Hyperbolicity and $\partial$-irreducibility of alternating tangles

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Abstract

We consider primeness, hyperbolicity, $\partial$-irreducibility and tangle sums of alternating tangles. We also study primeness and hyperbolicity of links and Dehn surgeries on knots admitting alternating tangle decompositions. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

A tangle is a pair $(B, T)$, where $B$ is a 3-ball and $T$ is a properly embedded 1-submanifold. Note that $T$ may have circle components. A tangle is called an $n$-string tangle if $T$ consists of $n$ properly embedded arcs (with no circle component). Two tangles are equivalent if they are homeomorphic as pairs. The trivial tangle is a tangle equivalent to $(D \times I, \{x_1, \ldots, x_n\} \times I)$, where $D$ is a disc, $I$ is an interval and the $x_i$'s are points lying in the interior of $D$. We draw a tangle diagram $E$ of $T$ on an equatorial disc $D$ of $B$.

An alternating tangle is a tangle which admits an alternating tangle diagram. W.B.R. Lickorish and M.B. Thistlethwaite [3] proved that a 2-string tangle admitting a strongly alternating diagram is non-trivial. Thistlethwaite [11] showed that a reduced alternating diagram for a 2-string tangle is a minimal diagram. C. Hayashi proved that if a tangle admits a reduced connected alternating diagram then it is non-split. See [1]. The author studied parallelism of two strings in alternating tangles and laminar surgeries on knots in $S^3$ which have some alternating tangle decompositions in [10].

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In this paper we consider primeness, hyperbolicity, $\partial$-irreducibility and tangle sums of alternating tangles by using methods developed by W. Menasco and Thistlethwaite in [4–7]. We show that if a tangle has an alternating diagram satisfying certain conditions, then it is prime, hyperbolic and $\partial$-irreducible. To prove primeness and hyperbolicity, by using results of Hayashi, Menasco and Y. Nakanishi [2,8], it follows that we have only to consider indivisibility and weak-anannularity of alternating tangles. As corollaries, we show primeness and hyperbolicity of some class of links in $S^3$ which have suitable alternating tangle decompositions. For example, we prove that a prime semi-alternating link [3] is hyperbolic. We also show if a knot has an alternating tangle decomposition satisfying certain conditions, then any non-trivial Dehn surgery on it yields a Haken 3-manifold. For a tangle obtained by a tangle sum of two tangles, we show that any of its alternating tangle diagrams either is a partial sum of two alternating tangle diagrams or satisfies a certain condition.

In the remainder of this section we state the results of this paper. First we deal with indivisibility and primeness. A tangle is indivisible if for any properly embedded disc $F$ in $B$ which intersects $T$ in one point, the closure of a component of $B - F$ meets $T$ in the trivial 1-string tangle. A tangle diagram $E$ on a disc $D \subset B$ is connected if each properly embedded arc in $D$ disjoint from $E$ cobounds a subdisc together with a subarc of $\partial D$ which does not meet $E$, and any properly embedded circle in $D$ disjoint from $E$ bounds a disc which does not meet $E$. A tangle diagram is reduced if it has no nugatory crossings as in Fig. 1. A tangle diagram $E$ is indivisible if each properly embedded arc in $D$ meeting $E$ in one point cobounds a pair consisting of a subdisc and a trivial arc together with a subarc of $\partial D$. A tangle diagram $E$ is locally trivial if each circle in $D$ meeting $E$ in two points bounds a pair consisting of a subdisc and a trivial arc.

If there is no ambiguity, for short we shall write $T$ instead of $(B, T)$.

**Theorem 1.1.** If a tangle $T$ has a reduced connected locally trivial indivisible alternating diagram, then $T$ is indivisible.

A tangle $T$ is non-split if for each properly embedded disc $F$ in $B$ disjoint from $T$, a component of $B - F$ does not meet $T$. A tangle $T$ (respectively a link $L$ in $S^3$) is locally
trivial if each 2-sphere in \( B \) (respectively \( S^3 \)) intersecting \( T \) (respectively \( L \)) in exactly two points bounds a 3-ball which meets \( T \) (respectively \( L \)) in the trivial 1-string tangle. A tangle is prime if it is non-split, locally trivial and indivisible. A link is prime if it is non-split and locally trivial.

Theorem 1.2. If a tangle \( T \) has a reduced connected locally trivial indivisible alternating diagram, then \( T \) is prime.

As a corollary of Theorem 1.2 we can show primeness of some class of links in \( S^3 \).

Corollary 1.3. Suppose a link \( L \) in \( S^3 \) has a tangle decomposition into two tangles admitting reduced connected locally trivial indivisible alternating diagrams. Then \( L \) is prime.

Next we consider weak-anannularity and hyperbolicity. A tangle is weakly-anannular if any properly embedded annulus in \( B \) disjoint from \( T \) is compressible in \( B \) and parallel to \( \partial B \) or \( \partial T \). Otherwise we say the tangle is strongly-annular. See [8].

Theorem 1.4. Suppose a tangle \( T \) is strongly-annular and has a reduced connected locally trivial indivisible alternating diagram \( E \). Then \( E \) is as in Fig. 2.

For 2-string tangles, we can show the following.

Corollary 1.5. If a 2-string tangle \( T \) has a reduced connected locally trivial alternating diagram, then \( T \) is weakly-anannular.
A torus in $B - T$ is essential if it is incompressible and it is not parallel to $\partial N(t)$, where $t$ is a component of $T$. A tangle is atoroidal if there is no essential torus in $B - T$. A tangle is hyperbolic if it is prime, weakly-annular and atoroidal.

**Theorem 1.6.** If a tangle $T$ has a reduced connected locally trivial indivisible alternating diagram which is not as in Fig. 2, then $T$ is hyperbolic.

**Corollary 1.7.** If a 2-string tangle $T$ has a reduced connected locally trivial alternating diagram, then $T$ is hyperbolic.

As corollaries, we can show hyperbolicity of some class of links in $S^3$. An annulus in $S^3 - \text{int} N(L)$ is essential if it is incompressible and it is not parallel to $\partial N(L)$. A link $L$ is anannular if there is no essential annulus in $S^3 - \text{int} N(L)$. Otherwise we say it is annular. A torus in $S^3 - L$ is essential if it is incompressible and it is not parallel to $\partial N(L)$. A link $L$ is atoroidal if there is no essential torus in $S^3 - L$. A link $L$ in $S^3$ is hyperbolic if it is prime, anannular and atoroidal.

**Corollary 1.8.** Suppose a link $L$ in $S^3$ has a tangle decomposition into two tangles admitting reduced connected locally trivial indivisible alternating diagrams which are not as in Fig. 2. Then $L$ is hyperbolic.

**Corollary 1.9.** Suppose a link $L$ in $S^3$ has a tangle decomposition into two 2-string tangles admitting reduced connected locally trivial alternating diagrams. Then $L$ is hyperbolic.

One is referred to the definition of a semi-alternating link in [3].

**Corollary 1.10.** A prime semi-alternating link is hyperbolic.

Next we consider tangle sums. A marked tangle is a triple $(B, T, \Delta)$, where $(B, T)$ is a tangle and $\Delta$ is a disc on $\partial B$ containing two endpoints of $T$. We call $\Delta$ a gluing disc. Given two marked tangles $(B_1, T_1, \Delta_1)$ and $(B_2, T_2, \Delta_2)$, we can obtain a new tangle $(B, T)$ as follows. Take a map $\phi: \Delta_1 \to \Delta_2$ with $\phi(\Delta \cap T_1) = \Delta \cap T_2$, and use it to glue two tangles to get $(B, T)$. This operation is called tangle sum, and we write $T = T_1 + T_2$. A tangle sum is non-trivial if $(B_1, T_1, \Delta_1)$ is neither $M[0]$ nor $M[\infty]$ (See [12].) This generalizes the definitions of marked tangle, tangle sum and non-trivial tangle sum in [12].

A marked tangle diagram is a triple $(D, E, \alpha)$, where $E$ is a tangle diagram on a disc $D$ and $\alpha$ a subarc in $\partial D$ which contains two endpoints of $E$. The arc $\alpha$ is called a gluing arc. Given two marked tangle diagrams $(D_1, E_1, \alpha_1)$ and $(D_2, E_2, \alpha_2)$, we can obtain a new tangle diagram $E$ on $D = D_1 \cap D_2$ as follows. Take a homeomorphism $f: \alpha_1 \to \alpha_2$ with $f(\alpha_1 \cap E_1) = \alpha_2 \cap E_2$. Then we glue two tangle diagrams using $f$ and get a new tangle diagram $E$. This operation is called partial sum and we write $E = E_1 + E_2$. This generalizes the notion defined in the 2-string case in [3]. A partial sum is non-trivial if both $E_1$ and $E_2$ are connected and contain crossings.
Theorem 1.11. Let $T$ be a tangle obtained by a non-trivial sum of two tangles $T_1$ and $T_2$. Suppose $T$ has a reduced connected alternating diagram $E$. Then either:

1. $E$ is a non-trivial partial sum of two alternating tangle diagrams, or
2. $E$ is as in Fig. 3 and one of the $T_i$ is a non-trivial sum of two tangles.

Moreover in both cases $T_1$ and $T_2$ admit alternating diagrams.

This generalizes a result in [11], where the 2-string case is studied. Note that if $T$ has a reduced alternating tangle diagram which is a non-trivial partial sum of two tangle diagrams, then $T$ is a non-trivial partial sum of two tangles. (See Proposition 4.1.)

Next we consider $\partial$-irreducibility. Let $E(T) = B - \text{int} N(T)$ be the exterior of $T$. A tangle is $\partial$-irreducible if $\partial E(T)$ is incompressible in $E(T)$. Otherwise it is $\partial$-reducible. In case an alternating diagram $E$ of $T$ is a partial sum of two tangle diagrams $E_1$ and $E_2$, let $T_1$ and $T_2$ denote the subtangles of $T$ whose diagrams are $E_1$ and $E_2$, respectively. The terminologies used below can be found in [12].

Theorem 1.12. Suppose a tangle $T$ is $\partial$-reducible and has a reduced connected locally trivial alternating diagram $E$. Then $E$ is a non-trivial partial sum of two alternating tangle diagrams $E_1$ and $E_2$. Furthermore if $T$ is a 2-string tangle, then $T_1$ is the 2-twist tangle, $T_2$ is a rational tangle, and $E(T)$ is a handlebody.

From Theorem 1.12 we can show the following results. Let $K(r)$ denote the 3-manifold obtained by Dehn surgery on $K$ along the slope $r$.

Corollary 1.13. Suppose a knot $K$ in $S^3$ has a decomposition into two $n$-string tangles $(B_1, T_1)$ and $(B_2, T_2)$ admitting a reduced connected alternating diagrams $E_1$ and $E_2$, and $E_1$ is not a non-trivial partial sum of two tangle diagrams. Then, for any non-meridional
slope $r$, $K(r)$ is Haken. Furthermore, if $E_2$ is not a non-trivial partial sum of two tangle diagrams and each of $E_1$ and $E_2$ is locally trivial and not as in Fig. 2, then $K(r)$ is a hyperbolic Haken manifold.

**Corollary 1.14.** Suppose a knot $K$ in $S^3$ has a decomposition into two 2-string tangles $(B_1, T_1)$ and $(B_2, T_2)$ admitting reduced connected locally trivial alternating diagrams $E_1$ and $E_2$, and $E_1$ is not a partial sum of diagrams of the 2-twist tangle and a rational tangle. Then $K(r)$ is a hyperbolic Haken manifold for any non-meridional slope $r$.

Note that from Corollary 1.9, a knot in Corollary 1.14 is hyperbolic. In [10] it is shown that if a knot has a tangle decomposition into two 2 or 3-string tangles admitting reduced connected locally trivial alternating diagrams, then $K(r)$ is laminar.

2. Preliminaries, local triviality, non-splittability, indivisibility and primeness

In this section we give some preparatory results and prove Theorems 1.1 and 1.2 and Corollary 1.3.

First we refer to results of local triviality and non-splittability of alternating tangles. Let $(B, T)$ be a tangle.

**Proposition 2.1.** If a tangle $T$ has a reduced connected locally trivial alternating diagram, then $T$ is locally trivial.

**Proof.** See [10, Theorem 4.1]. \(\square\)

Hayashi showed the following [1].

**Proposition 2.2.** If a tangle $T$ has a reduced connected alternating diagram, then $T$ is non-split.

Suppose $T$ has a reduced connected indivisible locally trivial alternating diagram $E$ on $D$, where $D$ is an equatorial disc of $B$. We assume that all endpoints of the strings lie in $\partial D$.

As in [4], we place a bubble at each crossing of the diagram $E$ and isotope $T$ so that the overstrand at the crossing runs on the upper hemisphere and the understrand runs on the lower hemisphere.

Let $D_+$ (respectively $D_-$) be the disc obtained from $D$ with each equatorial disc inside a bubble replaced by the upper (respectively lower) hemisphere of the bubble. We define $B_+$ (respectively $B_-$) to be the 3-ball cobounded by $D_+$ (respectively $D_-$) together with a hemisphere of $\partial B$ such that $B_\pm$ does not intersect the interiors of the bubbles. We use the notation $D_\pm$ (respectively $B_\pm$) to mean $D_+$ or $D_-$ (respectively $B_+$ or $B_-$).

Let $F$ be a surface properly embedded in $B$ meeting $T$ transversely. A properly embedded arc or a loop in $F$ is $T$-essential if it is disjoint from $T$ and essential in the
punctured surface \( F - T \). A surface \( F \) is \( T \)-compressible if there is a \( T \)-essential loop on \( F \) which bounds a disc in \( B - F \) disjoint from \( T \). The disc is called a \( T \)-compressing disc. Otherwise \( F \) is \( T \)-incompressible.

Let \( F \) be a \( T \)-incompressible surface. As in [5], we isotope \( F \) to be in a suitable position with respect to the tangle diagram \( E \).

**Proposition 2.3.** We can isotope \( F \) so that:

(1) \( F \) meets \( D_{+} \) transversely in a pairwise disjoint collection of simple closed curves and properly embedded arcs in \( F \).

(2) \( F \) meets each 3-ball bounded by a bubble in a collection of saddles.

For a \( T \)-incompressible surface \( F \) satisfying the conclusions of Proposition 2.3, we define the complexity \( c(F) \) of \( F \) to be the lexicographically ordered pair \((t, u)\), where \( t \) is the number of the saddles of \( F \) and \( u \) is the total number of components of \( F \cap D_{+} \) and \( F \cap D_{-} \). We say \( F \) has minimal complexity if \( c(F) \leq c(F') \) for any surface \( F' \) isotopic to \( F \).

From now on we assume \( F \) has minimal complexity. A point of \( F \cap T \) is called a puncture.

**Proposition 2.4.**

(1) Each component of \( F \cap B_{\pm} \) is a disc.

(2) Each loop of \( F \cap D_{\pm} \) meets a bubble or a puncture.

**Proof.** See [5, Lemma 4] and [4, Lemma 1]. \( \square \)

Now we shall begin to prove Theorem 1.1. We will show its contrapositive. Suppose \((B, T)\) is divisible. Then there is a properly embedded disc \( F \subset B \) such that \( F \) meets \( T \) in one point and that the closure of each component of \( B - F \) meets \( T \) in some tangle which is not the trivial 1-string tangle. We call such a disc a dividing disc. We will show that \( E \) is divisible.

We assume \( F \) has minimal complexity among all dividing discs.

**Proposition 2.5.** No arc or loop of \( F \cap D_{\pm} \) meets a bubble more than once.

**Proof.** Suppose not. There are two cases. If the component of \( F \cap D_{\pm} \) meets the same side of the bubble with respect to \( T \) at least twice, as in [4, Lemma 1(2)] we can reduce the number of saddles, which contradicts the minimal complexity of \( F \).

Hence we assume the component meets distinct sides of the bubble. Then there is a disc \( D_{1} \subset B \) meeting \( T \) in one point with \( D_{1} \cap F = \partial D_{1} \). Surger \( F \) with \( D_{1} \) and obtain a 2-sphere \( P \) meeting \( T \) in two points, and a properly embedded disc \( F' \) meeting \( T \) in one point. Since \( T \) is locally trivial by Proposition 2.1, it follows that \( F' \) is a dividing disc with less complexity, which is a contradiction. \( \square \)

**Lemma 2.6.** There is no loop of \( F \cap D_{\pm} \).
Proof. Since there is exactly one puncture, the argument of [4, Lemma 2] will do.

Lemma 2.7. An arc of $F \cap D_{\pm}$ meets a bubble or the puncture.

Proof. Suppose there is an arc, say $\gamma$, in $F \cap D_{\pm}$ which does not meet a bubble or the puncture. Since $E$ is connected, $\gamma$ cobounds a subdisc, say $D_1$, in $D$ together with a subarc of $\partial D$. We assume $D_1$ is an outermost one of such discs. On the other hand, $F$ is divided into two subdiscs $F_1$ and $F_2$ by $\gamma$. Since $D_1 \cap E$ is empty, either $D_1 \cup F_1$ or $D_1 \cup F_2$ cobounds a 3-ball disjoint from $T$ together with a subdisc of $\partial B$. Then we can isotope $F$ using this 3-ball to reduce the complexity of $F$, which is a contradiction.

Let $S$ denote the set of all saddles and $G = (F \cap (D_+ \cup D_-)) \cup S$. We consider $G$ as a kind of a graph on $F$ whose vertices are saddles and whose edges are arcs of $F \cap D$.

First we consider a part as shown in Fig. 4. That is, there is a saddle $s$ and there are two arcs of $F \cap D$ such that each arc connects $s$ and $\partial F$ and misses the puncture, and the arcs are not adjacent at $s$.

Lemma 2.8. $G$ does not contain a part as shown in Fig. 4.

Proof. See [10, Lemma 3.1].

Now we consider a particular part of $G$ as shown in Fig. 5. That is, there are two arcs $\alpha \subset F \cap D_+$ and $\beta \subset F \cap D_-$ and a saddle $s$ such that $\alpha$ and $\beta$ are incident to exactly one saddle $s$. We call this a fork.

Proof of Theorem 1.1. Suppose there is a saddle. By Lemma 2.6, there is no loop in $F \cap D_{\pm}$. Applying a standard outermost fork argument to the graph $G$ in $F$, we can find two forks in $G$. Since there is exactly one puncture, at least one of them contains a part violating Lemma 2.8.
Fig. 5.

Hence there is no saddle. By Proposition 2.4(2) and Lemma 2.7, \( F \cap D_\pm \) consists of an arc meeting the puncture. Then \( E \) is divisible by this arc. \( \Box \)

**Proof of Theorem 1.2.** It follows from Theorem 1.1 and Propositions 2.1 and 2.2. \( \Box \)

**Proof of Corollary 1.3.** It follows from Theorem 1.2 and [8, Theorem 2(2)]. \( \Box \)

### 3. Weak-anannularity, atoroidality and hyperbolicity

In this section we prove Theorems 1.4 and 1.6 and Corollaries 1.5, 1.7, 1.8, 1.9 and 1.10.

Suppose \( T \) is strongly-annular. That is, there is an annulus \( A \) properly embedded in \( B \) disjoint from \( T \) such that \( A \) is incompressible in \( B - T \), not isotopic to a component of \( \partial N(T) - \partial B \) and not parallel to \( \partial B - \partial T \). The annulus \( A \) intersects \( D \), otherwise \( A \) is contained in \( B_\pm \), which contradicts that \( A \) is incompressible. Since \( A \) is \( T \)-incompressible, we assume that \( A \) satisfies Propositions 2.3 and 2.4 and has minimal complexity.

Let \( F \subset B \) be a properly embedded surface meeting \( T \) transversely (if it meets). Then \( F \) is called *pairwise incompressible* if for each disc \( Q \subset B \) meeting \( T \) transversely in one point, with \( Q \cap F = \partial Q \), there is a disc \( Q' \subset F \) meeting \( T \) transversely in one point [4].

**Proposition 3.1.** No arc or loop of \( A \cap D_\pm \) meets a bubble more than once.

**Proof.** Suppose not. If the component of \( A \cap D_\pm \) meets the same side of the bubble with respect to \( T \) at least twice, as in Proposition 2.5 we have a contradiction.

Hence we assume the component meets distinct sides of the bubble. Then, as in [4, Lemma 1(2)], it follows that \( A \) is pairwise compressible. Then \( A \) can be meridionally surgered into two properly embedded discs \( D_1 \) and \( D_2 \). Let \( B_i \) \((i = 1 \text{ or } 2)\) be the 3-ball cobounded by \( D_i \) and a subdisc of \( \partial B \) such that \( B_1 \) and \( B_2 \) are disjoint. Let \( B_3 = \text{cl}(B - (B_1 \cup B_2)) \) and \( T_i = T \cap B_i \) \((i = 1, 2 \text{ or } 3)\), where \( \text{cl}(\cdot) \) means the closure.
Applying Theorem 1.1, it follows that at least one of \((B_1, T_1)\) and \((B_2, T_2)\) is the trivial 1-string tangle. Suppose \((B_1, T_1)\) and \((B_2, T_2)\) are trivial. From Proposition 2.1, \(T\) is locally trivial. It follows that \(A\) is isotopic to some component of \(\partial N(T) - \partial B\). This is a contradiction. Suppose \((B_1, T_1)\) is trivial and \((B_2, T_2)\) is non-trivial. Then by applying Theorem 1.1, \((B_3, T_3)\) is the trivial 1-string tangle, otherwise \(D_2\) would be a dividing disc. It follows that \(A\) is parallel to \(\partial B - \partial T\), which is a contradiction. This completes the proof.

**Lemma 3.2.** There is no loop of \(A \cap D_\pm\).

**Proof.** See [4, Lemma 1].

**Lemma 3.3.** An arc of \(A \cap D_\pm\) meets a bubble.

**Proof.** Suppose there is an arc, say \(\gamma\), in \(A \cap D_\pm\) which does not meet a bubble. Since \(E\) is connected, \(\gamma\) cobounds a disc, say \(D_1\), disjoint from \(E\) in \(D\) together with a subarc of \(\partial D\). We assume \(D_1\) is an outermost one of such discs.

Suppose both ends of \(\gamma\) lie in the same component of \(\partial A\). Then \(A\) is divided into a subdisc \(D_2\) and a subannulus \(A_1\) by \(\gamma\). The disc \(D_1 \cup D_2\) cobounds a 3-ball \(B_1\) together with a subdisc of \(\partial B\) such that \(B_1\) does not meet \(T\) nor \(\text{int} A_1\). Then we can isotope \(A\) so as to reduce the complexity using \(B_1\), which is a contradiction.

Hence the ends of \(\gamma\) lie in the distinct components of \(\partial A\). Then \(A\) is \(\partial\)-compressible by \(D_1\). It follows that either \(A\) is parallel to \(\partial B - \partial T\) or compressible in \(B - T\). In either case we have a contradiction.

Now we consider forks in \(A\). We do not require that arcs of \(A \cap D_\pm\) of a fork be boundary parallel in \(A\). Let \(G = (A \cap (D_+ \cup D_-)) \cup S\).

**Lemma 3.4.** There is no fork in \(G\).

**Proof.** If there is a fork, then we can find a part which violates Lemma 2.8.

**A cycle of** \(G\) **is a subgraph of** \(G\) **homotopic to** \(S^1\).

**Proof of Theorem 1.4.** As mentioned, \(A \cap D\) is not empty. By Lemmas 3.2 and 3.3, there is no loop in \(A \cap D_\pm\) and each arc of \(A \cap D_\pm\) meets a bubble. Hence there are saddles. From Lemma 3.4, there is no fork.

We show that \(G\) is as in Fig. 6. That is, for each saddle two adjacent arcs of \(A \cap D\) incident to it connect with the same component of \(\partial A\), and the others connect with other saddles.

First we show that there is a cycle in \(G\) which is an essential loop in \(A\). Suppose not. Let \(s\) be a saddle of \(G\). There are four arcs of \(A \cap D\) incident to \(s\). Since there is no loop in \(A \cap D_\pm\) by an outermost fork argument we can find a fork in \(G\), which contradicts
Lemma 3.4. If there is another cycle, then it contradicts Proposition 2.4(1) or Lemma 3.2. Hence there is exactly one cycle. Let $C$ be the cycle in $G$.

Let $s$ be a saddle $C$ meets. Note that there are two arcs, say $a$ and $b$, of $T \cap D$ incident to $s$ which are not contained in $C$. Suppose the end of $a$ or $b$ away from $s$ connects with another saddle. Then by applying an outermost fork argument and from Lemma 3.2, we can find a fork, which violates Lemma 3.4. Hence the end of these away from $s$ connects with $\partial A$. If their two ends are not adjacent at $s$, then $a$, $b$ and $s$ violate Lemma 2.8. Hence they are adjacent. Then $G$ is as in Fig. 6. From now on we consider $C$ consists of arcs of $T \cap D$ and subarcs of boundary of saddles between two ends of arcs.

There are two cases. First suppose that there is no arc of $A \cap D_\pm$ meeting exactly two saddles. Then $G$ is as in Fig. 7(1) and $E$ is as in Fig. 7(2). Hence the theorem follows.

So we assume there is an arc of $A \cap D_\pm$ meeting exactly two saddles. Let $\gamma$ be an outermost arc in $D$ which divides $D$ into two subdiscs $D_1$ and $D_2$ such that $D_1$ is
outermost. Then we can find subarcs of $C$ in $D_1$ and $D_2$. Let $B_1$ and $B_2$ be the bubbles which $\gamma$ meets and $s_1$ and $s_2$ saddles incident to $\gamma$ which are contained in $B_1$ and $B_2$ respectively. The subarc of $C$ which entered $D_1$ at $B_1$ exits at $B_2$ via a saddle, say $s_3$. Since $\gamma$ is outermost in the subarc of $C$ between $s_1$ and $s_3$ which is contained in $D_1$, two adjacent saddles lie in distinct sides of $C$ in $A$. See Fig. 8. Then $E$ in $D_1$ is as in Fig. 9. This completes the proof of Theorem 1.4. □

**Proof of Corollary 1.5.** It follows from Theorems 1.4. □

For a proof of Theorem 1.6, we need the following proposition.
Proposition 3.5 [10, Theorem 4.2]. If a tangle $T$ has a reduced connected locally trivial alternating diagram, then $T$ is atoroidal.

Proof of Theorem 1.6. It follows from Theorems 1.2 and 1.4 and Proposition 3.5. □

Proof of Corollary 1.7. It follows from Theorem 1.6. □

Proof of Corollary 1.8. From [2, Theorem 3.6.6], a link admitting a decomposition into two hyperbolic tangles is hyperbolic. Hence Corollary 1.8 follows from Theorem 1.6. □

Proof of Corollary 1.9. It follows from Corollary 1.8. □

See [3] for the definition of strongly alternating tangle diagrams.

Proof of Corollary 1.10. Let $L$ be a prime semi-alternating link. A strongly alternating tangle diagram is reduced and connected. Since $L$ is prime, two strongly alternating tangle diagrams of a semi-alternating diagram of $L$ are reduced, connected and locally trivial. Then the conclusion follows from Corollary 1.9. □

4. Tangle sums

In this section we prove Theorem 1.11. First we prove the following proposition.

Proposition 4.1. If a tangle $T$ admits a reduced alternating diagram $E$ which is a non-trivial partial sum of two tangle diagrams $E_1$ and $E_2$, then $T$ is a non-trivial sum of two tangles $T_1$ and $T_2$ such that $E_i$ is a diagram of $T_i$.

Proof. Let $T_1$ and $T_2$ denote the subtangles of $T$ whose diagrams are $E_1$ and $E_2$, respectively. From Proposition 2.1, $T$ is non-split. Hence neither $T_1$ nor $T_2$ is $M[\infty]$. Suppose, for a contradiction, that $T_1$ is $M[0]$. Let $T'_2$ be the 2-twist tangle (see [12]) and $E'_2$ a marked tangle diagram of $T'_2$ as in Fig. 10. Let $E'$ be a partial sum of $E_1$ and $E'_2$ and $T'$ the tangle whose diagram is $E'$. Then, by construction, $T'$ is the trivial 2-string tangle. On the other hand, as $E'$ is a strongly alternating diagram, $T'$ is non-trivial by [3, Corollary 5.1], which is a contradiction. □

Let $F$ be a surface properly embedded in $B$ meeting $T$ transversely. A surface $F$ is $T$-$\partial$-compressible if there is a $T$-essential arc on $F$ which cobounds a disc in $B - F$ disjoint from $T$ together with an arc in $\partial B$. The disc is called a $T$-$\partial$-compressing disc. Otherwise $F$ is $T$-$\partial$-incompressible.

Let $(B, T) = (B_1, T_1, \Delta_1) + (B_2, T_2, \Delta_2)$, where neither $(B_i, T_i, \Delta_i)$ is $M[0]$ nor $M[\infty]$. For terminologies see [12].

Let $F$ be the gluing disc in $B$. Then $F$ meets $T$ transversely in two points.

Lemma 4.2. $F$ is $T$-$\partial$-incompressible.
Fig. 10.

**Proof.** Suppose not. Then there is a $T$-$\partial$-compressing disc which separates two strings of $T_i$ ($i = 1$ or 2). Since $T$ is locally trivial by Proposition 2.1, $T_i$ is $M[0]$. This is a contradiction. □

We take $D_{\pm}$ and $B_{\pm}$ as in Section 2. We assume $F$ satisfies Propositions 2.3 and 2.4 and has minimal complexity among all dividing discs.

**Proposition 4.3.** $F \cap D_{\pm}$ satisfies the following properties:

1. no arc or loop of $F \cap D_{\pm}$ meets a bubble more than once.
2. no arc or loop of $F \cap D_{\pm}$ meets both a bubble and an arc of $T \cap D$ having an endpoint on that bubble in a puncture.

**Proof.** For (1), as in Proposition 2.5 we assume the component which meets a bubble more than once meets the distinct sides of the bubble. Then, as in Proposition 2.5, we surger $F$ and obtain a disc $F'$ and a sphere $P$ each of which meets $T$ in two points. Then $F'$ is a gluing disc with less complexity. Otherwise $F'$ is $T$-$\partial$-compressible. This implies that $F$ is $T$-$\partial$-compressible, which contradicts Lemma 4.2.

For (2) see [5, Lemma 3(a)]. □

Let $G = (F \cap (D_+ \cup D_-)) \cup S$.

**Lemma 4.4.** There is no fork in $G$ which does not contain a puncture.

**Proof.** It follows from Lemma 2.8. □

**Lemma 4.5.** If there is a loop in $F \cap D_{\pm}$, then $E$ is as in Fig. 3. Moreover, $T_1$ and $T_2$ have alternating tangle diagrams and one of the $T_i$’s is a non-trivial sum of two tangles.

**Proof.** From [4, Lemma 2] it follows that there is only one loop, say in $F \cap D_+$, and there is no loop in $F \cap D_-$. Moreover the loop in $F \cap D_+$ meets two punctures and two saddles. If there is another saddle, we can find a fork without a puncture in $G$. This contradicts
Lemma 4.4. Hence there are exactly two saddles. Then $G$ in $F$ is as in Fig. 11(1). Then we can show that $E$ is as in Fig. 11(2). Let $E_1$, $E_2$ and $E_3$ be subtangle diagrams as in Fig. 11(2). Let $T_1$ be a subtangle of $T$ whose diagram contains $E_1$ and $E_3$. Then obviously $T_2$ has an alternating tangle diagram. $T_1$ also has an alternating tangle diagram $E'$ which is a partial sum of two tangle diagrams as in Fig. 12. Since $E$ is reduced, $E'$ is also reduced. Then, by Proposition 4.1, $T_1$ is a non-trivial sum of two tangles. 

\[\text{Lemma 4.6. An arc of } F \cap D_{\pm} \text{ meets a bubble or a puncture.} \]

**Proof.** Suppose there is an arc, say $\gamma$, in $F \cap D_{\pm}$ which does not meet a bubble or a puncture. Since $E$ is connected, $\gamma$ cobounds a subdisc, say $D_1$, in $D$ together with a subarc of $\partial D$, which does not meet $E$. Note that arcs of $F \cap D_{\pm}$ in $D_1$ never meet bubbles or punctures. We assume $D_1$ is an outermost one of such discs. From Lemma 4.2, $\gamma$ is $T$-
inessential in $F$. Then, as in Lemma 2.7, we have a contradiction to the minimal complexity of $F$. □

**Proof of Theorem 1.11.** In case there is a loop in $F \cap D$, by Lemma 4.5 the theorem follows. Hence we assume there is no loop. First we assume that there are saddles. Then by an outermost fork argument, we can find at least two forks in $G$. If there is one which does not meet a puncture, it contradicts Lemma 4.4. So there are exactly two forks in $G$ and each contains a puncture. Then by Lemma 2.8, one of two arcs of $F \cap D$ of a nice fork meets a puncture and the other does not. Let $\gamma$ be the arc meeting the puncture and $B$ the bubble which $\gamma$ meets. Then $\gamma$ divides $D$ into two subdiscs. Let $D_1$ be the subdisc which does not contain the crossing of $B$. There are two arcs of $T \cap D$ entering $D_1$. If there is a crossing in $D_1$, then $E$ is a non-trivial partial sum of two tangle diagrams. So we assume there is no crossing in $D_1$. If two arcs of $T \cap D$ connect, then this contradicts Proposition 4.3(2). Hence two arcs go to $\partial D$. Then the crossing of $B$ is a nugatory crossing. See Fig 13.

Next we assume that there is no saddle. If there is an arc of $F \cap D$ which meets exactly one puncture, $F \cap D$ consists of two arcs. Then one of $(B_1, T_1, \Delta_1)$ is $M[0]$, which is a contradiction. Hence $F \cap D$ consists of just one arc meeting two punctures. Then $E$ is a non-trivial partial sum of two alternating tangle diagrams. This completes the proof of Theorem 1.11. □

5. $\partial$-irreducibility

In this section we prove Theorem 1.12 and Corollaries 1.13 and 1.14. Suppose $T$ is $\partial$-reducible. First we consider a special case where $T$ is a 2-string tangle.

**Lemma 5.1.** Suppose $T$ is a 2-string tangle and has a reduced connected locally trivial alternating diagram $E$. Furthermore, suppose $E$ is a non-trivial partial sum of two tangle diagrams $E_1$ and $E_2$. Then up to relabeling of $T_1$, $T_1$ is the 2-twist tangle and $T_2$ is a rational tangle, in which case $E(T)$ is a handlebody, where $T_1$ is a subtangle of $T$ whose diagram is $E_1$. 
Proof. From Proposition 3.5, \( T \) is atoroidal. Note that if one of \( T_i \) is toroidal, then \( T \) is also toroidal. Hence \( T_1 \) and \( T_2 \) are atoroidal. By Proposition 4.1, \( T \) is a non-trivial sum of \( T_1 \) and \( T_2 \). Then by [12, Lemma 3.3] the lemma follows. \( \square \)

Hence, to prove Theorem 1.12 we have only to show that \( E \) is a non-trivial partial sum of two tangle diagrams.

Let \( t_i \) be a string of \( T \) and \( c(t_i) \) the number of crossings of \( E \) incident to \( t_i \).

**Lemma 5.2.** For any \( t_i \subset T \), \( c(t_i) \geq 2 \).

**Proof.** Otherwise \( E \) would be split or have a nugatory crossing. \( \square \)

**Lemma 5.3.** If there is a string \( t_i \) of \( T \) with \( c(t_i) = 2 \), then \( E \) is a non-trivial partial sum of two alternating tangle diagrams.

**Proof.** If there is another crossing than those incident to \( t_i \), then since \( E \) is reduced, \( E \) is a non-trivial partial sum of two alternating tangle diagrams. If there is no other crossing, then \( E(T) \) is homeomorphic to \( T \times I \). But \( T \times I \) is \( \partial \)-irreducible, which is a contradiction. \( \square \)

Let \( F \) be a compressing disc for \( \partial E(T) \) in \( E(T) \). As we did so far, we consider the intersection of \( F \) and \( D_\pm \). Since \( \partial F \) meets \( \partial N(T) \), we reform \( D_\pm \) and \( B_\pm \) as in [7].

Let \( \tilde{D}_\pm \) be the disc we made in Section 2. We take a regular neighborhood \( N \) of \( T \) whose diameter is small in relation to bubbles. Let \( D_+ \) (respectively \( D_- \)) be the disc obtained from \( \tilde{D}_+ \) (respectively \( \tilde{D}_- \)) with the annulus \( N \cap \tilde{D}_+ \) (respectively \( N \cap \tilde{D}_- \)) replaced by the upper (respectively lower) annulus of \( \partial N \). We define \( B_+ \) (respectively \( B_- \)) to be the 3-ball bounded by \( S_+ \) (respectively \( S_- \)) which does not intersect the interiors of the bubbles.

We define a crossing-ball of a crossing of \( E \) to be a 3-ball which consists of the 3-ball we placed at the crossing and the regular neighborhood of the intersection of \( T \) and the bubble at the crossing. We use \( X_i \) to denote some crossing-ball. A component of \( \partial N - \bigcup X_i \) which is an annulus is called a segment. A subarc of \( \partial F \) which lies between two adjacent points of \( F \cap D_\pm \) is called a boundary edge. A boundary edge \( e \) is type I if \( e \) meets a crossing ball, and type II otherwise. The notions of crossing-balls, segments and boundary edges are defined in [7].

**Proposition 5.4.** We can isotope \( F \) so that:

1. \( F \) meets \( D_\pm \) transversely in a pairwise disjoint collection of arcs and simple closed curves.
2. \( F \) meets each 3-ball bounded by a bubble in a collection of saddles.
3. \( \partial F \) proceeds along \( \partial N \) monotonely with respect to the longitudinal coordinate of \( \partial N \); within each segment \( P \) of \( \partial N \), \( P \cap \partial F \) proceeds monotonely with respect to the meridional coordinate of \( \partial N \).
(4) There is a collar $C \cong I \times \partial F$ of $\partial F$ in $F$ and a projection $p : C \to \partial F$ such that for each $x \in \partial F \cap \partial X_i$, the fiber $p^{-1}(x)$ is a straight line segment which is normal to $\partial X_i$ and which does not meet the interior of $X_i$.

(5) No arc of $F \cap D_+ \cap X_i$ or $F \cap D_- \cap X_i$ meets $X_i - \partial N$.

Proof. See [7, Proposition 2.1].

For a compressing disc $F$ satisfying the conclusions of Proposition 5.4, we define the complexity $c(F)$ of $F$ to be the lexicographically ordered triple $(t, u, v)$, where $t$ is the number of the saddles of $F$, $u$ is the total number of components of $F \cap D_+$ and $F \cap D_-$, and $v$ is the number of the boundary edges. We say $F$ has minimal complexity if $c(F) \leq c(F')$ for any compressing disc $F'$ for $\partial E(T)$. From now on we assume $F$ has minimal complexity.

Proposition 5.5.

(1) Each component of $F \cap B_{\pm}$ is a disc.

(2) Each loop of $(\text{int} F) \cap D_{\pm}$ meets a bubble.

(3) No arc or loop of $F \cap D_{\pm}$ meets a crossing-ball or a segment more than once.

(4) A boundary edge which meets $\partial N(\partial t_i)$ is type I.

Proof. For (1), see [10, Proposition 2.5(1)].

For (2), see [4, Lemma 1].

For (3), see [7, Proposition 2.2(ii)].

For (4), if there is a type II boundary edge, say $e$, meeting $\partial N(\partial t_i)$. We can isotope $F$ near $\partial N(t_i)$ so as to remove $e$, which contradicts the minimal complexity of $F$. □

Lemma 5.6. There is no loop of $(\text{int} F) \cap D_{\pm}$.

Proof. See [6, Proposition 4]. □

Lemma 5.7. There is no loop of $F \cap D_{\pm}$ which contains exactly one boundary edge and meets just one saddle.

Proof. Suppose not. Let $\gamma \subset F \cap D_+$ be the loop. If the boundary edge is type II, $\gamma$ is as in Fig. 14. Then we can find a component of $F \cap D_-$ violating Proposition 5.5(3). For example, see Fig. 14. Hence it is type I. Then from the alternating property (for example, see [4]) $\gamma$ is as in Fig. 15. This contradicts the fact that $E$ is locally trivial. □

Let $G = (\text{int} F) \cap (D_+ \cup D_-) \cup S$. A face of $G$ is the closure of a component of $F - G$, that is, the closure of a component of $F \cap B_{\pm}$. We consider particular faces of $G$. A face $f$ of $G$ is simple if it does not meet a saddle and $|f \cap \partial F| = 1$, where $|\cdot|$ means the number of components. An arc of $F \cap D_{\pm}$ is simple if it is contained in the boundary of a simple face.
Lemma 5.8. A simple face contains exactly two boundary edges.

Proof. See [10, Lemma 3.2].

Lemma 5.9. If $G$ contains a simple face, then $E$ is a non-trivial partial sum of two alternating tangle diagrams.

Proof. Let $\gamma$ be a simple arc. Then from the alternating property, $\gamma$ is as in Fig. 16. Then we can find a proper arc in $D$ meeting $E$ in two points. Hence $E$ is a partial sum of two alternating tangle diagrams $E_1$ and $E_2$. Since $E_1$ and $E_2$ contains a crossing and $E$ is reduced, the partial sum is non-trivial.

A face $f$ of $G$ is nice if it meets exactly one saddle and $|f \cap \partial F| = 1$. An arc of $F \cap D_{\pm}$ is nice if it is contained in the boundary of a nice face.
Let $f$ be a nice face and $\alpha$ be the nice arc of $F \cap D_{\pm}$ in the boundary of $f$. We consider three kinds of nice faces as in Fig. 17: to be more precise, (1) $\alpha$ does not contain a boundary edge, (2) $\alpha$ contains exactly one boundary edge, (3) $\alpha$ contains exactly two boundary edges.

**Lemma 5.10.** A nice face is as in (1), (2) or (3) of Fig. 17.

**Proof.** See [10, Lemma 3.3]. □

**Lemma 5.11.** If $G$ contains a nice face as in (2) of Fig. 17, then $E$ is a non-trivial partial sum of two alternating tangle diagrams.

**Proof.** Let $f$ be a nice face as in (2) and $\alpha$ the nice arc in the boundary of $f$. From the alternating property, $\alpha$ is as in Fig. 18. Then, as in Lemma 5.9, we can show that $E$ is a non-trivial partial sum of two alternating tangle diagrams. □

A part of $(F \cap (D_+ \cup D_-)) \cup S$ is called a nice fork if it is the union of two nice arcs $\alpha \subset F \cap D_+$ and $\beta \subset F \cap D_-$ and a saddle $s$ such that $\alpha$ and $\beta$ meet exactly one saddle $s$.

**Lemma 5.12.** A nice fork is as in (1), (2), (3), (4) or (5) of Fig. 19.
Proof. See the proof of [10, Lemma 3.4]. □

Lemma 5.13. If there is a nice fork, then $E$ is a non-trivial partial sum of two tangle diagrams.

Proof. From Lemma 2.8 there is no nice fork as in (1). For a nice fork as in (2), (3) or (4), it contains a nice face as in (2) of Fig. 17. By Lemma 5.11, the lemma follows. For a nice fork as in (5), there is a component of $\partial F \cap N(T)$ which contains exactly two boundary edges. Since a boundary edge meets at most one crossing ball, $c(t_i) = 2$. By Lemma 5.3, the lemma follows. □
Proof of Theorem 1.12. If there is a simple face, the theorem follows from Lemma 5.9. Hence suppose there is no simple face. Since from Lemma 5.6 there is no loop in \((\text{int} F) \cap D_4\), by applying a standard outermost fork argument we can find a nice fork in \(G\). Then the theorem follows from Lemma 5.13. \(\square\)

To prove Corollaries 1.13 and 1.14, we need the following proposition, which is essentially proved in Section 3 of [13].

**Proposition 5.14.** Suppose a knot \(K\) in \(S^3\) has a decomposition into two non-split \(n\)-string tangles \((B_1, T_1)\) and \((B_2, T_2)\). Suppose \(T_1\) is \(\partial\)-irreducible. Then for any non-meridional slope \(r\), \(K(r)\) is Haken. Furthermore, if \(T_2\) is also \(\partial\)-irreducible and each of \(T_1\) and \(T_2\) is atoroidal and weakly-anannular, then \(K(r)\) is a hyperbolic Haken manifold.

**Proof.** Let \(H\) denote the genus \(n\) handlebody \(B_2 \cup N(T_1)\). Then \(S^3 = E(T_1) \cup H\). From the assumption, \(\partial H\) is incompressible in \(E(T_1)\). Let \(D_i\) be the meridian disc of a string \(t_i\) of \(T_1\). Then \(K\) intersects each \(D_i\) in one point. Let \(M = H - \text{int} N(K)\) and \(U_i = D_i \cap M\). Since \(T_2\) is non-split, \(\partial E(T_2) - N(T_2)\) is incompressible in \(E(T_2)\). Then it follows \(\partial H\) is incompressible in \(M\). Let \((H, K; r)\) denote the 3-manifold obtained from \(H\) by Dehn surgery on \(K\) along the slope \(r\). Since there are at least two non-parallel meridian discs, by Theorem 4 of [4], \(\partial H\) is also incompressible in \((H, K; r)\) after any non-trivial Dehn surgery.

Note that both \(E(T_1)\) and \(H\) are irreducible. Since \(K\) meets each \(D_i\) in one point, \(K\) is not cabled in \(H\). Then \((H, K; r)\) is irreducible by [9]. It follows that \(K(r)\) is irreducible. Hence \(K(r)\) is Haken.

To prove that \(K(r)\) is hyperbolic, by the argument in the second paragraph of the proof of [13, Theorem 3.3], we have only to show that \(K(r)\) is atoroidal. So suppose, for a contradiction, that \(K(r)\) is toroidal. Since \(E(T_1)\) is atoroidal, it follows that either \((H, K; r)\) is toroidal or an essential torus meets \(\partial H\). First suppose, for a contradiction, that \((H, K; r)\) is toroidal. Then as in the proof of [13, Lemma 3.4], it follows that \(E(T_2)\) is toroidal or strongly-annular, which is a contradiction. Hence an essential torus in \(K(r)\) meets \(\partial H\). Then by the argument in the proof of [13, Lemma 3.5] and the last paragraph of the proof of [13, Theorem 3.3], it follows that \(E(T_1)\) is strongly-annular or \(E(T_2)\) is \(\partial\)-reducible or strongly-annular, which contradicts the assumption. This completes the proof. \(\square\)

**Proof of Corollary 1.13.** From the assumption of the former half, Proposition 2.2 and Theorem 1.12, it follows that \(T_1\) and \(T_2\) are non-split and \(T_1\) is \(\partial\)-irreducible. Hence, by Proposition 5.14, \(K(r)\) is Haken.

From the assumption of the later half, Theorem 1.12, Proposition 3.5 and Theorem 1.4, it follows that \(T_2\) is also \(\partial\)-irreducible and \(T_1\) and \(T_2\) are atoroidal and weakly-anannular. Hence, by Proposition 5.14, \(K(r)\) is a hyperbolic Haken manifold. \(\square\)

**Proof of Corollary 1.14.** By Proposition 2.2, \(T_1\) and \(T_2\) is non-trivial, by Theorem 1.12, \(T_1\) is \(\partial\)-irreducible, and, by Proposition 3.5, \(T_1\) and \(T_2\) are atoroidal. Hence from [13, Theorem 3.3], \(K(r)\) is hyperbolic and Haken. \(\square\)
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References