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Algebraic computations in derived categories

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Abstract

This paper presents explicit algorithms for computations over a finite subspectroid of the bounded derived category of a finite spectroid. We will demonstrate methods for the construction of a projective resolution of a module and for finding the quiver of a finite spectroid given in terms of its radical spaces. This enables us to compute the endomorphism algebra of a tilting complex – or, in fact, any finite complex – in the derived category. In order to carry out these computations, we have to restrict to a finite base field or the field of rational numbers. We will show that it is possible to transfer the results to any extension of the base field, in particular to the algebraic closure.

Keywords: Auslander–Reiten quiver; Spectroid; Derived category; Tilting complex; Projective resolution; Algebraic computations; Gröbner bases

1. Introduction

Let k be a field and let $A = kQ/I$ be a finite-dimensional k-algebra with quiver Q and ideal of relations I. We are interested in the bounded derived category $\mathscr{D}^b(A)$ of A and derived equivalences of algebras.

By Rickard [7], if T is a tilting complex in $\mathscr{D}^b(A)$ then the endomorphism algebra of T is derived equivalent to A. This paper is devoted to a presentation of explicit algorithms for the computation of a finite subspectroid of $\mathscr{D}^b(A)$. Based on the concept of noncommutative Gröbner bases we present an algorithm for the construction

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of a projective resolution of a module. Such resolutions allow us to treat modules as objects of the homotopy category of complexes of projectives which is a full subcategory of the derived category.

If the endomorphism algebra of a tilting complex is again a factor of a path algebra modulo a so-called admissible ideal we are interested in the construction of the quiver with relations associated to it. We develop a general algorithm for finding the quiver with relations of a finite spectroid given by its radical spaces and show how it can be applied to tilting complexes or, more generally, to a finite subspectroid of the homotopy category. In order to do so, we provide various methods for dealing with chain complexes and their morphism spaces in the homotopy category.

The presented algorithms were implemented in a MuPAD[1] library pathalg[2]. Thus the computer provides a fast and reliable way to check examples in a short time. This helps to develop a good intuition and allows the researcher to support or contradict conjectures.

The base fields we consider are finite fields or the field of rational numbers because these fields allow for exact computations without rounding errors. Unfortunately, this means we cannot compute examples with an algebraically closed base field. However, this does not need to be a restriction since we will prove that it is possible to transfer results found by pathalg over a certain ground field to any field extension. In particular, our result states that if we have found a quiver with relations of the endomorphism algebra of a tilting complex, extension of the base field neither changes the quiver nor affects the relations.

We will first fix the notations and basic definitions used throughout this paper and recall some important facts about Gröbner bases, module categories and their derived categories. We show how an algorithm for finding the radical of a matrix algebra due to Cohen et al. [1] can be adapted for our purpose.

Section 3 is devoted to the construction of projective resolutions using Gröbner bases.

Then we turn to algebraic computations in the category of chain complexes and in the corresponding homotopy category. In Section 4 it is described how we can determine the morphism space of two complexes in those categories.

By then we have collected all necessary tools to complete in Section 5 the algorithm for finding the quiver with relations of the endomorphism algebra of a bounded complex in the homotopy category.

In Section 6 the we are concerned with the effect the extension of the base field of an algebra may have on the quiver with relations.

For further documentation of the library we refer the reader to the pathalg manual [4].

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1 "The Open Computer Algebra System", cf. [5,6] and www.mupad.de.

2 The library is available at www.mathematik.uni-bielefeld.de/~akrause/PATHALG.

2. Preliminaries

2.1. Modules and chain complexes

Let k be an arbitrary field. In the following A is always a finite dimensional factor of a path algebra over k, i.e., A is isomorphic to kQ/I for some finite quiver Q and an admissible ideal I of kQ. The set of points of a quiver Q will be denoted by Q_0 and the set of arrows by Q_1. For basic notations we refer the reader to [2,8].

We can view A also as finite k-spectroid (here, we denote the set of objects in A as A_0). By definition, a k-spectroid is a k-category such that no two distinct objects are isomorphic, the morphism spaces are finite dimensional k-vector spaces and the endomorphism rings of all objects are local.

The space of non-invertible morphisms $x \to y$, where x and y are objects of A, is called the (Jacobson) *radical* of $A(x, y)$ and will be denoted by $\operatorname{Rad} A(x, y)$.

In the categorical context, left A-modules are covariant functors from A to k-mod, the category of finite dimensional k-vector spaces. Here and in the following, an A-module will always be a finite dimensional left module over A. The category of finite dimensional left A-modules is denoted by A-mod.

The indecomposable projective modules over $A = kQ/I$ are the representable functors. Hence they are in bijective correspondence with the objects of A (as a spectroid). We denote an indecomposable projective by $P_x := A(x, -)$ for $x \in A_0$.

Throughout this paper a *chain complex* C of A-modules is a family $\{C_n\}_{n\in\mathbb{Z}}$ of A-modules together with module morphisms $d_n^C = d_n : C_n \to C_{n+1}$ such that $d_n d_{n-1} = 0$.

A chain complex C is called *bounded* if almost all C_n are zero. In this case, the *bounds* of $C \neq 0$ are defined as the pair (i, j) where i is the smallest integer such that C_i is non-zero and j is the largest integer such that C_j is non-zero. For convenience, the bounds of the zero complex are defined to be $(0, 0)$. Denote the category of bounded chain complexes of A-modules by $\mathscr{C}^b(A\text{-mod})$.

For an interval $[i, j] \subset \mathbb{Z}$ and a complex C we define the *truncated complex* $C_{[i,j]}$ to be the complex with $(C_{[i,j]})_n := C_n$ if $n \in [i, j]$ and $(C_{[i,j]})_n := 0$ otherwise; the differential in degree n is given by d_n^C for $n \in [i, j-1]$ and zero otherwise.

A chain map $f : C \longrightarrow D$ is called *null homotopic* if there is a sequence of morphisms $s_n : C_n \longrightarrow D_{n+1}$ such that $f = ds + sd$. Two chain maps f and g are *homotopic* if $f - g$ is null homotopic. The factor category of $\mathscr{C}^b(A\text{-mod})$ modulo chain homotopy is denoted by $\mathscr{K}^b(A\text{-mod})$. For details we refer the reader to [9, 10.1].

For a chain complex C we define the homology modules of C as the subquotients $H_n(C) := \operatorname{Ker} d_n / \operatorname{Im} d_{n-1}$ for $n \in \mathbb{Z}$. A morphism of chain complexes is called a *quasi-isomorphism* if it induces an isomorphism on the homology modules.

2.2. Derived categories and tilting complexes

The *bounded derived category* $\mathscr{D}^b(A)$ of chain complexes of A-modules is given by the localization of $\mathscr{K}^b(A\text{-mod})$ with respect to the set of quasi-isomorphisms. Together with the shift functor $-[1]$ it is equipped with the structure of a triangulated category.

Note that A-mod is a full subcategory of $\mathscr{D}^b(A)$ if we consider an A-module as a complex concentrated in degree zero.

Bounded chain complexes over A-proj, the category of finite dimensional projective modules over A, will be called *complexes of projectives*. They form the full subcategory $\mathscr{C}^b(A\text{-proj})$ of $\mathscr{C}^b(A\text{-mod})$. Its homotopy category is denoted by $\mathscr{K}^b(A\text{-proj})$ which is a full subcategory of $\mathscr{D}^b(A)$ (cf. [9,10.4]).

We say that two algebras A and B are *derived equivalent* if the derived categories $\mathscr{D}^b(A)$ and $\mathscr{D}^b(B)$ are equivalent as triangulated categories.

Definition 2.2.1. Let C be a complex in $\mathscr{K}^b(A\text{-proj}) \hookrightarrow \mathscr{D}^b(A)$. Then C is called a *tilting complex* or *tilting object* if

(i) $\mathscr{D}^b(A)(C, C[i]) = 0$ for all $i \neq 0$ and
(ii) add C generates $\mathscr{D}^b(A)$ as a triangulated category.

Here, add C is the set of summands of finite direct sums of copies of C.

Let $T = \bigoplus_{i=1}^n T_i$ be a tilting complex where T_i is indecomposable for all $i \in \{1, \dots, n\}$ and $T_i \ncong T_j$ for $i \neq j$. Then we can also consider T as a *tilting spectroid* with objects $T_1, \dots, T_n$. The set of morphisms $T_i \to T_j$ is given by the homomorphism space $\mathscr{D}^b(A)(T_i, T_j)$.

Tilting objects were introduced by Rickard in [7] to establish an analogue to the Morita theory of module categories. He showed the following result.

Theorem 2.2.2 [7]. *Let A be an algebra and $\mathscr{D}^b(A)$ its derived category. If T is a tilting complex in $\mathscr{D}^b(A)$, then the opposite endomorphism algebra* $\mathrm{End}(T)^{op}$ *of T is derived equivalent to A.*

2.3. Noncommutative Gröbner bases

For basic definitions and an introduction to Gröbner bases we refer the reader to [3].

A k-basis B of kQ is given by the (possibly infinite) set of paths in Q. Hence, $B \cup \{0\}$ is a pointed semigroup with set of generators the arrows of Q and the paths of length 0 which correspond to the points of Q. Choose total orders on the points $\{x_1, \dots, x_n\} = Q_0$ and on the arrows $\{\alpha_1, \dots, \alpha_m\} = Q_1$. Then we equip B with the length-lexicographic order generated by $x_1 < x_2 < \cdots < x_n < \alpha_1 < \cdots < \alpha_m$.

The *tip* of an element $a \in kQ$ is the largest basis element occurring in an expansion $a = \sum \lambda_i b_i$. The set of *nontips* of an ideal I of kQ is the set of basis elements which do not appear as the tip of some element of I. It is denoted by NonTip(I). The nontips of I form a k-basis of kQ/I. Since $kQ \cong I \oplus \mathrm{NonTip}(I)$ as a k-vector space, every element $a \in kQ$ can be written uniquely as a sum of some $i_a \in I$ and some $N(a) \in \mathrm{NonTip}(I)$. We call $N(a)$ the normal form of a. It can be computed using the division algorithm [3,2.3.2].

Remark 2.3.1. We have $N(f) + N(g) = N(f + g)$. Moreover, the multiplication of elements in kQ/I is given by $N(f \cdot g) = N(N(f) \cdot N(g))$. In particular, this means that $N(N(f)) = N(f)$.

The following theorem shows that in our setting there always exists a finite Gröbner basis of an ideal I and that it can be computed using Buchberger's algorithm.

Theorem 2.3.2 [3, 2.3]. *Let kQ/I be finite dimensional with I generated by a finite set of uniform, tip reduced elements. Then Buchberger's algorithm yields a finite Gröbner basis of I.*

2.4. The radical of an endomorphism algebra

An endomorphism f of a module M is a natural transformation $M \to M$, i.e., it is given by a collection of linear maps $f_{x_i} : M(x_i) \to M(x_i)$ (for $x_i \in A_0$) subject to a certain compatibility condition. If we fix bases of the k-vector spaces $M(x_i)$, the map f_{x_i} can be represented by a matrix f_{x_i}. Since $M \cong \bigoplus_{x_i \in A_0} M(x_i)$ as a k-vector space, f can be represented by a block diagonal matrix with f_{x_i} on the diagonal. In this way, we can view $\mathrm{End}_A(M)$ as a subalgebra of $\mathscr{M}_d(k)$, the algebra of $d \times d$-matrices over k (where $d = \dim_k M$).

Moreover, an endomorphism of a chain complex can also be represented in this fashion. Let C be a chain complex in $\mathscr{C}^b(A\text{-mod})$ and $f = (f^i)_{i \in \mathbb{Z}}$ an endomorphism of C. If (r, s) are the bounds of C then we can write f as a diagonal block matrix with blocks $f^i_{x_j}$ for $i \in \{r, \dots, s\}$ and $j \in \{1, \dots, n\}$. Thus we can view f as a square matrix of size d, where $d = \sum_{i=r}^{s} \dim_k C_i$.

This point of view allows us to apply known methods for computing the radical of a subalgebra of $\mathscr{M}_d(k)$. The algorithm we used in the implementation is described in [1].

3. Projective resolutions

3.1. Indecomposable projective modules

In the following let us view A as a finite k-spectroid. The indecomposable projective modules over A are the representable functors $P_x = A(x, -)$ for $x \in A_0$.

Assume that a finite Gröbner basis G of I and the set NonTip(I) have been constructed using the noncommutative analogue of Buchberger's algorithm [3,2.4.1].

An arbitrary module M is determined up to isomorphism by its dimension vector $\underline{\dim}M = (\dim_k M(x))_{x\in A_0}$ and a list of matrices $(M(\alpha))_{\alpha\in Q_1}$. In the following we will show how one can compute this data for $P_x = A(x, -)$ explicitly.

For any $y \in A_0$, a k-basis of $P_x(y) = A(x, y)$ is given by all b in NonTip(I) which (as paths in Q) have starting point x and target point y. We will denote this subset of NonTip(I) by NonTip(I)(x, y).

Next, we want to determine the matrices which represent the linear maps $P_x(\alpha) = A(x,\alpha) : P_x(z) \to P_x(y)$ for $\alpha : y \to z$ in Q_1. Such a map $P(\alpha)$ is given by left multiplication by α. The image of a basis element b of $P_x(y)$ under left multiplication by $\alpha : y \to z$ is the path $\alpha\, b$. We now consider $\alpha \cdot b$ as a product of elements of A. Then Remark 2.3.1 tells us that the normal form $N(\alpha\, b)$ is the product $\alpha \cdot b$ written as a linear combination of elements in $\mathrm{NonTip}(I)(x, z)$. This yields the image of the basis vector b in $P_x(z)$. Of course the images of the basis elements determine the linear maps $P_x(\alpha)$ for each arrow α.

3.2. Top and Rad

The Jacobson radical $\mathrm{rad}\, M$ of an A-module M is the submodule given by the intersection of all maximal submodules of M. It is the minimal submodule such that the quotient $M/\mathrm{rad}\, M$ is semisimple.

The Jacobson radical of the algebra A coincides with the radical of A as a left module over itself. It can be shown that a k-basis of the radical of A is the set of all paths of length $\geqslant 1$ in NonTip(I). Moreover, it is well known that $\mathrm{rad}\, M = (\mathrm{Rad}\, A)M$. It follows that $\mathrm{rad}\, M(y)$ is the k-space spanned by

$$\{M(n)(m) \mid n \in \mathrm{NonTip}(I)(x, y),\ l(n) \geqslant 1, m \in M(x)\}.$$

We assume that an A-module M is given by its dimension vector and linear maps represented by matrices. This means that we have fixed a k-basis for M. Therefore, when computing $\mathrm{rad}\, M(y)$ for $y \in A_0$, we run through all $x \in A_0$ and apply to each basis element b_x of $M(x)$ the image $M(n)$ of each element n of NonTip(I)(x, y) with length at least 1. In this way we obtain elements $M(n)(b_x) \in M(y)$ which span the vector space $\mathrm{rad}\, M(y)$.

The *top of* M is the factor module $\mathrm{top}\, M := M/\mathrm{rad}\, M$. It is a semisimple module, i.e., a direct sum of simple modules. For each $x \in A_0$ we have $\mathrm{top}\, M(x) = M(x)/\mathrm{rad}\, M(x)$. This means that we can compute $\mathrm{top}\, M$ explicitly.

If P_x is an indecomposable projective module corresponding to some $x \in A_0$, then $\mathrm{top}\, P_x$ is the simple module S_x given by $S_x(x) = k$ and zero otherwise. Moreover, P_x is a projective cover of S_x, hence a projective module is up to isomorphism determined by its top.

3.3. Construction of a projective resolution

We will determine a minimal projective resolution of an A-module M inductively: First we have to find the projective cover P_0 of M together with an epimorphism $\pi_0 : P_0 \to M$. The kernel of π_0 is the first syzygy module $\Omega_0(M)$ of M. Then we proceed in this fashion, i.e., in step i we construct a projective cover P_i of $\Omega_{i-1}(M)$ together with the epimorphism π_i and we obtain $\Omega_i(M)$ as the kernel of π_i. One may visualize the construction by the following diagram:

$$\begin{tikzcd}
\cdots P_3 \arrow[rr] \arrow[dr, "\pi_3"] & & P_2 \arrow[rr] \arrow[dr, "\pi_2"] & & P_1 \arrow[rr] \arrow[dr, "\pi_1"] & & P_0 \arrow[rr] \arrow[dr, "\pi_0"] & & 0 \\
& \Omega_2(M) \arrow[ur, "\iota_2"] \arrow[dr] & & \Omega_1(M) \arrow[ur, "\iota_1"] \arrow[dr] & & \Omega_0(M) \arrow[ur, "\iota_0"] \arrow[dr] & & M \arrow[dr] & \\
0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & & 0
\end{tikzcd}$$

The procedure stops when $\Omega_n(M)$ is the zero module for some $n \in \mathbb{N}_0$ in which case the projective dimension of M is n. If the procedure does not stop we say that M has infinite projective dimension.

Let us consider the construction of a projective cover of a module M. If $P \xrightarrow{\pi} M$ is a projective cover, then π restricts to an isomorphism of $\operatorname{top} P$ and $\operatorname{top} M$. We will use this to determine P. Recall that $\operatorname{top} M$ is a semisimple module, and $\operatorname{top} P_x$ is the simple module S_x corresponding to $x \in A_0$, if P_x is indecomposable projective. On the other hand, a projective module is uniquely determined by its top. We now decompose $\operatorname{top} M$ as a direct sum of simple modules

$$\operatorname{top} M = \bigoplus_{x \in A_0} S_x^{m_x}$$

with $m_x \in \mathbb{N}_0$. It follows that the projective cover of M is

$$P = \bigoplus_{x \in A_0} P_x^{m_x}.$$

By Yoneda's lemma, $\operatorname{Hom}(P_x, M)$ is isomorphic to $M(x)$ via $f \mapsto f_x(\mathrm{id}_x)$ and the projection $\pi : P \to M$ is determined by its restriction, the isomorphism $\operatorname{top} P \longrightarrow \operatorname{top} M$. Since A_0 is finite, we have

$$\operatorname{Hom}_A\left(\bigoplus_{x \in A_0} P_x^{m_x}, M\right) \cong \bigoplus_{x \in A_0} (\operatorname{Hom}_A(P_x, M))^{m_x}.$$

This provides us with a recipe to compute π by its restrictions to each of the direct summands of P. Fix a direct summand P_x of P and denote the restriction of π to P_x by $\tilde{\pi}$. S_x is a direct summand of $\operatorname{top} M$. Let $\{b\}$ be the k-basis of $S_x(x) \cong k$ which is contained in our given basis of $\operatorname{top} M$. Then the isomorphism of $\operatorname{top} P_x$ and $S_x \hookrightarrow \operatorname{top} M$ is given by $P_x(x) \ni \mathrm{id}_x \mapsto b \in S_x(x)$ and by Yoneda's lemma this determines the map $P_x \to M$. Explicitly, this means that $\tilde{\pi}_y$ maps an element v of $P_x(y)$ to $M(v)(b) \in M(y)$. This yields the module homomorphism $\tilde{\pi} = (\tilde{\pi}_y)_{y \in A_0} : P_x \to M$.

Once we have computed the projective cover P of M together with the projection π, we can compute the kernel K of π together with an inclusion $\iota : K \hookrightarrow P$ by solving systems of linear equations as follows.

The kernel map $\iota : K \to P$ is a natural transformation and thus for each $\alpha \in A(x, y)$ the following diagram is commutative with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & K(x) \arrow[r, "\iota_x"] \arrow[d, dashed, "K(\alpha)"'] & P(x) \arrow[r, "\pi_x"] \arrow[d, "P(\alpha)"'] & M(x) \arrow[r] \arrow[d, "M(\alpha)"'] & 0 \\ 0 \arrow[r] & K(y) \arrow[r, "\iota_y"] & P(y) \arrow[r, "\pi_y"] & M(y) \arrow[r] & 0 \end{tikzcd}$$

Each row is a split exact sequence of k-vector spaces and thus we can compute ι_x for all $x \in A_0$ by solving the matrix equation $\pi_x \iota_x = 0$. With this information we can calculate matrices $K(\alpha)$ making the left square of the diagram commutative, by solving another matrix equation $\iota_y K(\alpha) = P(\alpha)\iota_x$.

As we have seen before, $K = \Omega_0(M)$ and we can proceed with the construction of the projective resolution inductively by computing a projective cover of K.

4. Homomorphisms and homotopy

4.1. Computing the homomorphism space of two modules

Let M, N be A-modules. Our aim is to compute the homomorphism space of M and N. A homomorphism $f : M \to N$ is a natural transformation (here, we consider M and N as functors $A \to k$-mod). Since $M(x)$ and $N(x)$ are k-vector spaces for every $x \in A_0$, the linear map f_x can be represented as a matrix and also $M(\alpha)$ and $N(\alpha)$ are given by matrices. Thus we can calculate all homomorphisms $M \to N$ in A-mod as the solutions $(f_x)_{x \in A_0}$ to the system of matrix equations

$$f_y M(\alpha) = N(\alpha) f_x$$

for all arrows $\alpha : x \to y$ in Q_1.

If $M = A(x, -)$ is an indecomposable projective module, we can use a less time consuming method which avoids solving a system of linear equations. By Yoneda's lemma, every morphism $f : M \to N$ is uniquely determined by the image $M(x) \ni \mathrm{id}_x \mapsto f_x(\mathrm{id}_x) \in N(x)$. So we can find a basis of $\mathrm{Hom}_A(M, N)$ by mapping id_x successively to each basis element of $N(x)$ and extending to morphisms of modules $f : M \to N$ in the following way:

Let $z \in A_0$. The paths in NonTip(I) (x, z) form a basis of $M(z)$ since $M = A(x, -)$ is an indecomposable projective module. On the other hand, a path

$p \in \mathrm{NonTip}(I)(x, z)$ is a morphism $x \to z$ in A, and hence induces a map $M(p): M(x) \to M(z)$ which is given by left multiplication with p. Thus we have $M(p) \times (\mathrm{id}_x) = p \circ \mathrm{id}_x = p$ and by naturality of f it then follows that

$$f_z(p) = f_z M(p)(\mathrm{id}_x) = N(p) f_x(\mathrm{id}_x).$$

So when the image $b = f_x(\mathrm{id}_x)$ of id_x in $N(x)$ is fixed we can find the images of the basis of $M(z)$ by computing $N(p)(b)$ for all paths p in $\mathrm{NonTip}(I)(x, z)$. Obviously, this provides us with the linear map $f_z : M(z) \to N(z)$ and finally with the morphism of modules $f = (f_z)_{z \in A_0}$.

An easier method applies to the case of M and N both being indecomposable projective A-modules, $M := P_x$ and $N := P_y$, say. Then a basis of $\mathrm{Hom}_A(P_x, P_y) \cong P_y(x) = A(y, x)$ (by Yoneda's lemma) is given by all paths in $\mathrm{NonTip}(I)(y, x)$. For the composition of two morphisms $f : P_x \to P_y$ and $g : P_y \to P_z$, we do not need to compute the matrix product $g_v f_v$ for each $v \in A_0$ but it suffices to compute the product $g_y(\mathrm{id}_y) \cdot f_x(\mathrm{id}_x)$ as elements in A.

If M is a decomposable projective module, say $M = \bigoplus_{i=1}^s M_i$ with M_i indecomposable, we can decompose the homomorphism space accordingly. Thus, $\mathrm{Hom}_A (\bigoplus_{i=1}^s M_i, N) = \bigoplus_{i=1}^s \mathrm{Hom}_A(M_i, N)$ and we can compute a k-basis for $\mathrm{Hom}_A(M_i, N)$ for each i as shown above. This yields a k-basis for $\mathrm{Hom}_A(M, N)$.

4.2. The homomorphism space of two complexes

By definition, a chain map $f : C \to D$ of two complexes C and D in $\mathscr{C}^b(A\text{-mod})$ is given by a sequence of maps $f_n : C_n \to D_n$ compatible with the boundary maps.

We use the fact that C and D are bounded complexes. Let (m_C, n_C) and (m_D, n_D) be the bounds of C and D, respectively. Then $\mathscr{C}^b(A\text{-mod})(C, D)$ is zero, if $m_C > n_D$ or $m_D > n_C$.

Let $m = \max(m_C, m_D)$ and $n = \min(n_C, n_D)$. If $m > n$, then we have $\mathscr{C}^b(A\text{-mod}) (C, D) = 0$. Otherwise it suffices to compute the homomorphism space of the truncated complexes $C_{[m-1,n+1]}$ and $D_{[m-1,n+1]}$.

Algorithm 4.2.1. Let C and D be chain complexes. We compute a k-basis of $\mathscr{C}^b(A\text{-mod})(C, D)$.

Let m and n be defined as above. If $m > n$ we return $\emptyset$. If $m \leqslant n$, then we will proceed by induction on the degree and start by determining a basis B_m of $\mathrm{Hom}_A(C_m, D_m)$ as described in Section 4.1. Since $\mathrm{Hom}_A(C_{m-1}, D_{m-1}) = 0$ by definition of m, compatibility with the boundary maps yields the equation

$$\left(\sum_{b \in B_m} \lambda_b b\right) d^C_{m-1} = 0$$

which is in fact a system of matrix equations. Let Λ be a k-basis of the solution space.

In the following we will denote a sequence $(t_i)_{i \in \{m,\dots,n\}}$ of maps $t_i : C_i \to D_i$ by the vector $(t_m, \dots, t_n)$. Let

$$T_m := \left\{ \left(\sum_{b \in B_m} \lambda_b b, 0, \ldots, 0 \right) \middle| \lambda \in \Lambda \right\}.$$

For the induction step $m < i \leqslant n$ we compute a basis B_i of $\operatorname{Hom}_A(C_i, D_i)$. The compatibility with the boundary maps again yields a system of matrix equations

$$\left(\sum_{b \in B_i} \lambda_b b \right) d_{i-1}^C - d_{i-1}^D \left(\sum_{t \in T_{i-1}} \lambda_t t_{i-1} \right) = 0,$$

where t_{i-1} is the $(i-1)$th entry of a sequence $t \in T_{i-1}$ and therefore a module morphism $C_{i-1} \longrightarrow D_{i-1}$. Let Λ be a k-basis of the solution space to this equation. Then we define

$$T_i := \left\{ \sum_{b \in B_i} (0, \ldots, 0, \lambda_b b, 0, \ldots, 0) + \sum_{t \in T_{i-1}} \lambda_t t \;\middle|\; \lambda \in \Lambda \right\},$$

where $(0, \ldots, 0, \lambda_b b, 0, \ldots, 0)$ is the sequence with the only non-zero entry at position i. One may think of T_i as a set of "chain maps" $C \to D$ which are compatible with the boundary maps in the degrees $m-1, \ldots, i$.

Finally, we have to compute one last step: here, we have the equation

$$d_n^D \left(\sum_{t \in T_n} \lambda_t t_n \right) = 0,$$

where t_n is the nth entry of the sequence t. Let Λ be a k-basis of the solution space. Then we define

$$T_{n+1} := \left\{ \sum_{t \in T_n} \lambda_t t \;\middle|\; \lambda \in \Lambda \right\}.$$

After this step we return T_{n+1} which is a k-basis of $\mathscr{C}^b(A\text{-mod})(C, D)$. An element of T is a sequence $(t_j)_{j=m,\ldots,n}$, where $t_j \in \operatorname{Hom}_A(C_j, D_j)$ such that $t = (t_j)_{j \in \mathbb{Z}}$ with $t_j = 0$ for $j \notin \{m, \ldots, n\}$ is a chain map.

4.3. Null homotopic chain maps

For chain complexes C and D we need to compute $\mathscr{K}^b(A\text{-proj})(C, D)$ which, by definition, is $\mathscr{C}^b(A\text{-proj})(C, D)$ modulo null homotopic morphisms. We start by giving a useful characterization of null homotopic maps.

A *projective complex* in $\mathscr{C}^b(A\text{-proj})$ is a complex whose indecomposable direct summands are of the form P^i for an indecomposable projective P defined by $P_i^i := P$, $P_{i+1}^i := P$, and zero in all other degrees; the differential is $d_i := \mathrm{id}_P$ and $d_j = 0$ for $j \neq i$. So P^i is the chain complex $0 \to P = P \to 0$ in degrees i and $i+1$.

The proof of the following lemma will be left to the reader.

Lemma 4.3.1. *A morphism f in $\mathscr{C}^b(A\text{-}proj)$ is null homotopic if and only if f factors through a projective complex in $\mathscr{C}^b(A\text{-}proj)$.*

Thus we can compute the homomorphism space of two complexes in $\mathscr{K}^b(A\text{-proj})$ in the following way.

Algorithm 4.3.2. Let C, D be chain complexes in $\mathscr{K}^b(A\text{-proj})$. We compute a k-basis of $\mathscr{K}^b(A\text{-proj})(C, D)$.

As a first step we determine a k-basis of the space $\mathscr{C}^b(A\text{-proj})(C, D)$ using Algorithm 4.2.1.

Next, we have to find the set S of all indecomposable projective complexes P such that $\mathscr{C}^b(A\text{-proj})(P, D) \neq 0$ and all indecomposable projective complexes Q with $\mathscr{C}^b(A\text{-proj})(C, Q) \neq 0$. This is a finite set since C and D are bounded and there are only finitely many indecomposable projective complexes in these bounds.

Let (m_C, n_C) and (m_D, n_D) be the bounds of C and D, respectively. As before, we define $m := \max(m_C, m_D)$ and $n := \min(n_C, n_D)$. If $m > n$ then $\mathscr{K}^b(A\text{-proj})(C, D) = 0$ and we are finished. Otherwise we compute for each indecomposable projective complex P^i (where P is an indecomposable projective module and $i \in \{m-1, \dots, n\}$) a k-basis $V(P^i)$ of $\mathscr{C}^b(A\text{-proj})(C, P^i)$ and a k-basis $W(P^i)$ of $\mathscr{C}^b(A\text{-proj})(P^i, D)$ using Algorithm 4.2.1. Thus S is the set of all P^i where $V(P^i)$ or $W(P^i)$ is not empty.

Then we determine the subspace $\mathscr{P}(C, D)$ of $\mathscr{C}^b(A\text{-proj})(C, D)$ which contains all morphisms which factor through some projective complex. This is the k-vector space spanned by all compositions gf where $f \in V(P)$ and $g \in W(P)$, and P runs through all elements of S. We compute a basis of $\mathscr{P}(C, D)$ and then determine the factor space $\mathscr{K}^b(A\text{-proj})(C, D)$.

5. The quiver of an endomorphism algebra

5.1. The quiver of a spectroid

Let S be a finite k-spectroid. By a theorem of Gabriel (cf. [8,2.1]), if k is algebraically closed, we know that S is isomorphic to the path spectroid of a finite quiver modulo an admissible ideal.

Our aim is to construct this quiver and a set of generators of the ideal in the case that S is given by its radical spaces $\operatorname{Rad} S(x, y)$ for all objects x and y of S.

If k is not algebraically closed we cannot be sure that S is isomorphic to a path spectroid modulo an admissible ideal. In this case, the method described below will fail. This happens if and only if there exists an object x in S such that the k-dimension of the space $\operatorname{Rad}(\operatorname{End}(x))$ is not equal to $\dim_k \operatorname{End}(x) - 1$. We will test for this criterion and stop the computations if it is not satisfied.

So assume S is isomorphic to kQ_S/I_S. The points of the quiver Q_S are corresponding to the objects of S. Let $\{x_1, \dots, x_n\}$ be the objects of S. The arrows $\alpha_1, \dots, \alpha_r : x_i \to x_j$ of Q_S correspond to a k-basis $b_1, \dots, b_r$ of the space $\operatorname{Rad} S/\operatorname{Rad}^2 S(x_i, x_j)$. So the number of arrows $x_i \to x_j$ is given by the dimension of $\operatorname{Rad} S/\operatorname{Rad}^2 S(x_i, x_j)$.

This determines the quiver Q_S uniquely. In the following we describe how we find k-bases of the vector spaces $\operatorname{Rad} S/\operatorname{Rad}^2 S(x_i, x_j)$ for each pair of objects of S.

Let x_i, x_j be objects of S. Then $\operatorname{Rad}^2 S(x_i, x_j)$ is the space of all elements in $\operatorname{Rad} S(x_i, x_j)$ which are linear combinations of morphisms factoring through another object $x_s \in S_0$. Thus, $\operatorname{Rad}^2 S(x_i, x_j)$ is the vector space generated by all compositions gf where $f \in \operatorname{Hom}(x_i, x_s)$ and $g \in \operatorname{Hom}(x_s, x_j)$ for some $x_s \in S_0$. It suffices to consider only those f and g which are elements of a (fixed) basis of the respective spaces. Their compositions gf span the vector space $\operatorname{Rad}^2 S(x_i, x_j)$. Given a basis of $\operatorname{Rad}^2 S(x_i, x_j)$ we can compute a basis of a vector space complement in $\operatorname{Rad} S(x_i, x_j)$.

For later use, we define a map Φ which sends an arrow $\alpha : x_i \to x_j$ of Q_S to its corresponding basis element in $\operatorname{Rad} S/\operatorname{Rad}^2 S(x_i, x_j)$. We can extend Φ to a map from the set of finite paths in Q_S to the set of morphisms in S by defining $\Phi(p) = \Phi(\alpha_m) \cdots \Phi(\alpha_1)$ for a path $p = \alpha_m \cdots \alpha_1$ in Q_S.

Once we have found the quiver of S we want to compute an ideal I_S such that $S \cong kQ_S/I_S$. For this we have to find all linear combinations of paths in Q_S which lie in I_S, meaning that they are zero in S. But if Q_S has cycles there exists an infinite number of paths. On the other hand, I_S is an admissible ideal, so I_S contains all paths of length $\geqslant l$ for some $l \in \mathbb{N}$.

This allows us to determine I_S even if Q_S is not directed.

Algorithm 5.1.1 *Relations*. The input is a quiver Q_S of a spectroid S and the map Φ from the set of paths in Q_S to the set of morphisms in S. A generating set G for the ideal I_S is returned.

First, we check for zero relations, i.e., paths which lie in I_S and thus are zero in S. Initialize $Z := \emptyset$. We proceed by induction on the length of the paths in Q. We start with paths of length 2 since I_S does not contain shorter paths.

Length 2: Initialize $P_2 := \emptyset$. Let $\{p_1, \dots, p_r\}$ be the set of paths with length 2 in Q_S. For each p_i we determine $\Phi(p_i)$ and check if $\Phi(p_i)$ is zero in S. If so, we replace Z by $Z \cup \{p_i\}$. If not we extend P_2 by $\{p_i\}$.

Length *i*: Initialize $P_i := \emptyset$. By induction, P_{i-1} contains all non-zero paths in (Q_S, I_S) of length $i - 1$. We will go through the following procedure for each $p \in P_{i-1}$.

For each arrow α in Q_S which can be appended to the path p we determine $\Phi(\alpha p)$. If $\Phi(\alpha p)$ is zero then we enlarge Z by the path αp. If $\Phi(\alpha p)$ is not zero, then replace P_i by $P_i \cup \{\alpha p\}$.

```
INPUT : a quiver Q and
        a map f sending a path to the corresponding morphism
OUTPUT: G

Z:= {}
P:= {}
P[1]:= the set of all arrows in Q
i:= 2
WHILE P[i-1] <> {} DO
  P[i]:= {}
  FOR p IN P[i-1] DO
    FOR all arrows a such that (ap) is path in Q DO
      IF f(ap) = 0 THEN Z:= Z union {ap}
      ELSE P[i]:= P[i] union {ap}
      END
    END
  END
  P:= P union P[i]
  i:= i + 1
END
R:= {}
FOR i, j points of Q DO
  P[i,j]:= set of all paths in P which start in i and end in j
  B:= a basis of the space of solutions to
        x[1]*f(p[1]) + ... x[r]*f(p[r]) = 0
     where {p[1],..., p[r]} = P[i,j]
  R:= R union {b[1]*p[1] + ... + b[r]*p[r] | b in B}
END
```

Fig. 1. Finding the ideal of a spectroid.

This procedure must stop because I_S is an admissible ideal and so P_l will be empty for some $l \in \mathbb{N}$. We have constructed a set Z of generators for the zero-relations in S and we obtained $P := \bigcup_{i=2}^{l} P_i$, the finite set of non-zero paths in S.

We now partition P as $P = \bigcup_{i,j\in S_0} P_{ij}$, where P_{ij} contains all paths in P which start in i and end in j. For each P_{ij} we solve the linear equation $\sum_{p\in P_{ij}} \lambda_p \Phi(p) = 0$ and obtain a k-basis B_{ij} of the solution space. We define

$$R := \left\{ \sum_{p\in P_{ij}} \lambda_p p \;\middle|\; (\lambda_p)_{p\in P_{ij}} \in B_{ij} \right\}.$$

Finally we obtain $G := R \cup Z$ as the required set of generators for I_S.

For a description of this algorithm in pseudo-code see Fig. 1.

5.2. *Finding the quiver of an endomorphism algebra*

We now apply the algorithms described in the previous section to a special case: Given an object in the homotopy category $\mathscr{K}^b(A\text{-proj})$, we want to find the quiver with relations of its endomorphism algebra.

Let T be a complex in $\mathscr{K}^b(A\text{-proj})$ which decomposes as $T = \bigoplus_{i=1}^n T_i$ with T_i indecomposable for $i = 1, \dots, n$ and $T_i \ncong T_j$ for $i \neq j$. We will view T in the following as a full subspectroid of $\mathscr{K}^b(A\text{-proj})$ with the objects $T_1, \dots, T_n$ and try to construct a quiver Q and an admissible ideal I of kQ such that $T \cong kQ/I$.

First we compute the quiver Q as explained in the previous section.

Algorithm 5.2.1 *Quiver of a full subspectroid of* $\mathscr{K}^b(A\text{-}proj)$. We take as input a full subspectroid of $\mathscr{K}^b(A\text{-proj})$ given by its objects $T_1, \dots, T_n$. The quiver of the spectroid is returned.

We label the points of Q by $1, \dots, n$ corresponding to the objects $T_1, \dots, T_n$ of T.

Following Algorithm 4.3.2 we compute a k-basis of the homomorphism space $\mathscr{K}^b(A\text{-proj})(T_i, T_j)$ for any pair of summands T_i, T_j of T. These give us the morphism spaces $T(T_i, T_j)$ of the spectroid T. As a by-product we are also supplied with a basis for each space of null homotopic chain maps $T_i \to T_j$.

To determine the number of arrows $i \to j$ of Q, we have to compute a k-basis of the space $\operatorname{Rad} T/\operatorname{Rad}^2 T(T_i, T_j)$. If $T_i \neq T_j$, then $\operatorname{Rad} T(T_i, T_j)$ is equal to $\mathscr{K}^b(A\text{-proj})(T_i, T_j)$. Otherwise $\mathscr{K}^b(A\text{-proj})(T_i, T_i) = \operatorname{End}(T_i)$ is a local algebra because T_i is indecomposable. We compute a k-basis of $\operatorname{Rad}(\operatorname{End}(T_i))$ using [1] as described in Section 2.4.

At this point we can decide if T is a factor of a path spectroid: for all $i \in \{1, \dots, n\}$ let $d_i := \dim_k \operatorname{End}(T_i)$; if $\dim_k \operatorname{Rad}(\operatorname{End}(T_i)) < d_i - 1$ for some i then we cannot determine a quiver Q and ideal I such that $T \cong kQ/I$ and hence we stop the computations. If, however, we have $\dim_k \operatorname{Rad}(\operatorname{End}(T_i)) = d_i - 1$ for all $i \in \{1, \dots, n\}$, we know that T is a factor of a path spectroid and we can proceed with our calculations.

For all pairs $T_i, T_j \in T_0$ we fix a k-basis B_{ij} of the space $\operatorname{Rad} T(T_i, T_j)$. Then $\operatorname{Rad}^2 T(T_i, T_j)$ is the subspace generated by all compositions gf where f is an element of B_{is} and g an element of B_{sj} for $s = 1, \dots, n$. We compute a basis R_{ij} of a k-vector space complement of $\operatorname{Rad}^2 T(T_i, T_j)$ in $\operatorname{Rad} T(T_i, T_j)$. The number of arrows $i \to j$ in Q is the number of elements in R_{ij}.

So, by definition of Q, the elements of R_{ij} are in correspondence with the arrows $i \to j$ of Q, thus we can define a bijection $\Phi : Q_1 \to \bigcup_{i,j \in Q_0} R_{ij}$. We extend Φ to arbitrary paths of Q by mapping a path $\alpha_r \cdots \alpha_1$ to the composition of maps $\Phi(\alpha_r) \cdots \Phi(\alpha_1)$.

Finally, we have to determine I. As described in Algorithm 5.1.1, we start with searching for zero relations, by induction on the length of paths in Q.

Since T is a full subcategory of $\mathscr{K}^b(A\text{-proj})$ a morphism in T is zero if and only if it is null homotopic as a chain map. For a path p in Q we can easily determine if $\Phi(p)$ is zero as a chain map, so we test for this first. Only if this is not the case we must do further (and more time-consuming) checks to determine if $\Phi(p)$ is null homotopic.

Although the basis sets R_{ij} ($i, j \in Q_0$) do not contain null homotopic morphisms it may happen that the composition of two basis elements is null homotopic without being zero. This is due to the fact that we had to choose a complement of each subspace $\mathrm{Rad}^2 T(T_i, T_j)$ in $\mathrm{Rad}\, T(T_i, T_j)$, and the product fg of two elements $f \in R_{sj}$ and $g \in R_{is}$ may not be an element of the vector space spanned by R_{ij}.

But we have already computed a k-basis of the space of null homotopic chain maps $T_i \to T_j$ when we determined $\mathscr{K}^b(A\text{-proj})(T_i, T_j)$. Hence we can decide if $\Phi(p)$ is an element of this subspace. In fact, this amounts to solving a system of linear equations.

As described in Algorithm 5.1.1 we determine a set Z of zero relations in I as well as the finite set P of non-zero paths in kQ/I.

It remains to decide which linear combinations of elements of P are zero in S and therefore elements of I. Following Algorithm 5.1.1 we partition P into subsets P_{ij} containing the paths starting in i and ending in j. For each pair $i, j \in Q_0$ we must solve the linear equation $\sum_{p \in P_{ij}} \lambda_p \Phi(p) = 0$ in T meaning that we have to determine the null homotopic chain maps in the k-vector space spanned by $\{\Phi(p) \mid p \in P_{ij}\}$.

Let B_{ij} be a basis of the intersection of $\mathrm{span}\{\Phi(p) \mid p \in P_{ij}\}$ and the space of null homotopic morphisms $T_i \to T_j$ which we computed by solving a system of linear equations. Then we define

$$R := \left\{ \sum_{p \in P_{ij}} \lambda_p p \;\middle|\; \sum_{p \in P_{ij}} \lambda_p \Phi(p) \in B_{ij} \right\}$$

and obtain $G := Z \cup R$ as a set of generators for I.

We remark that G may not be minimal if Q has cycles, so it might be advisable to apply Buchberger's algorithm to G to get rid of redundant generators.

6. Extensions of the base field

In this chapter we will show that the quiver with relations of a k-algebra "remains the same" over any field extension of k. Moreover, this is also true for the quiver of the endomorphism algebra of a chain complex considered over k and over a field extension of k, respectively. Therefore it is possible to derive results over an algebraically closed base field from computations over a suitable subfield.

Throughout this chapter, L will be a field extension of k. Let A be a k-algebra. In this section, by an A-module we mean an arbitrary left A-module which may have infinite k-dimension. The category of all left A-modules is denoted by A-Mod.

6.1. The quiver of an algebra

The tensor product yields an endofunctor $L \otimes_k -$ of k-Mod, the category of k-vector spaces. Moreover, $L \otimes_k -$ is exact since L is free and therefore flat as a k-module.

Let M be an A-module. If k is central in A (this is obviously true if A is a factor of a path algebra) then $L \otimes_k M$ carries a left A-module structure given by $a \cdot (l \otimes m) := l \otimes (a \cdot m)$ for $a \in A$, $l \in L$ and $m \in M$. Therefore $L \otimes_k -$ can also be considered as a functor $A\text{-Mod} \longrightarrow A\text{-Mod}$. We note the following fact.

Lemma 6.1.1. *If k is central in A then $L\otimes_k -$ is an exact endofunctor of A-Mod.*

Let $A = kQ/I$. We are concerned with the question if the quiver of A stays the same if we change the base field of A to an extension field L of k, i.e., we want to know the quiver with relations of the L-algebra $L \otimes_k A$. Note that A is a subalgebra of $L \otimes_k A$ via the inclusion $A \cong k \otimes_k A \hookrightarrow L \otimes_k A$.

Proposition 6.1.2. *Let $A = kQ/I$. Then $L \otimes_k A \cong LQ/(L \otimes_k I)$ as L-algebras.*

Proof. We have the following short exact sequence:

$$0 \longrightarrow I \longrightarrow kQ \longrightarrow kQ/I \longrightarrow 0$$

Since $L\otimes_k$—is an exact functor by Lemma 6.1.1, the sequence

$$0 \longrightarrow L \otimes I \longrightarrow L \otimes kQ \longrightarrow L \otimes kQ/I \longrightarrow 0$$

is still exact. Therefore $L \otimes (kQ/I)$ is isomorphic to $(L \otimes kQ)/(L \otimes I)$ as an A-module. Moreover, this isomorphism respects the L-algebra structure.

It remains to show that $L \otimes kQ \cong LQ$. We define a map $L \otimes kQ \to LQ$ by $\sum_i (l_i \otimes \sum_j \mu_j p_j) \mapsto \sum_j \sum_i (l_i \mu_j) p_j$. One easily verifies that this is an isomorphism of L-algebras. □

This lemma shows that the quiver with relations of an algebra (if it exists) does not change when we extend the base field.

6.2. The endomorphism algebra of a chain complex

Lemma 6.2.1. *Let A be a k-algebra and let M, N be (not necessarily finite dimensional) A-modules. Then*

$$\begin{aligned} L \otimes_k \mathrm{Hom}_A(M, N) &\longrightarrow \mathrm{Hom}_{L\otimes_k A}(L \otimes_k M, L \otimes_k N) \\ l \otimes f &\longmapsto l \cdot (L \otimes f) \end{aligned}$$

is an isomorphism of L-vector spaces if M or L has finite k-dimension.

Proof. We have the following isomorphisms of L-vector spaces:

$$\begin{aligned} \mathrm{Hom}_{L\otimes_k A}(L \otimes_k M, L \otimes_k N) &\cong \mathrm{Hom}_{L\otimes_k A}(L \otimes_k A \otimes_A M, L \otimes_k N) \\ &\cong \mathrm{Hom}_A(M, L \otimes_k N). \end{aligned}$$

The last isomorphism follows from $(L \otimes_A A) \otimes_k -$ being left adjoint to the forgetful functor $(L \otimes_k A)$-Mod $\to$ $(k \otimes_k A)$-Mod $\cong$ A-Mod. So it suffices to show that $L \otimes_k \mathrm{Hom}_A(M, N)$ is isomorphic to $\mathrm{Hom}_A(M, L \otimes_k N)$.

Let $(l_j)_{j\in J}$ be a k-basis of L. Then

$$L \otimes N \cong \left(\bigoplus_{j\in J} l_j \cdot k\right) \otimes N \cong \bigoplus_{j\in J} (l_j \cdot k \otimes N) \cong \bigoplus_{j\in J} N$$

as A-modules. Now we define a morphism

$$\Phi : L \otimes \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_A(M, L \otimes N) \cong \mathrm{Hom}_A\left(M, \bigoplus_{j\in J} N\right)$$
$$l \otimes f \longmapsto (m \mapsto l \otimes f(m)).$$

Then an inverse of Φ is given by the map which sends $g : M \longrightarrow \bigoplus_{j\in J} N$ to $\sum_{j\in J} l_j \otimes g_j$ (where g_j is the composition of g with the jth projection $\bigoplus_{j\in J} N \xrightarrow{\pi_j} N$). We remark that the sum $\sum_{j\in J} l_j \otimes g_j$ is finite if $\dim_k L < \infty$ (so J is finite) or if M has finite k-dimension. In the latter case the image $g(M)$ in $\bigoplus_{j\in J} N$ is finite dimensional over k. Therefore almost all maps g_j must be zero. □

Let m and n be integers. We define a new quiver $Q^{[m,n]}$ by gluing together $n - m$ copies $Q^{(i)}$ of Q, indexed by $i = m, \ldots, n$.

We have two types of arrows in $Q^{[m,n]}$: All arrows $\alpha^{(i)}$ of $Q^{(i)}$ for $i = m, \ldots, n$, and arrows $\beta_x^{(i)} : x^{(i)} \to x^{(i+1)}$ for all $x \in Q_0$ and $i = m, \ldots, n - 1$ which connect two copies $Q^{(i)}$ and $Q^{(i+1)}$.

The relations are given by all relations in $Q^{(i)}$ for $i = m, \ldots, n$ together with all relations $\alpha^2 = 0$ and all relations $\beta\alpha = \alpha\beta$. We denote the ideal of relations by $I^{[m,n]}$.

Let C, D be two complexes in $\mathscr{C}^b(A\text{-mod})$ and let (m_C, n_C) and (m_D, n_D) be the bounds of C and D respectively. We define $m := \min(m_C, m_D)$ and $n := \max(n_C, n_D)$. Then we can view C and D as finite dimensional modules over $\bar{A} = kQ^{[m,n]}/I^{[m,n]}$. Hence, $\mathscr{C}^b(A\text{-mod})(C, D)$ and $\mathrm{Hom}_{\bar{A}}(C, D)$ are isomorphic as k-vector spaces.

Corollary 6.2.2. *Let $A = kQ/I$ and let C, D be chain complexes over A-mod. Then*

$$\Phi : L \otimes_k \mathscr{C}^b(A - \mathrm{mod})(C, D) \longrightarrow \mathscr{C}^b((L \otimes_k A) - \mathrm{mod}(L \otimes_k C, L \otimes_k D)$$
$$\lambda_j \otimes f_j \longmapsto \lambda_j \cdot (L \otimes f_j)$$

is an isomorphism of L-vector spaces.

Proof. Let $\bar{A} := kQ^{[m,n]}/I^{[m,n]}$ as defined above. Now we can apply proposition 6.2.1 to the finite dimensional $\bar{A}$-modules C and D: It follows that $L \otimes \mathrm{Hom}_{\bar{A}}(C, D) \cong \mathrm{Hom}_{L\otimes\bar{A}}(L \otimes C, L \otimes D)$.

By Lemma 6.1.2 we have $L \otimes \bar{A} \cong L\bar{Q}/(L \otimes \bar{I})$ as well as $L \otimes A \cong LQ/(L \otimes I)$, thus $L \otimes C$ and $L \otimes D$ can be considered as chain complexes in $\mathscr{C}^b((L \otimes A))$-mod. This yields the required isomorphism of L-vector spaces. □

6.3. The endomorphism algebra of a complex in $\mathscr{K}^b(A\text{-proj})$

We remark that $L \otimes_k -$ maps projective A-modules to projective $(L \otimes A)$-modules: If we decompose $A = \bigoplus P_i$ as an A-module then

$$L \otimes_k A = L \otimes_k \left(\bigoplus P_i\right) \cong \bigoplus (L \otimes_k P_i).$$

Therefore $L \otimes_k -$ is also a functor $\mathscr{C}^b(A\text{-proj}) \to \mathscr{C}^b((L \otimes_k A)\text{-proj})$.

Lemma 6.3.1. *Let $A = kQ/I$ and let C and D be complexes in $\mathscr{C}^b(A\text{-proj})$. Let*

$$\Phi : L \otimes_k \mathscr{C}^b(A\text{-proj})(C, D) \longrightarrow \mathscr{C}^b((L \otimes_k A)\text{-proj})(L \otimes_k C, L \otimes_k D)$$

be the isomorphism given by Corollary 6.2.2. If $h \in L \otimes_k \mathscr{C}^b(A\text{-proj})(C, D)$ such that $\Phi(h)$ is null homotopic, then there exist null homotopic chain maps $f_j \in \mathscr{C}^b(A\text{-proj})(C, D)$ and elements $\lambda_j \in L$ $(j \in J)$ with $h = \sum_{j\in J} \lambda_j \otimes f_j$.

Proof. We write $h = \sum_{s\in S} \mu_s \otimes h_s$ for some $\mu_s \in L$ and $h_s \in \mathscr{C}^b(A\text{-proj})(C, D)$. Let $(l_j)_{j\in J}$ be a k-basis of L. Then $\mu_s = \sum_{j\in J} \mu'_{sj} l_j$ for some $\mu'_{sj} \in k$ and an easy calculation shows that we can write h as $\sum_{j\in J} l_j \otimes f_j$, where $f_j = \sum_{s\in S} \mu'_{sj} h_s$.

By assumption $\Phi(h)$ is null homotopic, so it induces the zero map on the homology modules. For any $n \in \mathbb{Z}$ we infer

$$\begin{aligned} 0 = H_n(\Phi(h)) &= H_n\left(\sum_{j\in J} l_j (L \otimes f_j)\right) \\ &= \sum_{j\in J} l_j H_n(L \otimes f_j) \\ &= \sum_{j\in J} l_j (L \otimes H_n(f_j)). \end{aligned} \tag{1}$$

The last equality follows by exactness of $L \otimes_k -$. Now $\sum_{j\in J} l_j(L \otimes H_n(f_j))$ is an element of $\mathrm{Hom}_{L\otimes A}(L \otimes H_n(C), L \otimes H_n(D))$ which is, by Lemma 6.2.1, isomorphic to $L \otimes \mathrm{Hom}_A(H_n(C), H_n(D))$. Thus (1) is equivalent to

$$0 = \sum_{j\in J} l_j \otimes H_n(f_j).$$

Let $(b_t)_{t\in T}$ be a k-basis of $\mathrm{Hom}_A(H_n(C), H_n(D))$. We write each $H_n(f_j)$ as a linear combination $\sum_{t\in T} \nu_{tj} b_t$ and obtain

$$\begin{aligned} 0 &= \sum_{j\in J} l_j \otimes H_n(f_j) \\ &= \sum_{j\in J} l_j \otimes \sum_{t\in T} \nu_{tj} b_t \\ &= \sum_{j\in J} \sum_{t\in T} \nu_{tj} (l_j \otimes b_t). \end{aligned}$$

But $(l_j \otimes b_t)_{j\in J, t\in T}$ is a k-basis for $L \otimes \mathrm{Hom}_A(H_n(C), H_n(D))$ and thus $\nu_{tj} = 0$ for all $t \in T$ and all $j \in J$. Since $H_n(f_j) = \sum_{t\in T} \nu_{tj} b_j$ it follows that $H_n(f_j) = 0$ for all $j \in J$.

Finally, we conclude from $H_n(f_j) = 0$ that f_j is null homotopic: This is true since $\mathscr{K}^b(A\text{-proj})(C, D) \cong \mathscr{D}^b(A)(C, D)$ for the bounded complexes of projectives C and D. Then $H_n(f_j) = 0$ means that f_j is mapped to zero under this isomorphism. Therefore f_j must be zero in $\mathscr{K}^b(A\text{-proj})(C, D)$, so f_j is null homotopic as a chain map. □

Lemma 6.3.2. *Let $A = kQ/I$ and let C and D be complexes in $\mathscr{C}^b(A\text{-proj})$. Then $L \otimes_k \mathscr{K}^b(A\text{-mod})(C, D)$ and $\mathscr{K}^b((L \otimes_k A)\text{-mod})(L \otimes_k C, L \otimes_k D)$ are isomorphic as L-vector spaces.*

Proof. By Corollary 6.2.2 we know that $L \otimes \mathscr{C}^b(A\text{-proj})(C, D)$ is isomorphic to $\mathscr{C}^b((L \otimes A)\text{-proj})(L \otimes C, L \otimes D)$ via the isomorphism Φ which induces the map

$$\begin{aligned} \Phi' : L \otimes \mathscr{K}^b(A\text{-proj})(C, D) &\longrightarrow \mathscr{K}^b((L \otimes_k A)\text{-proj})(L \otimes C, L \otimes D) \\ \sum \lambda_j \otimes [f_j] &\longmapsto \sum \lambda_j [L \otimes f_j]. \end{aligned}$$

We check that Φ' is well defined. Let $f \in L \otimes_k \mathscr{C}^b(A\text{-mod})(C, D)$ with $f = \sum \lambda_j \otimes f_j$ such that all f_j are null homotopic. Then $\Phi'(\sum \lambda_j \otimes [f_j]) = \sum \lambda_j [L \otimes f_j]$. By exactness of $L \otimes_k -$ we have

$$H_n\left(\sum \lambda_j (L \otimes f_j)\right) = \sum \lambda_j (L \otimes H_n(f_j)) = 0$$

and a similar argument as in the Proof of Lemma 6.3.1 shows that $\sum \lambda_j (L \otimes f_j)$ is null homotopic.

Lemma 6.3.1 implies that Φ' is injective. So it remains to show that Φ' is surjective. This follows from the commutative diagram

$$\begin{tikzcd} L \otimes \mathcal{C}^b(A\text{-proj})(C, D) \arrow[r, "\Phi"] \arrow[d] & \mathcal{C}^b((L \otimes_k A)\text{-proj})(L \otimes C, L \otimes D) \arrow[d, "\pi"] \\ L \otimes \mathcal{K}^b(A\text{-proj})(C, D) \arrow[r, "\Phi'"] & \mathcal{K}^b((L \otimes_k A)\text{-proj})(L \otimes C, L \otimes D) \end{tikzcd}$$

since the composition of maps $\pi\Phi$ is surjective. □

Theorem 6.3.3. *Let $A = kQ/I$ and let T be a chain complex in $\mathscr{K}^b(A\text{-proj})$. If $\mathscr{K}^b(A\text{-proj})(T, T) \cong kQ'/I'$ as k-algebras then $\mathscr{K}^b((L \otimes_k A)\text{-proj})(L \otimes_k T, L \otimes_k T)$ is isomorphic to $LQ'/(L \otimes_k I')$ as an L-algebra.*

In other words, if the k-algebra $\mathrm{End}(T)$ *has the quiver with relations (Q', I') then the L-algebra* $\mathrm{End}(L \otimes_k T)$ *is given by the same quiver with the same relations.*

Proof. Application of Lemma 6.3.2 shows that there exists an isomorphism

$$\Phi' : \mathscr{K}^b((L \otimes_k A)\text{-proj})(L \otimes T, L \otimes T) \longrightarrow L \otimes_k \mathscr{K}^b(A\text{-mod})(T, T)$$

of L-vector spaces. It is easy to check that Φ' respects the L-algebra structure and therefore is an isomorphism of L-algebras.

Now it follows from $\mathscr{K}^b(A\text{-proj})(T, T) \cong kQ'/I'$ and proposition 6.1.2 that $L \otimes \mathscr{K}^b(A\text{-proj})(T, T)$ is isomorphic to $LQ'/(L \otimes_k I')$ as an L-algebra. Together this yields

$$\mathscr{K}^b((L \otimes_k A)\text{-proj})(L \otimes T, L \otimes T) \cong LQ'/(L \otimes_k I'). \qquad \square$$

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