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Note

The number of edges in a maximum cycle-distributed graph

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Abstract

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Let $f(n)$ ($f_2(n)$) be the maximum possible number of edges in a graph (2-connected simple graph) on n vertices in which no two cycles have the same length. In this note, we prove that, for every integer $n \geqslant 3$, $f(n) \geqslant n + k + [\frac{1}{2}(\sqrt{8n - 24k^2 + 8k - 7} - 1)]$, where $k = [\frac{1}{21}(\sqrt{21n - 26} + 11)]$, and obtain upper and lower bounds on $f_2(n)$.

1. Introduction

In this note, we consider finite undirected graphs. All definitions and notations not given here can be found in [1, 2].

A graph G is said to be a cycle-distributed graph if no two cycles in G have the same length. In particular, a graph G containing at most one cycle is a cycle-distributed graph.

Let $f(n)$ ($f^*(n)$, $f_2(n)$) be the maximum possible number of edges in a cycle-distributed graph (simple cycle-distributed graph, 2-connected simple cycle-distributed graph) on n vertices.

In 1975, Erdős raised the question of determining $f(n)$ (see [1, p. 247, Problem 11]). This problem remains unsettled. Till now, we have not known any good upper bound of $f(n)$. In [2], we obtained the following result:

$$f(n) \geqslant n + [\tfrac{1}{2}(\sqrt{8n - 23} + 1)] \quad \text{for each } n \geqslant 3$$

and the equality holds when $3 \leqslant n \leqslant 17$. In this note, we improve the lower bound of $f(n)$. Our main result is

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$$f(n) \geq n + k + [\tfrac{1}{2}(\sqrt{8n - 24k^2 + 8k - 7} - 1)], \quad \text{where } k = [\tfrac{1}{21}(\sqrt{21n - 26} + 11)].$$

2. New lower bound on $f(n)$

Lemma 2.1. *For every integer $n \geq 46$.*

$$f^*(n) \geq n + k + [\tfrac{1}{2}(\sqrt{8n - 24k^2 + 8k + 1} - 5)],$$

where $k = [\frac{1}{21}(\sqrt{21n - 5} + 11)]$.

Proof. We shall give a constructive proof of this lemma. For every integer $n \geq 46$, we can obtain integers k and t from the following equations:

$$k = [\tfrac{1}{21}(\sqrt{21n - 5} + 11)], \qquad t = [\tfrac{1}{2}(\sqrt{8n - 24k^2 + 8k + 1} - 12k + 7)].$$

Let $a_i = 6(k-1) + t + 3 - 2k + i$, $i = 0, 1, \ldots, 2k$. Let G_{a_j} be a graph obtained by the following method for $j = 1, 2, \ldots, k$.

We first take a cycle C of length $a_{2j-1} + 2(k+j) + 1$. Let x, y and z be vertices of C which split C into three paths $Q_1 = (y, x)$, $Q_2 = (x, z)$, $Q_3 = (z, y)$. We then join x to y by a new path P_1 and join x to z by a new path P_2 such that:

(1) P_1 and P_2 are internally disjoint;

(2) $Q_1 \cup P_1$ is a cycle of length a_{2j-1} and $Q_2 \cup P_2$ is a cycle of length a_{2j};

(3) $|V(P_1)| = [\frac{1}{2}(a_{2j-1} + 1)]$ and $|V(P_2)| = [\frac{1}{2}(a_{2j} + 1)]$.

Clearly G_{a_j} has exactly six cycles of length a_{2j-1}, a_{2j}, $a_{2j-1} + 2(k+j-1)$, $a_{2j-1} + 2(k+j-1) + 1$, $a_{2j-1} + 2(k+j-1) + 2$, $a_{2j-1} + 2(k+j-1) + 3$, respectively. Let

$$m = n - \left(3 + \sum_{i=3}^{a_{2k-1}} i + \sum_{j=1}^{k} 2(k+j-1)\right),$$

and let $G_{a_{k+1}} = K_{1,m}$. Let G_i be a cycle of length i, $i = 3, 4, \ldots, a_0$. We now form a graph G from the graph sequence $G_3, G_4, \ldots, G_{a_0}, G_{a_1}, \ldots, G_{a_k}, G_{a_{k+1}}$ by identifying one vertex of G_i and one vertex of G_{i+1} for every integer i, $3 \leq i \leq a_k$. Since every G_{a_j} has exactly six cycles, $\bigcup_{j=1}^{k} G_{a_j}$ has exactly $6k$ cycles of length $a_1, a_2, \ldots, a_{2k}, a_{2k} + 1, \ldots, a_{2k} + 4k$, respectively. Also $\bigcup_{i=3}^{a_0} G_i$ has exactly $a_0 - 2$ cycles of length $3, 4, \ldots, a_0$, respectively. Thus G is a simple cycle-distributed graph.

It is easily seen that G has exactly n vertices and $n + 6(k-1) + t + k$ edges. Therefore

$$\begin{aligned} f^*(n) &\geq n + 6(k-1) + t + k \\ &= n + 6(k-1) + k + [\tfrac{1}{2}(\sqrt{8n - 24k^2 + 8k + 1} - 12k + 7)] \\ &= n + k + [\tfrac{1}{2}(\sqrt{8n - 24k^2 + 8k + 1} - 5)], \end{aligned}$$

where $k = [\frac{1}{21}(\sqrt{21n - 5} + 11)]$. □

Theorem 2.2. *For every integer* $n \geq 2$,

$$f^*(n) \geq n + k + [\tfrac{1}{2}(\sqrt{8n - 24k^2 + 8k + 1} - 5)]$$

where $k = [\frac{1}{21}(\sqrt{21n - 5} + 11)]$.

Proof. In [2], we proved that, for every $n \geq 2$,

$$f^*(n) \geq n + [\tfrac{1}{2}(\sqrt{8n - 15} - 3)]. \tag{1}$$

It is easy to verify that, for every $2 \leq n \leq 45$,

$$n + [\tfrac{1}{2}(\sqrt{8n - 15} - 3)] = n + k + [\tfrac{1}{2}(\sqrt{8n - 24k^2 + 8k + 1} - 5)]. \tag{2}$$

The theorem follows immediately from (1), (2) and Lemma 2.1. □

Theorem 2.3. *For every integer* $n \geq 3$,

$$f(n) \geq n + k + [\tfrac{1}{2}(\sqrt{8n - 24k^2 + 8k - 7} - 1)].$$

where $k = [\frac{1}{21}(\sqrt{21n - 26} + 11)]$.

Proof. This follows directly from Theorem 2.2 and the following result obtained in [2]: For every $n \geq 3$, $f(n) = f^*(n - 1) + 3$. □

3. Bounds on $f_2(n)$

Theorem 3.1. *For every integer* $n \geq 3$,

$$f_2(n) \leq n + [\tfrac{1}{2}(\sqrt{8n - 15} - 3)].$$

Proof. This follows directly from the following result proved in [2]. Let m be the number of cycles contained in a 2-connected simple graph G and $j = |E(G)| - |V(G)|$, then $m \geq (j + 1)(j + 2)/2$. □

Theorem 3.2. *For every integer* $n \geq 3$,

$$f_2(n) \geq n + 1 + [\log_2((n - 2)/3)].$$

Proof. For every integer $k \geq 1$ and every integer n, $3 \cdot 2^{k-1} + 2 \leq n \leq 3 \cdot 2^k + 1$, take a n-cycle $C_n = (1\ 2\ 3 \cdots n)$ and let $x_1 = 1$ and $x_{i+1} = x_i + 2^{i-1} + 1$ for $i = 1, 2, \ldots, k$. Form a graph $G(n, k)$ from C_n by joining x_i to x_{i+1} for $i = 1, 2, \ldots, k$.

Clearly $G(n, k)$ has exactly $2^k + k$ cycles of length $2^j + 2$ $(j = 0, 1, \ldots, k - 1)$ and $n - t$ $(t = 0, 1, \ldots, 2^k - 1)$, respectively. $G(n, k)$ is clearly a 2-connected simple cycle-distributed graph with n vertices and $n + k$ edges, where $k = 1 + [\log_2((n - 2)/3)]$. Thus

$$f_2(n) \geq n + 1 + [\log_2((n - 2)/3)]. \quad \square$$

It is easily verified that, for every integer $3 \leq n \leq 11$,

$$n + [\tfrac{1}{2}(\sqrt{8n - 15} - 3)] = n + 1 + [\log_2((n - 2)/3)].$$

Thus, by Theorem 3.1 and Theorem 3.2, we have

$$f_2(n) = n + [\tfrac{1}{2}(\sqrt{8n - 15} - 3)] \quad \text{for} \quad 3 \leq n \leq 11.$$

Let $u_k = (((1 + \sqrt{5})/2)^k - ((1 - \sqrt{5})/2)^k)/\sqrt{5}$ be a Fibonacci number. The following theorem gives a better lower bound than Theorem 3.2.

Theorem 3.3. *Let k and n be integers such that $k \geq 4$ and $u_k \leq n < u_{k+1}$, then $f_2(n) \geq n + k - 4$.*

To prove this theorem, we need some properties of Fibonacci numbers (see [3]):

(a) $u_k + u_{k+1} = u_{k+2}$;
(b) $\sum_{i=1}^{n} u_{2i-1} = u_{2n}$;
(c) $\sum_{i=1}^{n} u_{2i} = u_{2n+1} - 1$.

Proof of Theorem 3.3. We consider two cases.

Case 1: $n = u_k$ $(k \geq 4)$.

Let $t = k - 4$. It is convenient to denote by C_i the cycle of length $u_{i+2} + 2$ in the proof. Also, let $1(C)$ denote the length of a cycle C.

Let G be a graph drawn on the plane and let C be a cycle of G, then C divides the plane into two regions. The bounded (unbounded) region is called the interior (exterior) of C and is denoted by int C (ext C).

Now form a graph G^t by the following method: We first take an n-cycle C^* drawn on the plane such that int C^* is a convex polygonal region. We then draw t diagonals meeting a common vertex v and divide int C^* into $t + 1$ regions such that the boundaries of these regions are cycles $C_0, C_1, C_2, \ldots, C_t$ and the $t + 1$ cycles are arranged in the order $C_0, C_2, C_4, \ldots, C_t, \ldots, C_3, C_1$, (see Fig. 1).

Since

$$1(C_0) + \sum_{i=1}^{t} (1(C_i) - 2) = u_2 + 2 + \sum_{i=1}^{t} u_{i+2} = u_{t+2} + u_{t+3} = u_{t+4} = u_k = n = 1(C^*),$$

the graph G^t can be formed.

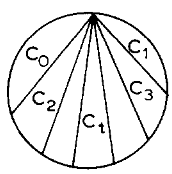

Fig. 1.

We proceed by induction on t. Clearly, G^0, G^1, G^2 and G^3 are 2-connected simple cycle-distributed graphs. Suppose that G^t $(t \geq 3)$ is a 2-connected simple cycle-distributed graph. Let C^t be the set of cycles in G^t. Let

$$M = \{C \mid C \in C^{t+1} \text{ and } \operatorname{int} C_{t+1} \nsubseteq \operatorname{int} C\},$$
$$N = \{C \mid C \in C^{t+1} \text{ and } \operatorname{int} C_{t+1} \subseteq \operatorname{int} C\}.$$

Then $C^{t+1} = M \cup N$. Clearly, there are no two cycles in M having the same length. With each cycle $C \in N - \{C_{t+1}\}$, we associate a unique cycle $C' \in C^t$ such that $1(C') = 1(C) - 1(C_{t+1}) + 2 = 1(C) - u_{t+3}$; different cycles of $N - \{C_{t+1}\}$ are associated different cycles of C^t. Therefore there are no two cycles in N having the same length. Using properties of Fibonacci numbers, for each $C \in M$, we obtain easily $1(C) < u_{t+3} + 2$. On the other hand, for each $C \in N$, we have $1(C) \geq u_{t+3} + 2$. Thus there are no two cycles in C^{t+1} having the same length. Consequently, G^{t+1} is a 2-connected simple cycle-distributed graph and $f_2(n) \geq n + t = n + k - 4$ follows.

Case 2: $u_k < n < u_{k+1}$ $(k \geq 4)$.

Let $t = k - 4$. Replacing the path $C_t \cap C^*$ of G drawn in Fig. 1 by a new path of length $u_{t+2} + (n - u_k)$ results in a new graph G^*. Clearly G^* is a 2-connected simple cycle-distributed graph on n vertices and $n + t$ edges. And hence $f_2(n) \geq n + t = n + k - 4$. □

Let $a_k = 3 \cdot 2^{k-1} + 2$, then Theorem 3.2 is equivalent to the following form: For any integer n, $a_k \leq n \leq a_{k+1}$ $(k \geq 1)$, $f_2(n) \geq n + k$.

It is easily seen that, for $5 \leq n < 8$, Theorem 3.3 is equivalent to Theorem 3.2. Also,

$$\lim_{k \to \infty} u_{k+1}/u_k = 1.618 \quad \text{and} \quad \lim_{k \to \infty} a_{k+1}/a_k = 2.$$

Therefore the lower bound of Theorem 3.3 is better than that of Theorem 3.2.

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