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ScienceDirect
Theoretical Computer Science 365 (2006) 258 – 282
Theoretical Computer Science
www.elsevier.com/locate/tcs

Towards a descriptive set theory for domain-like structures[☆]

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Abstract

This is a survey of results in descriptive set theory for domains and similar spaces, with the emphasis on the ω-algebraic domains. We try to demonstrate that the subject is interesting in its own right and is closely related to some areas of theoretical computer science. Since the subject is still in its beginning, we discuss in detail several open questions and possible future development. We also mention some relevant facts of (effective) descriptive set theory.

Keywords: Polish space; φ-Space; Algebraic domain; Effective space; Borel hierarchy; Difference hierarchy; Wadge reducibility; ω-Boolean operation

1. Introduction

Classical descriptive set theory (DST) [30,33,28] classifies definable sets and functions in Polish spaces by means of hierarchies, reducibilities and set-theoretic operations. This theory is old, well developed and has many applications e.g. to analysis and model theory. Different motivations require to consider problems typical to DST for spaces distinct from the Polish spaces, or for spaces with additional structure of some kind. E.g., the so-called effective DST [35,22,33], which is closely related to computability theory, studies effective versions of notions and results of the classical DST for different classes of effective spaces.

In this paper, we give an account of few attempts to develop a DST for some classes of T_0-spaces closely relevant to domain theory (we will refer to this area as "domain DST"). Note that all interesting spaces in domain theory are not Hausdorff, and consequently not Polish (we will recall some relevant definitions of topological notions in the next section). The reason for development of such a domain DST is the prominent role played by different classes of domains in some areas of theoretical computer science and the fact that definable sets of different kind are important in many cases. Though DST has a rather abstract and topological flavor, ideas, notions and results of (effective) DST appear again and again in different areas of theoretical computer science. The reason is that computability and complexity notions are intimately related to definability notions.

Though some earlier results of computability theory (say, the Rice–Shapiro theorem) are in the spirit of the (effective) domain DST, there are only few papers specially devoted to this field. The earliest papers known to me are Tang's papers [60,61] developing some DST for the well-known domain $P\omega$ and the author's papers [37–40,42] where some effective

☆ The work partially supported by a DAAD project within the program "Ostpartnerschaften".
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0304-3975/$ - see front matter
doi:10.1016/j.tcs.2006.07.053

domain DST was developed as a tool to solve some questions in computability theory. More recently, the author tried to develop the non-effective domain DST in a more systematic way [50–53]. Along with discussing the main results of the mentioned papers, we discuss also some applications, open problems, and the related material from (effective) classical DST. We omit almost all proofs and give only references to the source papers. A couple of exceptions is made for short proofs not presented explicitly in the literature.

Now a few words about our terminology in domain theory. In domain theory there are two terminological traditions. The first tradition (going back to Scott [36], see also [14,1,58] and references therein) tends to use the language of partially ordered sets (posets). The second tradition (going back to Ershov [7,8]) tends to use topological language. As is well-known (see e.g. [12]) the both approaches are closely interconnected and even, in a sense, almost equivalent. Though the poset terminology is now dominating in the literature, in this paper we use mainly the topological terminology for the following reasons: first, it is convenient when one treats domain DST in parallel to the classical DST, as we do here. As a result, some facts of the classical DST may be generalized to include also facts of domain DST; second, the topological terminology is not restricted to the directed complete posets (as is usual within the poset terminology), hence it is quite appropriate for considering effective spaces which are often non-complete.

Nevertheless, our choice of the topological terminology should make no problem for the readers used to the poset terminology. The reason is that, for simplicity of formulations, we confine ourselves here essentially to the well-known ω-algebraic domains which in the topological language correspond to the complete countably based φ-spaces.

In Section 2, we briefly recall definitions of spaces discussed in this paper, and in Section 3, we consider the effective versions of some of those spaces. Section 4 is devoted to the Borel hierarchy. In Section 5, we discuss analytic sets, while Section 6 is devoted to the difference hierarchy. In Section 7, we consider results on the Wadge reducibility, and in Section 8 some results on a natural class of set-theoretic operations. In Section 9, we discuss some applications and relations of the topic of this paper to some other fields, and we conclude in Section 10.

2. Some classes of spaces

In this section we briefly recall some well-known definitions, fix notation and define some less known classes of spaces studied in this paper.

A *metric space* is a pair (X, d) with X a set and d a function (called metric) from $X \times X$ to non-negative reals such that: $d(x, y) = 0$ iff $x = y$, $d(x, y) = d(y, x)$ and $d(x, y) \leqslant d(x, z) + d(z, y)$. If the last inequality is strengthened to $d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$ then d is an *ultrametric*. A sequence $\{x_n\}$ in a metric space is *Cauchy* if for every $\varepsilon > 0$ there is a k such that $d(x_m, x_n) < \varepsilon$ for all $m, n > k$. A metric space is *complete* if every Cauchy sequence in X converges to a point in X.

A *topological space* (or simply a *space*) is a pair $(X, \mathcal{T})$ with X a set and $\mathcal{T}$ a collection of subsets of X closed under arbitrary unions and finite intersections. Such a collection is called a *topology* on X and its elements *open* sets. A subset of X is *closed (clopen)* if its complement is open (resp., if it is both open and closed). The *closure* of a set $A \subseteq X$ is the intersection of all closed supersets of A. A subset of X is *dense* if its closure is X. A *basis* in X is a class $\mathcal{B}$ of open sets such that every open set is a union of sets from $\mathcal{B}$. When a metric (a topology) on X is clear from the context we do not mention it explicitly and refer to X as a metric (resp., a topological) space.

We denote spaces by letters $X, Y, \ldots$, elements of spaces (points) by $x, y, \ldots$ (for concrete examples of spaces also special notation may be used), subsets of spaces (pointsets) by $A, B, \ldots$ and classes of subsets of spaces (pointclasses) by $\mathcal{A}, \mathcal{B}, \ldots$. By $P(X)$ we denote the powerset of X, i.e. the class of subsets of X. By $\overline{A}$ we denote the complement of a set $A \subseteq X$, i.e. $\overline{A} = X \setminus A$ and by $co\text{-}\mathcal{A} = \{\overline{A} | A \in \mathcal{A}\}$—the dual of a pointclass $\mathcal{A}$. Let $\mathcal{A} \cdot \mathcal{B} = \{A \cap B | A \in \mathcal{A}, B \in \mathcal{B}\}$; in the case when $\mathcal{A} = \{A\}$ is a singleton we simplify the notation $\{A\} \cdot \mathcal{B}$ to $A \cdot \mathcal{B}$. The domain and range of a function f are denoted, respectively, by $dom(f)$ and $rng(f)$, the composition of functions f and g by $f \circ g$ or just by fg (thus, $(f \circ g)(x) = f(g(x))$), the value $f(x)$ of f on x is often simplified to f_x. We assume the reader to be acquainted with the notion of ordinal see e.g. [31]. The first non-countable ordinal is denoted ω_1.

A couple of times we will mention some properties of pointclasses popular in classical DST. Recall [28] that a class $\mathcal{A}$ has the *separation property* if for all disjoint $A, B \in \mathcal{A}$ there is $C \in \mathcal{A} \cap co\text{-}\mathcal{A}$ with $A \subseteq C \subseteq \overline{B}$ and that $\mathcal{A}$ has the *reduction property* if for all $A, B \in \mathcal{A}$ there are disjoint sets $A', B' \in \mathcal{A}$ with $A' \subseteq A$, $B' \subseteq B$ and $A' \cup B' = A \cup B$.

A space X is

- *zero-dimensional* if every open set is a union of clopen sets;
- *countably based* if there is a countable basis in X;

- *compact* if for every class $\mathcal{C}$ of open sets with $\cup\mathcal{C} = X$ there is a finite class $\mathcal{F} \subseteq \mathcal{C}$ with $\cup\mathcal{F} = X$;
- *Hausdorff* if for all distinct points $x, y \in X$ there exist disjoint open sets A, B with $x \in A$, $y \in B$;
- *a T_0-space* if for all two distinct points in X there exists an open set A that contains one of these points and does not contain the other;
- *metrizable* (*ultrametrizable*) if there is a metric (resp., an ultrametric) d on X such that every open set is a union of sets of the form $\{y \in X | d(x, y) < r\}$, where $x \in X$ and r is a positive real;
- *Polish* if it is countably based and metrizable with a metric d such that (X, d) is a complete metric space.

Note that every metrizable (and thus every Polish) space is Hausdorff. The classical DST is usually developed for the class of Polish spaces. As a reference to the classical DST we recommend [28]. The most important (for DST) examples of Polish spaces are Baire and Cantor spaces (their definitions are recalled below) and many spaces of interest in analysis, including of course the space $\mathbf{R}$ of reals.

Let X, Y be spaces. A function $f : X \to Y$ is

- *continuous* if the preimage $f^{-1}(A)$ of every open set A in Y is an open set in X;
- *a homeomorphism* if it is bijective, continuous and the inverse function $f^{-1} : Y \to X$ is continuous;
- *a retraction* if it is continuous and there is a continuous function $s : Y \to X$ (called *section*) with $fs = id_Y$, where id_Y is the identity function on Y;
- *a quasiretraction* if it is continuous and for every continuous function $g : Y \to Y$ there is a continuous function $\tilde{g} : X \to X$ such that $gf = f\tilde{g}$. Note that every retraction is a quasiretraction.

A *subspace* of a space $(X, \mathcal{T})$ is a subset $A \subseteq X$ equipped with the topology $A \cdot \mathcal{T}$. Spaces X and Y are *homeomorphic* if there is a homeomorphism of X onto Y; X is *a retract* (*a quasiretract*) of Y if there is a retraction (resp., a quasiretraction) $r : Y \to X$. It is well-known that if X is a retract of Y and s, r is a witnessing section–retraction pair then s is a homeomorphism of X onto the subspace $s(X)$ of Y.

There are many interesting constructions on spaces of which we mention only the cartesian product $X \times Y$ and the space Y^X of continuous functions from X to Y with the topology of pointwise convergence. For definitions see any standard text in topology, say [30].

Let ω^* be the set of finite sequences (strings) of natural numbers. The empty string is denoted by $\emptyset$, the concatenation of strings σ, τ by $\sigma\tau$. By $\sigma \sqsubseteq \tau$ we denote that the string σ is an initial segment of the string τ (please be careful in distinguishing $\sqsubseteq$ and $\subseteq$). Let ω^ω be the set of all infinite sequences of natural numbers (i.e., of all functions $\xi : \omega \to \omega$). For $\sigma \in \omega^*$ and $\xi \in \omega^\omega$, we write $\sigma \sqsubseteq \xi$ to denote that σ is an initial segment of the sequence ξ. Define a topology on ω^ω by taking arbitrary unions of sets of the form $\{\xi \in \omega^\omega | \sigma \sqsubseteq \xi\}$, $\sigma \in \omega^*$, as open sets. The space ω^ω with this topology known as the *Baire space* is of primary importance for DST.

For every n, $1 < n < \omega$, let n^* be the set of finite strings of elements of $\{0, \ldots, n-1\}$, $n^* \subseteq \omega^*$. E.g., 2^* is the set of finite strings of 0's and 1's. For $\sigma \in n^*$ and $\xi \in n^\omega$, the relation $\sigma \sqsubseteq \xi$ and the space n^ω are defined in the same way as in the previous paragraph. It is well-known that for each n, $2 \leqslant n < \omega$, the space n^ω is homeomorphic to the space 2^ω called the *Cantor space*. The Cantor space is a closed subspace of the Baire space. They are not homeomorphic because Cantor space is compact while Baire space is not.

Next we recall some definitions from domain theory. Let X be a T_0-space. For $x, y \in X$, let $x \leqslant y$ denote that $x \in U$ implies $y \in U$, for all open sets U. The relation $\leqslant$ is a partial order known as the *specialization order*. Let $F(X)$ be the set of *finitary elements* of X (known also as *compact elements*), i.e. elements $p \in X$ such that the upper cone $O_p = \{x | p \leqslant x\}$ is open. Such open cones are called *f-sets*. The space X is called a *φ-space* if every open set is a union of *f*-sets. A φ-space X is called a *φ_0-space* if $(X; \leqslant)$ contains a least element (denoted $\bot$). Note that every non-discrete φ-space is not Hausdorff. The φ-spaces were introduced in [42] under the name "generalized *f*-spaces". An effective version of φ-spaces (so called numbered sets with approximation, see Section 3) was introduced by Ershov in the context of the theory of numberings in the late sixties. The term 'φ-space' was coined in [12].

A φ-space X is *complete* if every non-empty directed set S without greatest element has a supremum $supS \in X$, and supS is a limit point of S (notice that $supS \notin F(X)$ and for each finitary element $p \leqslant supS$ there is $s \in S$ with $p \leqslant s$). As is well-known, every φ-space is canonically embeddable in a complete φ-space which is called the *completion* of X (see e.g. [12,1,14]).

For simplicity of formulations we state main results of this paper mostly for the complete countably based φ-spaces which are in a bijective correspondence with the ω-algebraic domains. Some results are valid only for more restricted classes of spaces. Important in this respect is the class of *f*-spaces introduced in [7]; these are φ-spaces with the property that if two finitary elements have an upper bound under the specialization order then they have a least upper bound. Bottomed *f*-spaces are called f_0-spaces. Complete f_0-spaces essentially coincide with the Scott domains [12]. From time to time we consider also topped φ-spaces (the top element is usually denoted by $\top$). Topped f_0-spaces are essentially the Scott continuous lattices. Standard references in domain theory are [1,58,14]. For correspondences between the poset and topological languages see [12].

Now we define two more special classes of spaces which are important for this paper. The notions and results studied below in this section are taken from [52]. The notions of reflective and 2-reflective spaces are non-effective versions of the corresponding effective notions introduced and studied in [40,42] (see also the next section).

Definition 1. By a reflective space we mean a complete φ_0-space X for which there exist continuous functions $q_0, e_0, q_1, e_1 : X \to X$ such that $q_0e_0 = q_1e_1 = id_X$ and $e_0(X), e_1(X)$ are disjoint open sets.

Define continuous functions $s_k, r_k (k < \omega)$ on X by $s_0 = e_0,\ s_{k+1} = e_1s_k$ and $r_0 = q_0,\ r_{k+1} = r_kq_1$. Let also $D_k = s_k(X)$. The following result shows that the reflective spaces look rather self-similar, i.e. their structure resembles the structure of fractals.

Proposition 2. *In each reflective space X, the following properties hold true*:
(i) *for every* $k < \omega, r_ks_k = id_X$;
(ii) *the sets* D_k *are open, pairwise disjoint and satisfy* $D_k = \{x|s_k(\bot) \leqslant x\}$. *Thus*, $\{\overline{\bigcup_k D_k}, D_0, D_1 \ldots\}$ *is a partition of X*.

Now we consider some examples of reflective spaces. Let $\omega^{\leqslant\omega}$ be the completion of the partial ordering $(\omega^*; \sqsubseteq)$. Of course, $\omega^{\leqslant\omega} = \omega^* \cup \omega^\omega$ consists of all finite and infinite strings of natural numbers. For every $2 \leqslant n < \omega$, let $n^{\leqslant\omega}$ be obtained in the same way from $(n^*; \sqsubseteq)$. Thus, $n^{\leqslant\omega} = n^* \cup n^\omega$ consists of all finite and infinite words over the alphabet $\{0, \ldots, n-1\}$. From the well-known properties of completions it follows that $\omega^{\leqslant\omega}$ and $n^{\leqslant\omega}$ are complete countably based f_0-spaces.

Let $\omega^\omega_\bot$ be the space of partial functions $g : \omega \rightharpoonup \omega$ with the usual structure of an *f*-space (as is usual in domain theory, we identify the partial function g with the total function $\tilde{g} : \omega \to \omega_\bot = \omega \cup \{\bot\}$ where $g(x)$ is undefined iff $\tilde{g}(x) = \bot$, for some "bottom" element $\bot \notin \omega$). For each n, $2 \leqslant n < \omega$, let $n^\omega_\bot$ be the space of partial functions $g : \omega \rightharpoonup \{0, \ldots, n-1\}$ defined similarly to $\omega^\omega_\bot$. As is well-known, $\omega^\omega_\bot$ and $n^\omega_\bot$ are complete countably based f_0-spaces.

Finally, let $\mathbf{U}$ be the space of all open subsets of the Cantor space 2^ω distinct from the biggest open set 2^ω. It is well-known that $(\mathbf{U}; \subseteq)$ is a complete countably based f_0-space, finitary elements being exactly the clopen subsets of 2^ω distinct from 2^ω.

Proposition 3. *The spaces* $\omega^{\leqslant\omega}, n^{\leqslant\omega}, \omega^\omega_\bot, n^\omega_\bot$ *and* $\mathbf{U}$ *are reflective*.

The next result states that the class of reflective spaces has some natural closure properties, hence there are many more natural examples of them than the last proposition suggests.

Theorem 4. (i) *If X is a reflective space and Y a complete* φ_0*-space then* $X \times Y$ *is a reflective space*.
(ii) *If X is an* f_0*-space and Y a reflective* f_0*-space then* Y^X *is a reflective f-space*.

Let us relate the introduced spaces one to another and to some other spaces. First we formulate a minimality property of the spaces $\omega^{\leqslant\omega}$ and $n^{\leqslant\omega}$ and a well-known maximality property of $\mathbf{U}$.

Theorem 5. (i) *The spaces* $\omega^{\leqslant\omega}$ *and* $n^{\leqslant\omega} (2 \leqslant n < \omega)$ *are retracts of an arbitrary reflective space X*.
(ii) *Every complete countably based* f_0*-space is a retract of* $\mathbf{U}$.

Next we relate the introduced spaces to the Baire and Cantor spaces ω^ω and n^ω $(2 \leqslant n < \omega)$.

Proposition 6. (i) $n^\omega_\bot$ *is a retract of* $(n+1)^\omega_\bot$ *and* $\omega^\omega_\bot$.
(ii) n^ω *is a retract of* ω^ω.
(iii) n^ω *is a subspace of* $n^\omega_\bot$, *and* ω^ω *is a subspace of* $\omega^\omega_\bot$
(iv) $\omega^{\leqslant\omega}$ *is a quasiretract of* ω^ω, $n^{\leqslant\omega}$ *is a quasiretract of* $(n+1)^\omega$, *and* $\omega^\omega_\bot, n^\omega_\bot$ *are quasiretracts of* ω^ω.

Now we define the second class of spaces properties of which are in a sense similar to the properties of reflective spaces.

Definition 7. By a 2-reflective space we mean a complete φ_0-space X with a top element $\top$ such that there exist continuous functions $q_0, e_0, q_1, e_1 : X \to X$ and open sets B_0, C_0, B_1, C_1 with the following properties:
(i) $q_0 e_0 = q_1 e_1 = id_X$;
(ii) $B_0 \supseteq C_0$ and $B_1 \supseteq C_1$;
(iii) $e_0(X) = B_0 \setminus C_0$ and $e_1(X) = B_1 \setminus C_1$;
(iv) $B_0 \cap B_1 = C_0 \cap C_1$.

Remarks. The classes of reflective and 2-reflective spaces are disjoint. The sections e_0, e_1 are embeddings and their ranges are disjoint.

Define continuous functions $s_k, r_k (k < \omega)$ on X by $s_0 = e_0$, $s_{k+1} = e_1 s_k$ and $r_0 = q_0$, $r_{k+1} = r_k q_1$. Let $D_k = s_k(X)$. Define also the sets $E_k, F_k (k < \omega)$ by $E_0 = B_0$, $E_{k+1} = e_1(E_k) \cup C_1$, $F_0 = C_0$, $F_{k+1} = e_1(F_k) \cup C_1$. The "self-similarity" property now looks as follows.

Proposition 8. *In each 2-reflective space X the following holds true*:
(i) *for each* $k < \omega$, $r_k s_k = id_X$;
(ii) *for each* $k < \omega$, E_k, F_k *are open*, $E_k \supseteq F_k$ *and* $D_k = E_k \setminus F_k$;
(iii) *for each* $k < \omega$, $D_k = \{x | s_k(\bot) \leqslant x \leqslant s_k(\top)\}$ *and* $s_k(\bot) \in F(X)$;
(iv) *for all* $k \neq m$, $E_k \cap E_m = F_k \cap F_m$;
(v) $(\overline{\bigcup_k E_k}, \bigcup_k F_k, D_0, D_1, \ldots)$ *is a partition of X*.

Now we look at some examples of 2-reflective spaces. Let $\omega_\top^{\leqslant\omega}$ be the completion of the partial ordering $(\omega^* \cup \{\top\}; \sqsubseteq)$ which is obtained from the ordering $(\omega^*; \sqsubseteq)$ by adding a top element $\top \notin \omega^*$ bigger than all the other elements. Let $n_\top^{\leqslant\omega}$ (for any $2 \leqslant n < \omega$) be defined in the same way from the partial ordering $(\omega^* \cup \{\top\}; \sqsubseteq)$.

Let $(C_\omega; \leqslant)$ be the completion of the partial ordering $(A_\omega; \leqslant)$ defined as follows:

$A_\omega = \{(0, \sigma), (1, \sigma) | \sigma \in \omega^*\}$;

$(0, \sigma) \leqslant (0, \tau)$ iff $\sigma \sqsubseteq \tau$; $(1, \sigma) \leqslant (1, \tau)$ iff $\sigma \sqsupseteq \tau$;

$(0, \sigma) \leqslant (1, \tau)$ iff $\sigma \sqsubseteq \tau \vee \tau \sqsubseteq \sigma$; $(1, \sigma) \nleqslant (0, \tau)$.

Let the space $(C_n; \leqslant)$ be defined in the same way from the partial ordering $(A_n; \leqslant)$ for every n, $2 \leqslant n < \omega$, which is defined just as above, only for $\sigma, \tau \in n^*$.

From the properties of completions it follows that $\omega_\top^{\leqslant\omega}, n_\top^{\leqslant\omega}, (C_\omega; \leqslant)$ and $(C_n; \leqslant)$ are topped complete countably based f_0-spaces (hence, continuous lattices).

Finally, let $(P\omega; \subseteq)$ be the well-known continuous lattice formed by the powerset of ω with the Scott topology, hence finitary elements of $P\omega$ are exactly the finite subsets of ω.

Proposition 9. *The spaces* $(C_\omega; \leqslant)$, $(C_n; \leqslant)$ *and* $P\omega$ *are 2-reflective.*

Next we state that the class of 2-reflective spaces has some natural closure properties, hence there are many more natural examples of them than the last proposition suggests.

Theorem 10. (i) *If X is a 2-reflective space and Y a topped complete* φ_0*-space then* $X \times Y$ *is a 2-reflective space.*
(ii) *If X is an* f_0*-space and Y a 2-reflective f-space then* Y^X *is a 2-reflective f-space.*

The last two results of this section relate the spaces introduced above to some other spaces. First we state a minimality property of the spaces C_ω and C_n and a well-known maximality property of $P\omega$.

Theorem 11. (i) *The spaces* $\omega_\top^{\leqslant\omega}, n_\top^{\leqslant\omega}$ C_ω *and* C_n $(2 \leqslant n < \omega)$ *are retracts of an arbitrary 2-reflective space X.*
(ii) *Every complete countably based continuous lattice is a retract of* $P\omega$.

Finally, we relate some of the 2-reflective spaces to some spaces considered above.

Proposition 12. (i) *$P\omega$ is a retract of $\omega_{\perp}^{\omega}$ and a quasiretract of ω^{ω}.*

(ii) *n^{ω} (ω^{ω}) is a subspace of $n_{\top}^{\leqslant\omega}$ (respectively, of $\omega_{\top}^{\leqslant\omega}$).*

3. Effective spaces

Topological considerations play an important role in several parts of theoretical computer science including semantics of programming languages, theory of infinite computations, model checking and computability in analysis. In some applications of the topological notions it is necessary to consider effective versions of them, e.g., effective topology instead of topology and computable functions instead of continuous functions. For this reason there is a big literature on such effective topological and domain-theoretic notions. The effective notions are usually based on ideas and results from the theory of numberings [9–11].

In DST, such effective notions are also important because they are inevitable for development, say, effective versions of the classical hierarchies which are used for classifications of different objects from computability theory (see e.g. [33,22,35]). Unfortunately, notions and terminology in the effective topology and effective domain theory are not completely established, there are too many different approaches sometimes incompatible with alternative ones (see e.g. [7,12,1,59,54]).

In this section we fix some effectivity notions suitable for the subsequent discussion. Our terminology bears on the fact that there are two different approaches to effective topology. The first approach, which we call here "constructive", considers only spaces containing computable points thus confining itself with countable structures. The second approach, which we call here "effective", is more liberal and applies to many "classical" spaces. Both approaches of course assume some effectivity conditions, say on basic open sets or on finitary elements. We attach the adjectives "constructive" and "effective" according to the point of view we choose, although our usage of these words sometimes contradicts to their meaning in some other papers. The effective and constructive approaches do not contradict each other because it is often possible to define constructive points within a given "effective" space, and form a "constructive" space from those points.

A basic notion of the effective classical DST is that of effective metric space. From several known variations of this notion we choose the following very general one [68,18]: an *effective metric space* is a triple (X, d, δ), where (X, d) is a complete metric space and $\delta : \omega \to X$ is a numbering of a dense subset $rng(\delta)$ of X such that the set $\{(i, j, k) | d(\delta(i), \delta(j)) < v_Q(k)\}$ is computably enumerable (c.e.). Here v_Q is a canonical computable numbering of the set Q of rationals. Let $B_{\langle m,n\rangle} = \{x \in X | d(x, \delta(m)) < v_Q(n)\}$ where $\langle m, n\rangle$ is a computable bijection between $\omega \times \omega$ and ω. Then $B_0, B_1, \ldots$ is a basis in X. The notions of a computable point of an effective metric space and of a computable function between such spaces are introduced in a natural way [68,18] so that every computable function is continuous, and the value of a computable function on a computable point is a computable point. The spaces 2^{ω}, ω^{ω} and $\mathbf{R}$ equipped with the standard metrics and with natural numberings of dense subsets are effective [68,18]. Note that most popular metric spaces are effective even in a stronger sense of [33].

In every space X with a fixed numbering of a basis $B_0, B_1, \ldots$ (in particular, in every effective metric space) we may define *effective open sets* as the sets $\cup\{B_n | n \in A\}$ where A is a c.e. subset of ω. Note that there is a natural numbering of effective open sets induced by the standard numbering $\{W_n\}_{n<\omega}$ of c.e. sets, see [35].

Among basic notions of effective domain DST there should be some notions of constructive and effective φ-space. We choose notions closely related to the notion of a numbered set with approximation [7,10,11] (below we call them simply approximable numberings). By an *effective φ-space* we mean a pair (X, δ) consisting of a complete φ-space X and a numbering $\delta : \omega \to F(X)$ of all the finitary elements such that the relation "$\delta_x \leqslant \delta_y$" is c.e. It is easy to check that all examples of φ-spaces from the previous section, equipped with natural numberings of the finitary elements, become effective φ-spaces. Let $B_n = \{x \in X | \delta_n \leqslant x\}$, then $B_0, B_1, \ldots$ is a basis. Thus, we have a notion of an effective open set in every effective φ-space. A point $x \in X$ is *computable* if the set $\{n | \delta_n \leqslant x\}$ is c.e. The set of all computable points in (X, δ) is denoted $con(X, \delta)$. In order to explain the relation of this set to the theory of numberings let us very briefly recall some well-known notions and results from that theory [9–11].

A *numbering* is a map with domain ω. A numbering μ is *reducible* to a numbering v ($\mu \leqslant v$) if $\mu = v \circ f$ for a computable function f on ω; μ is *equivalent* to v if $\mu \leqslant v$ and $v \leqslant \mu$. A *morphism* from μ to v is a function $g : rng(\mu) \to rng(v)$ such that $g \circ \mu \leqslant v$. Numberings and morphisms form the *category of numberings*. A set $A \subseteq rng(v)$ is *v-enumerable* if $v^{-1}(A)$ is c.e. The v-enumerable sets form a basis of a topology on $rng(v)$; let $\leqslant_v$ denote the corresponding specialization preorder.

A numbering ν is *approximable* if $rng(\nu)$ is a T_0-space and there is a numbering $\delta \leqslant \nu$ (called *approximation* of ν) such that the relation $\delta_x \leqslant_\nu \nu_y$ is c.e. and for all ν-enumerable sets $A \nsubseteq B$ there is an m with $\delta_m \in A \setminus B$. The approximation δ is unique up to equivalence. The relation $\delta_x \leqslant_\nu \delta_y$ is c.e. The space $rng(\nu)$ is a φ-space with the numbering of a basis defined similarly to the case of effective φ-spaces (thus, we have a notion of effective open set in $rng(\nu)$). A set $A \subseteq rng(\nu)$ is ν-enumerable iff it is effective open (the Rice-Shapiro theorem for approximable numberings).

Let ν be an approximable numbering with approximation δ; ν is *complete* if for every approximable numbering μ with the same approximation δ such that $\leqslant_\mu$ coincides with $\leqslant_\nu$ on $rng(\delta)$ there exists a morphism $g : rng(\mu) \to rng(\nu)$ identical on $rng(\delta)$ (such a morphism is unique).

Now let δ be an arbitrary numbering and $\leqslant$ be a c.e. partial order on $rng(\delta)$ (this means that the relation $\delta_x \leqslant \delta_y$ is c.e). Then there exists a unique (up to a natural equivalence) complete approximable numbering ν such that $\delta \leqslant \nu$ and $\leqslant_\nu$ coincides with $\leqslant$ on $rng(\delta)$. This is a constructive analog of the completion operation for φ-spaces.

Let us return to effective versions of φ-spaces and define *constructive φ-spaces* as just the complete approximable numberings. It is well-known (though, maybe, not published explicitly) that the constructive part $con(X, \delta)$ of an effective φ-space is a constructive φ-space, and every constructive φ-space is of this form, up to equivalence. E.g., $con(P\omega)$ is essentially the standard numbering W of c.e. sets while $con(\omega_\perp^\omega)$ is the standard numbering φ of the computable partial functions [35]. The reader is invited to formulate the nice descriptions of constructive parts for the other spaces introduced in the previous section.

A *morphism* $g : \mu \to \nu$ of constructive φ-spaces is just the morphism of numberings. A *morphism* $g : (X, \delta) \to (Y, \varepsilon)$ of effective φ-spaces is a computable function, i.e. a continuous function such that the relation "$\varepsilon_x \leqslant g(\delta_y)$" is c.e. Let $\mathcal{E}$ ($\mathcal{C}$) be the category of effective (resp., constructive) φ-spaces with the introduced morphisms. Note that every morphism of constructive φ-spaces is extentable to a unique $\mathcal{E}$-morphism of the corresponding completions, and restriction of every morphism $g : (X, \delta) \to (Y, \varepsilon)$ of effective φ-spaces to $con(X, \delta)$ is a $\mathcal{C}$-morphism from $con(X, \delta)$ to $con(Y, \varepsilon)$.

We will need also effective versions of some classes of spaces introduced in Section 2. Definitions of effective reflective and 2-reflective spaces are obtained from the corresponding definitions is Section 2 by requiring additionally that the functions q_i, e_i are $\mathcal{E}$-morphisms, and the sets B_i, C_i are effective open. By a similar modifications we obtain notions of constructive reflective and 2-reflective spaces. Similar notions were first introduced and studied in [40,42]. Effective versions of most of the results from Section 2 are easy to obtain. E.g., we have the following proposition which is proved by observing that the proof of the corresponding non-effective versions in [52] is valid also for the effective versions.

Proposition 13. *Let* $2 \leqslant n < \omega$.
(i) $n^{\leqslant\omega}$ *and* $\omega^{\leqslant\omega}$ *are* $\mathcal{E}$*-retracts of arbitrary effective reflective space.*
(ii) $con(n^{\leqslant\omega})$ *and* $con(\omega^{\leqslant\omega})$ *are* $\mathcal{C}$*-retracts of arbitrary constructive reflective space.*
(iii) C_n *and* C_ω *are* $\mathcal{E}$*-retracts of arbitrary effective* 2*-reflective space.*
(iv) $con(C_n)$ *and* $con(C_\omega)$ *are* $\mathcal{C}$*-retracts of arbitrary constructive* 2*-reflective space.*

4. Borel hierarchy

In this section we discuss the Borel hierarchy in φ-spaces. Most results of this section are taken from [51,53]. Let us recall definition of the Borel hierarchy in an arbitrary space X. Let ω_1 be the first non-countable ordinal.

Definition 14. Define a sequence $\{\mathbf{\Sigma}^0_\alpha\}_{\alpha<\omega_1}$ of classes of subsets of an arbitrary space X by induction on α as follows: $\mathbf{\Sigma}^0_0 = \{\emptyset\}$, $\mathbf{\Sigma}^0_1$ is the class of open sets, $\mathbf{\Sigma}^0_2$ is the class of countable unions of finite boolean combinations of open sets, and $\mathbf{\Sigma}^0_\alpha (\alpha > 2)$ is the class of countable unions of sets in $\bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta$, where $\mathbf{\Pi}^0_\beta = \{A | \overline{A} \in \mathbf{\Sigma}^0_\beta\}$.

The sequence $\{\mathbf{\Sigma}^0_\alpha\}_{\alpha<\omega_1}$ is called *Borel hierarchy* in X, the classes $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$ are called *levels* of the Borel hierarchy. If we want to stress the space X in which the levels are considered we can use a more complicated notation like $\mathbf{\Sigma}^0_\alpha(X)$. The class $\mathbf{B} = \mathbf{B}(X)$ of *Borel sets* in X is the union of all levels of the Borel hierarchy. Let us state the inclusions of levels of the Borel hierarchy which are well-known for the Polish spaces.

Proposition 15. *For all* α, β *with* $\alpha < \beta < \omega_1$, $\mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Delta}^0_\beta$.

Next we formulate some structural properties of the introduced classes which are well-known for the Polish spaces. For a simple proof see [45].

Proposition 16. *For every space X and every* α, $1 < \alpha < \omega_1$, *the class* $\mathbf{\Sigma}^0_\alpha$ *has the reduction property while the class* $\mathbf{\Pi}^0_\alpha$ *has the separation property. The class* $\mathbf{\Sigma}^0_1$ *may not have the reduction property.*

Remarks. Definition 14 applies to arbitrary topological space, and Propositions 15 and 16 hold true in the full generality. Note that Definition 14 differs from the classical definition for Polish spaces [28] only for the level 2, and that for the case of Polish spaces our definition of Borel hierarchy is equivalent to the classical one. The classical definition applied, say, to φ-spaces does not in general have the properties one expects from a hierarchy. E.g., Proposition 15 is true for our definition but is in general false for the classical one.

Let us now relate the corresponding levels of the hierarchies in different spaces. The next statement is obvious because the preimage map respects all boolean operations.

Proposition 17. *Let* X, Y *be arbitrary spaces and* $\alpha < \omega_1$

(i) *If* $f : X \to Y$ *is a continuous function then the map* $A \mapsto f^{-1}(A)$ *respects all levels of the Borel hierarchy, i.e.* $A \in \mathbf{\Sigma}^0_\alpha(Y)$ *implies* $f^{-1}(A) \in \mathbf{\Sigma}^0_\alpha(X)$ *and similarly for the other levels.*

(ii) *If Y is a subspace of X then* $\mathbf{\Sigma}^0_\alpha(Y) = Y \cdot \mathbf{\Sigma}^0_\alpha(X)$ *and* $\mathbf{\Pi}^0_\alpha(Y) = Y \cdot \mathbf{\Pi}^0_\alpha(X)$.

One of the most important questions about a hierarchy is the question of its non-triviality (or *non-collapse* which means that each level contains a set not belonging to the lower levels). The next corollary of Proposition 17(ii) relates the non-collapse property in a space to that in its subspace.

Corollary 18. *If Y is a subspace of X and the Borel hierarchy in Y does not collapse* (*which means that* $\mathbf{\Sigma}^0_\alpha(Y) \not\subseteq \mathbf{\Pi}^0_\alpha(Y)$ *for each* $\alpha < \omega_1$) *then the Borel hierarchy in X does not collapse.*

Proof. For each $\alpha < \omega_1$ we have to find a set B in $\mathbf{\Sigma}^0_\alpha(X) \setminus \mathbf{\Pi}^0_\alpha(X)$. By the non-collapse for Y, there is a set $A \subseteq Y$ in $\mathbf{\Sigma}^0_\alpha(Y) \setminus \mathbf{\Pi}^0_\alpha(Y)$. By Proposition 17(ii), $A = Y \cap B$ for some $B \in \mathbf{\Sigma}^0_\alpha(X)$. Again by Proposition 17(ii), $B \notin \mathbf{\Pi}^0_\alpha(X)$. This completes the proof. □

In classical DST the non-collapse question is settled in the full generality [28]: Borel hierarchy in arbitrary non-countable Polish space does not collapse. For domain DST, we currently do not have such a general fact. But we can establish the non-collapse property for some natural classes of spaces introduced in Section 2.

Theorem 19. *The Borel hierarchy does not collapse in all reflective and all* 2*-reflective spaces.*

Proof. Let X be a reflective space. The Cantor space 2^ω is subspace of $2^{\leqslant\omega}$. By Theorem 5, the space $2^{\leqslant\omega}$ is homeomorphic to a subspace of X, hence 2^ω is homeomorphic to a subspace of X. Borel hierarchy in 2^ω does not collapse, hence by Corollary 18 Borel hierarchy in X does not collapse. The same proof applies to the case when X is 2-reflective because 2^ω is a subspace of C_2 and by Theorem 11 the space C_2 is homeomorphic to a subspace of X This completes the proof. □

In classical DST many efforts are devoted to understanding the $\mathbf{\Delta}$-levels of the Borel hierarchy in Polish spaces. In the case of domains, we do not currently know a similar theory. Only the second level have been understood rather well in [50,51] (see also a particular case in [60]. Let us formulate notions relevant to this question and to some results in Section 6.

Definition 20. Let X be a φ-space and $A \subseteq X$.

(i) A is called approximable if for every $x \in A$ there is a finitary element $p \leqslant x$ with $\{y \in X | p \leqslant y \leqslant x\} \subseteq A$.

(ii) A is called weakly approximable if for every $x \in A$ there is a finitary element $p \leqslant x$ with $\{y \in F(X) | p \leqslant y \leqslant x\} \subseteq A$.

Theorem 21. *Let X be a complete countably based φ-space and $A \subseteq X$. Then A is $\mathbf{\Delta}^0_2$ iff both A and $\overline{A}$ are approximable iff both A and $\overline{A}$ are weakly approximable.*

Let us state a relationship between the Borel hierarchies in $P\omega$ and in the Cantor space 2^ω established in [51,52]. Note that as sets $P\omega$ and 2^ω coincide provided that we identify subsets of ω with their characteristic functions.

Proposition 22. (i) *For every $\alpha < \omega_1$, $\mathbf{\Sigma}^0_\alpha(P\omega) \subseteq \mathbf{\Sigma}^0_\alpha(2^\omega) \subseteq \mathbf{\Sigma}^0_{1+\alpha}(P\omega)$.*
(ii) $\bigcup_{n<\omega}\mathbf{\Sigma}^0_n(P\omega) = \bigcup_{n<\omega}\mathbf{\Sigma}^0_n(2^\omega)$.
(iii) *For every infinite ordinal $\alpha < \omega_1$, $\mathbf{\Sigma}^0_\alpha(P\omega) = \mathbf{\Sigma}^0_\alpha(2^\omega)$.*
(iv) $\mathbf{B}(2^\omega) = \mathbf{B}(P\omega)$.
(v) *For every n, $0 < n < \omega$, $\mathbf{\Sigma}^0_n(P\omega) \not\subseteq \mathbf{\Pi}^0_n(2^\omega)$ and $\mathbf{\Pi}^0_n(2^\omega) \not\subseteq \mathbf{\Sigma}^0_{n+1}(P\omega)$.*

We conclude this section with some results on an effective version of the Borel hierarchy developed in [40,42,43,45] (other related treatments of the effective Borel hierarchy see e.g. in [33,22,55,18]). Following a well-established tradition of DST, we denote levels of effective hierarchies in the same manner as levels of the corresponding classical hierarchies, using the lightface letters Σ, Π, Δ instead of the boldface $\mathbf{\Sigma}$, $\mathbf{\Pi}$, $\mathbf{\Delta}$ used in the classical case.

Let $\beta : \omega \to P(M)$ be a numbering of subsets of arbitrary set M such that $(rng(v); \cup, \cap, \bar{}, \emptyset, M)$ is a boolean algebra and the operations $\cup, \cap$ are presented by computable functions on β-numbers. The *finite effective Borel hierarchy* over β is a sequence $\{\Sigma^0_n\}_{n<\omega}$ defined as follows: $\Sigma^0_0 = \{\emptyset\}$; Σ^0_1 is the class of sets $\bigcup\{\beta_k | k \in W_x\}$, $x \in \omega$, equipped with the numbering induced by the standard numbering W of c.e. sets, Σ^0_n $(n > 1)$ is the class of sets $\bigcap\{\gamma_k | k \in W_x\}$, $x \in \omega$, equipped again with the numbering induced by W, where γ is the numbering of Π^0_{n-1} induced by the numbering of Σ^0_{n-1} (which exists by induction).

Transfinite extension of the hierarchy $\{\Sigma^0_n\}_{n<\omega}$ is also constructed in the natural way [35,42]. When speaking about an effective transfinite hierarchy we assume the reader to be familiar with the Kleene notation system $(O; <_O)$ for constructive ordinals which is a partial numbering $O \to \omega_1^{CK}$, where ω_1^{CK} is the first non-constructive ordinal. The ordinal denoted by $a \in O$ is $|a| = |a|_O$. The levels of the transfinite version (denoted $\Sigma^0_{(a)}$ $(a \in O)$) are defined in the same way as for the finite levels, using effective induction along the well-founded set $(O; <_O)$ [35]. In order to avoid some tedious technical details we omit the formal definition here. The simplest properties of the effective Borel hierarchy $\{\Sigma^0_{(a)}\}_{a\in O}$ are proved in a straightforward way [42].

Proposition 23. (i) *If $1 \leqslant |a| < \omega$ then $\Sigma^0_{(a)} = \Sigma^0_{|a|}$.*
(ii) *If $a <_O b$ then $\Sigma^0_{(a)} \subseteq \Delta^0_{(b)}$.*

Now we define the effective Borel hierarchy in arbitrary countably based space X with a fixed numbering $B_0, B_1, \ldots$ of basic open sets. Let $\beta : \omega \to P(X)$ be the numbering of the boolean algebra generated by the class of effective open sets equipped with the numbering induced by the numbering of effective open sets (see Section 3) and by the Gödel numbering of finite boolean terms. Let $\Sigma^0_0 = \{\emptyset\}$, let Σ^0_1 be the class of effective open sets with the numbering induced by W, and for $a \in O$, $|a| > 1$, let $\Sigma^0_{(a)}$ be the $(|a| - 1)$st level of the effective Borel hierarchy over the numbering β. In particular, Σ^0_2 is the class of effective unions of finite boolean combinations of effective open sets. Note that the inclusions of the introduced levels satisfy Proposition 23. This definition applies to all classes of effective spaces introduced in Section 3, including the approximable numberings (in the last case the classes look like $\Sigma^0_{(a)}(v)$, where v is the approximable numbering). An important example is the discrete space ω, in this space the effective Borel hierarchy coincides with the hyperarithmetical hierarchy which is very important in computability theory.

The next easy proposition relates the effective and classical Borel hierarchies. A similar fact is well-known in effective classical DST.

Proposition 24. *Let X be a countably based space with a numbering $B_0, B_1, \ldots$ of basic open sets.*
(i) *For every $a \in O$, $\Sigma^0_{(a)} \subseteq \mathbf{\Sigma}^0_{|a|}$.*
(ii) *For every $\alpha < \omega_1^{CK}$, $\mathbf{\Sigma}^0_\alpha = \bigcup\{\Sigma^{0,h}_{(a)} | h \in \omega^\omega, a \in O^h, |a|_{O^h} = \alpha\}$, where O^h is the Kleene system relativized to an oracle $h \in \omega^\omega$ and $\{\Sigma^{0,h}_{(a)}\}_{a\in O^h}$ is the effective Borel hierarchy relativized to h.*

Many problems about effective hierarchies are more subtle and complicated than their classical analogs. This applies in particular to the non-collapse problem. Some sufficient condition for the non-collapse of effective Borel hierarchy in effective metric spaces maybe found in [33,18]. Now we state effective analogs of the above-mentioned results about Borel hierarchy in φ-spaces. First we establish effective analogs of Proposition 17 and Corollary 18.

Proposition 25. *Let X, Y be effective φ-spaces and $a \in O$.*

(i) If $f : X \to Y$ is a morphism then the map $A \mapsto f^{-1}(A)$ respects all levels of the effective Borel hierarchies, i.e. $A \in \Sigma^0_{(a)}(Y)$ implies $f^{-1}(A) \in \Sigma^0_{(a)}(X)$ and similarly for the other levels. Moreover, $A \mapsto f^{-1}(A)$ is a morphism from $\Sigma^0_{(a)}(Y)$ to $\Sigma^0_{(a)}(X)$ in the category of numberings.

(ii) If $s : Y \to X$ and $r : X \to Y$ form a section–retraction pair in the category $\mathcal{E}$ of Section 3 then for all $A \subseteq Y$ and $a \in O$ we have: $A \in \Sigma^0_{(a)}(Y)$ iff $r^{-1}(A) \in \Sigma^0_{(a)}(X)$ and similarly for the Π-levels.

Similar assertions hold true for the constructive φ-spaces.

Proof. Consider only the effective case, the constructive being similar. The assertion (i) follows immediately from definitions. The assertion (ii) follows from and the equality $A = s^{-1}(r^{-1}(A))$, which follows from $r \circ s = id_Y$. This completes the proof. □

Corollary 26. *If Y is a retract of X in the category of effective φ-spaces and the effective Borel hierarchy in Y does not collapse (which means that $\Sigma^0_{(a)}(Y) \not\subseteq \Pi^0_{(a)}(Y)$ for each $a \in O$) then the effective Borel hierarchy in X does not collapse. The same is true for the constructive φ-spaces.*

Proof. For each $\alpha < \omega_1$ we have to find a set A in $\Sigma^0_{(a)}(X) \setminus \Pi^0_{(a)}(X)$. By the non-collapse for Y, there is a set $B \subseteq Y$ in $\Sigma^0_{(a)}(Y) \setminus \Pi^0_{(a)}(Y)$. Let $A = r^{-1}(B)$, where s and r are as in Proposition 25. By Proposition 25(ii), A has the desired property. This completes the proof. □

Theorem 27. *The effective Borel hierarchy does not collapse in all effective (and in all constructive) reflective and 2-reflective spaces.*

Proof. Consider again only the effective case the constructive case being similar. Let X be an effective reflective space. The Cantor space 2^ω is a subspace of $2^{\leqslant\omega}$, and it is easy to see that for every $a \in O$ we have $\Sigma^0_{(a)}(2^\omega) = 2^\omega \cdot \Sigma^0_{(a)}(2^{\leqslant\omega})$. Since the effective Borel hierarchy in 2^ω does not collapse [33], the effective Borel hierarchy in $2^{\leqslant\omega}$ does not collapse as well. By Proposition 13, $2^{\leqslant\omega}$ is an $\mathcal{E}$-retract of X. So the assertion follows from Corollary 26. The same proof applies to the case when X is 2-reflective because 2^ω is a subspace of C_2 and it is easy to see that for every $a \in O$ we have $\Sigma^0_{(a)}(2^\omega) = 2^\omega \cdot \Sigma^0_{(a)}(C_2)$. This completes the proof. □

In [42] the reader could find a bit different sufficient condition for the non-collapse of the effective Borel hierarchy.

The status of the effective analog of Theorem 21 is not clear, we can only formulate a conjecture which seems rather plausible, namely: for every constructive (or effective) φ-space (X, δ) and every $A \subseteq X$, $A \in \Delta^0_2$ iff both A and $\overline{A}$ are approximable and $\delta^{-1}(A) \in \Delta^0_2$ (the last Δ^0_2 is the second level of the arithmetical hierarchy in ω).

5. Analytic sets

In this section, we discuss the class of analytic sets playing a prominent role in the classical DST. Analytic sets are closely related to an infinitary set-theoretic operation $\mathcal{A}$ introduced by Alexandrov and studied by Suslin. Recall [28] that $\mathcal{A}$ sends a sequence $\{A_k\}_{k<\omega}$ of sets to the set $\mathcal{A}(\{A_k\}) = \bigcup_\xi \bigcap_n \alpha_{\hat{\xi}(n)}$, where ξ ranges over ω^ω, n ranges over ω and $\hat{\xi}(n)$ is the code of the string $(\xi(0), \ldots, \xi(n-1))$ in a computable bijective numbering of finite strings of numbers.

Definition 28. Let X be a space. The class $\mathbf{\Sigma}^1_1 = \mathbf{\Sigma}^1_1(X)$ of analytic sets in X consists of the sets $\mathcal{A}(\{A_k\}_{k<\omega})$, where all A_k are finite boolean combinations of open sets.

Note that taking the Lawson topology [14] instead of the Scott topology in a domain gives the same notion of analytic set in the domain. As usual, $\mathbf{\Pi}^1_1$ denotes the class of complements of $\mathbf{\Sigma}^1_1$-sets and $\mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$. The next result is well-known and easy to prove.

Proposition 29. *Let X be arbitrary space.*
(i) *Every Borel set is analytic, hence* $\mathbf{B} \subseteq \mathbf{\Delta}^1_1$.
(ii) *The class* $\mathbf{\Sigma}^1_1$ *coincides with the class of sets* $\mathcal{A}(\{A_k\}_{k<\omega})$, *where all* A_k *are analytic.*
(iii) *If* $A \subseteq X$ *then* $\mathbf{\Sigma}^1_1(A) = A \cdot \mathbf{\Sigma}^1_1(X)$.

According to a well-known result of classical DST [28], $\mathbf{\Sigma}^1_1 \not\subseteq \mathbf{\Pi}^1_1$ in every non-countable Polish space. From Proposition 29 we obtain the following result just in the same way as the non-collapse result in the previous section.

Proposition 30. *In all reflective and all* 2*-reflective spaces*, $\mathbf{\Sigma}^1_1 \not\subseteq \mathbf{\Pi}^1_1$.

The most interesting question about analytic sets is the status of the equality $\mathbf{B} = \mathbf{\Delta}^1_1$. One of the best results of classical DST is Suslin theorem stating that the equality is true in each Polish space. A generalization of this is Lusin separation theorem stating that in each Polish space every two disjoint analytic sets are separable by a Borel set. Closely relevant is Kuratowski theorem stating that the class $\mathbf{\Pi}^1_1$ has the reduction property in every Polish space (see [28]).

The author currently does not know whether results similar to results of the previous paragraph hold true for a broad enough class of domains. Although, the results hold true for some particular spaces, e.g. for the space $P\omega$ (this follows from Proposition 22 and the equality $\mathbf{\Sigma}^1_1(P\omega) = \mathbf{\Sigma}^1_1(2^\omega)$ which is easy to prove).

As the notation $\mathbf{\Sigma}^1_1$ could suggest, in classical DST the class of analytic sets is just the first level of a hierarchy $\{\mathbf{\Sigma}^1_n\}_{n<\omega}$ in every Polish space which is called *projective hierarchy* and actively studied. To my knowledge, a similar theory in the context of domain DST does not exist, and even the “right” definition of the projective hierarchy for the domain-like structures is not completely clear.

For the effective cases, definition of the class Σ^1_1 of *effective analytic* sets is natural: this is the class of sets $\mathcal{A}(\{A_k\}_{k<\omega})$ where $\{A_k\}$ range through numberings reducible to the numbering of finite boolean combinations of effective open sets (see the previous section). This definition applies to every space with a fixed numbering of basic open sets, and thus to all classes of spaces introduced in Section 3. In effective classical DST [33,22], the class of effective analytic sets is studied in detail, and analogs of the above-formulated Suslin, Luzin and Kuratowski theorems are obtained for a class of effective metric spaces containing ω, $\mathbf{R}$, ω^ω, 2^ω and closed under finite product. The effective version of Suslin theorem is known as Suslin–Kleene theorem.

For the case of effective φ-spaces, it is easy to establish analogs of the above-stated propositions. But the status of the effective analogs of theorems of Suslin, Luzin and Kuratowski is not clear. From results in [41] one could deduce some results on the (effective) hierarchy of the so-called *C*-sets obtained from finite boolean combinations of effective open sets by iterating the Alexandrov operation but we will not consider this in detail here.

6. Difference hierarchy

In this section, we discuss the difference hierarchy which is also a popular object in the classical DST and computability theory. Let us start with recalling the well-known definition of the Hausdorff difference operation. An ordinal α is called *even (odd)* if $\alpha = \lambda + n$ where λ is not a successor, $n < \omega$ and n is even (resp., odd). For an ordinal α, let $r(\alpha) = 0$ if α is even and $r(\alpha) = 1$, otherwise.

Definition 31. (i) For every ordinal α, define the operation D_α sending sequences of sets $\{A_\beta\}_{\beta<\alpha}$ to sets by

$$D_\alpha(\{A_\beta\}_{\beta<\alpha}) = \bigcup \left\{ A_\beta \setminus \bigcup_{\gamma<\beta} A_\gamma | \beta < \alpha,\ r(\beta) \neq r(\alpha) \right\}.$$

(ii) For all ordinals α and classes of sets $\mathcal{C}$, let $D_\alpha(\mathcal{C})$ be the class of all sets $D_\alpha(\{A_\beta\}_{\beta<\alpha})$, where $A_\beta \in \mathcal{C}$ for all $\beta < \alpha$.

Notice that if the class $\mathcal{C}$ above is closed under countable unions (as is e.g. the case for the classes $\mathcal{C} = \mathbf{\Sigma}^0_\beta$) then the class $D_\alpha(\mathcal{C})$ coincides with the class of all sets $D_\alpha(\{A_\beta\}_{\beta<\alpha})$, where $A_\beta \in \mathcal{C}$ for all $\beta < \alpha$ and $A_\beta \subseteq A_\gamma$ for $\beta < \gamma < \alpha$.

Next we shall define the Hausdorff difference hierarchy [16].

Definition 32. Let X be a space and $\{\mathbf{\Sigma}^0_\beta\}$ the Borel hierarchy in X.

(i) For each β, $0 < \beta < \omega_1$, the sequence $\{D_\alpha(\mathbf{\Sigma}^0_\beta)\}_{\alpha<\omega_1}$ is called the difference hierarchy over $\mathbf{\Sigma}^0_\beta$.
(ii) The difference hierarchy over $\mathbf{\Sigma}^0_1$ is called simply the difference hierarchy in X and is denoted by $\{\mathbf{\Sigma}^{-1}_\alpha\}_{\alpha<\omega_1}$.

As usual, let $\mathbf{\Pi}^{-1}_\alpha$ denote the dual class for $\mathbf{\Sigma}^{-1}_\alpha$, and $\mathbf{\Delta}^{-1}_\alpha = \mathbf{\Sigma}^{-1}_\alpha \cap \mathbf{\Pi}^{-1}_\alpha$. The next assertion is (for the Polish spaces) well-known and follows easily from the definitions above.

Proposition 33. *For all α, β and γ with $\alpha < \gamma < \omega_1$ and $0 < \beta < \omega_1$, $D_\alpha(\mathbf{\Sigma}^0_\beta) \cup co\text{-}D_\alpha(\mathbf{\Sigma}^0_\beta) \subseteq D_\gamma(\mathbf{\Sigma}^0_\beta)$. In particular, $\mathbf{\Sigma}^{-1}_\alpha \subseteq \mathbf{\Delta}^{-1}_\gamma$.*

The following relationship between the difference and Borel hierarchies also immediately follows from definitions.

Proposition 34. *For every β, $0 < \beta < \omega_1$, $\bigcup\{D_\alpha(\mathbf{\Sigma}^0_\beta) | \alpha < \omega_1\} \subseteq \mathbf{\Delta}^0_{\beta+1}$. In particular, $\bigcup\{\mathbf{\Sigma}^{-1}_\alpha | \alpha < \omega_1\} \subseteq \mathbf{\Delta}^0_2$.*

The last inclusion explains the meaning of the upper index -1 in the notation of the difference hierarchy (introduced in [6]). It stresses that the difference hierarchy is finer than the Borel hierarchy denoted traditionally with the upper index 0.

Let us now relate the corresponding levels of the hierarchies in different spaces. The next statement is obvious because the preimage map respects all boolean operations.

Proposition 35. *If $f : X \to Y$ is a continuous function then the map $A \mapsto f^{-1}(A)$ respects all levels of the difference hierarchies.*

The non-collapse problem is solved for the difference hierarchy in a similar way as for the Borel hierarchy. Namely, in arbitrary non-countable Polish space the difference hierarchy over each non-zero $\mathbf{\Sigma}$-level of the Borel hierarchy does not collapse [28]. In the domain DST, we have the following analog of Theorem 19 proved in the same way as in Section 4.

Theorem 36. *Let X be an arbitrary reflective or 2-reflective space and $0 < \beta < \omega_1$. Then the difference hierarchy over $\mathbf{\Sigma}^0_\beta$ does not collapse, i.e. $D_\alpha(\mathbf{\Sigma}^0_\beta) \not\subseteq co\text{-}D_\alpha(\mathbf{\Sigma}^0_\beta)$ for all $\alpha < \omega_1$. In particular, the difference hierarchy $\{\mathbf{\Sigma}^{-1}_\alpha\}_{\alpha<\omega_1}$ does not collapse.*

The next important question about the difference hierarchy is to understand the union of all its levels. In classical DST, the answer to this question is quite elegant and general: in arbitrary Polish space we have $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^{-1}_\alpha = \mathbf{\Delta}^0_2$ (Hausdorff theorem) and $\bigcup_{\alpha<\omega_1} D_\alpha(\mathbf{\Sigma}^0_\beta) = \mathbf{\Delta}^0_{\beta+1}$ for all β, $0 < \beta < \omega_1$ (Hausdorff–Kuratowski theorem). In domain DST, the following analog of the Hausdorff theorem was obtained in [50] (see also a particular case in [60]).

Theorem 37. *Let X be a complete countably based φ-space. Then $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^{-1}_\alpha = \mathbf{\Delta}^0_2$.*

Currently we do not know whether the Hausdorff–Kuratowski theorem holds true in arbitrary complete countably based φ-space for the difference hierarchy over $\mathbf{\Sigma}^0_\beta$ for $1 < \beta < \omega_1$.

The next result form [50] informally means that the difference hierarchy is in a sense the finest possible.

Theorem 38. *Let X be a complete countably based φ_0-space and let $\alpha < \omega_1$. Then $\bigcup\{\mathbf{\Sigma}^{-1}_\beta \bigcup \mathbf{\Pi}^{-1}_\beta | \beta < \alpha\} = \mathbf{\Delta}^{-1}_\alpha$.*

Theorems 36–38 follow rather easily from the next description of levels of the difference hierarchy obtained in [50, 52,53]. First we introduce some terminology. By a *tree* we mean a non-empty set $T \subseteq \omega^*$ closed downwards under $\sqsubseteq$.

A *path* through a tree T is a function $f : \omega \to \omega$ such that the string $(f(0), \ldots, f(n-1))$ is in T for every $n < \omega$. A tree T is *well-founded* if the partial ordering $(T; \sqsupseteq)$ is well-founded, i.e. there is no path through the tree T. As for each well-founded partial ordering, there is a canonical rank function rk_T from a well-founded tree T to ordinals defined by

$$rk_T(\tau) = sup\{rk_T(\sigma) + 1 | \sigma \in T \wedge \tau \sqsubset \sigma\}.$$

The rank $rk(T)$ of a well-founded tree T is by definition the ordinal $rk_T(\emptyset)$. It is well-known that rank of every well-founded tree is a countable ordinal, and every countable ordinal is the rank of a well-founded tree.

Definition 39. Let X be a φ-space and $A \subseteq X$. By alternating tree for A we mean a monotone function $f : (T; \sqsubseteq) \to (F(X); \leqslant)$ from a tree T to the finitary elements such that $f(\sigma) \in A$ iff $f(\sigma^n) \notin A$, for each $\sigma^n \in T$. Rank of f is the rank of the tree T (provided it is well-founded). An alternating tree f is called 1-alternating (0-alternating) if $f(\emptyset) \in A$ (resp., $f(\emptyset) \notin A$).

Theorem 40. *Let X be a complete countably based φ-space, $\alpha < \omega_1$, T a tree of rank α and $A \subseteq X$. Then the following assertions are equivalent*:

(i) $A \in \mathbf{\Sigma}^{-1}_{\alpha}$;

(ii) *both sets $A, \overline{A}$ are approximable and there is no alternating tree $f : T \to F(X)$ for A*;

(iii) *both sets $A, \overline{A}$ are approximable and there is no alternating tree for A of rank α.*

Next we give some information from [51,52] about *proper $\mathbf{\Sigma}^{-1}_{\alpha}$-sets*, i.e. sets in $\mathbf{\Sigma}^{-1}_{\alpha} \setminus \mathbf{\Pi}^{-1}_{\alpha}$. We consider the question whether such sets can contain the bottom $\perp$ or the top element $\top$ (provided these elements exist).

Proposition 41. (i) *Let X be a φ_0-space. For every $\alpha < \omega_1$, if A is proper $\mathbf{\Sigma}^{-1}_{\alpha}$ then $\perp \notin A$.*

(ii) *Let X be a φ-space with a top element $\top$. For every $n < \omega$ and every proper $\mathbf{\Sigma}^{-1}_{n}$-set A, $\top \in A$ iff n is odd.*

(iii) *In all 2-reflective spaces, for every infinite ordinal $\alpha < \omega_1$ there exist proper $\mathbf{\Sigma}^{-1}_{\alpha}$-sets A and B such that $\top \in A$ and $\top \notin B$.*

We conclude this section with remarks about an effective version of the difference hierarchy introduced and studied in [40,42]. Let $\{\Sigma^0_{(a)}\}_{a \in O}$ be the effective Borel hierarchy in a countably based space X with a fixed numbering $B_0, B_1, \ldots$ of basic open sets (see Section 4). Let $\Sigma^{-1}_{(a)}\}_{a \in O}$ be the effective difference hierarchy over Σ^0_1. Thus, $\Sigma^{-1}_{(a)}$ is the class of sets of the form $D_{|a|}(\{A_b\}_{b <_O a})$, where $\{A_b\}_{b <_O a}$ is a uniform sequence of effective open sets (naturally identified with a sequence $\{A_\beta\}_{\beta < |a|}$). The effective difference hierarchy $\{D_{(a)}(\Sigma^0_{(b)})\}_{a \in O}$ over each level $\Sigma^0_{(b)}$, $0 < |b|_O$, is defined in the same way, only the sequence $\{A_b\}_{b <_O a}$ is assumed to be a uniform (i.e. reducible to the natural numbering of $\Sigma^0_{(b)}$) sequence of $\Sigma^0_{(b)}$-sets. As in Section 4, this definition applies to all classes of effective spaces introduced in Section 3. It is easy to check that effective analogs of some results established above hold true (e.g. analogs of Propositions 23–25 and Corollary 26). Finally, the following analog of Theorem 36 is proved in the same way as in Section 4.

Theorem 42. *The effective difference hierarchy over each non-zero $\mathbf{\Sigma}$-level of the effective Borel hierarchy does not collapse in all effective reflective and all effective 2-reflective spaces. The same is true for the constructive reflective and 2-reflective spaces.*

Of course, proof of the last theorem uses the non-collapse property of the effective difference hierarchies in the Cantor space which is easy to show (see [48] for a particular case). In [42] the reader could find a bit different sufficient condition for the non-collapse of the effective difference hierarchies. The effective version of Proposition 41 is also true, almost with the same proof as for the non-effective case.

The status of effective analogs of other results on the difference hierarchy is not clear. In particular, we have no idea on how an effective analog of Theorem 40 could look like. It is clear that for describing the sets $A \in \Sigma^{-1}_{(a)}$ some effectivity condition on A has to be added. A plausible conjecture could look as follows: for every effective (or constructive) φ-space (X, δ) and for every $a \in O$, $A \in \Sigma^{-1}_{(a)}$ iff $A \in \mathbf{\Sigma}^{-1}_{|a|}$ and $\delta^{-1}(A) \in \Sigma^{-1}_{(a)}$ (where the last $\Sigma^{-1}_{(a)}$ denotes the ath level of the Ershov difference hierarchy of subsets of ω [6]).

It is easy to see that the conjecture is true for $|a| = 1$ (for the space $P\omega$ this is proved in [60], and the implication from left to right is true for every $a \in O$. But the conjecture is (in general) false for every $a \in O$, $|a| > 1$. This follows (with some modification of the construction) from counterexamples in [37,40,42,43] on the closely related problem of extensional characterization of index sets (see Section 9 below).

Without the effective analog of Theorem 40 the status of the effective analog of Theorem 38 is also not clear. For the effective analog of Theorem 37, we can only formulate a conjecture which seems rather plausible, namely: for every effective (or constructive) φ-space (X, δ), $\bigcup\{\Sigma^{-1}_{(a)} | a \in O\} = \Delta^0_2$. Till now, we were unable to prove or disprove this conjecture, though it is true in ω ([6]), in the Baire and Cantor spaces ([43,48], and in the finite-dimensional Euclidean spaces ([20]).

7. Wadge reducibility

Recall that $A \subseteq \omega^\omega$ is *Wadge reducible* to $B \subseteq \omega^\omega$ (in symbols $A \leqslant_W B$) if $A = f^{-1}(B)$ for some continuous function $f : \omega^\omega \to \omega^\omega$. Replacing the Baire space ω^ω by arbitrary space X, we get the preordering $\leqslant_W$ on the powerset $P(X)$ called the *Wadge reducibility in X*.

Wadge reducibility in the Baire and Cantor spaces is important in the classical DST because it subsumes important hierarchies including the Borel and difference hierarchies. Wadge and Martin showed (see [66]) that the structure $(\mathbf{B}; \leqslant_W)$ of Borel sets in the Baire and Cantor spaces is well-founded and for all $A, B \in \mathbf{B}$ it holds $A \leqslant_W B$ or $\overline{B} \leqslant_W A$ (we call structures satisfying these two properties *almost well-ordered*). Wadge also computed the corresponding (large) ordinal ν. In [64,57] it was shown that for every non-selfdual Borel Wadge class $\mathcal{C}$ (i.e., class of the form $\{B | B \leqslant_W A\}$ where A is Borel and $A \nleqslant_W \overline{A}$) exactly one of the classes $\mathcal{C}$, *co*-$\mathcal{C}$ has the separation property.

The results cited in the last paragraph give rise to the *Wadge hierarchy* (of Borel sets) which is, by definition, the sequence $\{\mathbf{\Sigma}_\alpha\}_{\alpha<\nu}$ of all non-selfdual Borel Wadge classes not having the separation property and satisfying for all $\alpha < \beta < \nu$ the strict inclusion $\mathbf{\Sigma}_\alpha \subset \mathbf{\Delta}_\beta$. As usual, we set $\mathbf{\Pi}_\alpha = co\text{-}\mathbf{\Sigma}_\alpha$ and $\mathbf{\Delta}_\alpha = \mathbf{\Sigma}_\alpha \cap \mathbf{\Pi}_\alpha$. Note that the classes $\mathbf{\Sigma}_\alpha \setminus \mathbf{\Pi}_\alpha$, $\mathbf{\Pi}_\alpha \setminus \mathbf{\Sigma}_\alpha$, $\mathbf{\Delta}_{\alpha+1} \setminus (\mathbf{\Sigma}_\alpha \cup \mathbf{\Pi}_\alpha)$, where $\alpha < \nu$, are exactly the equivalence classes induced by $\leqslant_W$ on $\mathbf{B}(2^\omega)$ (these equivalence classes are known as *Wadge degrees*).

In this section, we discuss Wadge reducibility in arbitrary spaces, with the emphasis on the φ-spaces. We will try to understand which properties (or their weaker versions) of the classical Wadge reducibility in ω^ω hold true in other spaces. E.g., we discuss when some substructures of the Wadge ordering are almost well ordered, which sets have a supremum (or a weak version of supremum) under Wadge reducibility, and consider relationship of the Wadge reducibility to hierarchies considered above. Most of results discussed in this section are taken from [52].

We start with results about the existence of supremums in the Wadge ordering. First we show that for many spaces the structure of Wadge degrees is not an upper semilattice.

Proposition 43. *Let X be a space such that every continuous function on X has a fixed point. Then for each $A \subseteq X$ the sets A, $\overline{A}$ have no supremum under the Wadge reducibility. This applies e.g. to all complete φ_0-spaces.*

A bit later we will see that the reflective and 2-reflective spaces have the stronger property that every sequence of sets with no greatest element under $\leqslant_W$ has no least upper bound. The next result gives a sufficient condition for existence of supremums under Wadge reducibility. We apply the well-known terminology about partial orderings also to preorderings meaning the correspondent quotient partial ordering.

Proposition 44. (i) *If the direct sum $X \oplus X$ is equivalent to X in the category of topological spaces then every two sets $A, B \subseteq X$ have a supremum in $(P(X); \leqslant_W)$.*

(ii) *If the direct sum of the infinite sequence $(X, X, \ldots)$ is equivalent to X in the category of topological spaces then every sequence $A_0, A_1, \ldots$ of subsets of X has a supremum in $(P(X); \leqslant_W)$.*

Remark. The statement (i) above applies both to the Cantor and Baire space, while the statement (ii) applies to the Baire space but does not apply to the Cantor space. This explains the well-known small differences in the structure of Wadge degrees in these spaces.

The next proposition relates quasiretractions and retractions from Section 2 to the Wadge reducibility.

Proposition 45. (i) *If $q : X \to Y$ is a quasiretraction then the map $A \mapsto q^{-1}(A)$ is a monotone function from $(P(Y); \leqslant_W)$ to $(P(X); \leqslant_W)$.*

(ii) *If $r : X \to Y$ is a retraction then $A \mapsto r^{-1}(A)$ is an embedding of $(P(Y); \leqslant_W)$ into $(P(X); \leqslant_W)$.*

From this and Propositions 5 and 11 we obtain the following corollary which shows that in order to understand the structure of Wadge degrees in reflective and 2-reflective spaces it is important to understand this structure for the simplest such spaces $\omega^{\leqslant\omega}$, $n^{\leqslant\omega}$, C_ω and C_n. The corollary also states some universality property of the spaces $\mathbf{U}$ and $P\omega$ with respect to Wadge reducibility.

Corollary 46. (i) *For every reflective space X, the structures $(P(\omega^{\leqslant\omega}); \leqslant_W)$ and $(P(n^{\leqslant\omega}); \leqslant_W)$, $2 \leqslant n < \omega$, are embeddable into $(P(X); \leqslant_W)$.*

(ii) *For every complete countably based f_0-space X, $(P(X); \leqslant_W)$ is embeddable into $(P(\mathbf{U}); \leqslant_W)$.*

(iii) *For every 2-reflective space X, the structures $(P(C_\omega); \leqslant_W)$ and $(P(C_n); \leqslant_W)$, $2 \leqslant n < \omega$, are embeddable into $(P(X); \leqslant_W)$.*

(iv) *For every complete countably based topped f_0-space X, the structure $(P(X); \leqslant_W)$ is embeddable into $(P(P\omega); \leqslant_W)$.*

Next we formulate a technical definition which is a version of the corresponding notion introduced in [38,39].

Definition 47. Let I be a non-empty set. By an I-discrete weak semilattice we mean a structure of the form $(P; \leqslant, \{P_i\}_{i\in I})$ with the following properties:

(i) $(P; \leqslant)$ is a preordering;

(ii) $P = \bigcup\{P_i | i \in I\}$;

(iii) for all $x_0, x_1, \ldots \in P$ and $i \in I$ there exists $u_i = u_i(x_0, x_1, \ldots) \in P_i$ which is a least upper bound for $x_0, x_1, \ldots$ in the set P_i, i.e. $\forall k < \omega(x_k \leqslant u_i)$ and for each $y \in P_i$ with $\forall k(x_k \leqslant y)$ it holds $u_i \leqslant y$;

(iv) for all $x_0, x_1, \ldots \in P$, $i \neq i' \in I$ and $y \in P_{i'}$, if $y \leqslant u_i(x_0, x_1, \ldots)$ then $y \leqslant x_k$ for some $k < \omega$.

Note that usually only the case when the set I is finite is of interest, hence the reader may assume I always to be finite. The following properties of the I-discrete weak semilattices are immediate (see also [38,39]).

Proposition 48. (i) *For each $i \in I$, every sequence $x_0, x_1, \ldots$ in P_i has a supremum $u_i(x_0, x_1, \ldots) \in P_i$. In particular, $(P_i; \leqslant)$ is an upper semilattice.*

(ii) *For all $y, x_0, x_1, \ldots \in P$, if $\forall k < \omega(x_k \leqslant y)$ then $\exists i \in I(u_i(x_0, x_1, \ldots) \leqslant y)$.*

(iii) *For all $y, x_0, x_1, \ldots \in P$, if $\forall i \in I(y \leqslant u_i(x_0, x_1, \ldots))$ then $\exists k < \omega(y \leqslant x_k)$.*

(iv) *If I has at least two elements and $(\{x_0, x_1, \ldots\}; \leqslant)$ has no greatest element then the set $\{x_0, x_1, \ldots\}$ has no supremum in $(P; \leqslant)$.*

Now we establish an interesting property of the Wadge reducibility in the reflective spaces.

Theorem 49. *Let X be a reflective space, $\mathcal{P}_0 = \{A \subseteq X | \bot \notin A\}$ and $\mathcal{P}_1 = \{A \subseteq X | \bot \in A\}$. Then $(P(X); \leqslant_W, \mathcal{P}_0, \mathcal{P}_1)$ is a $\{0, 1\}$-discrete weak semilattice (and, consequently, if a sequence in $P(X)$ has no greatest element under $\leqslant_W$ then it has no supremum under $\leqslant_W$).*

Now we discuss the Wadge reducibility in 2-reflective spaces. We start with a definition which again is a version of the corresponding notion in [38,39].

Definition 50. Let I be a non-empty set. By a 2-I-discrete weak semilattice we mean a structure of the form $(P; \leqslant, \{P_i^j\}_{i,j\in I})$ with the following properties:

(i) $(P; \leqslant)$ is a preordering;

(ii) $P = \bigcup\{P_i^j | i, j \in I\}$;

(iii) for all $x_0, x_1, \ldots \in P$ and $i, j \in I$ there exists $u_i^j = u_i(x_0, x_1, \ldots) \in P_i^j$ which is a least upper bound for $x_0, x_1, \ldots$ in the set P_i^j, i.e. $\forall k < \omega(x_k \leqslant u_i^j)$ and for each $y \in P_i^j$ with $\forall k(x_k \leqslant y)$ it holds $u_i^j \leqslant y$;

(iv) for all $x_0, x_1, \ldots \in P$, $i \neq i' \in I$, $j \neq j' \in I$ and $y \in P_{i'}^{j'}$, if $y \leqslant u_i^j(x_0, x_1, \ldots)$ then $y \leqslant x_k$ for some $k < \omega$.

The following properties of the 2-I-discrete weak semilattices are immediate (see also [38,39]).

Proposition 51. (i) *For each* $i, j \in I$, *every sequence* $x_0, x_1, \ldots$ *in* P_i^j *has a supremum* $u_i^j(x_0, x_1, \ldots) \in P_i^j$.

(ii) *For all* $y, x_0, x_1, \ldots \in P$, *if* $\forall k < \omega(x_k \leqslant y)$ *then* $\exists i, j \in I(u_i^j(x_0, x_1, \ldots) \leqslant y)$.

(iii) *For all* $y, x_0, x_1, \ldots \in P$, *if* $\forall i, j \in I(y \leqslant u_i^j(x_0, x_1, \ldots))$ *then* $\exists k < \omega(y \leqslant x_k)$.

(iv) *If* I *has at least two elements and* $(\{x_0, x_1, \ldots\}; \leqslant)$ *has no greatest element then the set* $\{x_0, x_1, \ldots\}$ *has no supremum in* $(P; \leqslant)$.

The next result is parallel to Theorem 49.

Theorem 52. *Let* X *be a* 2*-reflective space,* $\mathcal{P}_0^0 = \{A \subseteq X | \bot \notin A, \top \notin A\}$, $\mathcal{P}_1^0 = \{A \subseteq X | \bot \in A, \top \notin A\}$ *and similarly for* $\mathcal{P}_0^1, \mathcal{P}_1^1$. *Then* $(P(X); \leqslant_W, P_i^j)$ *is a* 2-$\{0, 1\}$*-discrete weak semilattice* (*and, consequently, if a sequence in* $P(X)$ *has no greatest element under* $\leqslant_W$ *then it has no supremum under* $\leqslant_W$).

The notion of 2-I-discrete weak semilattice looks similar to the notion of I^2-discrete weak semilattice, but actually it is easy to see that already the structure $(P(X); \leqslant_W, \mathcal{P}_i^j)$ above is not a $\{0, 1\}^2$- discrete weak semilattice.

Next we discuss relation of the Wadge reducibility to the hierarchies considered above. From Propositions 17 and 35 it follows that all levels of these hierarchies are closed downwards under the Wadge reducibility. Now we formulate several facts about the difference hierarchy.

Proposition 53. *Let* X *be a complete* φ*-space. For each* $n < \omega$, *every* $\mathbf{\Sigma}_n^{-1}$*-set is Wadge reducible to every set from* $\mathbf{\Delta}_2^0 \setminus \mathbf{\Pi}_n^{-1}$.

The next result is an immediate corollary of the last proposition.

Theorem 54. *Let* X *be a complete* φ*-space having chains of finitary elements of arbitrary finite length.*

(i) *For each* $n < \omega$, *the class of proper* $\mathbf{\Sigma}_n^{-1}$*-sets forms a Wadge degree.*

(ii) *For each* $n < \omega$, $G_n <_S G_{n+1}$ *and* $G_n <_S \overline{G}_{n+1}$, *where* G_n *is a proper* $\mathbf{\Sigma}_n^{-1}$*-set* $(n < \omega)$.

In [52] a close relations of the operations u_s and u_s^t, $s, t < 2$, on subsets of reflective and 2-reflective spaces to the difference hierarchy were established (namely, the levels of the difference hierarchy are closed under these operations with suitable indices). These relations and Proposition 41 imply the following three theorems.

Theorem 55. *Let* X *be a countably based reflective space. For each* $\alpha < \omega_1$, *the structures* $(\mathbf{\Sigma}_\alpha^{-1} \setminus \mathbf{\Pi}_\alpha^{-1}; \leqslant_W)$ *and* $(\mathbf{\Pi}_\alpha^{-1} \setminus \mathbf{\Sigma}_\alpha^{-1}; \leqslant_W)$ *are upper semilattices with least elements.*

For the 2-reflective spaces the analog of the last result looks a bit more complicated. By a *least pair* of a preordering $(P; \leqslant)$ we mean a pair x_0, x_1 of incomparable elements of P such that $\forall y \in P(x_0 \leqslant y \vee x_1 \leqslant y)$.

Theorem 56. *Let* X *be a countably based* 2*-reflective space and let* $\mathcal{P}^t = \mathcal{P}_0^t \cap \mathcal{P}_1^t$, $t < 2$ (*see Theorem* 52).

(i) *For all* α, $\omega \leqslant \alpha < \omega_1$ *and* $t < 2$, *there exists a least element in* $\mathcal{P}^t \cap (\mathbf{\Delta}_2^0 \setminus \mathbf{\Pi}_\alpha^{-1}; \leqslant_W)$.

(ii) *For each* α, $\omega \leqslant \alpha < \omega_1$, *there exists a least pair in* $(\mathbf{\Sigma}_\alpha^{-1} \setminus \mathbf{\Pi}_\alpha^{-1}; \leqslant_W)$.

The next result provides some additional information on the structure $(\mathbf{\Sigma}_\alpha^{-1} \setminus \mathbf{\Pi}_\alpha^{-1}; \leqslant_W)$ in 2-reflective spaces. By Theorem 54, this structure for $\alpha < \omega$ is trivial. In contrast to this, the last theorem shows that for $\alpha \geqslant \omega$ the structure is non-trivial. The next result strengthens this by showing that it contains an isomorphic copy of the ordering $\omega_1 \times \{0, 1\}$ obtained from the ordering $(\omega_1; <)$ by replacing every point by two incomparable points; in particular, the structure is uncountable. From [66] it follows that $\omega_1 \times \{0, 1\}$ is the order type of non-selfdual Wadge degrees of Δ_2^0-sets in the Baire (or Cantor) space.

Theorem 57. *Let* X *be a countably based* 2*-reflective space and* $\omega \leqslant \alpha < \omega_1$. *Then there exist proper* $\mathbf{\Sigma}_\alpha^{-1}$*-sets* B_γ^0, B_γ^1 $(\gamma < \omega_1)$ *with the following properties*:

(i) *if* $\gamma < \delta$ *then* $B_\gamma^s \leqslant_W B_\delta^t$ *for all* $s, t \leqslant 1$;

(ii) B_γ^0 *and* B_γ^1 *are Wadge incomparable*;

(iii) *if* $\gamma > 0$ *and a proper* $\mathbf{\Sigma}_\alpha^{-1}$*-set* C *is below* (*under* $\leqslant_W$) *both* B_γ^0, B_γ^1 *then it is below at least one of* B_δ^0, B_δ^1 *for some* $\delta < \gamma$;
(iv) *if* $\gamma > 0$ *and a proper* $\mathbf{\Sigma}_\alpha^{-1}$*-set* C *is above* (*under* $\leqslant_W$) *both* B_δ^0, B_δ^1 *for all* $\delta < \gamma$ *then it is above at least one of* B_γ^0, B_γ^1;
(v) *if* $B_\gamma^s \leqslant_W C \leqslant_W B_{\gamma+1}^{1-s}$ *then* $C \equiv_W B_\gamma^s$ *or* $C \equiv_W B_{\gamma+1}^{1-s}$.

Next we give some additional information on the Wadge reducibility in some concrete spaces introduced in Section 2. The last result yields some information on Wadge reducibility in the space $\omega_\perp^\omega$.

Theorem 58. *For each infinite ordinal* $\alpha < \omega$, *the structure* $(\mathbf{\Sigma}_\alpha^{-1} \setminus \mathbf{\Pi}_\alpha^{-1}; \leqslant_W)$ *in* $\omega_\perp^\omega$ *has a substructure of order type* $\omega_1 \times \{0, 1\}$.

The most important property of the Wadge reducibility in the Baire and Cantor spaces is the almost well-ordered property on the class of Borel sets. Which other spaces have this property (or its weaker version)? For the Polish zero-dimensional spaces the property is true [28]. In [21] (see also [19]), Hertling has shown that there are infinite ascending and descending chains, as well as infinite antichains within $(P(\mathbf{R}); \leqslant_W)$, where $\mathbf{R}$ is again the space of reals. Moreover, his examples use only very simple subsets of $\mathbf{R}$, namely finite boolean combinations of intervals. Thus, the Wadge degrees in $\mathbf{R}$ are much more complicated than in the Baire and Cantor spaces. The next result shows that the structure of Wadge degrees in $\omega^{\leqslant\omega}$ behaves better.

Theorem 59. *The order type of the quotient structures* $(\Delta_2^0(\omega^{\leqslant\omega}); \leqslant_W)$ *and* $(\Delta_2^0(n^{\leqslant\omega}); \leqslant_W)$ (*for each* $n, 2 \leqslant n < \omega$) *is* $\omega_1 \times \{0, 1\}$. *In particular*, *these structures are isomorphic*.

Remark. The structure $(\Sigma_2^0 \cup \Pi_2^0(n^{\leqslant\omega}); \leqslant_W)$ contains four pairwise incomparable elements. E.g., one easily checks that $A, \overline{A}, B, \overline{B}$ are pairwise Wadge incomparable, where $A = \{\sigma : |\sigma| = 1\}$ and $B = n^\omega$.

We know very little about similar results in other spaces. Many natural questions remain open. E.g., we do not currently know whether the structure $(\Delta_2^0(P\omega); \leqslant_W)$ contains infinite antichains or infinite descending chains.

Another interesting open question is the existence of Wadge complete (i.e., biggest under Wadge reducibility) sets in classes of the hierarchies considered above. The complete sets exist for the Baire and Cantor spaces [66]. But what about other spaces? The next result from [52] answers the question for the space $P\omega$. For most of other natural spaces (e.g., for the space $\omega_\perp^\omega$) the question is open.

Theorem 60. *For all* $\alpha, \beta < \omega_1$, $\beta > 0$, *each of the classes* $\mathbf{\Sigma}_\beta^0$, $D_\alpha(\mathbf{\Sigma}_\beta^0)$ *has a Wadge complete set*.

Let us mention a result from [53] which characterizes in terms of the Wadge reducibility a rather important class of sets introduced in [60]. Recall that a set $A \subseteq P\omega$ is *closed under chain* [60] if for every chain $\zeta_0 \subseteq \zeta_1 \subseteq \cdots$ of sets from A the union $\bigcup_n \zeta_n$ is in A.

Proposition 61. *A set* $A \subseteq P\omega$ *is not closed under chain iff* $\{\omega\} \leqslant_W \overline{A}$.

We conclude this section with a couple of remarks on effective versions of the Wadge reducibility (i.e., the reducibility by morphisms in the categories $\mathcal{E}$ and $\mathcal{C}$ from Section 3). In fact, this direction was initiated by the author in the context of the theory of numberings [38,39,42] when he did not know about the existence of the Wadge reducibility. From general facts of computability theory it easily follows that the structures under these effective reducibilities are extremely complicated. Nevertheless, papers [38,39,42] contain effective analogs of several results formulated above, e.g. there are non-trivial connections of the effective Wadge reducibility with the effective difference hierarhcies. In one respect the effective Wadge reducibility behaves even better than the non-effective one: in [42] we have shown that all levels of the effective difference hierarchies have complete sets under the reducibility by morphisms.

In this section we discussed only the Wadge reducibility of sets, though its generalization to the case of maps $\nu : X \to S$ to arbitrary set S is also of interest (the case of sets is obtained for $S = \{0, 1\}$), even for the Baire or Cantor space X. Some results about this generalization of the Wadge reducibility may be found in [39,32,52,21,49].

8. ω-Boolean operations

In this section we mention a couple of facts on the so-called ω-boolean operations which play an important role in the classical DST and are related to some results of the previous sections.

Recall that many levels of hierarchies in the classical DST may be obtained from the open sets by means of suitable (in general, infinitary) set-theoretic operations. For example, the operations of countable union and intersection and their iterates were first considered by Borel and Lebesgue in their study of the Borel hierarchy. Alexandrov and Suslin introduced and studied the A-operation which is important for the investigation of analytic sets. Kolmogorov [29] and Hausdorff [16] independently introduced the so-called positive analytic (or δs-) operations which generalize all the above-mentioned operations. Hausdorff defined the difference operations which were probably the first systematically considered non-positive operations. Kantorovich and Livenson [27] introduced operations that generalize all the operations mentioned above. Let us recall their definition.

Relate to each $A \subseteq 2^\omega$ the infinitary term d_A with variables $v_k (k < \omega)$ as follows:

$$d_A = d_A(v_k) = \bigcup_{\xi \in X} c_\xi \quad \textit{where } c_\xi = \left(\bigcap_{\xi(k)=1} v_k \right) \cap \left(\bigcap_{\xi(k)=0} \overline{v}_k \right).$$

Note that c_ξ are infinitary analogs of "elementary conjunctions" and d_A—of "disjunctive normal forms" in propositional logic. The term d_A induces in the evident way an ω-ary operation $d_A : P(2^\omega)^\omega \to P(2^\omega)$ on $P(2^\omega)$ (actually on any complete boolean algebra, in particular on $P(X)$ for every space X). Following Wadge [66], we call these operations here *ω-ary boolean operations*.

The ω-ary boolean operations are closely related to ω_1-*terms*, defined by induction as follows: constants 0, 1 and variables $v_k (k < \omega)$ are ω_1-terms; if $t_i (i < \omega)$ are ω_1-terms, then so are the expressions $\overline{t}_0$, $t_0 \cup t_1$, $t_0 \cap t_1$, $\bigcup_{i<\omega} t_i$ and $\bigcap_{i<\omega} t_i$. If $t = t(v_k)$ is an ω_1-term, let $t(\{A_k\})$ denote the value of t when each variable $v_k (k < \omega)$ is interpreted as some set $A_k \subseteq 2^\omega$. Let $t(\mathbf{\Sigma}^0_1)$ be the set of all values $t(\{A_k\})$, when $A_k \in \mathbf{\Sigma}^0_1$ for every $k < \omega$. We use similar notation $t(\mathcal{C})$ also for other kinds of terms t and classes of sets $\mathcal{C}$. We call two infinitary boolean terms *equivalent* if they define the same infinitary operation in every complete boolean algebra. The next easy fact from [45] relates ω_1-terms to a natural class of ω-ary boolean operations.

Theorem 62. *Each ω_1-term is equivalent to the term d_A for some Borel set $A \subseteq 2^\omega$, and vice versa.*

It turns out that the classes $t(\mathbf{\Sigma}^0_1)$ have a very natural description in terms of the Wadge reducibility in 2^ω. Recall that a *Wadge class* is a principal ideal of the form $\{B | B \leqslant_W A\}$, for a given $A \subseteq 2^\omega$. Such a class is *Borel* if A is Borel, and is *non-selfdual* if $A \not\leqslant_W \overline{A}$. The next fact was proved in [65,66] (the equivalence of (ii) and (iii) follows from the last theorem).

Theorem 63. *For every $\mathcal{C} \subseteq P(2^\omega)$ the following assertions are equivalent*:
(i) *$\mathcal{C}$ is a non-selfdual Borel Wadge class*;
(ii) *$\mathcal{C} = d_A(\mathbf{\Sigma}^0_1)$ for some Borel set $A \subseteq 2^\omega$*;
(iii) *$\mathcal{C} = t(\mathbf{\Sigma}^0_1)$ for some ω_1-term t.*

The next simple fact, attributed in [64] to Miller, yields a similar description of the Wadge classes (not only Borel), using the class $\mathbf{\Delta}^0_1$ of clopen sets in place of $\mathbf{\Sigma}^0_1$.

Theorem 64. *For every $A \subseteq 2^\omega$, $d_A(\mathbf{\Delta}^0_1) = \{B | B \leqslant_W A\}$.*

Corollary 65. *The map $A \mapsto d_A(\mathbf{\Delta}^0_1)$ is an isomorphism between the structure of all Wadge degrees in 2^ω and the structure of all Wadge classes under inclusion.*

The next result from [53] is an analog of the last theorem for the space $P\omega$.

Theorem 66. *For every $A \subseteq P\omega$, $d_A(\mathbf{\Sigma}^0_1(P\omega)) = \{B | B \leqslant_W A\}$.*

Corollary 67. *The preorderings* $(\{d_A(\mathbf{\Sigma}^0_1(P\omega))|A \subseteq P\omega\}; \subseteq)$ *and* $(P(P\omega); \leqslant_W)$ *are equivalent.*

The last corollary and results of Section 7 show that the structure $(\{d_A(\mathbf{\Sigma}^0_1(P\omega))|A \in P\omega\}; \subseteq)$ is not almost well-ordered. In contrast to this, Theorem 6.5 in [45] implies that the structure $(\{d_A(\mathbf{\Sigma}^0_\alpha(P\omega))|A \in P\omega\}; \subseteq)$ for every $\alpha \geqslant 2$ is almost well-ordered.

Finally, we state a result from [51,53] which relates the Wadge reducibility in $P\omega$ to the structure of non-selfdual Wadge classes in 2^ω.

Theorem 68. *For all* $A, B \subseteq P\omega$, $A \leqslant_W B$ *implies* $d_A(\mathbf{\Sigma}^0_1) \subseteq d_B(\mathbf{\Sigma}^0_1)$.

9. Applications and connections

In this section we mention connections of the topic discussed above with some branches of mathematics and theoretical computer science. By application we mean such a connection of the domain DST with some field that yields results in the field obtained by using results and/or techniques discussed above.

9.1. Application to classical DST

Here we apply some results established above to some natural questions about the ω-boolean operations from the previous section. The general question is formulated as follows: for a given space X, describe the class of sets $A \subseteq P\omega$ such that $d_A(\mathbf{\Sigma}^0_1(X)) \subseteq \mathcal{C}$ where $\mathcal{C}$ is a level of some hierarchy in X discussed above (Borel, difference or even the Wadge hierarchy). There are some variations of this question. E.g., we could take the level $\mathbf{\Delta}^0_1(X)$ instead of $\mathbf{\Sigma}^0_1(X)$ or try to describe the class of sets $A \subseteq P\omega$ with $d_A(\mathbf{\Sigma}^0_1(X)) = \mathcal{C}$. Note that solution of the last question (for $=$) often easily follows from the solution of the first one (for $\subseteq$); an example is Corollary 70. The results of this subsection were obtained in [51,53].

First we settle the problem for some classes related to higher levels of the Borel hierarchy in the Cantor space $X = 2^\omega$. Let again $\mathbf{B}$ denote the class of Borel sets in 2^ω.

Theorem 69. *Let* $\alpha < \omega_1$, $\omega \leqslant \beta < \omega_1$ *and* $A \subseteq 2^\omega$.

(i) $A \in \mathbf{B}$ *iff* $d_A(\mathbf{\Sigma}^0_1) \subseteq \mathbf{B}$.

(ii) $A \in \mathbf{\Sigma}^0_\beta$ *iff* $d_A(\mathbf{\Sigma}^0_1) \subseteq \mathbf{\Sigma}^0_\beta$.

(iii) $A \in \bigcup_{k<\omega} \mathbf{\Sigma}^0_k$ *iff* $d_A(\mathbf{\Sigma}^0_1) \subseteq \bigcup_{k<\omega} \mathbf{\Sigma}^0_k$.

(iv) *For every* $B \subseteq P\omega$, $A \in d_B(\mathbf{\Sigma}^0_\beta)$ *iff* $d_A(\mathbf{\Sigma}^0_1) \subseteq d_B(\mathbf{\Sigma}^0_\beta)$. *In particular,* $A \in D_\alpha(\mathbf{\Sigma}^0_\beta)$ *iff* $d_A(\mathbf{\Sigma}^0_1) \subseteq D_\alpha(\mathbf{\Sigma}^0_\beta)$.

From the inclusions of levels of hierarchies one easily obtains characterizations of some conditions of the form $d_A(\mathbf{\Sigma}^0_1) = \mathcal{C}$, in place of conditions of the form $d_A(\mathbf{\Sigma}^0_1) \subseteq \mathcal{C}$ considered above. E.g., we have:

Corollary 70. *For all* $\alpha, \beta < \omega_1$ *with* $\omega \leqslant \beta < \omega_1$, $A \in D_\alpha(\mathbf{\Sigma}^0_\beta(2^\omega)) \setminus co\text{-}D_\alpha(\mathbf{\Sigma}^0_\beta(2^\omega))$ *iff* $d_A(\mathbf{\Sigma}^0_1(2^\omega)) = D_\alpha(\mathbf{\Sigma}^0_\beta(2^\omega))$.

The next result settles the problem for a lower level of the Borel hierarchy. Note that, in contrast to the previous theorem, we have to use a hierarchy in $P\omega$.

Theorem 71. *For every* $A \subseteq P\omega$, $A \in \Delta^0_2(P\omega)$ *iff* $d_A(\mathbf{\Sigma}^0_1(2^\omega)) \subseteq \mathbf{\Delta}^0_2(2^\omega)$.

From the proof of the last theorem we obtain the following.

Corollary 72. *For every* $A \subseteq P\omega$, *if* $A \notin \mathbf{\Delta}^0_2(P\omega)$ *then* $\mathbf{\Sigma}^0_2(2^\omega) \subseteq d_A(\mathbf{\Sigma}^0_1(2^\omega))$ *or* $\mathbf{\Pi}^0_2(2^\omega) \subseteq d_A(\mathbf{\Sigma}^0_1(2^\omega))$.

The next result settles the problem for levels of the difference hierarchy in the Cantor space. Again, the description uses the difference hierarchy in $P\omega$.

Theorem 73. *For every* $\alpha < \omega_1$, $A \in \Sigma^{-1}_\alpha(P\omega)$ *iff* $d_A(\mathbf{\Sigma}^0_1(2^\omega)) \subseteq \mathbf{\Sigma}^{-1}_\alpha(2^\omega)$.

The problem discussed above remains open for many levels of the Wadge hierarchy, and even of the Borel hierarchy. From Theorems 69 and 73 we immediately obtain the following.

Corollary 74. *Let* $\alpha = 1$ *or* $\omega \leqslant \alpha < \omega_1$, *and* $A \subseteq P\omega$. *Then* $d_A(\mathbf{\Sigma}^0_1(2^\omega)) \subseteq \mathbf{\Sigma}^0_\alpha(2^\omega)$ *iff* $A \in \mathbf{\Sigma}^0_\alpha(P\omega)$.

For all other levels $\mathbf{\Sigma}^0_n$ of the Borel hierarchy, $2 \leqslant n < \omega$, the problem remains open. We guess that Corollary 74 is true also for the levels $\mathbf{\Sigma}^0_n$, $2 \leqslant n < \omega$. If this is really the case there is a hope to obtain a complete solution of the problem for all levels of the Wadge hierarchy.

We conclude this subsection with a couple of results about some variations of the main question. In the case when we consider the class $\mathbf{\Delta}^0_1$ in place of $\mathbf{\Sigma}^0_1$ the answer easily follows from Theorem 64 and looks as follows.

Corollary 75. *Let* $A \subseteq 2^\omega$ *and* $\mathcal{C}$ *be a Wadge class in* 2^ω. *Then* $d_A(\mathbf{\Delta}^0_1(2^\omega)) \subseteq \mathcal{C}$ *iff* $A \in \mathcal{C}$.

The last Corollary of course applies to the case when $\mathcal{C}$ is a level of the Borel or difference hierarchies in the Cantor space.

Above we considered only the space $X = 2^\omega$. The next result which follows immediately from Theorem 66 settles the problem for the space $X = P\omega$.

Corollary 76. *Let* $A \subseteq P\omega$ *and* $\mathcal{C}$ *be a Wadge class in* $P\omega$. *Then* $d_A(\mathbf{\Sigma}^0_1(P\omega)) \subseteq \mathcal{C}$ *iff* $A \in \mathcal{C}$.

The last Corollary applies to the case when $\mathcal{C}$ is a level of the Borel or difference hierarchies in $P\omega$.

9.2. Application to computability theory

Here we describe an application of the effective domain DST to the problem of extensional description of index sets. The material is taken from [40,42].

Let v be a numbering. Recall that a *v-index set* of $A \subseteq rng(v)$ is the preimage $v^{-1}(A)$. For natural numberings index sets represent decision problems. Here we consider the problem of extensional (i.e. not using explicitly the names n of objects v_n) characterization of the sets A for which $v^{-1}(A)$ belongs to a given class $\mathcal{C}$ of sets. An example is the Rice-Shapiro Theorem for approximable numberings, see Section 3: $v^{-1}(A)$ is c.e. (i.e. belongs to the level Σ^0_1 of the arithmetical hierarchy) iff A is effective open (i.e., belongs to the level $\Sigma^0_1(v)$ of the effective Borel hierarchy in v). We consider the problem for the approximable numberings and some of their subclasses.

Let us first consider the problem of extensional characterization for the levels $\mathcal{C} = \Sigma^0_n$, $n > 1$, of the arithmetical hierarchy (and also for the transfinite levels of the hyperarithmetical hierarchy). This problem was mentioned in [35]. Our main idea in solving this problem is to use the effective hierarchies in v considered above.

Theorem 77. *Let* v *be an arbitrary approximable numbering,* $|a|_O \geqslant 3$ *and* $A \subseteq rng(v)$. *Then* $v^{-1}(A) \in \Sigma^0_{(a)}$ *iff* $A \in \Sigma^0_{(a)}(v)$.

This result proved in [40,42] solves the problem for all levels of the hyperarithmetical hierarchy (except the second level) because the classes $\Sigma^0_{(a)}(v)$ are defined extensionally. What about the second level of the arithmetical hierarchy? It turns out that for this level the situation is quite different: an extensional characterization similar to Theorem 77 is impossible. To see this, note that the class I_n of v-index sets from $\Sigma^0_n (n = 1, n > 2)$ (and also from the transfinite levels) is Σ^0_n-computable, i.e. $I_n = \mu(U)$ for some c.e. set U, where μ is the acceptable numbering of Σ^0_n. This fact was observed in [40] (for $n > 2$ it is almost evident: one should only note that $A \mapsto v^{-1}v(A)$ is a morphism from Σ^0_n into itself, and this is immediate by the Tarski–Kuratowski algorithm). For the level Σ^0_2 the situation is opposite. Call a set $A \subseteq rng(\mu)$ μ*-productive*, if there is a computable function p such that $\mu(W_x) \subseteq A$ implies $\mu p(x) \in A \setminus \mu(W_x)$. Of course, μ-productive sets are not μ-computable.

For simplicity we formulate the next result (and Theorem 81 below) only for the standard numbering W of c.e. sets, though it is true for a broad enough class of approximable numberings including φ, W and closed under product and taking the (effective) functional space (see [42]).

Theorem 78. *Let $\nu = W$ be the standard numbering of c.e. sets. The class of ν-index sets from Σ^0_2 is Σ^0_2-productive (hence it is not Σ^0_2-computable).*

This result from [40,42] shows that there is no description of ν-index sets from Σ^0_2 constructive enough to induce a Σ^0_2-computable numbering of them. A constructive extensional description of $\{A \subseteq rng(\nu) | \nu^{-1}(A) \in \Sigma^0_2\}$ would probably give such a numbering, hence it is impossible. Moreover, Theorem 78 gives an algorithm which computes a counterexample to any such constructive candidate for the description. E.g., a natural analog of Theorem 77 is the equality $\{A \subseteq rng(\nu) | \nu^{-1}(A) \in \Sigma^0_2\} = \Sigma^0_2(\nu)$. But the natural numbering $\{A_n\}$ of $\Sigma^0_2(\nu)$ induces a Σ^0_2-computable numbering $\{\nu^{-1}(A_n)\}$ of index sets, so one can compute an index set $\nu^{-1}(A) \in \Sigma^0_2$ with $A \notin \Sigma^0_2(\nu)$. In particular, the inclusion $\Sigma^0_2(\nu) \subset \{A \subseteq rng(\nu) | \nu^{-1}(A) \in \Sigma^0_2\}$ is strict.

Now let us consider the problem for levels of other hierarchies considered above. A clear description exists for the first level $\mathcal{C} = \Sigma^1_1$ of analytical hierarchy.

Theorem 79. *Let ν be an arbitrary approximable numbering. For each $A \subseteq rng(\nu)$, $\nu^{-1}(A) \in \Sigma^1_1$ iff $A \in \Sigma^1_1(\nu)$.*

It is an open question to find an extensional characterization of the classes $A \subseteq rng(\nu)$ with $\nu^{-1}(A) \in \Sigma^1_n$ for higher levels $n > 1$ of the analytical hierarchy.

Next we consider the problem for the levels $\mathcal{C} = \Sigma^{-1}_n$ of the difference hierarchy (for simplicity of notation we consider only finite levels). In this case the natural candidate for the description is the difference hierarchy $\Sigma^{-1}_k(\nu)$. The next result from [42] generalizes a similar fact obtained in [15,17] for the case $\nu = W$.

Theorem 80. *Let ν be a complete approximable numbering, δ its approximation, $n > 1$ and $A \subseteq rng(\nu)$ be such that $\delta^{-1}(A)$ is computable. Then $\nu^{-1}(A) \in \Sigma^{-1}_n$ iff $A \in \Sigma^{-1}_n(\nu)$.*

In [42] we obtained the following analog of Theorem 78 which shows that the condition of computability of $\delta^{-1}(A)$ is essential, and that for the levels Σ^{-1}_n, $n > 1$, the simple extensional description is impossible.

Theorem 81. *Let $\nu = W$ be the standard numbering of c.e. sets and $n > 1$. The class of ν-index sets from Σ^{-1}_n is Σ^{-1}_n-productive (hence it is not Σ^{-1}_n-computable and the inclusion $\Sigma^{-1}_n(\nu) \subset \{A \subseteq rng(\nu) | \nu^{-1}(A) \in \Sigma^{-1}_n\}$ is strict).*

The paper [42] contains several variations of the last result. E.g., a similar fact holds true for all levels of the difference hierarchy over Σ^0_2, and the set $\{A \subseteq rng(\nu) | \nu^{-1}(A) \in \Sigma^{-1}_2\}$ is contained in $\Sigma^0_2(\nu)$ but is not contained in $\Pi^0_2(\nu)$).

9.3. Connections with computability in analysis

Here we briefly mention connections of the effective DST to computability in analysis. Computable analysis is a branch of computability theory dealing with computability on the reals and other spaces relevant to analysis and functional analysis. The topic is important because it is intended to serve as a theoretical foundation of numeric analysis. Research in computable analysis is developing very actively, a standard reference is [69].

The effective DST is of course fundamental for computable analysis, similar to the well-known fact that classical DST is fundamental for classical analysis. Nevertheless, up to now there are only few publications specially devoted to this field. In our opinion, there are two main reasons for this. First, computable analysis is still in its early stage and there are many interesting open questions which do not require deep considerations of effective DST. Second, effective DST is itself still in the very beginning and many natural questions remain open, as we have seen above.

In [18] the author considers the effective difference hierarchy on the reals and applies it to define some new concepts of computability for sets of reals. In [3], an effective theory of Borel measurable functions is applied to investigation of computability issues for discontinuous functions on the reals. Both papers consider also the notion of degree of discontinuity of a function introduced and studied in [21] which is closely related to the Wadge reducibility of sets of reals. Along with the Wadge reducibility, people working in computable analysis began to consider some of its weaker variants [21,69]. A search for such useful variants and their applications seems reasonable because it yields interesting and computationally relevant classifications of discontinuous functions.

The research mentioned above tries to apply the effective classical DST. This is because that work uses the so-called TTE-approach to computability in analysis [69] which does not use the domain theory. Another popular approach to computable analysis based on domain theory tries to embed the spaces relevant to analysis (like the space of reals) into some domains and then apply the computability in domains. There are several interesting approaches to construction of such embeddings, and some of them seem to be relevant to the domain DST developed above. An example is the paper [63] where some computationally interesting embeddings were invented. We believe that similar embeddings are relevant to the domain DST, in particular they may help to solve some open questions about the Wadge reducibility in domains left open in Section 7.

9.4. Connections with infinite computations

Here we briefly discuss relations of DST to the theory of infinite computations. The behavior of computing devices working indefinitely are often modeled by ω-languages (i.e., subsets of the Cantor space n^ω, $2 \leqslant n < \omega$) recognized by such a device. Much information on the subject may be found in [62,5,56,34].

DST in the Cantor space provides tools to classify "natural" classes of ω-languages according to their "complexity". In particular, the Borel and difference hierarchies were employed in development of this subject. A long series of papers culminated with the paper [67] where Wagner determined the order type of Wadge degrees of regular ω-languages (i.e. ω-languages recognized by finite automata) to be ω^ω. (Note that in this subsection we use a couple of times standard notation from ordinal arithmetic [31] which, unfortunately, conflicts with notation used above. E.g., the last ω^ω denotes the ordinal exponentiation, not the Baire space. In spite of this inconvenience, we decided not to change the notation used in other sections which is also quite standard.) Interestingly, Wagner that time knew nothing about the results of Wadge and thus defined the Wadge reducibility independently.

In [44,46,47] the Wagner hierarchy of regular ω-languages was related to the Wadge hierarchy and to the author's fine hierarchy [45]. This provided new proofs of results in [67] and yielded some new results on the Wagner hierarchy. In [4] a description of the Wadge degrees containing regular ω-languages was obtained (this description is also implicitly contained in [44], if one takes into account the relationship of the fine hierarchy to the Wadge hierarchy [43]). In 1999 the author has proved that the Wadge degrees of regular star-free ω-languages (for the last notion see e.g. [62]) coincide with the Wadge degrees of regular ω-languages (this result is still unpublished though it was reported at several conferences and seminars). In [4] the Wadge degrees of deterministic context-free ω-languages were determined; the corresponding ordinal is $(\omega^\omega)^\omega$. In the same paper a conjecture about the structure of Wadge degrees of ω-languages recognizable by deterministic Turing machines was formulated (for the Muller acceptance condition, see [56]) implying that the corresponding ordinal is $(\omega_1^{CK})^\omega$. This conjecture was proved in [48]. Meanwhile, in [56] it was shown that the class of ω-languages recognized by deterministic Turing machines coincides with the boolean closure of the second level Σ_2^0 of the effective Borel hierarchy in the Cantor space.

The results mentioned in the previous section essentially finished the study of Wadge degrees of ω-languages recognized by deterministic devices. Nevertheless, some interesting questions related to effectivity issues remain open. For instance, the results and proofs in [67,47] are constructive while the description in [4] is not. It is currently an open problem whether it is possible to develop an effective version of the Wagner hierarchy of deterministic context-free ω-languages parallel to the effective theory in [67,47].

For the case of non-deterministic accepting devices, Staiger [56] has shown that the class of ω-languages recognized by non-deterministic Turing machines coincides with the class Σ_1^1 of effective analytic sets. In a series of papers (see [13] and references therein), Finkel obtained much information on Wadge degrees of non-deterministic context-free ω-languages.

The results mentioned above relate the theory of infinite computations to the (effective) classical DST in the Cantor space. We believe that the domain DST discussed in this paper is also relevant to that field. The reason is that it is also very natural to study computations which may terminate or not. Such considerations lead to the theory of so-called ∞-languages, i.e. to the study of sets in $n^{\leqslant\omega}$ for $2 \leqslant n < \omega$ (see e.g. [2] and references therein). Though the theory of ∞-languages seems to differ considerably from the theory of ω-languages (e.g. the analysis of possibly infinite computations in [2] leads to three different topologies on $n^{\leqslant\omega}$ instead of one Cantor topology for the case of ω-languages) we think that the domain DST is relevant to this case as well. A concrete open problem in this field is to describe the analog of the Wagner hierarchy for the regular ∞-languages (the last notion is well established).

9.5. Connections with labeled transition systems

In this section we very briefly and informally discuss a relation of domain theory to labeled transition systems (LTS) which may lead to a new interesting application of the domain DST. The relation was discovered in a series of recent publications (see [23–26] and references therein).

As is well-known, the notion of LTS is one of the central notions of theoretical computer science. It is for example central in the practically important field of model checking, where people use different temporal logics (Hennessy–Milner logic, linear temporal logic, μ-calculus and so on) for specification and verification of behavior of LTSs. The behavioral equivalence of two LTSs is captured by the notion of bisimulation.

A drawback of the LTS-formalism is that it is not adequate when there is a need to refine a given system in order to obtain a more concrete system which is closer to the real implementation (the process of subsequent refinements is the usual procedure in the practical design of hardware and software systems). The desire to capture the notion of refinement was a reason to weaken the notion of LTS to that of modal transition system (MTS).

Let *Act* be a finite set of *events*. A *modal transition system* (over *Act*) is a triple $M = (S, R^a, R^c)$ where S is a set of *states* and $R^a \subseteq R^c \subseteq S \times Act \times S$; elements of R^a are called *must-transitions* while elements of $R^c \setminus R^a$—*may-transitions*. A *pointed MTS* is a pair (M, s) consisting of an MTS M and a state s of M. For pointed MTSs (M, s) and (N, t) the notion $(M, s) \preccurlyeq (N, t)$ meaning that (N, t) is a refinement of (M, s) is defined in a natural way. LTSs correspond to MTSs without may-transition, and two LTSs are bisimilar iff they refine each other as MTSs.

The main invention in the cited papers was the construction of an ω-algebraic domain $\mathbf{D}$ which may also be interpreted as an MTS $\mathcal{D}$ such that:

- for all $d, e \in \mathbf{D}$, $d \leqslant e$ iff $(\mathcal{D}, d) \preccurlyeq (\mathcal{D}, e)$;
- there is an embedding $(M, s) \mapsto [M, s]$ of the pointed finitely branching MTSs into $\mathbf{D}$ such that the pointed MTSs (M, s) and $(\mathcal{D}, [M, s])$ are refinement-equivalent;
- the subspace $\max(\mathbf{D})$ of $\mathbf{D}$ formed by the maximal points in $(\mathbf{D}; \leqslant)$ is a Stone space (i.e. compact, zero-dimensional and ultrametrizable);
- the set $\max(\mathbf{D})$ is in a bijective correspondence (induced by the map $(M, s) \mapsto [M, s]$) with the pointed LTSs modulo bisimulation;
- the set of finitely branching pointed LTSs is dense in $\max(\mathbf{D})$.

This approach unifies several known approaches to semantics of LTSs and suggests many new developments. For example, in the cited papers the above-mentioned temporal logics were somehow interpreted in every pointed MTS, which induces a definability theory in $\mathbf{D}$. Preliminary results and discussions in those papers show that there is probably fruitful interrelations of that definability theory with the domain DST described above. This direction is similar to the well-known application of the classical DST to model theory [28], through considering the class of countable structures of a given finite relational signature as a Polish space.

10. Conclusion

We hope that this survey may convince the reader that the domain DST is an interesting and deep field with some interesting applications and connections to several branches of theoretical computer science. It has its own flavor as compared say with the well-developed classical DST. The theory is still in its beginning and there are many open questions related to results reported in this paper. Many open questions were mentioned above. Another general question is to find for the main theorems of DST the broadest possible classes of topological spaces in which they still hold true.

Acknowledgments

Some results reported above were obtained during several research visits of the author to the University of Siegen. I am grateful to the university for supporting my stays and to Dieter Spreen for making the stays possible. I thank also an anonymous referee for the careful reading and useful remarks. I am grateful to the editors of this volume R. Kopperman and D. Spreen for inviting me to submit this paper.

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