{\mathrm{wex}(\sigma)}q^{\mathrm{cr}(\sigma)}$ the left-hand side of (5) reads $A_n(-1,q)$. Using another combinatorial description (see Section 2), the coefficients $\hat{E}_{k,n}(q)$ such that

$$A_n(y,q)=\sum_{k=1}^{n}y^k\hat{E}_{k,n}(q),\tag{7}$$

were explicitly calculated by Williams [29, Corollary 6.3] in the form

$$\hat{E}_{k,n}(q)=\sum_{i=0}^{k-1}(-1)^i[k-i]_q^n q^{ki-k^2}\left(\binom{n}{i}q^{k-i}+\binom{n}{i-1}\right).$$

These coefficients must be regarded as q-Eulerian numbers since the preceding formula becomes the classical formula for Eulerian numbers when $q=1$.

Another expression for $A_n(y,q)$ was conjectured by Corteel and Rubey and proved in two different ways, first by the author [19] and second by Corteel, Rubey, Prellberg [6]:

$$\begin{aligned}A_n(y,q)&=\frac{1}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{n-k}y^j\left(\binom{n}{j}\binom{n}{j+k}-\binom{n}{j-1}\binom{n}{j+k+1}\right)\right)\\&\quad\times\left(\sum_{i=0}^{k}y^iq^{i(k+1-i)}\right).\end{aligned}\tag{8}$$

The two different proofs of (8) use respectively the combinatorics of permutation tableaux [26] and Laguerre histories [5]. There is no direct analytic proof that the two formulas for $A_n(y,q)$ are equal.

The second purpose of this article is to show that with similar methods, we can obtain closed formulas for $E_n(q)$ which differ from the ones coming from [17].

Theorem 1.4.

$$E_{2n+1}(q)=\frac{1}{(1-q)^{2n+1}}\sum_{k=0}^{n}\left(\binom{2n+1}{n-k}-\binom{2n+1}{n-k-1}\right)\sum_{i=0}^{2k+1}(-1)^{i+k}q^{i(2k+2-i)}.\tag{9}$$

Theorem 1.5.

$$E_{2n}(q)=\frac{1}{(1-q)^{2n}}\sum_{k=0}^{n}\left(\binom{2n}{n-k}-\binom{2n}{n-k-1}\right)\sum_{i=0}^{2k}(-1)^{i+k}q^{i(2k-i)+k}.\tag{10}$$

Eq. (14) in the article of Han, Randrianarivony, Zeng [17] is an expression for the generating function of some polynomials which generalize both $E_{2n}(q)$ and $E_{2n+1}(q)$ (see Eq. (7) in this reference). By expanding the ratios in their identity it is thus possible to derive expressions for $E_{2n}(q)$ and $E_{2n+1}(q)$; this is done explicitly in [25].

The formulas in (9) and (10) are different from the ones in [25], and one can notice similarities with the Touchard–Riordan formula [20] which gives the distribution of crossings in the set $\mathfrak{I}_{2n}\subset\mathfrak{S}_{2n}$ of fixed-point free involutions. If $\sigma\in\mathfrak{I}_{2n}$, then $\frac{1}{2}\mathrm{cr}(\sigma)$ is the number of pairs (i,j) such that $i<j<\sigma(i)<\sigma(j)$, so we recover a more classical definition of crossings. The formula is

$$\sum_{I\in\mathfrak{I}_{2n}}q^{\frac{1}{2}\mathrm{cr}(I)}=\frac{1}{(1-q)^n}\sum_{k=0}^{n}\left(\binom{2n}{n-k}-\binom{2n}{n-k-1}\right)(-1)^kq^{\frac{k(k+1)}{2}}.\tag{11}$$

Note that the left-hand side of (11) is also the sum of weights of Dyck paths such that the weight of a step $\nearrow$ is always 1, and the weight of a step $\searrow$ starting at height h is q^i for some $i \in \{0, \dots, h-1\}$. So this is a subset of the paths for $E_{2n}(q)$.

This article is organized as follows. Section 2 contains preliminaries that are mostly contained in the references. Section 3 contains the proofs of Theorems 1.2 and 1.4. Section 4 contains the proofs of Theorems 1.3 and 1.5. Section 5 contains alternative proofs of Theorems 1.4 and 1.5.

2. Definitions, conventions, preliminaries

2.1. Permutations and Laguerre histories

Throughout this article, we use the convention that $\sigma(0) = 0$ and $\sigma(n+1) = n+1$ if $\sigma \in \mathfrak{S}_n$. Let us have $i \in \{1, \dots, n\}$; it is a *weak exceedance* of σ if $\sigma(i) \geq i$ and an *ascent* of σ if $\sigma(i) < \sigma(i+1)$ (note that n is always an ascent with our conventions). We denote by $\mathrm{wex}(\sigma)$ and $\mathrm{asc}(\sigma)$ the total numbers of weak exceedances and ascents in σ. A *descent* of σ is an i such that $\sigma(i) > \sigma(i+1)$. A pattern 31-2 of σ is a triple $(i, i+1, j)$ such that $i+1 < j$ and $\sigma(i+1) < \sigma(j) < \sigma(i)$. We denote by 31-2$(\sigma)$ the total number of patterns 31-2 in σ.

A *Laguerre history* (or "histoire de Laguerre") of size n is a weighted Motzkin path of n steps such that:

- the weight of a step $\nearrow$ starting at height h is yq^i for some $i \in \{0, \dots, h\}$,
- the weight of a step $\rightarrow$ starting at height h is either yq^i for some $i \in \{0, \dots, h\}$ or q^i for some $i \in \{0, \dots, h-1\}$,
- the weight of a step $\searrow$ starting at height h is q^i for some $i \in \{0, \dots, h-1\}$.

They are in bijection with other kinds of Motzkin paths with one fewer step. A *large Laguerre history* of size n is a weighted Motzkin path of $n-1$ steps such that:

- the weight of a step $\nearrow$ starting at height h is yq^i for some $i \in \{0, \dots, h\}$,
- the weight of a step $\rightarrow$ starting at height h is either yq^i or q^i for some $i \in \{0, \dots, h\}$,
- the weight of a step $\searrow$ starting at height h is q^i for some $i \in \{0, \dots, h\}$.

There are several known bijections between $\mathfrak{S}_n$ and the set of Laguerre histories of size n. The Foata–Zeilberger bijection Ψ_{FZ} has the property that the weight of $\Psi_{FZ}(\sigma)$ is $y^{\mathrm{wex}(\sigma)}q^{\mathrm{cr}(\sigma)}$. The Françon–Viennot bijection Ψ_{FV}, first given in [14], has the property that the weight of $\Psi_{FV}(\sigma)$ is $y^{\mathrm{asc}(\sigma)}q^{31\text{-}2(\sigma)}$. All this is present in [5] and references therein, we recall here briefly the definition of Ψ_{FV} since we use it later. Let us have $\sigma \in \mathfrak{S}_n, j \in \{1, \dots, n\}$ and $k = \sigma(j)$. Then the kth step of $\Psi_{FV}(\sigma)$ is:

- a step $\nearrow$ if k is a *valley*, *i.e.* $\sigma(j-1) > \sigma(j) < \sigma(j+1)$,
- a step $\searrow$ if k is a *peak*, *i.e.* $\sigma(j-1) < \sigma(j) > \sigma(j+1)$,
- a step $\rightarrow$ if k is a *double ascent*, *i.e.* $\sigma(j-1) < \sigma(j) < \sigma(j+1)$, or a *double descent*, *i.e.* $\sigma(j-1) > \sigma(j) > \sigma(j+1)$.

Moreover the weight of the kth step is $y^\delta q^i$ where $\delta = 1$ if j is an ascent and 0 otherwise, and i is the number of $u \in \{1, \dots, j-2\}$ such that $\sigma(u) > \sigma(j) > \sigma(u+1)$ (*i.e.* the number of patterns 31-2 such that j correspond to the 2). See Fig. 1 for an example.

The continued fractions in Definition 1.1 are the natural q-analogs of the ones appearing in Flajolet's celebrated article [10]. The methods in this reference give a combinatorial interpretation of $E_n(q)$ in terms of weighted Dyck paths. Indeed, if $\delta \in \{0, 1\}$ then $E_{2n+\delta}(q)$ is the sum of weights of Dyck paths of length $2n$ such that:

- the weight of a step $\nearrow$ starting at height h is q^i for some $i \in \{0, \dots, h\}$,
- the weight of a step $\searrow$ starting at height h is q^i for some $i \in \{0, \dots, h-1+\delta\}$,

and the weight of the path is the product of the weights of its steps.

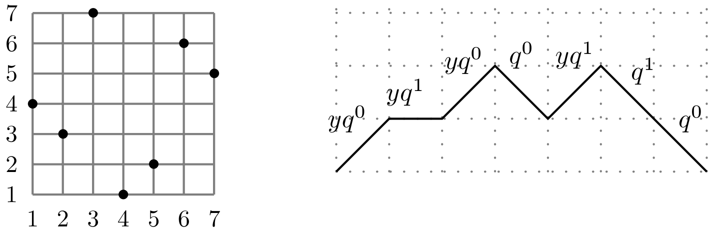

Fig. 1. Example of the permutation 4 371 265 and its image under the Françon–Viennot bijection.

The set of alternating permutations $\mathfrak{A}_n$ consists of permutations $\sigma \in \mathfrak{S}_n$ such that $\sigma(2i-1) > \sigma(2i) < \sigma(2i+1)$ for any $i \in \left\{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$. When n is even, we have $\sigma \in \mathfrak{A}_n$ if and only if σ has no double descent, and no double ascent. When n is odd, we have $\sigma \in \mathfrak{A}_n$ if and only if σ has no double descent, and only one double ascent at position n. Another possible definition is that σ is alternating if $\sigma(1) < \sigma(2) > \sigma(3) < \cdots$, but with the convention that $\sigma(0) = 0$ and $\sigma(n+1) = n+1$ the present definition is more natural.

Remark 2.1. When n is even, the fact that $\sigma \in \mathfrak{A}_n$ if and only if σ has no double descent and no double ascent implies a known combinatorial interpretation [4,17]:

$$E_n(q) = \sum_{\sigma \in \mathfrak{A}_n} q^{\text{31-2}(\sigma)}. \tag{12}$$

Indeed, if σ is alternating then the corresponding Laguerre history $\Psi_{FV}(\sigma)$ has no horizontal step, and when $y = 1$ these Laguerre histories are precisely the weighted Dyck paths corresponding to the continued fraction (4) defining q-secant numbers. This interpretation (12) was first given by Han, Randrianarivony, and Zeng (see Eq. (42) in [17]). The number of patterns 2-13 is called the right-embracing number in this reference, and clearly the number of 2-13 on $\mathfrak{A}_n$ is equidistributed with the number of 31-2. The same result was proved by Chebikin [4] with a refinement of the Françon–Viennot bijection. For the case when n is odd, see Remark 3.3.

2.2. Eulerian polynomials and their q-analogs

We define

$$A_n(y,q) = \sum_{\sigma \in \mathfrak{S}_n} y^{\operatorname{wex}(\sigma)} q^{\operatorname{cr}(\sigma)} \quad \text{and} \quad B_n(y,q) = \sum_{\sigma \in \mathfrak{D}_n} y^{\operatorname{wex}(\sigma)} q^{\operatorname{cr}(\sigma)}, \tag{13}$$

so the left-hand sides of (5) and (6) are respectively $A_n(-1,q)$ and $B_n\left(-\frac{1}{q}, q\right)$. From the bijections Ψ_{FV} and Ψ_{FZ} mentioned above, we know that the bistatistic (wex, cr) is equidistributed with (asc, 31-2) in $\mathfrak{S}_n$ (but not in $\mathfrak{D}_n$). It is elementary to show that $\sum_{\sigma \in \mathfrak{S}_n} y^{\operatorname{wex}(\sigma)} = y \sum_{\sigma \in \mathfrak{S}_n} y^{\operatorname{exc}(\sigma)}$, and $A_n(y,q)$ could be defined in terms of the statistic exc with another definition of crossings. The distinction is not relevant for derangements.

Proposition 2.2. *We have inversion formulas:*

$$A_n(y,q) = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} B_k(y,q) \quad \textit{and} \quad B_n(y,q) = \sum_{k=0}^{n} \binom{n}{k} (-y)^{n-k} A_k(y,q). \tag{14}$$

Proof. This is perhaps the most common application of the inclusion–exclusion principle. To get this refined version, we have to check how the statistics are changed when adding a fixed point to a permutation. The number of crossings does not change, and the number of weak exceedances increases by 1 due to the added fixed point. This explains the powers of y in the formulas. $\square$

From the formula for $A_n(y,q)$ given in Eq. (8) and the previous proposition, we obtain that

$$B_n(y,q) = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\sum_{j=0}^{n-k} y^j C(n,k,j) \right) \left(\sum_{i=0}^{k} y^i q^{i(k+1-i)} \right), \tag{15}$$

$$\text{where } C(n,k,j) = \binom{n}{j} \sum_{i=0}^{j} q^{j-i} \binom{j}{i} \binom{n-j}{i+k} - \binom{n}{j-1} \sum_{i=0}^{j} q^{j-i} \binom{j-1}{i-1} \binom{n-j+1}{i+k+1}.$$

Unfortunately it seems that there is no further simplification; it is just a straightforward rearrangement of $\sum_{k=0}^{n} \binom{n}{k} (-y)^{n-k} A_k(y,q)$. We omit details. Moreover the two different proofs of (8) in [19,6] can be modified so as to give a direct derivation of (15).

2.3. Permutation tableaux

Let λ be a Young diagram in English notation, eventually with empty rows. A *permutation tableau* T of shape λ is a filling of λ with 0s and 1s such that there is at least a 1 in each column, and for any 0 either all entries to its left are also 0s or all entries above are also 0s. We refer the reader to [26] for more details. We denote by $r(T)$ the number of rows of T, by $c(T)$ the number of columns, by $o(T)$ the number of 1s, by $so(T) = o(T) - c(T)$ the number of *superfluous* 1s (*i.e.* all 1s except the topmost of each column).

Via the Steingrímsson–Williams bijection [26], the number of fixed points (respectively of weak exceedances, of crossings) in a permutation is the number of zero-rows (respectively of rows, of superfluous 1s) in the corresponding permutation tableau. Let PT_n be the set of permutation tableaux of half-perimeter n, and $DT_n \subset PT_n$ the subset of permutation tableaux with no zero-row (that we call *derangement tableaux*). Then we have

$$A_n(y,q) = \sum_{T \in PT_n} y^{r(T)} q^{so(T)}, \qquad B_n(y,q) = \sum_{T \in DT_n} y^{r(T)} q^{so(T)}, \tag{16}$$

$$B_n\left(-\frac{1}{q}, q\right) = \sum_{T \in DT_n} (-1)^{r(T)} q^{o(T)-n}. \tag{17}$$

Note that the combinatorial interpretation of $A_n(y,q)$ in terms of permutation tableaux explains the link (7) between our definition (13) and the numbers $\hat{E}_{k,n}(q)$ from [29].

3. Eulerian numbers and q-tangent numbers

First we give a characterization of Laguerre histories corresponding to permutations $\sigma \in \mathfrak{S}_{n+1}$ such that $\sigma(n+1) = 1$. It is particularly similar to Viennot's criterion [28, Chapter II, page 24], but since the settings are different we believe that it is worth writing a direct proof.

Lemma 3.1. *Let us have $\sigma \in \mathfrak{S}_{n+1}$ and H its image under the Françon–Viennot bijection as described in Section 2. Then the following conditions are equivalent:*

- $\sigma(n+1) = 1$.
- *Except the first step, H has no step starting at height h with weight yq^h.*

In particular, if this condition is true there is no return to height 0 before the last step.

Proof. To begin, suppose that $\sigma(n+1) \neq 1$. Let k be the minimal integer such that $\sigma^{-1}(k) > \sigma^{-1}(1)$, and h the starting height of the kth step in $\Psi_{FV}(\sigma)$. We will show that the weight of this kth step is yq^h.

Let $j = \sigma^{-1}(k)$. We have $\sigma(j) < \sigma(j+1)$; otherwise the inequalities $\sigma(j+1) < \sigma(j) = k$ and $j+1 > 1$ would imply that the integer $k' = \sigma(j+1)$ contradicts the minimality of k. This shows that j is an ascent, so the kth step in $\Psi_{FV}(\sigma)$ is either $\rightarrow$ or $\nearrow$ and has weight yq^i for some $i \in \{0, \dots, h\}$.

We have to show that $i = h$. By definition of Ψ_{FV}, this means that we have to find distinct integers $u_1, \dots, u_h \in \{1, \dots, j-2\}$ such that $\sigma(u_l) > \sigma(j) > \sigma(u_l+1)$ for any $l \in \{1, \dots, h\}$.

Let $a < k$ be such that the ath step of $\Psi_{FV}(\sigma)$ is a step $\nearrow$. Let $c = \sigma^{-1}(a)$ and b the minimum integer such that $\sigma(b) > \sigma(b+1) > \cdots > \sigma(c)$. We have $b < c < j$. We distinguish two cases.

- If $\sigma(b) > k$, then there is $u \in \{b, \dots, c-1\}$ such that $\sigma(u) > k > \sigma(u+1)$. So we have found one of the u_l.
- Otherwise, we have $\sigma(b-1) < \sigma(b) > \sigma(b+1)$ and $\sigma(b) < k$. So the $\sigma(b)$th step in $\Psi_{FV}(\sigma)$ is a step $\searrow$ before the kth step.

Thus, with each step $\nearrow$ among the first $k-1$ ones in $\Psi_{FV}(\sigma)$, we can associate either a u_l satisfying the desired properties, or a step $\searrow$ among the first $k-1$ ones. This is indeed sufficient for finding the $u_1, \dots, u_h$ and this completes the first part of the proof. We can use similar arguments to prove the reverse implication. $\square$

Remark 3.2. It would be possible to consider two other parameters counting the right-to-left minima and the left-to-right maxima. Indeed, they respectively correspond to steps starting at height h with weight yq^h, and to steps $\rightarrow$ with weight yq^0 or $\searrow$ with weight q^0. In this broader context the previous lemma is immediate because permutations satisfying $\sigma(n+1) = 1$ are exactly the ones with only one right-to-left minima.

We now give the proof of Theorem 1.2. Shin and Zeng [25] gave other proofs of this theorem in the spirit of [12].

Proof. For any $\sigma \in \mathfrak{S}_n$, we define $\tilde{\sigma} \in \mathfrak{S}_{n+1}$ by $\tilde{\sigma}(i) = \sigma(i) + 1$ if $i \in \{1, \dots, n\}$ and $\tilde{\sigma}(n+1) = 1$. It is clear that $\mathrm{asc}(\sigma) = \mathrm{asc}(\tilde{\sigma})$ and $31\text{-}2(\sigma) = 31\text{-}2(\tilde{\sigma})$.

Then for any $\sigma \in \mathfrak{S}_n$, Let $f(\sigma) = \Psi_{FV}(\tilde{\sigma})$. From the previous lemma, f is a bijection between $\mathfrak{S}_n$ and the set of Laguerre histories of $n+1$ steps such that:

- The weight of a horizontal step at height h is q^i or yq^i for some $i \in \{0, \dots, h\}$.
- Except for the first step which has weight yq^0, the weight of a step $\nearrow$ (resp. $\searrow$) starting at height h is yq^i (resp. q^i) for some $i \in \{0, \dots, h-1\}$.

Moreover the weight of $f(\sigma)$ is $y^{\mathrm{asc}(\sigma)} q^{31\text{-}2(\sigma)}$.

In such a Laguerre history, we can remove the first and last steps, and obtain a large Laguerre history of $n-1$ steps with respect to the shifted origin (this is not weight-preserving; there is a factor y coming from the first step). In the large Laguerre history, when $y = -1$ the weights on horizontal steps cancel each other, and so $A_n(-1, q)$ is a sum of weights of Dyck paths of length $n-1$. When n is even, there is no such Dyck path, so $A_n(-1, q) = 0$.

When n is odd, each Dyck path has $\frac{n-1}{2}$ steps $\nearrow$. So we can factorize the sum by $y^{\frac{n-1}{2}} = (-1)^{\frac{n-1}{2}}$, and the remaining weighted Dyck paths are precisely the ones given by the combinatorial interpretation of the q-tangent numbers. So, taking into account the factor y coming from the removed step, in this case we have $A_n(-1, q) = (-1)^{\frac{n+1}{2}} E_n(q)$. $\square$

Remark 3.3. Through the previous proof we have seen a bijection between permutations and large Laguerre histories. We easily see that when n is odd, $\sigma \in \mathfrak{S}_n$ is alternating if and only if $\tilde{\sigma}$ is alternating, and this is equivalent to the fact that $f(\sigma)$ has no horizontal step. When $y = 1$ we obtain exactly the weighted Dyck paths corresponding to the continued fraction (3) defining the q-tangent numbers, so this proves the combinatorial interpretation (12) when n is odd. This was also obtained by Chebikin [4]. Han, Randrianarivony, Zeng [17] also gave a combinatorial interpretation of $E_n(q)$ when n is odd, but it differs from the one we have here.

Now we prove Theorem 1.4.

Proof. From Theorem 1.2 we have $E_{2n+1}(q) = (-1)^{n+1} A_{2n+1}(-1, q)$. There are interesting cancellations if we directly set $y = -1$ in the proofs of (8). However, since it is not particularly useful to rewrite these proofs with $y = -1$, we only show how to simplify $A_{2n+1}(-1, q)$ and get the right-hand side of (9). From (8) we have

$$A_{2n+1}(-1,q) = \frac{1}{(1-q)^{2n+1}} \sum_{k=0}^{2n+1} (-1)^k \left(g(n,k) + g(n,k+2)\right) \sum_{i=0}^{k} (-1)^i q^{i(k+1-i)}, \tag{18}$$

where $g(n, k)$ is the sum

$$g(n,k) = \sum_{j=0}^{2n+1-k} (-1)^j \binom{2n+1}{j} \binom{2n+1}{j+k}.$$

The main step is to simplify this sum. The first term is $\binom{2n+1}{k}$ and the quotient of two successive terms is

$$\frac{(-1)^{j+1}\binom{2n+1}{j+1}\binom{2n+1}{j+1+k}}{(-1)^j\binom{2n+1}{j}\binom{2n+1}{j+k}} = -\frac{(2n+1-j)(2n+1-k-j)}{(j+1)(j+1+k)}$$
$$= -\frac{(-2n-1+j)(-2n-1+k+j)}{(1+j)(k+1+j)},$$

so $g(n, k)$ is the hypergeometric series $\binom{2n+1}{k}\,{}_2F_1(-2n-1+k, -2n-1; k+1; -1)$. We can use Kummer's summation formula [16, Chapter 1], which reads

$${}_2F_1(a,b;1+a-b;-1) = \frac{\Gamma(1+a-b)\Gamma\left(1+\frac{a}{2}\right)}{\Gamma(1+a)\Gamma\left(-b+1+\frac{a}{2}\right)}.$$

We cannot rigorously apply the summation formula here because the function Γ is singular at non-positive integers, but we can handle this since $\Gamma(-m+\epsilon) \sim \frac{(-1)^m}{m!\epsilon}$ for any $m \in \mathbb{N}$ and small ϵ. We write formally

$$g(n,k) = \frac{\Gamma(2n+2)}{\Gamma(k+1)\Gamma(2n+2-k)} \cdot \frac{\Gamma(k+1)\Gamma\left(-n+\frac{1}{2}+\frac{k}{2}\right)}{\Gamma(k-2n)\Gamma\left(n+\frac{3}{2}+\frac{k}{2}\right)}.$$

When k is even, the only singular value is $\Gamma(k-2n)$ in the denominator so $g(n, k) = 0$. When k is odd, the singular values compensate:

$$\lim_{\epsilon\to 0} \frac{\Gamma\left(-n+\frac{1}{2}+\frac{k}{2}+\epsilon\right)}{\Gamma(k-2n+2\epsilon)} = 2(-1)^{n+\frac{k-1}{2}} \frac{\Gamma(2n+1-k)}{\Gamma\left(n+\frac{1}{2}-\frac{k}{2}\right)}.$$

After some simplification it emerges that when n is odd,

$$g(n,k) = \frac{2(-1)^{n+\frac{k-1}{2}}\Gamma(2n+2)}{(2n+1-k)\Gamma\left(n+\frac{1}{2}-\frac{k}{2}\right)\Gamma\left(n+\frac{3}{2}+\frac{k}{2}\right)} = (-1)^{n+\frac{k-1}{2}} \binom{2n+1}{n-\frac{k-1}{2}}.$$

Going back to (18), we can restrict the sum to odd k and reindex it so that k becomes $2k + 1$. Since $A_{2n+1}(-1, q) = (-1)^{n+1}E_{2n+1}(q)$ this gives the formula claimed in Theorem 1.4. □

4. Derangements and *q*-secant numbers

First we give the proof of Theorem 1.3. As in the case of Theorem 1.2, others proofs are as given in [25].

Proof. The image of a derangement by the Foata–Zeilberger bijection (as defined in [5]) is a Laguerre history such that there is no horizontal step with weight yq^0. This just means that a fixed point cannot be part of a crossing in a permutation. So $B_n(y,q)$ is the sum of weights of Motzkin paths of length n such that:

- the weight of a step $\nearrow$ starting at height h is $y[h+1]_q$,
- the weight of a step $\searrow$ starting at height h is $[h]_q$,
- the weight of a step $\rightarrow$ starting at height h is $(1+yq)[h]_q$.

When $y=-1/q$, the weights cancel each other on the horizontal steps. So in this case we can restrict the sum to Dyck paths instead of Motzkin paths, and the end of the proof is similar to that of Theorem 1.2. If n is odd, there is no Dyck path of length n so $B_n\left(-\frac{1}{q},q\right)=0$.

When n is even, each Dyck path has $\frac{n}{2}$ steps $\nearrow$. So we can factorize the sum by $y^{\frac{n}{2}}=\left(-\frac{1}{q}\right)^{\frac{n}{2}}$, and the remaining weighted Dyck paths are precisely the ones given by the combinatorial interpretation of the q-secant numbers. So in this case we have $B_n\left(-\frac{1}{q},q\right)=\left(-\frac{1}{q}\right)^{\frac{n}{2}}E_n(q)$. $\square$

The fact that $B_n\left(-\frac{1}{q},q\right)=0$ when n is odd is also a consequence of (17), because the transposition of derangement tableaux changes the parity of the number of rows and does not change the number of 1s.

To prove Theorem 1.5, it should be possible to use (15) and specialize at $y=-\frac{1}{q}$. However it seems simpler to prove it directly with the methods and previous results of [19], where the first proof of (8) was given.

A method for computing $A_n(y,q)$ is using a Matrix Ansatz. This is a consequence of the combinatorial interpretation of the PASEP partition function, given by Corteel and Williams in terms of permutation tableaux [7]. There is a similar result for computing $B_n(y,q)$.

Proposition 4.1. *Suppose that we have matrices D, E, a row vector $\langle W|$, and a column vector $|V\rangle$ such that*

$$DE-qED=I+qE+D,\qquad \langle W|E=0,\qquad D|V\rangle=0,\qquad \langle W|V\rangle=1,\tag{19}$$

where I is the identity matrix. Then we have

$$B_n(y,q)=\langle W|(yD+E)^n|V\rangle.$$

This is to be understood as follows. Via the commutation relation, $(yD+E)^n$ can be uniquely written as a linear combination $\sum_{i,j\ge 0}c_{ij}E^iD^j$, and then $B_n(y,q)$ is the constant term c_{00}. We give two proofs of this.

Proof. The first proof relies on the combinatorial interpretation of the three-parameter partition function of the PASEP [7]. If we have

$$D'E'-qE'D'=D'+E',\qquad \langle W|E'=0,\qquad D'|V\rangle=|V\rangle,$$

then by [7, Theorem 3.1], $\langle W|(yD'+E')^n|V\rangle$ is the generating function of permutation tableaux of half-perimeter $n+1$, with no 1 in the first row, where y counts the number of rows minus 1, and q counts the number of superfluous 1s. In such permutation tableaux, we can remove the first row (which is filled with 0s) and obtain any permutation tableaux of half-perimeter n. So we get

$$A_n(y,q)=\langle W|(yD'+E')^n|V\rangle.$$

Together with Proposition 2.2, this gives

$$B_n(y,q)=\langle W|(yD'-yI+E')^n|V\rangle.$$

But it is straightforward to check that $D=D'-I$ and $E=E'$ satisfy conditions (19). $\square$

Proof. In the second proof, we sketch how a recursive enumeration of derangement tableaux directly leads to the matrix ansatz (19). The method is the same as in [19, Section 2], and also [27]. For any word w of size n in D and E, we define a Young diagram $\lambda(w)$ by the following rule: the south-east boundary of $\lambda(w)$ is obtained by reading w from left to right, and drawing a step east for each letter D in w, and drawing a step north for each letter E in w. Let $T_w = \sum_T q^{\mathrm{so}(T)}$ where the sum is over derangement tableaux of shape $\lambda(w)$. Then we have

$$T_{w_1DEw_2} = qT_{w_1EDw_2} + T_{w_1w_2} + qT_{w_1Ew_2} + T_{w_1Dw_2}, \quad \text{and} \quad T_{Ew} = T_{wD} = 0.$$

This is the same kind of relation as the one given by Williams for Ⅎ-diagrams [29]. It is obtained by examining a particular corner of the Young diagram, and distinguishing four different cases whether this corner contains a topmost or non-topmost 1, and a leftmost or non-leftmost 1. We can translate these relations in terms of operators D and E satisfying conditions (19), and such that $T_w = \langle W|w|V\rangle$. Since $(yD+E)^n$ expands into a (weighted) sum of all words in D and E we get all possible shapes for a derangement tableau, so $\langle W|(yD+E)^n|V\rangle$ is the generating function $\sum_w y^{w_D}T_w = B_n(y,q)$, where w_D is the number of letters D in the word w. See the references for more details about the method. $\square$

Now we prove Theorem 1.5.

Proof. We define

$$\hat{D} = \frac{q-1}{q}\left(D + \frac{q}{q-1}\right), \qquad \hat{E} = (q-1)\left(E + \frac{1}{q-1}\right)$$

so an immediate computation gives

$$\hat{D}\hat{E} - q\hat{E}\hat{D} = \frac{1-q}{q}, \qquad \langle W|\hat{E} = \langle W|, \qquad \hat{D}|V\rangle = |V\rangle,$$

and

$$\left(-\frac{1}{q}D + E\right)^n = \frac{1}{(q-1)^n}(-\hat{D}+\hat{E})^n.$$

With the results of [19], it is known that $\langle W|(-\hat{D}+\hat{E})^n|V\rangle$ is a generating function for weighted involutions and has the following expression:

$$\langle W|(-\hat{D}+\hat{E})^n|V\rangle = \sum_{0\le k\le j\le n} (-1)^j \left(\binom{n}{k} - \binom{n}{k-1}\right) q^{(j-k)(n-j-k)-k}.$$

Since $E_{2n}(q) = (-q)^n B_{2n}\left(-\frac{1}{q}, q\right)$, we obtain

$$E_{2n}(q) = \frac{1}{(1-q)^{2n}} \sum_{0\le k\le j\le 2n} (-1)^{n+j} \left(\binom{2n}{k} - \binom{2n}{k-1}\right) q^{(j-k)(2n-j-k)+n-k}.$$

Discarding some negative powers of q, we can restrict the sum to k such that $k \le n$, and then substitute k with $n-k$, which gives

$$E_{2n}(q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \sum_{j=n-k}^{2n} (-1)^{n+j} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right) q^{(j+k-n)(n+k-j)+k}.$$

Then we substitute j with $n-k+j$ and obtain

$$E_{2n}(q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \sum_{j=0}^{n+k} (-1)^{k+j} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right) q^{j(2k-j)+k}.$$

Discarding other negative powers of q, we can restrict the sum to j such that $j \le 2k$ and this gives the desired expression. $\square$

Remark 4.2. The idea of introducing the operators $\hat{D}$ and $\hat{E}$ and their combinatorial interpretation is at the origin of one of the proof of (8). Similarly, the computation above can be refined to obtain another proof of (15).

5. The second proof of the *q*-secant and *q*-tangent formulas

A second proof of (8), inspired by Penaud's bijective proof of the Touchard–Riordan formula [20], was given in [6]. Actually, a slight modification of the method can give a direct proof of (15). In this section, we show that this can be used to obtain Theorems 1.4 and 1.5 directly from the definition of $E_n(q)$ in terms of weighted Dyck paths. We begin with the q-secant numbers because this case is simpler.

5.1. The q-secant numbers

First, we notice that $(1-q)^{2n}E_{2n}(q)$ is the sum of weights of Dyck paths H of length $2n$ such that:

- the weight of a step $\nearrow$ starting at height h is 1 or $-q^{h+1}$,
- the weight of a step $\searrow$ starting at height h is 1 or $-q^h$.

Proposition 5.1. *There is a weight-preserving bijection between these Dyck paths H and pairs (H_1, H_2) such that for some $k \in \{0, \dots, n\}$,*

- *H_1 is a left factor of a Dyck path, of $2n$ steps and final height $2k$,*
- *H_2 is a weighted Dyck path of length $2k$, with the same weights as H and also the condition that there are no two consecutive steps $\nearrow\searrow$ both weighted by 1.*

Proof. This is a direct adaptation of [6, Lemma 1] so we only sketch the proof. The idea is to look for the maximal factors of H being Dyck paths and having only steps with weight 1. To obtain H_1 from H we transform into a step $\nearrow$ any step which is not inside one of these maximal factors. And to obtain H_2 from H we just remove these maximal factors. □

By an elementary recurrence, the number of left factors of Dyck paths of $2n$ steps and final height $2k$ is $\binom{2n}{n-k} - \binom{2n}{n-k-1}$. So to obtain Theorem 1.5 it just remains to prove the following proposition.

Proposition 5.2. *The sum of weights of Dyck paths of length $2k$, satisfying conditions as H_2 above, is*

$$M_k(q) = \sum_{j=0}^{2k} (-1)^{j+k} q^{j(2k-j)+k}.$$

Proof. It is possible to adapt the proof of [6, Proposition 5]. However we take a slightly different point of view here, and show that we can exploit some properties of T-fractions. The combinatorial theory for T-fractions was developed by Roblet and Viennot in [23], and gives generating functions for the kinds of paths we are dealing with here. Indeed, this reference gives immediately that

$$\sum_{k=0}^{\infty} M_k(q)t^k = \cfrac{1}{1+t-\cfrac{(1-q)^2t}{1+t-\cfrac{(1-q^2)^2t}{\ddots}}}. \tag{20}$$

To make this article more self-contained, we briefly show how to obtain (20). Recall that a Schröder path of length n is a path starting at $(0,0)$, ending at $(n,0)$, with steps $(1,1)$, $(1,-1)$ and $(2,0)$, and remaining above the horizontal axis. They are similar to Dyck paths except that we authorize the horizontal step $(2,0)$ (note that the length of the step does not determine the number of steps). Let us consider weighted Schröder paths, such that:

- the weight of a step $\nearrow$ starting at height h is 1 or $-q^{h+1}$,
- the weight of a step $\searrow$ starting at height h is 1 or $-q^h$,
- the weight of a step $\longrightarrow$ is -1.

The sum of weights of these Schröder paths of length $2k$ is $M_k(q)$. Indeed there is a sign-reversing involution on Schröder paths such that the fixed points are the Dyck paths with no two consecutive steps $\nearrow\searrow$ both weighted by 1. The involution is obtained by exchanging the first occurrence of $\longrightarrow$ with weight -1, or $\nearrow\searrow$ with weight 1, with (respectively) $\nearrow\searrow$ with weight 1, or $\longrightarrow$ with weight -1. For the generating function of Schröder paths, standard arguments [10] give the continued fraction in (20).

We need to extract the coefficients of t^k in the continued fraction. To do this, there are well-established methods linking such continued fractions with quotients of contiguous basic hypergeometric series [8, Chapter 19]. In the present case, we can use a limit case of (19.2.11a) in [8]. If we consider the more general continued fraction

$$M(z) = \cfrac{1}{1+t-\cfrac{(1-qz)^2t}{1+t-\cfrac{(1-q^2z)^2t}{\ddots}}},$$

then [8, (19.2.11a)] with $c = tq$ and $a = -b = \mathrm{i}\sqrt{qt}$ gives that

$$M(z) = \frac{1}{1-z}\cdot\frac{{}_2\phi_1(a,b;ab;q;qz)}{{}_2\phi_1(a,b;ab;q,z)}.$$

It is the same series as in [6] but with different values for a and b, so we proceed similarly with a Heine transformation [16, Chapter 1]:

$${}_2\phi_1(a,b;ab;q,z) = \frac{(az,b;q)_\infty}{(ab,z;q)_\infty}{}_2\phi_1(a,z;az;q,b).$$

So

$$M(z) = \frac{(aqz,b,ab,z;q)_\infty}{(1-z)(ab,qz,az,b;q)_\infty}\cdot\frac{{}_2\phi_1(a,qz;aqz;q,b)}{{}_2\phi_1(a,z;az;q,b)} = \frac{1}{1-az}\cdot\frac{{}_2\phi_1(a,qz;aqz;q,b)}{{}_2\phi_1(a,z;az;q,b)},$$

and

$$\begin{aligned} M(1) &= \frac{1}{1-a}{}_2\phi_1(a,q;aq;q,b) = \sum_{n=0}^{\infty}\frac{b^n}{1-aq^n} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} b^n a^j q^{jn} = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} t^{\frac{n+j}{2}}(-\mathrm{i})^n \mathrm{i}^j q^{jn+\frac{n+j}{2}}. \end{aligned}$$

To obtain the coefficient of t^k in $M(1)$, we just have to restrict the sum to the pairs (n, j) such that $n = 2k - j$. This gives the desired formula. $\square$

5.2. *The q-tangent numbers*

To prove Theorem 1.5 with the same method, we first need to slightly modify our expression for $E_{2n+1}(q)$. Let P_k be the polynomial $\sum_{j=0}^{2k+1}(-1)^{j+k}q^{j(2k+2-j)}$. By elementary properties of the binomial coefficients we have

$$E_{2n+1}(q) = \frac{\sum_{k=0}^{n}\left(\binom{2n+1}{n-k}-\binom{2n+1}{n-k-1}\right)P_k}{(1-q)^{2n+1}} = \frac{\sum_{k=0}^{n}\left(\binom{2n}{n-k}-\binom{2n}{n-k-1}\right)\frac{P_k+P_{k-1}}{1-q}}{(1-q)^{2n}}.$$

The latter expression is the one that we can prove with the previous method. We decompose the weighted paths exactly as in the case of q-secant numbers, *i.e.* as in Proposition 5.1. The sole difference is treated in the following proposition, which is therefore the last step of the second proof of Theorem 1.5.

Proposition 5.3. *The sum of weights of Dyck paths of length* $2k$, *with weights* 1 *or* $-q^{h+1}$ *for any step* ($\nearrow$ *or* $\searrow$) *starting at height* h, *and no two consecutive steps* $\nearrow\searrow$ *both weighted by* 1, *is*

$$N_k(q) = \frac{P_k + P_{k-1}}{1-q}.$$

Proof. We follow the scheme of Proposition 5.2. The Schröder paths in this case give the following T-fraction:

$$\sum_{k=0}^{\infty} N_k(q)t^k = \cfrac{1}{1+t-\cfrac{(1-q)(1-q^2)t}{1+t-\cfrac{(1-q^2)(1-q^3)t}{\ddots}}}. \tag{21}$$

We need to consider the more general continued fraction

$$N(z) = \cfrac{1}{1+t-\cfrac{(1-qz)(1-q^2z)t}{1+t-\cfrac{(1-q^2z)(1-q^3z)t}{\ddots}}}.$$

And then [8, (19.2.11a)] with $c = tq$ and $a = -b = \mathrm{i}\sqrt{tq}$ gives that

$$N(z) = \frac{1}{1-z} \cdot \frac{{}_2\phi_1(a, b; tq; q; qz)}{{}_2\phi_1(a, b; tq; q, z)}.$$

In this case Heine transformation gives

$${}_2\phi_1(a, b; tq; q, z) = \frac{(-\mathrm{i}\sqrt{t}q, \mathrm{i}\sqrt{t}qz; q)_\infty}{(z, tq; q)_\infty} {}_2\phi_1(\mathrm{i}\sqrt{t}, z; \mathrm{i}\sqrt{t}qz; q; -\mathrm{i}\sqrt{t}q),$$

and hence, after some simplification,

$$N(1) = \frac{1}{1-\mathrm{i}\sqrt{t}q} {}_2\phi_1(\mathrm{i}\sqrt{t}, q; \mathrm{i}\sqrt{t}q^2; q; -\mathrm{i}\sqrt{t}q) = \sum_{n=0}^{\infty} \frac{(1-\mathrm{i}\sqrt{t})(-\mathrm{i}\sqrt{t}q)^n}{(1-\mathrm{i}\sqrt{t}q^n)(1-\mathrm{i}\sqrt{t}q^{n+1})}.$$

The usual method for reducing degrees in denominators leads to

$$(1-q)N(1) = -\sum_{n=0}^{\infty} \frac{1-\mathrm{i}\sqrt{t}}{1-\mathrm{i}\sqrt{t}q^n}(-\mathrm{i}\sqrt{t})^{n-1} + \sum_{n=0}^{\infty} \frac{1-\mathrm{i}\sqrt{t}}{1-\mathrm{i}\sqrt{t}q^{n+1}}(-\mathrm{i}\sqrt{t})^{n-1}.$$

Some terms in this sum cancel each other and there remains

$$(1-q)N(1) = -\sum_{n=0}^{\infty} \frac{1}{1-\mathrm{i}\sqrt{t}q^n}(-\mathrm{i}\sqrt{t})^n + \sum_{n=0}^{\infty} \frac{1}{1-\mathrm{i}\sqrt{t}q^{n+1}}(-\mathrm{i}\sqrt{t})^{n-1} + \frac{(\mathrm{i}\sqrt{t})^{-1}}{1-\mathrm{i}\sqrt{t}}.$$

At this point we can expand the fractions as in the previous case, and we readily obtain that the coefficient of t^k in $(1-q)N(1)$ is $P_k + P_{k-1}$. □

6. Final remarks

It might be possible to give a combinatorial proof of Proposition 5.2 or 5.3, with a sign-reversing involution whose fixed points account for the terms in the right-hand side (as was done for the Touchard–Riordan formula in [20]). In either case, we could not even conjecture a set of possible fixed points for such an involution.

There is a parity-independent formula for the classical Euler numbers $E_n = E_n(1)$ given by Shin and Zeng [25]. They also show that their formula has an analog that gives $E_n(q)$. One might ask whether there is a parity-independent formula for $E_n(q)$ in the style of our Theorems 1.4 and 1.5. Actually from our results we get that for any even or odd n,

$$(-1)^{\frac{n-1}{2}} A_n(-1,q) = \frac{(-1)^{\lfloor n/2 \rfloor}}{(1-q)^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n}{k} - \binom{n}{k-1} \right) \sum_{i=0}^{n-2k} (-1)^{k+i} q^{i(n-2k-i)+i}$$

(when n is even the right-hand side is 0 because $\sum_{i=0}^{n-2k} (-1)^{i+k} q^{i(n+1-2k-i)} = (-1)^{n-k}$ and $\sum_{k=0}^{n} (-1)^k \left(\binom{2n}{k} - \binom{2n}{k-1} \right) = 0$), and also

$$(-1)^{\frac{n}{2}} B_n\left(-\frac{1}{q}, q\right) = \frac{(-1)^{\lfloor n/2 \rfloor}}{(1-q)^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n}{k} - \binom{n}{k-1} \right) \sum_{i=0}^{n-2k} (-1)^{k+i} q^{i(n-2k-i)+\frac{n}{2}-k}$$

(when n is odd the right-hand side is 0 because $\sum_{i=0}^{n-2k} (-1)^{i+k} q^{i(n-2k-i)} = 0$), and adding the previous two identities gives

$$E_n(q) = \frac{(-1)^{\lfloor n/2 \rfloor}}{(1-q)^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n}{k} - \binom{n}{k-1} \right) \sum_{i=0}^{n-2k} (-1)^{k+i} q^{i(n-2k-i)} \left(q^i + q^{\frac{n}{2}-k} \right).$$

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