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On Quotient Measures and Induced Representations

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We construct continuous positive ρ-functions on locally compact groups. As a corollary, we have "nice" measures on quotient spaces. We give a simple proof to Mackey's intertwining theorem on induced representations.

Our aim is to use continuity instead of measurability to make simpler some aspects of Mackey's theory on induced representations. For the relations between the present note and earlier works see our various remarks. For the convenience of the reader we shall cite several standard arguments instead of bare references.

Notations. Let G be a locally compact group and H be a closed subgroup of G. Let $X = \{Ha; a \in G\}$ be the set of right cosets equipped with the factor topology and $p: G \to X$ be the canonical mapping, i.e., $p(g) = Hg$. Fix a right invariant Haar measure on G and H, respectively, and denote it by $\int_G \cdots d$ and $\int_H \cdots d$. Let Δ and δ be the modular functions of G and H, respectively (i.e., $\int_G f(ba)\, \Delta(b)\, da = \int_G f(a)\, da$ and similarly for H). Then $\int_G f(g^{-1})\, \Delta(g^{-1})\, dg = \int_G f(g)\, dg$.

It T is a locally compact Hausdorff space then $C_c(T)$ will denote the space of complex valued continuous compactly supported functions on T and we shall identify the positive linear functionals on $C_c(T)$ with the Baire measures and the regular Borel measures as usual. We refer to this using the term "Radon measure."

If m is a Radon measure on X then let m_g be the translation of m by g, i.e., $\int_X f(xg)\, dm_g(x) = \int_X f(x)\, dm(x)$.

For $f \in C_c(G)$ let f' be the "conditional expectation" of f, i.e., $f'(p(g)) = \int_H f(hg)\, dh$. Then $f' \in C_c(X)$ and for any $u \in C_c(X)$ one can find an $f \in C_c(G)$ such that $f' = u$; moreover, if $u \geqslant 0$ then we can choose $f \geqslant 0$ (see, e.g., [3, Vol. I, p. 205]).

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DEFINITION. We call a function $\rho: G \to \mathbb{C}$ a ρ-function if it satisfies the condition

$$\rho(hg) = \delta(h)\,\Delta(h)^{-1}\rho(g) \qquad \text{for all} \quad h \in H,\ g \in G. \tag{1}$$

We deviate from Mackey's notation here because the above version will be more convenient for our purposes. Of course, a continuous positive ρ-function *is* a ρ-function in the sense of [6] as well.

PROPOSITION 1. *There exist continuous positive ρ-functions.*

Proof. First we construct a cone of continuous ρ-functions which are only non-negative. If $b \in C_c(G)$ then we set

$$f_b(g) = \int_H b(hg)\,\Delta(h)\,\delta(h)^{-1}\,dh.$$

These f_b's are clearly ρ-functions and if $b \geqslant 0$ then $f_b \geqslant 0$. Now if the variable g varies on a compact then the integration can be restricted to a compact subset of H, and b is uniformly continuous, hence we get f_b is continuous.

Observe that $(r \circ p) \cdot f$ is a ρ-function whenever f is a ρ-function and r is an arbitrary function on X.

Now assume that $\{u_\alpha\}_\alpha$ is a locally finite family of continuous ρ-functions (i.e., the supports of u_α form a locally finite family of subsets of G). Then clearly $\sum_\alpha u_\alpha$ is a continuous ρ-function.

Choose $b_\alpha \in C_c(G)$ such that $b_\alpha \geqslant 0$ for all α and $\bigcup_\alpha G_\alpha = G$, where $G_\alpha = \{g \in G; b_\alpha(g) > 0\}$. Then the sets $p(G_\alpha)$ form an open covering of X, and since X is paracompact (we sketch the proof of this known result below) we can find a locally finite family $r_\alpha \in C_c(X)$ such that $r_\alpha \geqslant 0$, $\sum_\alpha r_\alpha = 1$, and $\operatorname{supp} r_\alpha \subset p(G_\alpha)$ for all α. Then the family $u_\alpha = (r_\alpha \circ p) \cdot f_{b_\alpha}$ is locally finite because p is continuous. On the other hand, for any $g \in G$, $\exists \alpha: r_\alpha(p(g)) > 0$ (because $\sum_\alpha r_\alpha = 1$) and for such an α we have $p(g) \in p(G_\alpha)$ and therefore $f_{b_\alpha}(g) > 0$, since the integrand is non-negative, continuous, and not 0 in that case. Thus $\sum_\alpha u_\alpha$ is a continuous positive ρ-function.

It remains to show the paracompactness of X. Let G_0 be the identity component of G. It is well known that G_0 is a σ-compact open normal subgroup of G. Since G_0 is a normal subgroup, hence HG_0 is a subgroup of G, and therefore the translations of the σ-compact open set $M = p(G_0)$ in X form a partition of X. In other words, X is a "direct sum" of σ-compact locally compact Hausdorff spaces, hence paracompact.

Remark 1. If G is a Lie group then we can use $b_\alpha \in C_c^\infty(G)$ and $r_\alpha \in C_c^\infty(X)$ in the above proof to get the following result:

In case of a Lie group there exist smooth positive ρ-functions.

Question. Are there analytic positive ρ-functions?

Remark 2. The construction of f_b is closely related to the "$\varepsilon(f, v)$ construction" from [1] and even to that from [6]. Moreover, if we have a look at the construction in [1] then we realize that $\|\varepsilon(f, v)\|^2$ is a ρ-function.

PROPOSITION 2. *We have non-zero positive Radon measures m on X such that the Radon–Nikodym derivatives dm_g/dm exist for all g and can be chosen so that $(dm_g/dm)(x)$ be a continuous function on $G \times X$. Namely, if ρ is a continuous positive ρ-function then the formula*

$$\int_X f' \, dm = \int_G f \cdot \rho$$

defines a Radon measure m for which $(dm_g/dm)(x) = \rho(sg)/\rho(s)$, where $x = p(s)$.

Proof. If ρ is any locally integrable ρ-function and u, v are compactly supported bounded measurable functions on G then using the properties of the Haar measure and applying Fubini's theorem to the function $\rho(g)\, v(g)\, u(hg)$ on $H \times G$ we get

$$\int_G \rho \cdot (u' \circ p) \cdot v = \int_G \rho \cdot u \cdot (v' \circ p). \tag{2}$$

(Obtaining such a formula was the motivation of introducing ρ-functions.) Note that we need (2) only for continuous ρ, u, v in which case "a simpler Fubini's theorem" can be applied.

Now we see from (2) and the properties of the mapping $f \to f'$ that our construction $\int_X f' \, dm = \int_G f \cdot \rho$ is correct (use that for the compact set $A = p(\operatorname{supp} f)$ we can find $r \in C_c(X)$ such that $r|_A = 1$ and r can be written in the form $r = v'$ with some $v \in C_c(G)$).

If ρ is a continuous positive ρ-function then (2) applies and $\rho(sg)\, \rho(s)^{-1}$ has meaning and is continuous on $G \times G$; further, this function depends only on $p(s)$ and g. Therefore it can be considered as a functions on $G \times X$, and it remains continuous because X carries the factor topology. Now an easy calculation completes the proof of the proposition.

Remark 3. The existence of such an m for which $(dm_g/dm)(x)$ is continuous on $G \times X$ was discovered by L. H. Loomis in 1960. This result made it possible to simplify greatly the proof of Mackey's imprimitivity theorem (see [5]). About further developments see the remarks at the end of the present note.

Now we turn to the induced representations and their intertwining theorem. In essence, we adopt the version of R. J. Blattner for the notion of the induced representation. The details are the following. Consider a strongly continuous unitary representation L of H in the Hilbert space $\mathscr{K}$. Then let $\mathscr{A} = \{f\colon G \to \mathscr{K}\,; f$ is strongly measurable, $\|f(\cdot)\|$ is in L^2loc, and $f(hg) = \delta(h)^{1/2}\,\Delta(h)^{-1/2} L_h f(g)\ \forall g \in G,\ h \in H\}$. Then (using the polarization identity) we get for any $f_1, f_2 \in \mathscr{A}$ that $\langle f_1(\cdot), f_2(\cdot)\rangle$ is a locally integrable ρ-function and therefore gives rise to a complex Radon measure m_{f_1, f_2} as in Proposition 2. Now let $\mathscr{A}_1 = \{f \in \mathscr{A}\,; m_{f,f}(X) < +\infty\}$, $\mathscr{A}_0 = \{f \in \mathscr{A}_1\,; f$ is continuous$\}$ and Let $\mathscr{H}^L$ be the closure of $\mathscr{A}_0$ in $\mathscr{A}_1$ with respect to the seminorm $|f| = m_{f,f}(X)^{1/2}$ (or more precisely the closure of $\mathscr{A}_0$ in the space $\mathscr{A}_1/\{f \in \mathscr{A}_1\,; |f| = 0\}$). Then $\mathscr{H}^L$ is a Hilbert space with the scalar product $(f_1, f_2) := m_{f_1, f_2}(X)$. For $f \in \mathscr{H}^L$, $g, g' \in G$, and $r \in C_c(X)$ we define $U^L_g f(g') := f(g'g)$ and $P^L(r)\, f(g) := r(p(g)) \cdot f(g)$. Then (U^L, P^L) is called the induced system of L (U^L is called the representation induced by L).

It can be shown that $\mathscr{H}^L$ consists of all functions from $\mathscr{A}_1$, and also that the subspace $\mathscr{A}_0$ is large enough. See [1] for the details but we mention that the heart of the construction is the integral transform

$$\varepsilon(b, v)(g) = \int_H \delta(h)^{-1/2}\,\Delta(h)^{1/2} b(hg)\, L_h v\, dh \tag{3}$$

with $g \in G$, $b \in C_c(G)$, and $v \in \mathscr{K}$ which yields $\varepsilon(b, v) \in \mathscr{A}_0$.

Observe that the whole construction of the induced system can be done inside $\mathscr{A}_0$ (not concerning measurability), and then simply take the completion of $\mathscr{A}_0$ in the end.

Now using our Proposition 1 we see that the original definition of an induced system (due to G. W. Mackey) goes through in the non-separable case as well and gives us an equivalent notion. Namely, we can pick out a continuous positive ρ-function ρ, write $f(hg) = L_h f(g)$ to define $\mathscr{A}'$ instead of $\mathscr{A}$, and write $(f, f) = \int_X \|f\|^2\, dm$, where m is associated with ρ as in Proposition 1 and, eventually, we see that f in the old system is the same as $\rho^{1/2} \cdot f$ in the new system. Observe that the factor $\rho^{1/2}$ does not affect the continuity of f, nor the local integrability of $\|f\|^2$. One must use, of course, the fact that if r is a locally integrable function (with respect to the measure m) then the measure $r \cdot dm$ is associated to the ρ-function $(r \circ p) \cdot \rho$. This fact can be proved easily by applying (2).

Again one might restrict one's attention to continuous f.

THE INTERTWINING THEOREM. *Let L and M be strongly continuous unitary representations of H in the Hilbert spaces $\mathscr{K}$ and $\mathscr{K}_1$, respectively, and $\mathscr{R}$ be the set of operators which intertwine L and M, i.e., $\mathscr{R} = \{T \in B(\mathscr{K}, \mathscr{K}_1);\ TL_h = M_h T$ for all $h \in H\}$. Then any intertwining* *operator of* (U^L, P^L) *and* (U^M, P^M) *is of the form* $f \to T \circ f$ *with some fixed* $T \in \mathscr{R}$ *(these ones trivially intertwine the two systems).*

Now we present a proof to this theorem which seems to be even simpler than the proof given by R. J. Blattner (see [2]). Of course, our proof is closely related to Blattner's proof.

LEMMA. *Let* $b \in C_c(G)$ *and* $f \in \mathscr{H}^L$. *Set* $U^L(b) f := \int_G b(g)\, U^L_{g^{-1}} f\, dg$ *(this is an ordinary Riemann integral in the Hilbert space* $\mathscr{H}^L$*). Then* $U^L(b) f \in \mathscr{A}_0$, *i.e., continuous as a function* $G \to \mathscr{K}$.

Proof of the Lemma. We assert that $U^L(b) f(s) = \int_G b(g)\, f(sg^{-1})\, dg = \int_G b(gs)\, f(g^{-1})\, dg$. The second equality is clear because dg is right invariant. But $\|f(g^{-1})\|$ is locally integrable (since $\|f\|$ is even in L^2loc and local integrability with respect to dg or $d(g^{-1})$ is the same) so we get the continuity of the function $q(s) := \int_G b(g)\, f(sg^{-1})\, dg$. Hence for any $r \in C_c(X)$ we have $r \circ p \cdot q \in \mathscr{A}_0$. Apply Fubini's theorem to $\int_{G \times X} b(g) r(x)\, u(x)\, dg\, dm(x)$, where $u(p(s)) = \langle f(sg^{-1}), v(s) \rangle / \rho(s)$ with some $v \in \mathscr{H}^L$. Here m and ρ are taken from Proposition 2. Then we get $(r \circ p \cdot q, v) = (U^L(b) f, \bar{r} \circ p \cdot v) = (r \circ p \cdot U^L(b) f, v)$, and hence $r \circ p \cdot q = r \circ p \cdot U^L(b) f$ for all $r \in C_c(X)$, and the lemma is proved.

We note that it will be enough to know this lemma for continuous f, and in the above proof it is also enough to take all $v \in \mathscr{A}_0$, thus we may argue with "continuous Fubini's theorem."

Now let S be an intertwining operator of (U^L, P^L) and (U^M, P^M). We say that $k \sim k_1$ if there is an $f \in \mathscr{A}_0(L)$, i.e., a continuous element of the space $\mathscr{H}^L$ such that $Sf \in \mathscr{A}_0(M)$, $f(e) = k$, and $Sf(e) = k_1$. Since S is intertwining thus we have

if f and Sf are continuous then

$$f(g) = U^L_g f(e) \sim U^M_g Sf(e) = Sf(g) \qquad \forall g \in G, \tag{4}$$

$$SU^L(b) f = U^M(b)\, Sf \qquad \text{for all} \quad f \in \mathscr{H}^L,\ b \in C_c(G), \tag{5}$$

and

$$P^M(r)\, Sf = SP^L(r) f \qquad \text{for all} \quad f \in \mathscr{H}^L,\ r \in C_c(X). \tag{6}$$

Choose a net r_n from $C_c(X)$ such that the support of r_n tends to $p(e)$ and $\int_X r_n\, dm = \rho(e)$ for all n. Then (6) yields $\|k_1\| \leqslant \|S\| \cdot \|k\|$ whenever $k \sim k_1$. Since the relation "$\sim$" is clearly linear, we see it is a bounded linear operator T_0 on its domain D. We deduce from (4) and our lemma that $k \in D$ whenever $k = U^L(b) f(e)$, and hence $\bar{D} \supset \{f(e); f \in \mathscr{A}_0\}$ (this is seen simply by tending with b to the Dirac δ). But the latter set is dense in $\mathscr{K}$ because of the construction (3).

Thus the closure T of T_0 is an ordinary bounded operator. By (4), (5) and the lemma we have $Sf = T \circ f$ whenever f is of the form $U^L(b)v$, i.e., on a dense set, and hence for all f. In particular, for $h \in H$ and $f \in \mathscr{A}_0$ we have $TL_hf(e) = \Delta(h)^{1/2}\,\delta(h)^{-1/2}Tf(h) = \Delta(h)^{1/2}\,\delta(h)^{-1/2}Sf(h) = M_hSf(e) = M_hTf(e)$, i.e., T interwines L and M.

Remarks. The author has to apologize because the results of this paper can be considered as corollaries of the ideas due to R. J. Blattner (see [1, 2]); moreover, these ideas are closely related to the classical paper of G. W. Mackey (see [6]). But we must notice that Mackey in [6] and even Loomis in [5] did not observe the existence of a continuous positive ρ-function though it would have simplified their proofs. Later contributors seem to have avoided the problem by modifying the definition of induced representation (cf. [1, 2, 7]). They did not point out that this is equivalent to the older one in the non-separable case, too. The cause of this might be the fact these authors were able to give much simpler proofs to the classical theorems of Mackey. Moreover, the proof given by B. Ørsted to the Imprimitivity Theorem goes through even if we do not assume P to be a projection valued measure, instead, we just assume P to be a positive linear mapping from $C_c(X)$ to $B(\mathscr{H})$ satisfying $\sup_{r \leqslant 1} P(r) = I$. In that case the system of imprimitivity may not be induced but anyway a subsystem of an induced one in the sense that U_g is simply the restriction of U_g^L to a U^L-invariant subspace $\mathscr{H}$ of $\mathscr{H}^L$ but $P(r) = QP^L(r)|_{\mathscr{H}}$, where Q is the projection onto $\mathscr{H}$. The latter facts are pointed out in [4].

The present note is intended to show these notions are useful for better understanding of the theory of induced representations.

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