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On Groupoid Graded Rings

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1. INTRODUCTION

Our investigation is motivated by analogy with classical $\mathbb{Z}$-graded rings. There are many theorems connecting the structure of a $\mathbb{Z}$-graded ring $R = \bigoplus_{n \in Z} R_n$ and the initial component R_0. In particular, Camillo and Fuller [3] proved that a finitely $\mathbb{Z}$-graded ring is semiprimary or right or left perfect if and only if its initial subring is.

Let S be a groupoid, and let R be an associative ring not necessarily with identity. Then R is said to be *graded by S*, or *S-graded*, if R is a direct sum of additive subgroups R_s indexed by the elements of S and such that $R_s R_t \subseteq R_{st}$ for any $s, t \in S$. This concept was first mentioned in [26]. Its two special cases, group and semigroup graded rings, have been actively investigated recently (see [13, 15, 20]).

Several interesting results establish connections between the properties of a graded ring R and homogeneous components R_e, where e is an idempotent of S. Denote by $E(S)$ the set of idempotents of S. Let $\mathscr{K}$ be a class of rings. Some authors have considered the implication

$$R_e \in \mathscr{K} \qquad \text{for all } e \in E(S) \Rightarrow R \in \mathscr{K} \tag{1}$$

while other authors have examined the equivalence

$$R_e \in \mathscr{K} \qquad \text{for all } e \in E(S) \Leftrightarrow R \in \mathscr{K}. \tag{2}$$

For rings graded by finite groups results of this sort are deducible with the use of duality theory of Cohen and Montgomery [8]. Let G be a finite group with identity e, and let R be a G-graded ring. Beattie and Jespers [1] and Jensen and Jöndrup [12] proved that a ring R graded by a finite group is right or left perfect or semiprimary if and only if the component R_e

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satisfies the same property, where e is the identity of the group (see [20] for other relevant references). The results of Cohen and Rowen [7] and Cohen and Montgomery [8] show that the Jacobson radical of R is Baer radical (nilpotent, locally nilpotent) if and only if the radical of R_e is Baer radical (nilpotent, locally nilpotent), as was noted by Okniński [21, Lemma 1.1]. Also, Theorem 3.5 of [8] implies that R is a PI-ring if and only if R_e is a PI-ring, see [17, Lemma 4].

Similar problems have been considered for semigroup graded rings. Wauters [27] proved that a ring graded by a finite semilattice is semilocal, or semiprimary, or left or right perfect if and only if all the homogeneous components satisfy the same property. Okniński and Wauters [23, Lemma 4.1] showed that a ring R graded by a finite semigroup S is quasiregular if and only if R_e is quasiregular for every e in $E(S)$. Clase and Jespers proved that the same can be said of semilocal, right and left perfect, semiprimary rings [4], Jacobson rings, and rings with nilpotent or locally nilpotent Jacobson radicals [5].

In a number of papers it was established that various facts concerning rings graded by finite groups or semigroups can be generalized for group or semigroup graded rings with finite supports. A graded ring R is said to have a *finite support* if only a finite number of the homogeneous components of R_s are nonzero. In particular, Beattie and Jespers [1] proved that a group graded ring R with finite support is semilocal or right or left perfect or semiprimary if and only if R_e satisfies the same property, where e is the identity of the group. Clase, Jespers, and del Rio [6] proved that a semigroup graded ring R with finite support is semilocal or right or left or semiprimary or PI if and only if R_e satisfies the same property, for every e in $E(S)$.

Groupoid graded rings include all the constructions mentioned above as special cases. Our main theorem shows that, for ring classes $\mathscr{X}$ with certain natural closure properties, as soon as relation (1) or relation (2) has been verified for rings graded by finite groups, it immediately holds for rings graded by finite groupoids (Theorem 1). This generalizes several known facts on group or semigroup graded rings (see Corollaries 1, 3). Thereby our main theorem provides new unified proofs for several different previous results.

Our theorem cannot be generalized further by introducing larger classes of sets with binary operations as indexing sets of graded rings. Indeed, the definition of a groupoid graded ring naturally appears when one considers a ring R which is just a direct sum of its additive subgroup R_s, indexed by the elements s of a set S (this situation is connected to the results of several authors; see [18] for references). In such generality there is very little relation between the properties of R and those of R_s, and so some extra restrictions are needed to obtain positive results. Following [26], we shall consider the restriction that the product of every two homogeneous elements of R be homogenous again. In this case we say that R is *graded by* the set S.

Then it easily follows that, for any $s, t \in S$, there exists $u \in S$ such that $R_s R_t \subseteq R_u$. Therefore we can introduce an operation on S and make R a groupoid graded ring. For every groupoid S it is easy to construct an S-graded ring R such that $R_s R_t \neq 0$ for each pair $s, t \in S$, and so the multiplication of S is uniquely determined by R. Therefore, it is impossible to "regrade" such R by a semigroup, if S is not associative. Thus, groupoid graded rings are *a priori* interesting; they are equivalent to set graded rings and cannot be immediately reduced to semigroup graded ones.

As we see, for groupoid graded rings the concepts of a "ring graded by a finite groupoid" and a "graded ring with finite support" coincide. Since our main theorem transfers results from groups to groupoids, it shows that these results are immediately true for graded rings with finite supports (see Remark 1). Thus the concept of a groupoid graded ring explains why so many facts valid for rings graded by finite groups or semigroups remain true also for group or semigroup graded rings with finite supports.

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2. MAIN THEOREM

A class $\mathscr{X}$ of rings is said to be *closed under finite sums of one-sided ideals* if and only if, for every ring with right (or left) ideals, $A, B \in \mathscr{X}$, it follows that $A + B \in \mathscr{X}$. Our proof applies to both (1) and (2), and so we combine "if and only if" and "provided that" parts in one theorem.

THEOREM 1. *Let $\mathscr{X}$ be a class of rings which contains all rings with zero multiplication and is closed for homomorphic images, right and left ideals, ring extensions. Then the following assertions are equivalent*:

(i) *for each finite groupoid S, an S-graded ring $R = \oplus_{s \in S} R_s$ belongs to $\mathscr{X}$ provided that (if and only if) R_e belongs to $\mathscr{X}$ for every idempotent e of S*;

(ii) *for each finite semigroup S, an S-graded ring $R = \oplus_{s \in S} R_s$ belongs to $\mathscr{X}$ provided that (if and only if) R_e belongs to $\mathscr{X}$ for every idempotent e of S.*

If, moreover, $\mathscr{X}$ is also closed for finite sums of one-sided ideals, then the following is equivalent to the above assertions:

(iii) *for every finite group G with identity e, a G-graded ring $R = \oplus_{g \in G} R_g$ is in $\mathscr{X}$ provided that (if and only if) $R_e \in \mathscr{X}$.*

3. PROOF OF THE MAIN THEOREM

The implication (i) $\Rightarrow$ (ii) is trivial.

Assume that (ii) holds. Then Lemma 3 of [15] tells us that $\mathscr{X}$ is closed for finite sums of one-sided ideals. For convenience we include a proof of this fact. Clearly, it suffices to consider a ring A which is the sum of its two right ideals M and N from $\mathscr{X}$. Let $L = \{a, b\}$ be the two-element semigroup such that $ab = a = a^2$, $ba = b = b^2$. In the semigroup ring AL consider subrings $R_a = Ma$ and $R_b = Nb$. Then $R = R_a + R_b$ is L-graded. Since $R_a \cong M$ and $R_b \cong N$ belong to $\mathscr{X}$, it follows from (ii) that $R \in \mathscr{X}$. It is easily seen that $I = \{k(a-b) | k \in M \cap N\}$ is an ideal of R and $R/I \cong A$. Given that $\mathscr{X}$ is closed for homomorphic images, we get $A \in \mathscr{X}$, as required.

Since (ii) $\Rightarrow$ (iii) is also trivial, it remains to prove (iii) $\Rightarrow$ (i) with the extra hypothesis that $\mathscr{X}$ is closed for finite sums of one-sided ideals.

Recall that a subset I of $R = \oplus_{s \in S} R_s$ is said to be *homogeneous* if $I = \oplus_{s \in S} I_s$, where $I_s = I \cap R_s$. The *support* of I is the set $\mathrm{Supp}(I) = \{s \in S | I_s \neq 0\}$. If I is a homogeneous ideal of R, then $R/I = \oplus_{s \in S} R_s/I_s$ is S-graded, as well.

Let S be any finite groupoid, $R = \oplus_{s \in S} R_s$ an S-graded ring, and I a homogeneous two-sided ideal of R. Given that $\mathscr{X}$ is closed for ideals and homomorphic images, it is very easy to prove that R is a counterexample for (i) if, and only if, either I or R/I is a counterexample for (i).

Suppose now that $R = \oplus_{s \in S} R_s$ is a counterexample for (i) with $|S|$ minimal. We claim that, given any $s \in S$ and any additive subgroup A or R_s such that $\mathrm{Supp}(AR) \neq S$, there is a two-sided homogeneous ideal I of R such that $A \subseteq I$, $I \in \mathscr{X}$, and $I_e \in \mathscr{X}$ for every idempotent $e \in E(S)$. Obviously, I is not a counterexample for (i). As a consequence, we shall be able to factor I out, and a new counterexample for (i) will appear in which $A = 0$.

If $AR = 0$, we take $I = R^1 A$. Then $I^2 = 0$, and so $I \in \mathscr{X}$ and $I_t \in \mathscr{X}$, for every $t \in S$.

If $AR = P \neq 0$ then, by the minimality of $|S|$, P cannot be a counterexample to (i). In the "provided that" part of our theorem, that R is a counterexample implies that R_e is in $\mathscr{X}$, for every $e \in E(S)$. Then $P_e \in \mathscr{X}$ for every $e \in E(S)$, because P_e is a right ideal in R_e. Since P satisfies (i), we get $P \in \mathscr{X}$. (In the "if and only if" version of the theorem, that R is a counterexample implies that either $R \in \mathscr{X}$ or R_e is in $\mathscr{X}$, for every

$e \in E(S)$. If $R \in \mathscr{X}$, then $P \in \mathscr{X}$ because P is a right ideal of R. If R_e is in $\mathscr{X}$, for every $e \in E(S)$, then P_e is also in $\mathscr{X}$, and so $P \in \mathscr{X}$, again.)

But $\mathrm{Supp}(R_x P) \neq S$, for every $x \in S$, and the same argument applied to the additive subgroup $R_x A$ contained in R_{xs} tells us that $R_x A$ and $(R_x A)_e$ are in $\mathscr{X}$, for every idempotent $e \in S$. Given that $\mathscr{X}$ is closed for finite sums of two-sided ideals, it follows that $I = R^1 P = P + \Sigma_{x \in S} R_x P$ is a homogeneous two-sided ideal with the desired properties.

If now $s \in S$ and $sS \neq S$, then $\mathrm{Supp}(R_s R) \subseteq sS \neq S$ and, putting $A = R_s$ above, we obtain a counterexample for (i) whose sth homogeneous component is zero and hence can be graded by the set $T = S \setminus \{s\}$. As in Section 1, we can introduce a multiplication on this set, make it a groupoid, and obtain a contradiction to the minimality of $|S|$. Therefore $sS = S$ and, by changing sides in this argument, we get $Ss = S$. Thus S is a left and right simple groupoid.

We claim that it is also associative and it is thereby a semigroup. In deed, if $(st)x \neq s(tx)$ then $R_s R_t R x \subseteq R_{(st)x} \cap R_{s(tx)}$ implies that $R_s R_t R_x = 0$ and so $\mathrm{Supp}(R_s R_t R) \neq S$, because S is finite. By applying the above paragraph with $A = R_s R_t$, we may assume $R_s R_t = 0$. Again using the same reduction with $A = R_s$, we can also assume $R_s = 0$, because $\mathrm{Supp}(R_s R) \neq S$. This yields a contradiction with the minimality of $|S|$. So our claim has been established. It is well known and easily seen that every finite left and right simple semigroup is a group. Thus S is a group, and we obtain a contradiction with (iii), which completes the proof.

Note also that the implication (iii) $\Rightarrow$ (ii) easily follows from [23, proof of Lemma 4.1]. This was pointed out in [14, Lemma 1; 17, Lemma 5]. We do not use the proofs of these lemmas; they all follow from our main theorem.

Next we give an example which shows that the closedness restrictions on $\mathscr{J}$ are essential in the main theorem.

EXAMPLE 1. The class $\mathscr{M}$ of Brown–McCoy radical rings satisfies implication (1) in (iii), but not in (ii).

It is well known that $\mathscr{M}$ contains all rings with zero multiplication and is closed for ring extensions, homomorphic images, and ideals [19, Sect. 37]. Theorem 5 of [9] says that, for every finite group G with identity e, each G-graded ring R belongs to $\mathscr{M}$ provided that $R_e \in \mathscr{M}$.

Take a simple non-Artinian domain R with unity (for example, the Weyl algebra A_1). Pick two different maximal right ideals M and N in R. It is proved in [2, Lemma 2], that M and N are simple. Since R has no nonzero idempotents, M and N are rings without identities, and so they are Brown–McCoy radical rings. However, $R = M + N$ is Brown–McCoy semisimple. Therefore $\mathscr{M}$ is not closed for sums of two right ideals. It follows from the second paragraph of the proof of Theorem 1 that $\mathscr{M}$ does not satisfy (ii).

4. MODIFICATIONS OF THE MAIN THEOREM

Remark 1. For groupoid gradings the terms "graded ring with finite support" and "ring graded by a finite groupoid" are the same. As a consequence, conditions (i), (ii), and (iii) of Theorem 1 are equivalent, under the given hypotheses, to the corresponding statements obtained by replacing "ring graded by a finite groupoid (semigroup, group)" by "groupoid (semigroup, group)-graded ring with finite support," and so the theorem can be rewritten for graded rings with finite supports.

Remark 2. In the "if and only if" version we can slightly weaken the closedness restrictions imposed on the class of the hypothesis of Theorem 1.

Indeed, the example in the last paragraph of the proof of necessity of [16, Theorem 1] shows that every class $\mathscr{X}$ which contains all rings with zero multiplication, is closed for ring extensions and homomorphic images, and satisfies the "if and only if" version of (ii) is also closed for one-sided ideals. For completeness we include a proof of this fact. Consider a ring $A \in \mathscr{X}$ with a right ideal I. Let $L = \{c, d\}$ be a semigroup such that $cd = c = c^2$, $dc = d = d^2$. Then $R = Ac + Id$ is L-graded. It is readily verified that $N = \{i(c - d) | i \in I\}$ is an ideal of R and $N^2 = 0$. Since $N \in \mathscr{X}$ and $R/N \cong A \in \mathscr{X}$, we get $R \in \mathscr{X}$. The "if and only if" version of (ii) yields $I \cong R_d \in \mathscr{X}$, as required. Thus, in the "if and only if" case of the main theorem the closedness of $\mathscr{X}$ for right and left ideals can be moved from the hypothesis of the theorem to the extra restrictions before (iii).

Remark 3. Suppose that R is an S-graded ring and R_s is a subring of R. If $s' \notin E(S)$, then $R_s^2 \subseteq R_s \cap R_{s^2} = 0$. Since every class of Theorem 1 contains all rings with zero multiplication, our main theorem and several corollaries can be rewritten by replacing "every e in $E(S)$" by "every R_s which is a ring."

5. COROLLARIES

For the class of all quasiregular (locally nilpotent; Baer radical) rings the properties in the hypothesis of Theorem 1 are well known. In particular, it is known that the Jacobson (Levitzki; Baer) radical of a ring contains all quasiregular (locally nilpotent, Baer radical) left and right ideals of the ring, and therefore these classes are closed for sums of one-sided ideals (see [10, Sects. 1.6, 8.1, 8.3]). For PI-rings all the necessary closedness properties are obvious except for the very difficult fact that every sum of two right (or left) ideals satisfying polynomial identities is a PI-ring. This was proved by Rowen [25]. Jaegermann and Sands established the fact that the class of Jacobson rings is an N-radical class, and that all such classes are closed for left and right ideals (see [11, p. 348, Theorem 11]). For semilocal, right or left perfect, semiprimary rings all properties which are not straightforward were proved by Clase and Jespers [4].

For the classes of semilocal, right or left perfect, semiprimary, nilpotent, locally nilpotent, T-nilpotent, Baer radical, quasiregular, and PI-rings the "if and only if" versions of (iii) are also known: see [12] for right or left perfect and semiprimary rings; [1] for right or left perfect, semilocal, and semiprimary rings; [7] for nilpotent rings; [8] for quasiregular, Baer radical, and locally nilpotent rings; [17] for PI-rings. Note that for nilpotent and T-nilpotent rings assertion (iii) (and even Corollary 1 below) follows from the Ramsey theorem (see [18]). Combinatorial proofs are also possible for the classes of locally nilpotent and Baer radical rings [18]. A "provided that" version of (iii) for Jacobson rings is contained in [24]; see also [5].

COROLLARY 1. *Let $\mathscr{X}$ be the class of all semilocal (right perfect; left perfect; semiprimary; nilpotent; locally nilpotent; T-nilpotent; Baer radical; quasiregular; PI) rings, S a finite groupoid, $R = \oplus_{s \in S} R_s$ an S-graded ring. Then $R \in \mathscr{X}$ if and only if all R_e belong to $\mathscr{X}$ for all $e \in E(S)$. In addition, R is a Jacobson ring provided that R_e is a Jacobson ring for every $e \in E(S)$.*

COROLLARY 2. *Let $\mathscr{X}$ be a class of rings which contains all rings with zero multiplication and is closed under subrings, homomorphic images, ring extensions, and finite sums of one-sided ideals. Then, for any finite groupoid S, each S-graded ring $R = \oplus_{s \in S} R_s$ belongs to $\mathscr{X}$ if and only if R_e is in $\mathscr{X}$ for every e in $E(S)$.*

Proof. The result is true for S a group by [8, Theorem 3.5], as was noted in [17, Lemma 4]. Hence the corollary follows from Theorem 1.

COROLLARY 3. *Let $\mathscr{X}$ be the class of all semilocal (right perfect; left perfect; semiprimary; nilpotent; locally nilpotent; T-nilpotent; Baer radical; quasiregular; PI) rings, S a semigroup, $R = \oplus_{s \in S} R_s$ an S-graded ring with finite support. Then $R \in \mathscr{X}$ if and only if $R_e \in \mathscr{X}$ for every e in $E(S)$.*

Corollary 3 was known earlier in the following special cases:

(a) for semiprimary or left perfect or right perfect $\mathbb{Z}$-graded rings with finite supports [3, Proposition 10];

(b) for a finite semigroup S and quasiregular rings [23, Lemma 4.1] (the same proof holds for nilpotent, locally nilpotent, Baer radical rings, as noted in [14], and for PI-rings, see [17]);

(c) for a semilattice (i.e., a commutative semigroup entirely consisting of idempotents) S and semilocal, right perfect, left perfect, semiprimary rings [27];

(d) for a group S and semilocal, right perfect, left perfect, semiprimary rings [1];

(e) for a finite semigroup S and semilocal, right perfect, left perfect, semiprimary rings [4];

(f) for PI-algebras and certain periodic semigroups [17];

(g) for an arbitrary semigroup S and semilocal, right perfect, left perfect, semiprimary rings [6].

Our proof of Theorem 1 is independent of the proofs in the papers indicated in (a) to (g). Note that [1] uses reflected radicals, and [4, 17, 23, 27] use the structure theory of semigroups. The proof in [6] uses ideas similar to ours, but was obtained independently.

Denote the Jacobson radical of a ring R by $\mathcal{J}(R)$. The following analog of [3, Theorem 1] immediately follows from Corollary 1.

COROLLARY 4. *Let S be a semigroup, $R = \oplus_{s \in S} R_s$ an S-graded ring. If R_e is semilocal for every e in $E(S)$ and if $R_s \subseteq \mathcal{J}(R)$ for all finitely many s in S, then R is semilocal, too.*

Our main theorem can also be applied to some graded rings with not necessarily finite supports. We record only one corollary of this sort.

COROLLARY 5. *Let S be a periodic semigroup with a finite number of idempotents and only finite subgroups, and such that all nil factors of S are nilpotent. Let $R = \oplus_{s \in S} R_s$ be an S-graded ring. Then R is semiprimary if and only if R_e is semiprimary for every e in $E(S)$.*

Proof. It follows from [22, Theorem 3.3] or from [17, Lemma 11] that S has a finite ideal chin with finite or nilpotent factors. If I is an ideal of S, then R is an extension of R_I by R/R_I, and R/R_I is graded by S/I. As in [17, proof of Theorem 1], it easily follows by induction on the length of the ideal chain of S that it suffices to prove the corollary for all factors of S. For finite factors Theorem 1 in the present paper gives the result. For nilpotent factors the claim is obvious.

Analogous corollaries can be written for other properties from Theorem 1. Known examples of semigroup rings show that all the restrictions on semigroup S in Corollary 5 are essential (cf. [17, 22]).

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