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Cycle systems in the complete bipartite graph minus a one-factor

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Dedicated to Curt Lindner on the occasion of his 65th birthday

Abstract

Let $K_{n,n} - I$ denote the complete bipartite graph with n vertices in each part from which a 1-factor I has been removed. An *m-cycle system* of $K_{n,n} - I$ is a collection of m-cycles whose edges partition $K_{n,n} - I$. Necessary conditions for the existence of such an m-cycle system are that $m \geqslant 4$ is even, $n \geqslant 3$ is odd, $m \leqslant 2n$, and $m \mid n(n-1)$. In this paper, we show these necessary conditions are sufficient except possibly in the case that $m \equiv 0 \pmod 4$ with $n < m < 2n$.

Keywords: Decomposition; Cycle; Complete bipartite graph

1. Introduction

Throughout this paper, $K_{n,n}$ will denote the complete bipartite graph with n vertices in each partite set; $K_{n,n} - I$ will denote the complete bipartite graph with a 1-factor I removed; and C_m will denote the m-cycle $(v_1, v_2, \ldots, v_m)$. An *m-cycle system* of a graph G is set T of m-cycles whose edges partition the edge set of G. Several obvious necessary conditions for an m-cycle system T of a graph G to exist are immediate: $m \leqslant |V(G)|$, the degrees of the vertices of G must be even, and m must divide the number of edges in G.

There have been many results regarding the existence of m-cycle systems of the complete graph K_v (see, for example, [8]). In this case, the necessary conditions imply that $m \leqslant v$, v is odd, and that m divides $v(v-1)/2$. In [1,9], it is shown that these necessary conditions are also sufficient. In the case that v is even, m-cycle systems of $K_v - I$, where I denotes a 1-factor, have been studied. Here, the necessary conditions are that $m \leqslant v$ and that m divides $v(v-2)/2$. These conditions are also known to be sufficient [1,9].

Cycle systems of complete bipartite graphs have also been studied. The necessary conditions for the existence of an m-cycle system of $K_{n,k}$ are that m, n, and k are even, $n, k \geqslant m/2$, and m must divide nk. In [10], these necessary conditions were shown to be sufficient. To study m-cycle systems of $K_{n,k}$ when n and k are odd, it is necessary to remove a 1-factor and hence $n = k$. Then, the necessary conditions are that m is even, $n \geqslant m/2$ with n odd, and m must divide $n(n-1)$. As a consequence of the main result of [6], it is known that $(2n)$-cycle systems of $K_{n,n} - I$ exist. Other results involving cycle systems of $K_{n,n} - I$ are given in [4], and other authors have considered cycle systems of complete multipartite graphs [2,3,5–7].

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The main result of this paper is the following.

Theorem 1. *Let m and n be positive integers with* $m \geqslant 4$ *even and* $n \geqslant 3$ *odd. If* $m \equiv 0 \pmod 4$ *and* $m \leqslant n$, *or if* $m \equiv 2 \pmod 4$ *and* $m \leqslant 2n$, *then the graph* $K_{n,n} - I$ *has an m-cycle system if and only if the number of edges in* $K_{n,n} - I$ *is a multiple of m.*

Our methods involve Cayley graphs and difference constructions. In Section 2, we give some basic definitions while the proof of Theorem 1 is given in Section 3. We shall see that the case $m \equiv 2 \pmod 4$ is fairly easy to handle using known results, but the case $m \equiv 0 \pmod 4$ is more involved.

2. Notation and preliminaries

Let us begin with a few basic definitions. We write $G = H_1 \oplus H_2$ if G is the edge-disjoint union of the subgraphs H_1 and H_2. If $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, where $H_1 \cong H_2 \cong \cdots \cong H_k \cong H$, then the graph G can be *decomposed* into subgraphs isomorphic to H and we say that G is H*-decomposable.* We also shall write $H \mid G$.

The proof of Theorem 1 uses Cayley graphs, which we now define. Let S be a subset of a finite group Γ satisfying

(1) $1 \notin S$, where 1 denotes the identity of Γ, and
(2) $S = S^{-1}$; that is, $s \in S$ implies that $s^{-1} \in S$.

A subset S satisfying the above conditions is called a *Cayley subset.* The *Cayley graph* $X(\Gamma; S)$ is defined to be that graph whose vertices are the elements of Γ, with an edge between vertices g and h if and only if $h = gs$ for some $s \in S$. We call S the *connection set* and say that $X(\Gamma; S)$ is a *Cayley graph on the group* Γ.

The graph $K_{n,n}$ is a Cayley graph by selecting the appropriate group; that is, $K_{n,n} = X(\mathbb{Z}_n \times \mathbb{Z}_2; \{(0,1),(1,1),(2,1),\ldots,(n-1,1)\})$. Equivalently, for a positive integer n, let $S \subseteq \{0,1,2,\ldots,n-1\}$ and let $X(n;S)$ denote the graph whose vertices are $u_0, u_1, \ldots, u_{n-1}$ and $v_0, v_1, \ldots, v_{n-1}$ with an edge between u_i and v_j if and only if $j - i \in S$. Clearly, $K_{n,n} = X(n; \{0,1,\ldots,n-1\})$, and we will often write $-s$ for $n - s$ when n is understood.

Many of our decompositions arise from the action of a permutation on a fixed subgraph. Let ρ be a permutation of the vertex set V of a graph G. For any subset U of V, ρ acts as a function from U to V by considering the restriction of ρ to U. If H is a subgraph of G with vertex set U, then $\rho(H)$ is a subgraph of G provided that for each edge $xy \in E(H)$, $\rho(x)\rho(y) \in E(G)$. In this case, $\rho(H)$ has vertex set $\rho(U)$ and edge set $\{\rho(x)\rho(y) : xy \in E(H)\}$. Note that $\rho(H)$ may not be defined for all subgraphs H of G since ρ is not necessarily an automorphism. In this paper, however, ρ will be an automorphism, so $\rho(H)$ will be defined for all subgraphs H.

For a set D of integers and an integer x, we define the sets $\pm D = \{\pm d \mid d \in D\}$, $D + x = \{d + x \mid d \in D\}$, and $x - D = \{x - d \mid d \in D\}$.

3. The proof of the main theorem

In this section, we shall prove Theorem 1. It turns out that when $m \equiv 2 \pmod 4$, an m-cycle system of $K_{n,n} - I$ can be found from an $(m/2)$-cycle system of K_n as we now show.

Lemma 2. *For positive integers m and n with* $m \equiv 2 \pmod 4$, *n odd, and* $6 \leqslant m \leqslant 2n$, *the graph* $K_{n,n}$ *has a decomposition into m-cycles and a* 1*-factor if and only if* $m \mid n(n-1)$.

Proof. Let m and n be integers with $m \equiv 2 \pmod 4$, n odd, and $6 \leqslant m \leqslant 2n$. Let the partite sets of $K_{n,n}$ be denoted by $\{u_0, u_1, \ldots, u_{n-1}\}$ and $\{v_0, v_1, \ldots, v_{n-1}\}$. Since $m \equiv 2 \pmod 4$, we have $m = 2k$ for some odd integer k. Then $k \leqslant n$ and $k | n(n-1)/2$. Hence, by [1,9], K_n has a decomposition into k-cycles. Let the vertices of K_n be labelled with $w_0, w_1, \ldots, w_{n-1}$ and let T be a decomposition of K_n into k-cycles. Suppose that $C = (w_{i_0}, w_{i_1}, w_{i_2}, w_{i_3}, \ldots, w_{i_{k-1}})$ is a k-cycle in T. Then the cycle

$$C' = (u_{i_0}, v_{i_1}, u_{i_2}, v_{i_3} \ldots, u_{i_{k-1}}, v_{i_0}, u_{i_1}, v_{i_2}, u_{i_3}, \ldots, v_{i_{k-1}})$$

is of length $2k$ in $K_{n,n}$. Furthermore, for each edge $w_i w_j$ of C, the edges $u_i v_j$ and $v_i u_j$ appear on C'. Thus, the collection

$$T' = \{(u_{i_0}, v_{i_1}, u_{i_2}, v_{i_3} \ldots, u_{i_{k-1}}, v_{i_0}, u_{i_1}, v_{i_2}, u_{i_3}, \ldots, v_{i_{k-1}}) | (w_{i_0}, w_{i_1}, w_{i_2}, w_{i_3}, \ldots, w_{i_{k-1}}) \in T\}$$

together with $\{u_i v_i | 0 \leqslant i \leqslant n-1\}$ is a decomposition of $K_{n,n}$ into m-cycles and a 1-factor. □

The case $m \equiv 0 \pmod 4$ cannot be obtained by using a similar argument as in Lemma 2. Suppose that $m \equiv 0 \pmod 4$, say $m = 2k$ with k even and let $n \geqslant 3$ be odd with $m \leqslant 2n$ and $m \mid n(n-1)$. As before, $k \mid n(n-1)/2$ and $k \leqslant n$ so that a k-cycle system T of K_n exists. However, for each cycle $C = (w_{i_0}, w_{i_1}, w_{i_2}, w_{i_3}, \ldots, w_{i_{k-1}})$ in T, we obtain the two k-cycles

$$C' = (u_{i_0}, v_{i_1}, u_{i_2}, v_{i_3} \ldots, v_{i_{k-1}})$$

and

$$C'' = (v_{i_0}, u_{i_1}, v_{i_2}, u_{i_3}, \ldots, u_{i_{k-1}})$$

in $K_{n,n}$ rather than one $2k$-cycle. Thus, we need more elaborate constructions for the case $m \equiv 0 \pmod 4$.

To help guide the reader, we will now give a rough outline of these constructions. Suppose that $m < n$ and $n(n-1)$ is a multiple of m. Let $n = qm + r$. The first construction, given in Lemma 3, generates n cycles, each of length m. Collectively, these cycles contain all edges $u_i v_j$ where $j - i \in \pm D$ for a given set D of $m/2$ nonzero differences. This construction will be applied q times, leaving r differences. If $r = 1$, then this will give the required 1-factor, while if $r > 2$, we proceed as follows. In Lemma 6, we show that $r - 1 = s(m/g)$, where $g = \gcd(m, n)$. Lemma 4 generates $2n/g$ cycles where these cycles contain all edges $u_i v_j$ where $j - i \in \pm(D \cup (D + n/g))$ for a given set D of $m/(2g)$ differences. This construction will be applied $\lfloor s/2 \rfloor$ times, leaving either 1 difference (the missing 1-factor) or $m/g + 1$ differences. In the latter case, we apply the construction of Lemma 5. The details of how the difference sets are chosen are given in Lemma 6.

Lemma 3. *Let m and n be positive integers with $m \equiv 0 \pmod 4$, n odd, and $4 \leqslant m < n$. If $D = \{d_1, d_2, \ldots, d_{m/2}\}$, where $d_1, d_2, \ldots, d_{m/2}$ are positive integers satisfying $d_1 < d_2 < \cdots < d_{m/2} \leqslant (n-1)/2$, then $C_m | X(n; \pm D)$.*

Proof. Label the vertices of $X(n; \pm D)$ with $u_0, u_1, \ldots, u_{n-1}$ and $v_0, v_1, \ldots, v_{n-1}$. We have $u_i v_j \in E(X(n; \pm D))$ if and only if $j - i \in \pm D$. Let ρ denote the permutation

$$(u_0 u_1 \cdots u_{n-1})(v_0 v_1 \cdots v_{n-1}).$$

Observe that $\rho \in \mathrm{Aut}(X(n; \pm D))$, so for any subgraph L of $X(n; \pm D)$, $\rho(L)$ is also a subgraph. Similarly, let τ denote the permutation $(u_0\, v_0)(u_1\, v_1) \cdots (u_{n-1} v_{n-1})$. Let $e_k = \sum_{i=1}^{k} (-1)^{i+1} d_i$, and let P be the trail of length $(m-2)/2$ given by

$$P : u_{e_1}, v_{e_2}, u_{e_3}, v_{e_4}, \ldots, u_{e_{(m-2)/2}}, v_{e_{m/2}}.$$

Now, the lengths of the edges of P, in the order that they are encountered, are $-d_2, -d_3, \ldots, -d_{m/2}$. Since $e_1, e_3, \ldots, e_{(m-2)/2}$ is a strictly increasing sequence while $n + e_2, n + e_4, \ldots, n + e_{m/2}$ is a strictly decreasing sequence, it follows that the vertices of P are distinct so that P is a path. Let $P' = \rho^{-d_1}(\tau(P))$ so that P' begins at v_0 and ends at $u_{e_{m/2} - d_1}$ and the edges of P' have lengths $d_2, d_3, \ldots, d_{m/2}$. Since $d_1, d_{m/2} \leqslant (n-1)/2$, we see that $u_{e_{(m-2)/2}} \neq u_{e_{m/2} - d_1}$ and $v_{e_{(m-2)/2}} \neq v_{e_{m/2} - d_1}$. Therefore, the vertices of P' are distinct from the vertices of P.

Next, we form a cycle C of length m by taking

$$C = \{u_{e_1} v_0, u_{e_{m/2} - d_1} v_{e_{m/2}}\} \cup P \cup P'.$$

Observe that these two additional edges have difference $\pm d_1$. From the above remarks, it follows that

$$C, \rho(C), \rho^2(C), \ldots, \rho^{n-1}(C)$$

is a partition of the edge set of $X(n; \pm D)$ into m-cycles. □

Suppose n is odd, $m \equiv 0 \pmod 4$ with $4 \leqslant m < n$ and $D = \{d_1, d_2, \ldots, d_{m/2}\}$ is a set of positive integers with $n - 1 \geqslant d_1 > d_2 > \cdots > d_{m/2} > (n-1)/2$. Then, applying Lemma 3 to $-D$, we find a decomposition of $X(n; \pm D)$ into m-cycles. Another consequence of Lemma 3 is the following. Suppose that A is a set of $mq/2$ distinct positive integers for some positive integer q, such that all elements of A are either at most $(n-1)/2$ or at least $(n+1)/2$. Then, applying Lemma 3 q times, we have that $X(n; \pm A)$ decomposes into m-cycles.

In Lemma 3, we found a cycle with m distinct differences, and used ρ to create n cycles that collectively covered all edges with those differences. We now consider cycles that have repeated differences.

Lemma 4. *Let m and n be positive integers with $m \equiv 0 \pmod 4$, n odd, $4 \leqslant m < n$, and let $g = gcd(m, n) > 1$. Let $D = \{d_1, d_2, \ldots, d_{m/(2g)}\}$ be a set of $m/(2g)$ positive integers, and let $\bar{d}_i \equiv d_i \pmod{n/g}$. Suppose either*

(1) $0 < d_1 < d_2 < \cdots < d_{m/(2g)} \leqslant (n-1)/2 - n/g$ and $0 < \bar{d}_1 < \bar{d}_2 < \cdots < \bar{d}_{m/(2g)} \leqslant (n-g)/(2g)$, or
(2) $(n-1)/2 - n/g \geqslant d_1 > d_2 > \cdots > d_{m/(2g)} > 0$ and $n/g - 1 \geqslant \bar{d}_1 > \bar{d}_2 > \cdots > \bar{d}_{m/(2g)} > (n-g)/(2g)$.

Then $C_m | X(n; \pm(D \cup (D + n/g)))$.

Proof. Label the vertices of $X(n;\pm(D\cup(D+n/g)))$ as in Lemma 3 and let ρ,τ be as defined in Lemma 3. Suppose first $0<d_1<d_2<\cdots<d_{m/(2g)}\leqslant(n-1)/2-n/g$ and $0<\bar{d}_1<\bar{d}_2<\cdots<\bar{d}_{m/(2g)}\leqslant(n-g)/(2g)$. Let $e_k=\sum_{i=1}^{k}(-1)^{i+1}d_i$. Let P_1 be the trail of length $m/(2g)-1$ given by

$$P_1: u_{e_1}, v_{e_2}, u_{e_3}, v_{e_4},\ldots, u_{e_{m/(2g)-1}}, v_{e_{m/(2g)}}.$$

Letting $\bar{e}_k=\sum_{i=1}^{k}(-1)^{i+1}\bar{d}_i$, we have that $\bar{e}_1,\bar{e}_3,\ldots,\bar{e}_{m/(2g)-1}$ is a strictly increasing sequence while $n/g+\bar{e}_2, n/g+\bar{e}_4,\ldots,n/g+\bar{e}_{m/(2g)}$ is a strictly decreasing sequence. Hence, the subscripts of vertices in P_1 lie in different nonzero congruence classes modulo n/g so that P_1 is a path. Let $P_1'=\rho^{-d_1}(\tau(P_1))$ and note that the vertices of P_1' are distinct from P_1 as in the proof of Lemma 3.

Form a path W_1 of length m/g by taking

$$W_1=\{u_{e_1}v_{-n/g}, u_{e_{m/(2g)}-d_1}v_{e_{m/(2g)}}\}\cup P_1 \cup P_1'.$$

Observe that these two additional edges have differences d_1 and $-(d_1+n/g)$, so W_1 is a path from v_0 to $v_{-n/g}$. Moreover, the first and last vertices are the only ones whose subscripts are congruent modulo n/g. It follows that

$$C_1=W_1\cup\rho^{n/g}(W_1)\cup\rho^{2n/g}(W_1)\cup\cdots\cup\rho^{(g-1)n/g}(W_1)$$

is a cycle of length m. Each difference occurs exactly g times, and the subscripts of the u_is incident with edges of difference k are all congruent modulo n/g. From the above remarks, it follows that

$$C_1,\rho(C_1),\rho^2(C_1),\ldots,\rho^{n/g-1}(C_1)$$

is a partition of the edge set of $X(n;\pm D\cup\{-(d_1+n/g)\}\setminus\{-d_1\})$ into m-cycles.

We form a second set of cycles in a similar manner. We define P_2 analogously to P_1, except that, d_i is replaced by d_i+n/g and $-d_i$ by $-(d_i+n/g)$ in e_k. Let $P_2'=\rho^{-(d_1+n/g)}(\tau(P_2))$. Form W_2 by adding the edges $u_{e_1+n/g}v_{n/g}$ and $u_{e_{m/(2g)}-(d_1+n/g)}v_{e_{m/(2g)}}$ with differences $-d_1$ and d_1+n/g.

The cycles

$$C_2,\rho(C_2),\rho^2(C_2),\ldots,\rho^{n/g-1}(C_2)$$

are a partition of the edge set of $X(n;\pm(D+n/g)\cup\{-d_1\}\setminus\{-(d_1+n/g)\})$ into m-cycles. Taken with the first set of cycles, we have our desired partition of $X(n;\pm(D\cup(D+n/g)))$ into m-cycles.

Now suppose $(n-1)/2-n/g\geqslant d_1>d_2>\cdots>d_{m/(2g)}>0$ and $n/g-1\geqslant\bar{d}_1>\bar{d}_2>\cdots>\bar{d}_{m/(2g)}>(n-g)/(2g)$. In this case, let $e_k=\sum_{i=1}^{k}(-1)^i d_i$. Let P_1 be as defined above and note that if $\bar{e}_k=\sum_{i=1}^{k}(-1)^i\bar{d}_i$, again $\bar{e}_1,\bar{e}_3,\ldots,\bar{e}_{m/(2g)-1}$ is a strictly increasing sequence while $n/g+\bar{e}_2,n/g+\bar{e}_4,\ldots,n/g+\bar{e}_{m/(2g)}$ is a strictly decreasing sequence. Hence, the subscripts of vertices in P_1 lie in different nonzero congruence classes modulo n/g so that P_1 is a path. Let $P_1'=\rho^{d_1}(\tau(P_1))$ and note that the vertices of P_1' are distinct from P_1 as in the proof of Lemma 3.

Form a path W_1 of length m/g by taking

$$W_1=\{u_{e_1}v_{n/g}, u_{e_{m/(2g)}+d_1}v_{e_{m/(2g)}}\}\cup P_1 \cup P_1',$$

where these two additional edges have differences $-d_1$ and d_1+n/g, so W_1 is a path from v_0 to $v_{n/g}$. Again, the first and last vertices are the only ones whose subscripts are congruent modulo n/g so that

$$C_1=W_1\cup\rho^{n/g}(W_1)\cup\rho^{2n/g}(W_1)\cup\cdots\cup\rho^{(g-1)n/g}(W_1)$$

is a cycle of length m and

$$C_1,\rho(C_1),\rho^2(C_1),\ldots,\rho^{n/g-1}(C_1)$$

is a partition of the edge set of $X(n;\pm D\cup\{d_1+n/g\}\setminus\{d_1\})$ into m-cycles.

Form a second set of cycles as before, defining P_2 analogously to P_1 by replacing d_i with d_i+n/g and $-d_i$ with $-(d_i+n/g)$ in e_k. Let $P_2'=\rho^{d_1+n/g}(\tau(P_2))$. Form W_2 by adding the edges $u_{e_1-n/g}v_{-n/g}$ and $u_{e_{m/(2g)}+d_1+n/g}v_{e_{m/(2g)}}$ with differences d_1 and $-(d_1+n/g)$.

The cycles

$$C_2,\rho(C_2),\rho^2(C_2),\ldots,\rho^{n/g-1}(C_2)$$

are a partition of the edge set of $X(n;\pm(D+n/g)\cup\{d_1\}\setminus\{d_1+n/g\})$ into m-cycles. As in the previous case, we have our desired partition of $X(n;\pm(D\cup(D+n/g)))$ into m-cycles. □

The previous lemma used $2m/g$ differences. The following lemma will use m/g differences and will give a 1-factor.

Lemma 5. *Let m and n be positive integers with $m \equiv 0 \pmod 4$, n odd, $4 \leqslant m < n$, and let $g = \gcd(m,n) > 1$. Let $D = \{d_1, d_2, \ldots, d_{m/(2g)-1}\}$ be a set of positive integers and let $\bar{d}_i \equiv d_i\, (\mathrm{mod}(n/g))$. Suppose either*

(1) $0 < d_1 < d_2 < \cdots < d_{m/(2g)-1} \leqslant (n-1)/2$ *and* $0 < \bar{d}_1 < \bar{d}_2 < \cdots < \bar{d}_{m/(2g)-1} \leqslant (n-g)/(2g)$; *or*
(2) $(n-1)/2 \geqslant d_1 > d_2 > \cdots > d_{m/(2g)-1} > 0$ *and* $n/g - 1 \geqslant \bar{d}_1 > \bar{d}_2 > \cdots > \bar{d}_{m/(2g)-1} > (n-g)/(2g)$.

Then $X(n; \pm D \cup \{0, \pm n/g\})$ decomposes into m-cycles and a 1-factor.

Proof. The proof is similar to that of Lemma 4 and uses the same notation. Suppose first that $0 < d_1 < d_2 < \cdots < d_{m/(2g)-1} \leqslant (n-1)/2$ and $0 < \bar{d}_1 < \bar{d}_2 < \cdots < \bar{d}_{m/(2g)-1} \leqslant (n-g)/(2g)$. Let $e_k = \sum_{i=1}^{k} (-1)^i d_i$. Let P be the trail of length $m/(2g) - 1$ given by

$$P : u_0, v_{e_1}, u_{e_2}, v_{e_3}, \ldots, u_{e_{m/(2g)-2}}, v_{e_{m/(2g)-1}}.$$

Clearly, P is a path and the lengths of the edges of P, in the order they are encountered and reduced modulo n/g, are $-\bar{d}_1, -\bar{d}_2, \ldots, -\bar{d}_{m/(2g)-1}$. Hence, as in Lemma 4, the subscripts of vertices in P lie in different nonzero congruence classes modulo n/g.

Form a path W of length m/g by taking

$$W = \{u_0 v_{n/g}, u_{e_{m/(2g)-1}} v_{e_{m/(2g)-1}}\} \cup P \cup \tau(P).$$

Observe that these two additional edges have differences n/g and 0, respectively, so W is a path from v_0 to $v_{n/g}$. Moreover, the first and last vertices are the only ones whose subscripts are congruent modulo n/g. As before,

$$C = W \cup \rho^{n/g}(W) \cup \rho^{2n/g}(W) \cup \cdots \cup \rho^{(g-1)n/g}(W)$$

is a cycle of length m, and

$$C, \rho(C), \rho^2(C), \ldots, \rho^{n/g-1}(C)$$

is a partition of the edge set of $X(n; \pm D \cup \{0, n/g\})$ into m-cycles. The edges with difference $-n/g$ form the 1-factor, completing the construction.

Now suppose $(n-1)/2 \geqslant d_1 > d_2 > \cdots > d_{m/(2g)-1} > 0$ and $n/g - 1 \geqslant \bar{d}_1 > \bar{d}_2 > \cdots > \bar{d}_{m/(2g)-1} > (n-g)/(2g)$. Let $e_k = \sum_{i=1}^{k} (-1)^{i+1} d_i$. Let P, W, and C be defined as above so that

$$C, \rho(C), \rho^2(C), \ldots, \rho^{n/g-1}(C)$$

is a partition of the edge set of $X(n; \pm D \cup \{0, n/g\})$ into m-cycles. As before, let the edges with difference $-n/g$ form the 1-factor. □

We now have all of the constructions needed for the proof of Theorem 1 in the case $m \equiv 0 \pmod 4$ and $m < n$.

Lemma 6. *For positive integers m and n with $m \equiv 0 \pmod 4$ and n odd with $4 \leqslant m < n$, the graph $K_{n,n}$ can be decomposed into m-cycles and a 1-factor if and only if $m \mid n(n-1)$.*

Proof. Let m and n be positive integers with $m \equiv 0 \pmod 4$, n odd, $4 \leqslant m < n$, and $m \mid n(n-1)$, say $n(n-1) = mt$. If t is even, then $m \mid n(n-1)/2$. Thus, since $m < n$, an m-cycle system T of K_n exists [9]. We have already noted that T will give rise to a collection T' of m-cycles in $K_{n,n}$ so that what remains when T' is removed from $K_{n,n}$ is a 1-factor. Therefore, it suffices to consider the case when t is odd.

Let $n = qm + r$, where $q \geqslant 1$ and $0 \leqslant r < m$ with r odd. Let $S = \{1, 2, \ldots, (n-1)/2\}$ so that $K_{n,n} = X(n; \pm S \cup \{0\})$, and let $g = \gcd(m, n)$. Suppose first that $g = 1$, and observe that this implies that $m \mid (n-1)$ so that $n - 1 = qm$. Thus $|S| = mq/2$, and by Lemma 3, the graph $X(n; \pm S)$ decomposes into m-cycles. Since the edges of difference 0 form a 1-factor, this completes the construction when $g = 1$.

We may now assume that $g > 1$ and let $r - 1 = s(m/g)$ for some positive integer s, say $s = 2k + \varepsilon$ for some nonnegative integer k and with $\varepsilon = 0$ or $\varepsilon = 1$. If $s = 1$, then let $D = \{1, 2, \ldots, m/(2g) - 1\}$. Now $X(n; \pm D \cup \{0, \pm n/g\})$ decomposes into m-cycles and 1-factor by Lemma 5. Next, the set $A = S \setminus (D \cup \{n/g\})$ consists of $mq/2$ positive integers and thus $X(n; \pm A)$ decomposes into m-cycles by Lemma 3. Therefore, we have found the required decomposition of $K_{n,n}$ in this case.

Now suppose that $s > 1$. Let

$$D_1 = \left\{1, 2, \ldots, \frac{m}{2g}\right\} \text{ and } D_2 = \frac{n}{g} - D_1.$$

For a positive integer i, let

$$D_{2i+1} = D_1 + 2i\left(\frac{n}{g}\right) \text{ and } D_{2i+2} = D_2 + 2i\left(\frac{n}{g}\right).$$

Suppose first that k is even. Consider the sets $D_1, D_2, \ldots, D_k$ (so $i = 1, \ldots, k/2 - 1$). Note that

- for each $j=1,2,\ldots,k$, the set $D_j=\{d_{j,1}, d_{j,2}, \ldots, d_{j,m/(2g)}\}$ consists of $m/(2g)$ positive integers, and if $\bar{d}_{j,i} \equiv d_{j,i} \pmod{n/g}$, then either

(1) $0 < d_{j,1} < d_{j,2} < \cdots < d_{j,m/(2g)}$ and $0 < \bar{d}_{j,1} < \bar{d}_{j,2} < \cdots < \bar{d}_{j,m/(2g)} \leqslant (n-g)/(2g)$, or
(2) $d_{j,1} > d_{j,2} > \cdots > d_{j,m/(2g)} > 0$ and $n/g - 1 \geqslant \bar{d}_{j,1} > \bar{d}_{j,2} > \cdots > \bar{d}_{j,m/(2g)} > (n-g)/(2g)$;

- the sets $D_1, D_2, \ldots, D_k$ are pairwise disjoint;
- if $d \in D_1 \cup D_2 \cup \cdots \cup D_k$, then $d + n/g \notin D_1 \cup D_2 \cup \cdots \cup D_k$;
- $(D_1 \cup (D_1 + n/g)) \cup (D_2 \cup (D_2 + n/g)) \cup \cdots \cup (D_k \cup (D_k + n/g)) \subset \{1, 2, \ldots, nk/g\}$.

Let

$$D = \left\{1 + \frac{nk}{g}, 2 + \frac{nk}{g}, \ldots, \frac{m}{2g} - 1 + \frac{nk}{g}\right\},$$

and let

$$S' = \left(D_1 \cup \left(D_1 + \frac{n}{g}\right)\right) \cup \left(D_2 \cup \left(D_2 + \frac{n}{g}\right)\right) \cup \cdots \cup \left(D_k \cup \left(D_k + \frac{n}{g}\right)\right).$$

Now $D \cap S' = \emptyset$ and the largest difference in $D \cup S'$ is $m/(2g) - 1 + nk/g$. We now show $m/(2g) - 1 + nk/g \leqslant (n-1)/2$ so that these difference sets satisfy the hypotheses of Lemmas 4 and 5. Since $r < m$, we have $r - 1 = s(m/g) < g(m/g) - 1$, so that $s < g - g/m$. Since s is an integer, it follows that $s \leqslant g - 1$. Hence

$$\begin{aligned} \frac{m}{2g} - 1 + \frac{nk}{g} &\leqslant \frac{m}{2g} - 1 + \frac{n}{g}\left(\frac{s}{2}\right) \\ &\leqslant \frac{m}{2g} - 1 + \frac{n}{g}\left(\frac{g-1}{2}\right) \\ &= \frac{n}{2} - \left(\frac{n}{2g} - \frac{m}{2g}\right) - 1 \\ &\leqslant \frac{n-1}{2}. \end{aligned}$$

For each j with $1 \leqslant j \leqslant k$, the graph $X(n; \pm(D_j \cup (D_j + n/g)))$ has a decomposition into m-cycles by Lemma 4. If $\varepsilon = 1$, then $X(n; \pm D \cup \{0, \pm n/g\})$ decomposes into m-cycles and a 1-factor by Lemma 5. Let $A = S \setminus S'$ if $\varepsilon = 0$ or let $A = S \setminus (D \cup S')$ if $\varepsilon = 1$. Then, A consists of $mq/2$ differences and Lemma 3 gives a decomposition of $X(n; \pm A)$ into m-cycles, completing the construction in the case that k is even.

Now suppose that k is odd. Consider the sets D_1, D_2, …, D_{k+1} (so $i=1, \ldots, (k-1)/2$). As before, the sets $D_1, D_2, \ldots, D_{k+1}$ satisfy the same properties as in the case when k is even except that

$$\left(D_1 \cup \left(D_1 + \frac{n}{g}\right)\right) \cup \cdots \cup \left(D_k \cup \left(D_k + \frac{n}{g}\right)\right) \cup D_{k+1} \subset \left\{1, 2, \ldots, \frac{m}{2g} + \frac{nk}{g}\right\}.$$

Let $D = D_{k+1} \setminus \{nk/g - m/(2g)\}$. Let S' be defined as above and note that the largest positive integer in $D \cup S'$is $m/(2g) + nk/g$, and we have seen that $m/2g + nk/g < n/2 - (n-m)/(2g)$. Since $m/(2g) + nk/g$ is an integer, it follows that $m/(2g) + nk/g \leqslant (n-1)/2$. Thus, as was done in the case when k is even, the graph $X(n; \pm(D_j \cup (D_j + n/g)))$ has a decomposition into m-cycles by Lemma 4 for each $j = 1, 2, \ldots, k$. If $\varepsilon = 1$, then $X(n; \pm D \cup \{0, \pm n/g\})$ decomposes into m-cycles and a 1-factor by Lemma 5. Thus, letting A be defined as in the case when k is even, we have that $X(n; \pm A)$ decomposes into m-cycles by Lemma 3, completing the construction in the case that k is odd. □

Theorem 1 now follows from Lemmas 2 and 6, and we have shown that the necessary conditions for an m-cycle system of $K_{n,n} - I$ are sufficient for many values of m and n. The remaining open case is to show that an m-cycle system exists when $m \equiv 0 \pmod 4$ and $n < m < 2n$.

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