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On cellular algebras with Jucys Murphy elements ☆

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ABSTRACT

We study analogues of Jucys–Murphy elements in cellular algebras arising from repeated Jones basic constructions. Examples include Brauer and BMW algebras and their cyclotomic analogues.

Contents

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1. Introduction

We recently developed a framework for proving cellularity of a tower of algebras $(A_n)_{n\geqslant 0}$ that is obtained from another tower of cellular algebras $(Q_n)_{n\geqslant 0}$ by repeated Jones basic constructions [17]. A key idea in this work is that of a tower of algebras with coherent cellular structures; coherence means that the cellular structures are well–behaved with respect to induction and restriction. This paper continues our work on the themes of [17]; here we refine our framework by taking into account the role played by Jucys–Murphy elements.

Before restricting to the setting of [17], we first obtain some simple general results regarding coherent towers. We show the existence of special cellular bases, called *path bases* which are distinguished by a restriction rule for the action of subalgebras on the basis elements. We then give an axiomatization of Jucys–Murphy elements in coherent towers; our assumptions imply that the Jucys–Murphy elements act via triangular matrices on a path basis, as in Andrew Mathas's axiomatization [29] of cellular algebras with Jucys–Murphy elements.

Passing to the setting of [17], we use the general results mentioned above to give conditions which allow lifting Jucys–Murphy elements from Q_n to A_n. Examples of algebras covered by this theory are Jones–Temperley–Lieb algebras, Brauer algebras, BMW algebras, and their cyclotomic analogues. Our method yields an easy and uniform proof of the triangularity property of the action of the Jucys–Murphy elements in these examples, recovering theorems of Enyang [11] and of Rui and Si [40] and [39].

1.1. Antecedents and motivations

Aside from our own paper [17], the most immediate antecedent and inspiration for this work was the paper of Andrew Mathas [29] on Jucys–Murphy elements in cellular algebras. As [17] is about lifting cellular structures from Hecke–like algebras to BMW–like algebras, our intention was to find a way to lift Jucys–Murphy elements as well. In order to do this, we needed a new axiomatization of Jucys–Murphy elements well adapted to the context of coherent towers. The axiomatization that we propose does not replace that of Mathas, but compliments it; a set of Jucys–Murphy elements in our sense is also a set of Jucys–Murphy elements in the sense of Mathas.

The Jucys–Murphy elements in $\mathbb{C}S_n$ were introduced by Murphy [34]in order to give a new construction of Young's seminormal representations. The Jucys–Murphy elements of $\mathbb{C}S_n$ generate the "Gelfand–Zeitlin algebra" for the sequence $(\mathbb{C}\mathfrak{S}_k)_{k\leqslant n}$, see Section 3.1; this is a maximal abelian subalgebra of $\mathbb{C}\mathfrak{S}_n$ containing a canonical family of mutually orthogonal minimal idempotents $F_\mathfrak{t}$ indexed by Young tableaux of size n. The seminormal basis of a simple module Δ^λ is obtained by a particular choice of one non–zero vector in the range of each $F_\mathfrak{t}$ for $\mathfrak{t}$ of shape λ. This interpretation of the seminormal representations has been stressed by Ram [38] and by Okounkov and Vershik [37,42]. Likewise, Nazarov emphasized this point of view in his treatment of Jucys–Murphy elements and seminormal representations of the Brauer algebras [36].

The Jucys–Murphy elements in our theory duplicate this behavior of the classical Jucys–Murphy elements; in a "generic" setting, when the Jucys–Murphy elements satisfy the separating condition of Mathas (and the algebras are in particular semisimple), our Jucys–Murphy elements generate the Gelfand–Zeitlin subalgebra for $(A_k)_{k\leqslant n}$ for each n, see Proposition 3.11.

2. Preliminaries

2.1. Algebras with involution, and their bimodules

Let R be a commutative ring with identity. Recall that an involution i on an R-algebra A is an R-linear algebra anti–automorphism of A with $i^2 = \mathrm{id}_A$. If A and B are R-algebras and Δ is an A–B bimodule, then we define a B–A bimodule $i(\Delta)$ as follows. As an R-module, $i(\Delta)$ is a copy of Δ with elements marked with the symbol i. The B–A bimodule structure is defined by $bi(x)a = i(i(a)xi(b))$. Then i is a functor from the category of A–B bimodules to the category of B–A bimodules. By the same token, we have a functor i from the category of B–A bimodules to the category of A–B bimodules, and for an A–B bimodule Δ, we can identify $i \circ i(\Delta)$ with Δ.

Suppose that A, B, and C are R-algebras with involutions i_A, i_B, and i_C. Let ${}_BP_A$ and ${}_AQ_C$ be bimodules. Then

$$i(P \otimes_A Q) \cong i(Q) \otimes_A i(P),$$

as C–B-bimodules. Note that if we identify $i(P \otimes_A Q)$ with $i(Q) \otimes_A i(P)$, then we have the formula $i(p \otimes q) = i(q) \otimes i(p)$. In particular, let M be a B–A-bimodule, and identify $i \circ i(M)$ with M, and $i(M \otimes_A i(M))$ with $i \circ i(M) \otimes_A i(M) = M \otimes_A i(M)$. Then we have the formula $i(x \otimes i(y)) = y \otimes i(x)$.

2.2. Cellularity

The definition of cellularity that we use is slightly weaker than the original definition of Graham and Lehrer in [21], see Remark 2.2.

Definition 2.1. Let R be an integral domain and A a unital R-algebra. A *cell datum* for A consists of an algebra involution i of A; a finite partially ordered set $(\Lambda, \geqslant)$ and for each $\lambda \in \Lambda$ a finite set $\mathscr{T}(\lambda)$; and a subset $\mathscr{C} = \{c^\lambda_{\mathfrak{s},\mathfrak{t}}\colon\ \lambda \in \Lambda 0 \text{ and } \mathfrak{s},\mathfrak{t} \in \mathscr{T}(\lambda)\} \subseteq A$; with the following properties:

(1) $\mathscr{C}$ is an R-basis of A.
(2) For each $\lambda \in \Lambda$, let $\breve{A}^\lambda$ be the span of the $c^\mu_{\mathfrak{s},\mathfrak{t}}$ with $\mu > \lambda$. Given $\lambda \in \Lambda$, $\mathfrak{s} \in \mathscr{T}(\lambda)$, and $a \in A$, there exist coefficients $r^{\mathfrak{s}}_{\mathfrak{v}}(a) \in R$ such that for all $\mathfrak{t} \in \mathscr{T}(\lambda)$:

$$a c^\lambda_{\mathfrak{s},\mathfrak{t}} \equiv \sum_{\mathfrak{v}} r^{\mathfrak{s}}_{\mathfrak{v}}(a) c^\lambda_{\mathfrak{v},\mathfrak{t}} \mod \breve{A}^\lambda.$$

(3) $i(c^\lambda_{\mathfrak{s},\mathfrak{t}}) \equiv c^\lambda_{\mathfrak{t},\mathfrak{s}} \mod \breve{A}^\lambda$ for all $\lambda \in \Lambda$ and $\mathfrak{s},\mathfrak{t} \in \mathscr{T}(\lambda)$.

A is said to be a *cellular algebra* if it has a cell datum.

For brevity, we will write that $(\mathscr{C}, \Lambda)$ is a cellular basis of A.

Remark 2.2.

(1) The original definition in [21] requires that $i(c^\lambda_{\mathfrak{s},\mathfrak{t}}) = c^\lambda_{\mathfrak{t},\mathfrak{s}}$ for all $\lambda, \mathfrak{s}, \mathfrak{t}$. However, one can check that all of [21] remains valid with our weaker axiom.
(2) In case $2 \in R$ is invertible, one can check that our definition is equivalent to the original; see [17, Remark 2.4].
(3) One reason for using the weaker definition is that it allows a more graceful treatment of extensions of cellular algebras; see [17, Remark 2.6]. Another reason is that it becomes trivial to lift bases of cell modules to cellular bases of the algebra; see Lemma 2.3 and Remark 2.4 below.

We recall some basic structures related to cellularity, see [21]. Given $\lambda \in \Lambda$, let A^λ denote the span of the $c^\mu_{\mathfrak{s},\mathfrak{t}}$ with $\mu \geqslant \lambda$. It follows that both A^λ and $\breve{A}^\lambda$ (defined above) are i-invariant two sided ideals of A. The *left cell module* Δ^λ is defined as follows: as an R-module, Δ^λ is free with basis indexed by $\mathscr{T}(\lambda)$, say $\{c^\lambda_{\mathfrak{s}}\colon\ \mathfrak{s} \in \mathscr{T}(\lambda)\}$; for each $a \in A$, the action of a on Δ^λ is defined by $a c^\lambda_{\mathfrak{s}} = \sum_{\mathfrak{v}} r^{\mathfrak{s}}_{\mathfrak{v}}(a) c^\lambda_{\mathfrak{v}}$ where $r^{\mathfrak{s}}_{\mathfrak{v}}(a)$ is as in Definition 2.1(2).

For each $\lambda \in \Lambda$, we have an A–A-bimodule isomorphism $\alpha^\lambda : A^\lambda/\breve{A}^\lambda \to \Delta^\lambda \otimes_R i(\Delta^\lambda)$ determined by $\alpha^\lambda(c^\lambda_{\mathfrak{s},\mathfrak{t}} + \breve{A}^\lambda) = c^\lambda_{\mathfrak{s}} \otimes i(c^\lambda_{\mathfrak{t}})$ satisfying $i \circ \alpha^\lambda = \alpha^\lambda \circ i$, using the remarks at the end of Section 2.1 and point (2) of Definition 2.1.

2.3. Globalizing bases of cell modules

A given cellular algebra can have many cellular bases yielding the same cell modules and ideals A^λ. The following lemma shows that an arbitrary collection of bases of the cell modules can be globalized to a cellular basis of the algebra.

Lemma 2.3. *Let A be a cellular algebra, with cell datum denoted as above. For each $\lambda \in \Lambda$, fix an A–A-bimodule isomorphism $\alpha^\lambda : A^\lambda/\breve{A}^\lambda \to \Delta^\lambda \otimes_R i(\Delta^\lambda)$ satisfying $i \circ \alpha^\lambda = \alpha^\lambda \circ i$. For each $\lambda \in \Lambda$, let $\mathscr{B}^\lambda = \{b^\lambda_{\mathfrak{s}}\colon \mathfrak{s} \in \mathscr{T}(\lambda)\}$ be an arbitrary R-basis of Δ^λ. For $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}(\lambda)$, let $b^\lambda_{\mathfrak{s},\mathfrak{t}}$ be an arbitrary lifting of $(\alpha^\lambda)^{-1}(b^\lambda_{\mathfrak{s}} \otimes b^\lambda_{\mathfrak{t}})$ to A^λ. Then*

$$\mathscr{B} = \left\{b^\lambda_{\mathfrak{s},\mathfrak{t}}\colon \lambda \in \Lambda;\ \mathfrak{s}, \mathfrak{t} \in \mathscr{T}(\lambda)\right\}$$

is a cellular basis of A.

Proof. It is easy to check that for each $\lambda \in \Lambda$, $\{b^\mu_{\mathfrak{s},\mathfrak{t}}\colon\ \mu \geqslant \lambda;\ \ \mathfrak{s}, \mathfrak{t} \in \mathscr{T}(\mu)\}$ spans A^λ. In fact, if λ is maximal in Λ, then $A^\lambda \cong A^\lambda/\breve{A}^\lambda$, and $\{b^\lambda_{\mathfrak{s},\mathfrak{t}}\colon\ \mathfrak{s}, \mathfrak{t} \in \mathscr{T}(\lambda)\}$ is a basis of A^λ. Now fix $\lambda \in \Lambda$ and assume inductively that for each $\lambda' > \lambda$, $\{b^\mu_{\mathfrak{s},\mathfrak{t}}\colon\ \mu \geqslant \lambda';\ \ \mathfrak{s}, \mathfrak{t} \in \mathscr{T}(\mu)\}$ spans $A^{\lambda'}$. This means that $\{b^\mu_{\mathfrak{s},\mathfrak{t}}\colon\ \mu > \lambda;\ \ \mathfrak{s}, \mathfrak{t} \in \mathscr{T}(\mu)\}$ spans $\breve{A}^\lambda$. Now if $x \in A^\lambda$, then $x \in \operatorname{span}\{b^\lambda_{\mathfrak{s},\mathfrak{t}}\colon\ \mathfrak{s}, \mathfrak{t} \in \mathscr{T}(\lambda)\} + \breve{A}^\lambda$ and hence $x \in \operatorname{span}\{b^\mu_{\mathfrak{s},\mathfrak{t}}\colon\ \mu \geqslant \lambda;\ \ \mathfrak{s}, \mathfrak{t} \in \mathscr{T}(\mu)\}$.

Now it follows that $\{b^\lambda_{\mathfrak{s},\mathfrak{t}}\colon\ \lambda \in \Lambda;\ \ \mathfrak{s}, \mathfrak{t} \in \mathscr{T}(\lambda)\}$ spans A. Since R is an integral domain, and this set has the same cardinality as the basis $\mathscr{C}$ of A, it follows that the set is an R-basis of A. Moreover, we have checked that $\breve{A}^\lambda$ (defined in terms of the original basis $\mathscr{C}$) is the span of the $b^\mu_{\mathfrak{s}\mathfrak{t}}$ with $\mu > \lambda$, and A^λ is the span of the $b^\mu_{\mathfrak{s}\mathfrak{t}}$ with $\mu \geqslant \lambda$.

Properties (2) and (3) of Definition 2.1 (with $\mathscr{C}$ replaced by $\mathscr{B}$) follow from the properties of the maps α^λ. $\square$

Remark 2.4. Note that the proof only yields the weaker property (3) of Definition 2.1 rather than the stronger requirement $i(b^\lambda_{\mathfrak{s},\mathfrak{t}}) = b^\lambda_{\mathfrak{t},\mathfrak{s}}$ of [21], so this lemma would not be valid with the original definition of [21].

Definition 2.5. If $\mathscr{B}^\lambda = \{b^\lambda_{\mathfrak{s}}\colon\ \mathfrak{s} \in \mathscr{T}(\lambda)\}$, $\lambda \in \Lambda$ is a family of bases of the cell modules Δ^λ, and $\mathscr{B} = \{b^\lambda_{\mathfrak{s},\mathfrak{t}}\colon\ \lambda \in \Lambda;\ \ \mathfrak{s}, \mathfrak{t} \in \mathscr{T}(\lambda)\}$, is a cellular basis of A such that $\alpha^\lambda(b^\lambda_{\mathfrak{s},\mathfrak{t}} + \breve{A}^\lambda) = b^\lambda_{\mathfrak{s}} \otimes b^\lambda_{\mathfrak{t}}$ for each $\lambda, \mathfrak{s}, \mathfrak{t}$, then we call $\mathscr{B}$ a *globalization* of the family of bases $\mathscr{B}^\lambda$, $\lambda \in \Lambda$.

2.4. Coherent towers of cellular algebras

In [17], we defined a coherent tower of cellular algebras as follows:

Definition 2.6. Let $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ be an increasing sequence of cellular algebras over an integral domain R. Let Λ_n denote the partially ordered set in the cell datum for A_n. We say that $(A_n)_{n \geqslant 0}$ is a *coherent tower of cellular algebras* if the following conditions are satisfied:

(1) The involutions are consistent; that is, the involution on A_{n+1}, restricted to A_n, agrees with the involution on A_n.
(2) For each $n \geqslant 0$ and for each $\lambda \in \Lambda_n$, the induced module $\operatorname{Ind}_{A_n}^{A_{n+1}}(\Delta^\lambda)$ has a filtration by cell modules of A_{n+1}. That is, there is a filtration

$$\operatorname{Ind}_{A_n}^{A_{n+1}}\left(\Delta^\lambda\right) = M_t \supseteq M_{t-1} \supseteq \cdots \supseteq M_0 = (0)$$

such that for each $j \geqslant 1$, there is a $\mu_j \in \Lambda_{n+1}$ with $M_j/M_{j-1} \cong \Delta^{\mu_j}$.

(3) For each $n \geqslant 0$ and for each $\mu \in \Lambda_{n+1}$, the restriction $\mathrm{Res}^{A_{n+1}}_{A_n}(\Delta^{\mu})$ has a filtration by cell modules of A_n. That is, there is a filtration

$$\mathrm{Res}^{A_{n+1}}_{A_n}\left(\Delta^{\mu}\right) = N_s \supseteq N_{s-1} \supseteq \cdots \supseteq N_0 = (0)$$

such that for each $i \geqslant 1$, there is a $\lambda_i \in \Lambda_n$ with $N_j/N_{j-1} \cong \Delta^{\lambda_i}$.

The modification of the definition for a *finite* tower of cellular algebras is obvious. We call a filtration as in (2) and (3) a *cell filtration*. In the examples of interest to us, we will also have *uniqueness of the multiplicities* of the cell modules appearing as subquotients of the cell filtrations, and *Frobenius reciprocity* connecting the multiplicities in the two types of filtrations. We did not include uniqueness of multiplicities and Frobenius reciprocity as requirements in the definition, as they will follow from additional assumptions that we will impose later.

We introduce a stronger notion of coherence:

Definition 2.7. Say that a coherent tower of cellular algebras $(A_n)_{n\geqslant 0}$ is *strongly coherent* if $A_0 \cong R$ and in the cell filtrations (2) and (3) in Definition 2.6, we have

$$\mu_t < \mu_{t-1} < \cdots < \mu_1$$

in the partially ordered set Λ_{n+1}, and

$$\lambda_s < \lambda_{s-1} < \cdots < \lambda_1$$

in the partially ordered set Λ_{n-1}.

2.5. Inclusions of split semisimple algebras and branching diagrams

A finite dimensional split semisimple algebra over a field F is one which is isomorphic to a finite direct sum of full matrix algebras over F.

Suppose $A \subseteq B$ are finite dimensional split semisimple algebras over F (with the same identity element). Let $A(i)$, $i \in I$, be the minimal ideals of A and $B(j)$, $j \in J$, the minimal ideals of B. We associate a $J \times I$ *inclusion matrix* Ω to the inclusion $A \subseteq B$, as follows. Let W_j be a simple $B(j)$-module. Then W_j becomes an A-module via the inclusion, and $\Omega(j,i)$ is defined to be the multiplicity of a simple $A(i)$-module in the decomposition of W_j as an A-module.

It is convenient to encode an inclusion matrix by a bipartite graph, called the *branching diagram*; the branching diagram has vertices labeled by I arranged on one horizontal line, vertices labeled by J arranged along a second (higher) horizontal line, and $\Omega(j,i)$ edges connecting $j \in J$ to $i \in I$.

If $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ is a (finite or infinite) sequence of inclusions of finite dimensional split semisimple algebras over F, then the branching diagram for the sequence is obtained by stacking the branching diagrams for each inclusion, with the upper vertices of the diagram for $A_i \subseteq A_{i+1}$ being identified with the lower vertices of the diagram for $A_{i+1} \subseteq A_{i+2}$. For two vertices λ on level ℓ of a branching diagram and μ on level $\ell + 1$, write $\lambda \nearrow \mu$ if λ and μ are connected by an edge.

Notation 2.8. Let R be an integral domain with field of fractions F. Let A be a cellular algebra over R and Δ an A-module. Write A^F for $A \otimes_R F$ and Δ^F for $\Delta \otimes_R F$.

Lemma 2.9. *(See [17, Lemma 2.20].) Let R be an integral domain with field of fractions F. Suppose that $(A_n)_{n\geqslant 0}$ is a coherent tower of cellular algebras over R and that A_n^F is split semisimple for all n. Let Λ_n denote the partially ordered set in the cell datum for A_n. Then*

(1) *$\{(\Delta^{\lambda})^F\colon \lambda \in \Lambda_n\}$ is a complete family of simple A_n^F-modules.*

(2) *Let* $[\omega(\mu,\lambda)]_{\mu\in\Lambda_{n+1},\,\lambda\in\Lambda_n}$ *denote the inclusion matrix for* $A_n^F \subseteq A_{n+1}^F$. *Then for any* $\lambda \in \Lambda_n$ *and* $\mu \in \Lambda_{n+1}$, *and any cell filtration of* $\mathrm{Res}_{A_n}^{A_{n+1}}(\Delta^\mu)$, *the number of subquotients of the filtration isomorphic to* Δ^λ *is* $\omega(\mu,\lambda)$.

(3) *Likewise, for any* $\lambda \in \Lambda_n$ *and* $\mu \in \Lambda_{n+1}$, *and any cell filtration of* $\mathrm{Ind}_{A_n}^{A_{n+1}}(\Delta^\lambda)$, *the number of subquotients of the filtration isomorphic to* Δ^μ *is* $\omega(\mu,\lambda)$.

Corollary 2.10. *Under the hypotheses of Lemma* 2.9, *the multiplicity of a cell module as a subquotient of a cell filtration of* $\mathrm{Res}_{A_n}^{A_{n+1}}(\Delta^\mu)$ *or of* $\mathrm{Ind}_{A_n}^{A_{n+1}}(\Delta^\lambda)$ *is independent of the choice of the cell filtration. Moreover, The multiplicity of* Δ^λ *in* $\mathrm{Res}_{A_n}^{A_{n+1}}(\Delta^\mu)$ *equals the multiplicity of* Δ^μ *in* $\mathrm{Ind}_{A_n}^{A_{n+1}}(\Delta^\lambda)$.

Definition 2.11. A tower of split semisimple algebras $(A_n)_{n\geqslant 0}$ over a field F is *multiplicity free* if all entries in the inclusion matrices are 0 or 1 and $A_0 \cong F$. Equivalently, there are no multiple edges in the branching diagram of the tower, and there is a unique vertex (denoted $\emptyset$) at level 0. We will also say that the branching diagram is multiplicity free.

Corollary 2.12. *Under the hypotheses of Lemma* 2.9, *if* $(A_n)_{n\geqslant 0}$ *is strongly coherent, then* $(A_n^F)_{n\geqslant 0}$ *is a multiplicity free tower of split semisimple algebras.*

Example 2.13. Fix an integral domain S and an invertible $q \in S$. The Hecke algebra $H_n(q) = H_{n,S}(q)$ is the associative, unital S-algebra with generators T_j for $1 \leqslant j \leqslant n-1$, satisfying the braid relations and the quadratic relation $(T_j - q)(T_j + 1) = 0$ for all j. $H_n(q)$ has an algebra involution $x \mapsto x^*$ uniquely determined by $(T_j)^* = T_j$. $H_n(q)$ has a cellular basis due to Murphy [35]

$$\{m^\lambda_{\mathfrak{s},\mathfrak{t}}\colon \lambda \in Y_n;\ \mathfrak{s},\mathfrak{t} \in \mathscr{T}(\lambda)\},$$

where Y_n is the partially ordered set of all Young diagrams of size n, with dominance order $\trianglerighteq$, and $\mathscr{T}(\lambda)$ is the set of all standard Young tableaux of shape λ. By results of Murphy [35], Dipper and James [7,8], and Jost [25], the sequence of Hecke algebras $(H_{n,S}(q))_{n\geqslant 0}$ is strongly coherent.

The generic ground ring for the Hecke algebras is $R = \mathbb{Z}[\boldsymbol{q}, \boldsymbol{q}^{-1}]$, where $\boldsymbol{q}$ is an indeterminant over $\mathbb{Z}$; the Hecke algebra $H_{n,S}(q)$ over any S is a specialization of $H_{n,R}(\boldsymbol{q})$. If $F = \mathbb{Q}(\boldsymbol{q})$ denotes the field of fractions of R, then $H_{n,F}(\boldsymbol{q})$ is split semisimple for all n and the branching diagram for the tower of Hecke algebras $(H_{n,F}(\boldsymbol{q}))_{n\geqslant 0}$ is Young's lattice $\mathscr{Y}$, which is multiplicity free.

2.6. Remark on the role of generic ground rings

In the examples of interest to us (Hecke algebras, BMW algebras, etc.) there is a generic ground ring R with the properties that:

(1) R is an integral domain and the algebras A_n^F over the field of fractions of R are split semisimple, and
(2) the algebras over any ground ring S are specializations of those over R, $A_n^S = A_n^R \otimes_R S$.

Certain properties of the algebras over the generic ground ring R carry over to any specialization. For example, if the algebras over R are cellular, so are all of the specializations. For another example, in the next section, we show the existence of certain bases, called path bases, in strongly coherent towers of cellular algebras over an integral domain R, assuming the algebras over the field of fractions of R are semisimple. This hypothesis would apply to the generic ground ring in our examples. But then the path bases in the cell modules over R can be specialized to cell modules over any ground ring S.

2.7. Path bases in strongly coherent towers

In this section, we discuss *path bases* in strongly coherent towers of cellular algebras.

Assumption 2.14. In Section 2.7, let R be an integral domain with field of fractions F, $(A_n)_{n\geqslant 0}$ a strongly coherent tower of cellular algebras over R, such that A_n^F is semisimple for all n. Let $\mathfrak{B}$ denote the branching diagram of $(A_n^F)_{n\geqslant 0}$ and Λ_n the partially ordered set in the cell datum for A_n.

Definition 2.15. A *path* on $\mathfrak{B}$ from $\lambda \in \Lambda_\ell$ to $\mu \in \Lambda_m$ ($\ell < m$) is a sequence $(\lambda = \lambda^{(\ell)}, \lambda^{(\ell+1)}, \dots, \lambda^{(m)} = \mu)$ with $\lambda^{(i)} \nearrow \lambda^{(i+1)}$ for all i. A path $\mathfrak{s}$ from λ to μ and a path $\mathfrak{t}$ from μ to ν can be concatenated in the obvious way; denote the concatenation $\mathfrak{s} \circ \mathfrak{t}$. If $\mathfrak{t} = (\emptyset = \lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(n)} = \lambda)$ is a path from $\emptyset$ to $\lambda \in \Lambda_n$, and $0 \leqslant k < \ell \leqslant n$, write $\mathfrak{t}(k) = \lambda^{(k)}$, $\mathfrak{t}_{[k,\ell]}$ for the path $(\lambda^{(k)}, \dots, \lambda^{(\ell)})$, and write $\mathfrak{t}'$ for $\mathfrak{t}_{[0,n-1]}$.

For $\lambda \in \Lambda_n$, the rank of the cell module Δ^λ of A_n is the same as the dimension of the simple A_n^F module $(\Delta^\lambda)^F$, namely the number of paths on $\mathfrak{B}$ from $\emptyset$ to λ. It follows that we can assume without loss of generality that the index set $\mathscr{T}(\lambda)$ in the cell datum for A_n is equal to the set of paths on $\mathfrak{B}$ from $\emptyset$ to λ. We set $\mathscr{T}(n) = \bigcup_{\lambda\in\Lambda_n} \mathscr{T}(\lambda)$, the set of paths on $\mathfrak{B}$ from $\emptyset$ to some $\lambda \in \Lambda_n$.

Definition 2.16. (Partial orders on the set of paths.) We introduce two natural partial orders on $\mathscr{T}(n)$. Let $\mathfrak{s} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(n)})$ and $\mathfrak{t} = (\mu^{(0)}, \mu^{(1)}, \dots, \mu^{(n)})$ be two paths with $\lambda^{(i)}, \mu^{(i)} \in \Lambda_i$. Say that $\mathfrak{s}$ precedes $\mathfrak{t}$ in *dominance order* (denoted $\mathfrak{s} \trianglelefteq \mathfrak{t}$) if $\lambda^{(i)} \leqslant \mu^{(i)}$ for all i ($0 \leqslant i \leqslant n$). Say that $\mathfrak{s}$ precedes $\mathfrak{t}$ in *reverse lexicographic order* (denoted $\mathfrak{s} \preceq \mathfrak{t}$) if $\mathfrak{s} = \mathfrak{t}$, or if for the last index j such that $\lambda^{(j)} \neq \mu^{(j)}$, we have $\lambda^{(j)} < \mu^{(j)}$ in Λ_j. Similarly, we can order the paths going from level k to level n on $\mathfrak{B}$ by dominance or by reverse lexicographic order.

Example 2.17. Take $\mathfrak{B}$ to be Young's lattice. For a Young diagram λ, standard Young tableaux of shape λ can be identified with paths on $\mathfrak{B}$ from the empty diagram to λ. Dominance order on paths, as defined in Definition 2.16, agrees with dominance order on standard tableaux as usually defined. Reverse lexicographic order coincides with the "last letter order", see for example [34, p. 288].

We will now construct certain bases $\mathscr{B}^\lambda = \{b_\mathfrak{s}^\lambda\colon \mathfrak{s} \in \mathscr{T}(\lambda)\}$ of the cell modules Δ^λ, $\lambda \in \cup_n \Lambda_n$, each indexed by the set of paths $\mathscr{T}(\lambda)$, by induction on n. For $\lambda \in \Lambda_0$ or $\lambda \in \Lambda_1$, the cell module Δ^λ is free of rank one, and we choose any basis. Suppose now that $n > 1$, and a basis $\{b_\mathfrak{s}^\mu\colon \mathfrak{s} \in \mathscr{T}(\mu)\}$ for Δ^μ has been obtained for each $\mu \in \Lambda_k$ for $k \leqslant n-1$. Let $\lambda \in \Lambda_n$, and consider the filtration

$$\operatorname{Res}^{A_n}_{A_{n-1}}\left(\Delta^\lambda\right) = N_s \supseteq N_{s-1} \supseteq \cdots \supseteq N_0 = (0), \tag{2.1}$$

with $N_j/N_{j-1} \cong \Delta^{\mu_j}$ and $\mu_s < \mu_{s-1} < \cdots < \mu_1$. For each j, let $\{\bar{b}_\mathfrak{s}^{\mu_j}\colon \mathfrak{s} \in \mathscr{T}(\mu_j)\}$ be any lifting to N_j of the basis $\{b_\mathfrak{s}^{\mu_j}\colon \mathfrak{s} \in \mathscr{T}(\mu_j)\}$ of $N_j/N_{j-1} \cong \Delta^{\mu_j}$. Then $\cup_j\{\bar{b}_\mathfrak{s}^{\mu_j}\colon \mathfrak{s} \in \mathscr{T}(\mu_j)\}$ is a basis of Δ^λ. Note that $\mathfrak{t} \mapsto \mathfrak{t}'$ is a bijection from $\mathscr{T}(\lambda)$ to $\cup_j \mathscr{T}(\mu_j)$. We define $b_\mathfrak{t}^\lambda$ to be $\bar{b}_{\mathfrak{t}'}^{\mu_j}$ if $t' \in \mathscr{T}(\mu_j)$, so our basis is now denoted by $\{b_\mathfrak{s}^\lambda\colon \mathfrak{s} \in \mathscr{T}(\lambda)\}$. The bases $\mathscr{B}^\lambda = \{b_\mathfrak{s}^\lambda\colon \mathfrak{s} \in \mathscr{T}(\lambda)\}$ of the cell modules Δ^λ have the following property.

Proposition 2.18. *Fix $0 \leqslant k < n$, $\lambda \in \Lambda_n$, and $\mathfrak{t} \in \mathscr{T}(\lambda)$. Write $\mu = \mathfrak{t}(k)$, $\mathfrak{t}_1 = \mathfrak{t}_{[0,k]}$, and $\mathfrak{t}_2 = \mathfrak{t}_{[k,n]}$. Let $x \in A_k$, and let $xb_{\mathfrak{t}_1}^\mu = \sum_\mathfrak{s} r(x; \mathfrak{s}, \mathfrak{t}_1) b_\mathfrak{s}^\mu$. Then*

$$xb_\mathfrak{t}^\lambda \equiv \sum_\mathfrak{s} r(x; \mathfrak{s}, \mathfrak{t}_1) b_{\mathfrak{s}\circ\mathfrak{t}_2}^\lambda,$$

modulo $\operatorname{span}\{b_\mathfrak{v}^\lambda\colon \mathfrak{v}_{[k,n]} \succ \mathfrak{t}_{[k,n]}\}$, *where $\succ$ denotes reverse lexicographic order.*

Proof. We prove this by induction on $n-k$. Consider the case $n-k=1$. Consider the filtration (2.1). If $\mathfrak{t}' \in \mathcal{T}(\mu_j)$, then by the construction of the basis $\{b^\lambda_\mathfrak{t}: \mathfrak{t} \in \mathcal{T}(\lambda)\}$, we have

$$xb^\lambda_\mathfrak{t} \equiv \sum_\mathfrak{s} r(x; \mathfrak{s}, \mathfrak{t}_1) b^\lambda_{\mathfrak{s} \circ \mathfrak{t}_2},$$

modulo N_{j-1}, while N_{j-1} equals the R-span of $\{b^\lambda_\mathfrak{v}: \mathfrak{v}_{[n-1,n]} \succ \mathfrak{t}_{[n-1,n]}\}$.

Now suppose that $n-k > 1$, and $\mathfrak{t}' \in \mathcal{T}(\mu_j)$. Then $xb^\lambda_\mathfrak{t} = x\bar{b}^{\mu_j}_{\mathfrak{t}'}$. By a suitable induction hypothesis,

$$xb^{\mu_j}_{\mathfrak{t}'} \equiv \sum_\mathfrak{s} r(x; \mathfrak{s}, \mathfrak{t}_1) b^{\mu_j}_{\mathfrak{s} \circ \mathfrak{t}_{[k,n-1]}},$$

modulo the span of $\{b^{\mu_j}_\mathfrak{v}: \mathfrak{v}_{[k,n-1]} \succ \mathfrak{t}_{[k,n-1]}\}$. But then

$$xb^\lambda_\mathfrak{t} \equiv \sum_\mathfrak{s} r(x; \mathfrak{s}, \mathfrak{t}_1) b^\lambda_{\mathfrak{s} \circ \mathfrak{t}_2},$$

modulo

$$\operatorname{span}\{b^{\mu_j}_\mathfrak{v}: \mathfrak{v}_{[k,n-1]} \succ \mathfrak{t}_{[k,n-1]}\} + N_{j-1} = \operatorname{span}\{b^\lambda_\mathfrak{v}: \mathfrak{v}_{[k,n]} \succ \mathfrak{t}_{[k,n]}\}. \qquad \square$$

With the family of bases $\mathcal{B}^\lambda = \{b^\lambda_\mathfrak{s}: \mathfrak{s} \in \mathcal{T}(\lambda)\}$ of the cell modules Δ^λ, as above, for each $n \geqslant 0$, let $\mathcal{B}_n = \{b^\lambda_{\mathfrak{s},\mathfrak{t}}: \lambda \in \Lambda_n,\ \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$ be a cellular basis of A_n globalizing the bases $\mathcal{B}^\lambda$, $\lambda \in \Lambda_n$; see Lemma 2.3 and Definition 2.5.

The cellular bases $\mathcal{B}_n = \{b^\lambda_{\mathfrak{s},\mathfrak{t}}: \lambda \in \Lambda_n,\ \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$ have the property:

Corollary 2.19. *Fix $0 \leqslant k < n$, $\lambda \in \Lambda_n$, and $\mathfrak{t} \in \mathcal{T}(\lambda)$. Write $\mu = \mathfrak{t}(k)$, $\mathfrak{t}_1 = \mathfrak{t}_{[0,k]}$, and $\mathfrak{t}_2 = \mathfrak{t}_{[k,n]}$. Let $x \in A_k$, and let $xb^\mu_{\mathfrak{t}_1,\mathfrak{v}} \equiv \sum_\mathfrak{s} r(x; \mathfrak{s}, \mathfrak{t}_1) b^\mu_{\mathfrak{s},\mathfrak{v}}$ modulo $\breve{A}^\mu_k$ for all $\mathfrak{v} \in \mathcal{T}(\mu)$. Then, for all $\mathfrak{v} \in \mathcal{T}(\lambda)$,*

$$xb^\lambda_{\mathfrak{t},\mathfrak{v}} \equiv \sum_\mathfrak{s} r(x; \mathfrak{s}, \mathfrak{t}_1) b^\lambda_{\mathfrak{s} \circ \mathfrak{t}_2, \mathfrak{v}},$$

modulo $\operatorname{span}\{b^\lambda_{\mathfrak{w},\mathfrak{v}}: \mathfrak{w}_{[k,n]} \succ \mathfrak{t}_{[k,n]}\} + \breve{A}^\lambda_n$,

Definition 2.20. A family of bases $\mathcal{B}^\lambda$ of the cell modules Δ^λ, $\lambda \in \bigcup_n \Lambda_n$, having the property described in Proposition 2.18 will be called a *family of path bases* of the cell modules.

A family of cellular bases $\mathcal{B}_n$ of A_n, $n \geqslant 0$, globalizing a family of path bases $\mathcal{B}^\lambda$ of the cell modules will also be called a *family of path bases* of the cellular algebras.

3. JM elements in coherent towers

Example 3.1. We recall the classical Jucys–Murphy elements in the Hecke algebra $H_n(q)$, and some of their properties. The (multiplicative) Jucys–Murphy elements in $H_n(q)$ are the elements $\{L_1, \dots, L_n\}$ defined by $L_1 = 1$ and $L_{j+1} = q^{-1} T_j L_j T_j$ for $1 \leqslant j \leqslant n-1$. The elements L_k are mutually commuting; in fact, $L_k \in H_k(q) \subseteq H_n(q)$ for $1 \leqslant k \leqslant n$, and for $k \geqslant 2$, L_k commutes with H_{k-1}. Symmetric polynomials in the $\{L_k\}$ are in the center of $H_n(q)$. The Jucys–Murphy elements act on the Murphy bases of the cell module Δ^λ as follows. Let $\kappa(j, \mathfrak{t}) = c(j, \mathfrak{t}) - r(j, \mathfrak{t})$, where $c(j, \mathfrak{t})$ is the column of j in the standard tableau $\mathfrak{t}$ and $r(j, \mathfrak{t})$ is the row of j in $\mathfrak{t}$. Then

$$L_j m^\lambda_\mathfrak{t} = q^{\kappa(j,\mathfrak{t})} m^\lambda_\mathfrak{t} + \sum_{\mathfrak{s} \rhd \mathfrak{t}} r_\mathfrak{s} m^\lambda_\mathfrak{s}. \tag{3.1}$$

For a cell x in the Young diagram λ, let $\kappa(x)$ denote its content, namely the column of x minus the row of x. It follows from (3.1) that the product $p = \prod_{j=1}^n L_j$ acts as a scalar $\alpha_\lambda = q^{\sum_{x \in \lambda} \kappa(x)}$ on the cell module Δ^λ. Namely, if $\mathfrak{t}_0$ is the most dominant standard tableaux of shape λ then $pm^\lambda_{\mathfrak{t}_0} = \alpha_\lambda m^\lambda_{\mathfrak{t}_0}$, by (3.1). But p is central and Δ^λ is a cyclic module with generator $m^\lambda_{\mathfrak{t}_0}$.

Abstracting from the Hecke algebra example, Mathas [29] defined a family of JM–elements in a cellular algebra as follows.

Definition 3.2. (See [29].) Let A be a cellular algebra over R; let Λ denote the partially ordered set in the cell datum for A, and, for each $\lambda \in \Lambda$, let $\{a^\lambda_\mathfrak{t}\colon\ \mathfrak{t} \in \mathscr{T}(\lambda)\}$ denote the basis of the cell module Δ^λ (derived from the cellular basis of A.) Suppose that for each $\lambda \in \Lambda$, the index set $\mathscr{T}(\lambda)$ is given a partial order $\succeq$.

A finite family of elements $\{L_j\colon\ 1 \leqslant j \leqslant M\}$ in A is a *JM–family in the sense of Mathas* if the elements L_j are mutually commuting and invariant under the involution of A, and, for each $\lambda \in \Lambda$, there is a set of scalars $\{\kappa(j, \mathfrak{t})\colon\ 1 \leqslant j \leqslant n,\ \mathfrak{t} \in \mathscr{T}(\lambda)\}$ such that for $1 \leqslant j \leqslant n$ and $\mathfrak{t} \in \mathscr{T}(\lambda)$,

$$L_j a^\lambda_\mathfrak{t} = \kappa(j, \mathfrak{t}) a^\lambda_\mathfrak{t} + \sum_{\mathfrak{s} \succ \mathfrak{t}} r_\mathfrak{s} a^\lambda_\mathfrak{s},$$

for some $r_\mathfrak{s} \in R$, depending on j and $\mathfrak{t}$. In addition, the family $\{L_j\}$ is said to be *separating* if $\mathfrak{t} \mapsto (\kappa(j, \mathfrak{t}))_{1 \leqslant j \leqslant n}$ is injective on $\mathscr{T} = \bigcup_{\lambda \in \Lambda} \mathscr{T}(\lambda)$.[1]

We are going to introduce a different abstraction of Jucys–Murphy elements that is appropriate for strongly coherent towers of cellular algebras. We will see that our concept implies that of Mathas.

Definition 3.3. Let $(A_n)_{n \geqslant 0}$ be a strongly coherent tower of cellular algebras over R. Let Λ_n denote the partially ordered set in the cell datum for A_n.

A family of *invertible* elements $\{L_n\colon\ n \geqslant 1\}$ is a *multiplicative JM–family* if for all $n \geqslant 1$,

(1) $L_n \in A_n$, L_n is invariant under the involution of A_n, and, for $n \geqslant 1$, L_n commutes with A_{n-1}. In particular, the elements L_j are mutually commuting.
(2) For each $n \geqslant 1$ and each $\lambda \in \Lambda_n$, there exists an invertible $\alpha(\lambda) \in R$ such that the product $L_1 \cdots L_n$ acts as the scalar $\alpha(\lambda)$ on the cell module Δ^λ.

For convenience, we will set $\alpha(\emptyset) = 1$, where $\emptyset$ is the unique element of Λ_0.

Definition 3.4. An *additive JM–family* is defined similarly, except that the elements L_j are not required to be invertible and (2) is replaced by

(2′) For each $n \geqslant 1$ and each $\lambda \in \Lambda_n$, there exists $d(\lambda) \in R$ such that the sum $L_1 + \cdots + L_n$ act as the scalar $d(\lambda)$ on the cell module Δ^λ.

For convenience, we will set $d(\emptyset) = 0$.

Assumption 3.5. For the remainder of Section 3, let R be an integral domain with field of fractions F, $(A_n)_{n \geqslant 0}$ a strongly coherent tower of cellular algebras over R, such that A^F_n is split semisimple for all n. Let $\mathfrak{B}$ denote the branching diagram of $(A^F_n)_{n \geqslant 0}$ and Λ_n the partially ordered set in the cell datum for A_n. Let $\mathscr{B}^\lambda = \{b^\lambda_\mathfrak{s}\colon\ \mathfrak{s} \in \mathscr{T}(\lambda)\}$ be a family of path bases of the cell modules Δ^λ, $\lambda \in \bigcup_n \Lambda_n$ (Definition 2.20). We employ the reverse lexicographic order $\preceq$ on paths (Definition 2.16).

Proposition 3.6. *Suppose that $\{L_n\colon\ n \geqslant 0\}$ is a multiplicative JM–family for the strongly coherent tower $(A_n)_{n \geqslant 0}$.*

[1] Mathas' definition of separating is slightly weaker.

(1) *For $n \geqslant 1$ and $\lambda \in \Lambda_n$, let $\alpha(\lambda) \in R^\times$ be such that $L_1 \cdots L_n$ acts by the scalar $\alpha(\lambda)$ on the cell module Δ^λ. Then for all $n \geqslant 1$, $\lambda \in \Lambda_n$, $\mathfrak{t} \in \mathscr{T}(\lambda)$, and $1 \leqslant j \leqslant n$, we have*

$$L_j b_{\mathfrak{t}}^\lambda = \kappa(j, \mathfrak{t}) b_{\mathfrak{t}}^\lambda + \sum_{\mathfrak{s} \succ \mathfrak{t}} r_{\mathfrak{s}} b_{\mathfrak{s}}^\lambda, \tag{3.2}$$

for some elements $r_{\mathfrak{s}} \in R$ (depending on j and $\mathfrak{t}$), with $\kappa(j, \mathfrak{t}) = \frac{\alpha(\mathfrak{t}(j))}{\alpha(\mathfrak{t}(j-1))}$.

(2) *For each $n \geqslant 1$, $L_1 \cdots L_n$ is in the center of A_n.*

Proof. We prove (1) by induction on n. For $n = 1$, the statement follows from (2) of Definition 3.3. Assume $n > 1$ and adopt the appropriate induction hypothesis. For $j < n$, $\lambda \in \Lambda_n$, and $\mathfrak{t} \in \mathscr{T}(\lambda)$, (3.2) holds by the induction hypothesis and Proposition 2.18, while

$$\begin{aligned} L_n b_{\mathfrak{t}}^\lambda &= (L_1 \cdots L_{n-1})^{-1} (L_1 \cdots L_n) b_{\mathfrak{t}}^\lambda \\ &= \alpha(\lambda) (L_1 \cdots L_{n-1})^{-1} b_{\mathfrak{t}}^\lambda \\ &= \alpha(\lambda) \alpha\big(\mathfrak{t}(n-1)\big)^{-1} b_{\mathfrak{t}}^\lambda + \sum_{\mathfrak{s} \succ \mathfrak{t}} r_{\mathfrak{s}} b_{\mathfrak{s}}^\lambda, \end{aligned}$$

using point (2) of Definition 3.3 and Proposition 2.18.

For all $x \in A_n$, $x(L_1 \cdots L_n) = (L_1 \cdots L_n)x$ on each cell module. But the direct sum of all cell modules is faithful. This proves (2). □

The additive version of the proposition is the following; the proof is similar. Recall that Assumption 3.5 is still in force.

Proposition 3.7. *Suppose that $\{L_n \colon n \geqslant 0\}$ is an additive JM–family for the tower $(A_n)_{n \geqslant 0}$.*

(1) *For $n \geqslant 1$ and $\lambda \in \Lambda_n$, let $d(\lambda) \in R$ be such that $L_1 + \cdots + L_n$ acts by the scalar $d(\lambda)$ on the cell module Δ^λ. Then for all $n \geqslant 1$, $\lambda \in \Lambda_n$, $\mathfrak{t} \in \mathscr{T}(\lambda)$, and $1 \leqslant j \leqslant n$, we have*

$$L_j b_{\mathfrak{t}}^\lambda = \kappa(j, \mathfrak{t}) b_{\mathfrak{t}}^\lambda + \sum_{\mathfrak{s} \succ \mathfrak{t}} r_{\mathfrak{s}} b_{\mathfrak{s}}^\lambda, \tag{3.3}$$

for some elements $r_{\mathfrak{s}} \in R$ (depending on j and $\mathfrak{t}$), with $\kappa(j, \mathfrak{t}) = \alpha(\mathfrak{t}(j)) - \alpha(\mathfrak{t}(j-1))$.

(2) *For each $n \geqslant 1$, $L_1 + \cdots + L_n$ is in the center of A_n.*

Remark 3.8. The techniques employed here give triangularity of the action of the JM elements only with respect to the reverse lexicographic order on paths, and not with respect to the dominance order. Our techniques cannot recover the result on triangularity with respect to the dominance order for the Hecke algebras (see Example 3.1).

3.1. The separated case – Gelfand–Zeitlin algebras

3.1.1. Generalities on Gelfan–Zeitlin subalgebras

Let us recall the following notion pertaining to a finite multiplicity free tower $(A_k)_{0 \leqslant k \leqslant n}$ of split semisimple algebras over a field F. The terminology is from Vershik and Okounkov [37,42].

Definition 3.9. The *Gelfand–Zeitlin subalgebra G_n* of A_n is the subalgebra generated by the centers of $A_0, A_1, \ldots, A_n$.

The Gelfand–Zeitlin subalgebra is a maximal abelian subalgebra of A_n and contains a remarkable family of idempotents indexed by paths on the branching diagram $\mathfrak{B}$ of $(A_k)_{0\leqslant k\leqslant n}$. For each j let $\{z_\lambda\colon\ \lambda \in \Lambda_j\}$ denote the set of minimal central idempotents in A_j. For $k \leqslant n$ and $\mathfrak{t}$ a path on $\mathfrak{B}$ of length k, let $F_\mathfrak{t} = \prod_j z_{\mathfrak{t}(j)}$. Then the elements $F_\mathfrak{t}$ for $\mathfrak{t}$ of length k are mutually orthogonal minimal idempotents whose sum is the identity; moreover the sum of those $F_\mathfrak{t}$ such that $\mathfrak{t}(k) = \lambda$ is z_λ. If s is a path of length k and t is a path of length ℓ, with $k \leqslant \ell$, then $F_\mathfrak{s} F_\mathfrak{t} = \delta_{\mathfrak{s},\mathfrak{t}[0,k]} F_\mathfrak{t}$. Evidently, the set of $F_\mathfrak{t}$ as $\mathfrak{t}$ varies over paths of length $k \leqslant n$ generate G_n. Let us call the set $\{F_\mathfrak{t}\}$ the family of Gelfand–Zeitlin idempotents for $(A_k)_{0\leqslant k\leqslant n}$. The properties listed above characterize this family of idempotents:

Lemma 3.10. *Consider a finite multiplicity free tower $(A_k)_{0\leqslant k\leqslant n}$ of split semisimple algebras over a field F. Let $F'_\mathfrak{t}$ be a family of idempotents indexed by paths of length $k \leqslant n$ on the branching diagram $\mathfrak{B}$ of $(A_k)_{0\leqslant k\leqslant n}$ with the following properties*:

(1) *For t of length k, $F'_\mathfrak{t}$ is a minimal idempotent in A_k. The sum of those $F'_\mathfrak{t}$ such that t has length k and $t(k) = \lambda$ is z_λ.*
(2) *If $\mathfrak{s}$ is a path of length k and $\mathfrak{t}$ is a path of length ℓ, with $k \leqslant \ell$, then $F'_s F'_\mathfrak{t} = \delta_{\mathfrak{s},\mathfrak{t}[0,k]} F'_\mathfrak{t}$.*

Then $F'_\mathfrak{t} = F_\mathfrak{t}$ for all paths t.

Proof. Let $\mathfrak{t}$ be a path of length $k \geqslant 1$ let $\mathfrak{t}' = \mathfrak{t}[0, k-1]$ and $\lambda = \mathfrak{t}(k)$. It follows from the assumptions that $F'_\mathfrak{t} = F'_{\mathfrak{t}'} z_\lambda$. Using this, the conclusion $F'_\mathfrak{t} = F_\mathfrak{t}$ follows by induction on the length of the path. □

3.1.2. JM elements and GZ subalgebras

We return to our Assumption 3.5. Suppose that $(L_n)_{n\geqslant 0}$ is a multiplicative or additive JM family in $(A_n)_{n\geqslant 0}$. According to Propositions 3.6 and 3.7, for each $n \geqslant 0$, $\{L_1, \dots, L_n\}$ is a JM family for A_n in the sense of Mathas, with respect to the reverse lexicographic order and any path basis. Suppose now, in addition, that Mathas' separation property is satisfied, namely that for each n, $\mathfrak{t} \mapsto (\kappa(\mathfrak{t}, j))_{1\leqslant j\leqslant n}$ is injective on $\mathscr{T}(n)$.

Proposition 3.11. *Suppose that for each k, $\mathfrak{t} \mapsto (\kappa(\mathfrak{t}, j))_{1\leqslant j\leqslant k}$ is injective on $\mathscr{T}(k)$. Then for each n, $\{L_1, \dots, L_n\}$ generates the Gelfand–Zeitlin subalgebra of the finite tower $(A_k^F)_{0\leqslant k\leqslant n}$.*

Proof. Fix n. For $j \leqslant k \leqslant n$, let $K(j) = \{\kappa(\mathfrak{t}, j)\colon\ \mathfrak{t} \in \mathscr{T}(k)\}$; note that $K(j)$ does not depend on k as long as $j \leqslant k$. For $\mathfrak{t}$ a path on $\mathfrak{B}$ of length k, define

$$F'_\mathfrak{t} = \prod_{j=1}^{k} \prod_{\substack{c\in K(j)\\ c\neq\kappa(\mathfrak{t},j)}} \frac{L_j - c}{\kappa(\mathfrak{t}, j) - c}.$$

Then Mathas [29] shows that $F'_\mathfrak{t}$ is a minimal idempotent in A_k^F and the sum of those $F'_\mathfrak{t}$ such that $\mathfrak{t}$ has length k and $\mathfrak{t}(k) = \lambda$ is z_λ. Moreover, for $j \leqslant k$, $L_j F'_\mathfrak{t} = \kappa(\mathfrak{t}, j) F'_\mathfrak{t}$. It follows from this that if $\mathfrak{s}$ is a path of length k and $\mathfrak{t}$ is a path of length ℓ, with $k \leqslant \ell$, then $F'_s F'_\mathfrak{t} = \delta_{\mathfrak{s},\mathfrak{t}[0,k]} F'_\mathfrak{t}$. Hence, by Lemma 3.10, Mathas' idempotents $F'_\mathfrak{t}$ are the Gelfand–Zeitlin idempotents for the finite tower $(A_k^F)_{0\leqslant k\leqslant n}$. This shows that the Gelfand–Zeitlin algebra is contained in the algebra generated by the JM elements; on the other hand, the JM elements are in the linear span of the idempotents F'_t, which gives the opposite inclusion. □

4. Framework axioms and a theorem on cellularity

We describe the framework axioms and main theorem of [17]. Let R be an integral domain with field of fractions F. We consider two towers of R-algebras

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots, \quad \text{and} \quad Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \cdots.$$

The framework axioms of [17] are the following:

(1) $(Q_n)_{n\geqslant 0}$ is a coherent tower of cellular algebras.
(2) There is an algebra involution i on $\cup_n A_n$ such that $i(A_n) = A_n$.
(3) $A_0 = Q_0 = R$, and $A_1 = Q_1$ (as algebras with involution).
(4) For all n, $A_n^F := A_n \otimes_R F$ is split semisimple.
(5) For $n \geqslant 2$, A_n contains an essential idempotent e_{n-1} such that $i(e_{n-1}) = e_{n-1}$ and $A_n/(A_n e_{n-1} A_n) \cong Q_n$, as algebras with involution.
(6) For $n \geqslant 1$, e_n commutes with A_{n-1} and $e_n A_n e_n \subseteq A_{n-1} e_n$.
(7) For $n \geqslant 1$, $A_{n+1} e_n = A_n e_n$, and the map $x \mapsto x e_n$ is injective from A_n to $A_n e_n$.
(8) For $n \geqslant 2$, $e_{n-1} \in A_{n+1} e_n A_{n+1}$.

Say that the pair of towers of algebras $(Q_k)_{k\geqslant 0}$ and $(A_k)_{k\geqslant 0}$ satisfy the *strong framework axioms*, if they satisfy the axioms with (1) replaced by

(1′) $(Q_n)_{n\geqslant 0}$ is a strongly coherent tower of cellular algebras.

In the following theorem, point (4) we use the notion of a branching diagram obtained by reflections from another branching diagram. We refer the reader to [17, Section 2.5] for this notion.

Theorem 4.1. *(See [17, Theorem 3.2].) Let R be an integral domain with field of fractions F. Let $(Q_n)_{n\geqslant 0}$ and $(A_n)_{n\geqslant 0}$ be two towers of R-algebras satisfying the framework axioms (resp. the strong framework axioms). Then*

(1) *$(A_n)_{n\geqslant 0}$ is a coherent tower of cellular algebras (resp. a strongly coherent tower of cellular algebras).*
(2) *For all n, the partially ordered set in the cell datum for A_n can be realized as*

$$\Lambda_n = \coprod_{\substack{i \leqslant n \\ n-i \text{ even}}} \Lambda_i^{(0)} \times \{n\},$$

with the following partial order: Let $\lambda \in \Lambda_i^{(0)}$ and $\mu \in \Lambda_j^{(0)}$, with i, j, and n all of the same parity. Then $(\lambda, n) > (\mu, n)$ if, and only if, $i < j$, or $i = j$ and $\lambda > \mu$ in $\Lambda_i^{(0)}$.

(3) *Suppose $n \geqslant 2$ and $(\lambda, n) \in \Lambda_i^{(0)} \times \{n\} \subseteq \Lambda_n$. Let $\Delta^{(\lambda,n)}$ be the corresponding cell module.*
  (a) *If $i < n$, then $\Delta^{(\lambda,n)} = A_{n-1} e_{n-1} \otimes_{A_{n-2}} \Delta^{(\lambda,n-2)}$. Moreover,*

$$\left(A_n e_{k-1} A_n \Delta^{(\lambda,k)}\right) \otimes_R F = \Delta^{(\lambda,k)} \otimes_R F.$$

  (b) *If $i = n$ then $\Delta^{(\lambda,n)}$ is a Q_n module, and $A_n e_{n-1} A_n \Delta^{(\lambda,n)} = 0$.*
(4) *The branching diagram $\mathfrak{B}$ for $(A_n^F)_{k\geqslant 0}$ is that obtained by reflections from the branching diagram $\mathfrak{B}_0$ for $(Q_n^F)_{n\geqslant 0}$.*

Proof. The theorem for coherent towers is proved in [17]. The modification for strongly coherent towers is straightforward. □

Remark 4.2. At first sight, it may seem that to apply Theorem 4.1 requires verifying a formidable list of axioms, but in fact the theorem is always easy to apply. All of the axioms except (1′) and (4) are elementary. Axiom (1′) is generally a substantial theorem, which however is already available in the literature in many interesting examples. Axiom (4) can generally be verified by use of Wenzl's method, applying the Jones basic construction. For examples, see Section 6 of this paper and [17, Section 5].

5. JM elements in algebras arising from the basic construction

Theorem 5.1. *Consider two towers of R-algebras $(A_n)_{n\geqslant 0}$ and $(Q_n)_{n\geqslant 0}$ satisfying the strong framework axioms of Section 4. Suppose that $\{L_j^{(0)}\colon\ j\geqslant 1\}$ is a multiplicative JM–family for the tower $(Q_n)_{n\geqslant 0}$, in the sense of Section 3, and that $\{L_n\colon\ n\geqslant 1\}$ is a family of elements in $(A_n)_{n\geqslant 0}$ satisfying the following conditions:*

(1) *$L_n \in A_n$, and L_n commutes with A_{n-1}.*
(2) *$\pi_j(L_j) = L_j^{(0)}$, where $\pi_j : A_j \to Q_j$ is the quotient map.*
(3) *For each $j\geqslant 1$, there exists $\gamma_j \in R^\times$ such that*

$$L_jL_{j+1}e_j = e_jL_jL_{j+1} = \gamma_j e_j.$$

Then $\{L_j\colon\ j\geqslant 1\}$ is a multiplicative JM–family for the tower $(A_n)_{n\geqslant 0}$.

Proof. Write $\Lambda_n^{(0)}$ for the partially ordered set in the cell datum for Q_n and Λ_n for that in the cell datum for A_n. Recall that Λ_n is realized as the set of ordered pairs (λ, n), where $\lambda \in \Lambda_k^{(0)}$ for some $k \leqslant n$ with $n-k$ even. For $n\geqslant 1$ and $\lambda \in \Lambda_n^{(0)}$, let $\alpha(\lambda) \in R^\times$ be such that the product $L_1^{(0)}\cdots L_n^{(0)}$ acts by the scalar $\alpha(\lambda)$ on the cell module Δ^λ of Q_n.

To show that $\{L_j\colon\ j\geqslant 1\}$ is a multiplicative JM–family for the tower $(A_n)_{n\geqslant 0}$, we need only verify point (2) of Definition 3.3. We do this by induction on n. For $n=0$, we interpret $L_1\cdots L_n$ to be the identity, and we observe that the statement is trivial. For $n=1$, $A_1 = Q_1$, so again there is nothing to prove. Suppose that $n>1$, and that for all $m<n$ and all $(\mu, m)\in\Lambda_m$, with $\mu\in\Lambda_k^{(0)}$, $(L_1\cdots L_m)$ acts as the scalar

$$\beta\big((\mu, m)\big) := \gamma_{m-1}\gamma_{m-3}\cdots\gamma_{k+1}\alpha(\mu)$$

on the cell module $\Delta^{(\mu,m)}$ of A_m.

If $\lambda \in \Lambda_n^{(0)}$, then the cell module $\Delta^{(\lambda,n)}$ is actually the Q_n-module Δ^λ, so

$$(L_1\cdots L_n)y = \big(L_1^{(0)}\cdots L_n^{(0)}\big)y = \alpha(\lambda)y,$$

for $y\in\Delta^{(\lambda,n)}$.

Let $\lambda\in\Lambda_k^{(0)}$ for some $k<n$. Then $\Delta^{(\lambda,n)} = A_{n-1}e_{n-1}\otimes_{A_{n-2}}\Delta^{(\lambda,n-2)}$. For $x\in A_{n-1}$ and $y\in\Delta^{(\lambda,n-2)}$, we have

$$\begin{aligned}
(L_1\cdots L_n)xe_{n-1}\otimes y &= (L_1\cdots L_{n-1})xL_ne_{n-1}\otimes y\\
&= x(L_1\cdots L_{n-1})L_ne_{n-1}\otimes y\\
&= x(L_{n-1}L_n)e_{n-1}\otimes (L_1\cdots L_{n-2})y\\
&= \gamma_{n-1}xe_{n-1}\otimes \gamma_{n-3}\cdots\gamma_{k+1}\alpha(\lambda)y\\
&= \gamma_{n-1}\cdots\gamma_{k+1}\alpha(\lambda)xe_{n-1}\otimes y,
\end{aligned}$$

where the first equality is valid since L_n commutes with A_{n-1}, the second follows from the induction hypothesis and Proposition 3.6 (2), the third follows because $L_1 \cdots L_{n-2}$ is an element of A_{n-2}, and so commutes with e_{n-1}, and the fourth comes from the induction hypothesis and hypothesis (3) of the theorem statement. $\square$

Corollary 5.2. *If γ_j is independent of j, say $\gamma_j = \gamma$ for all j, then $\beta((\lambda, n)) = \gamma^{(n-k)/2}\alpha(\lambda)$ when $\lambda \in \Lambda_k^{(0)}$.*

The additive version of the theorem is the following. The proof is similar.

Theorem 5.3. *Consider two towers of R-algebras $(A_n)_{n\geqslant 0}$ and $(Q_n)_{n\geqslant 0}$ satisfying the strong framework axioms of Section 4. Suppose that $\{L_j^{(0)}\colon\ j \geqslant 1\}$ is an additive JM–family for the tower $(Q_n)_{n\geqslant 0}$, in the sense of Section 3, and that $\{L_n\colon\ n \geqslant 1\}$ is a family of elements in $(A_n)_{n\geqslant 0}$ satisfying the following conditions:*

(1) *$L_n \in A_n$, and L_n commutes with A_{n-1}.*
(2) *$\pi_j(L_j) = L_j^{(0)}$, where $\pi_j : A_j \to Q_j$ is the quotient map.*
(3) *For each $j \geqslant 1$, there exists $\gamma_j \in R$ such that*

$$(L_j + L_{j+1})e_j = e_j(L_j + L_{j+1}) = \gamma_j e_j.$$

Then $\{L_j\colon\ j \geqslant 1\}$ is an additive JM–family for the tower $(A_n)_{n\geqslant 0}$.

The additive analogue of the formula for β developed in the proof of Theorem 5.1 is the following. For $n \geqslant 1$ and $\lambda \in \Lambda_n^{(0)}$, let $d(\lambda) \in R$ be such that $L_1^{(0)} + \cdots + L_n^{(0)}$ acts by the scalar $d(\lambda)$ on the cell module Δ^λ of Q_n. Then for $(\lambda, n) \in \Lambda_n$, with $\lambda \in \Lambda_k^{(0)}$, $L_1 + \cdots + L_n$ acts by the scalar

$$\beta\big((\lambda, n)\big) = \gamma_{n-1} + \cdots + \gamma_{k+1} + d(\lambda).$$

If γ_j is independent of j, say $\gamma_j = \gamma$ for all j, then

$$\beta\big((\lambda, n)\big) = \frac{n-k}{2}\gamma + d(\lambda).$$

6. Examples

6.1. Preliminaries on tangle diagrams

Several of our examples involve *tangle diagrams* in the rectangle $\mathscr{R} = [0, 1] \times [0, 1]$. Fix points $a_i \in [0, 1]$, $i \geqslant 1$, with $0 < a_1 < a_2 < \cdots$. Write $\boldsymbol{i} = (a_i, 1)$ and $\bar{\boldsymbol{i}} = (a_i, 0)$.

Recall that a *knot diagram* means a collection of piecewise smooth closed curves in the plane which may have intersections and self-intersections, but only simple transverse intersections. At each intersection or crossing, one of the two strands (curves) which intersect is indicated as crossing over the other.

An (n, n)-*tangle diagram* is a piece of a knot diagram in $\mathscr{R}$ consisting of exactly n topological intervals and possibly some number of closed curves, such that: (1) the endpoints of the intervals are the points $\mathbf{1}, \ldots, \boldsymbol{n}, \bar{\mathbf{1}}, \ldots, \bar{\boldsymbol{n}}$, and these are the only points of intersection of the family of curves with the boundary of the rectangle, and (2) each interval intersects the boundary of the rectangle transversally.

An (n, n)-*Brauer diagram* is a "tangle" diagram containing no closed curves, in which information about over and under crossings is ignored. Two Brauer diagrams are identified if the pairs of boundary points joined by curves is the same in the two diagrams. By convention, there is a unique $(0, 0)$-Brauer diagram, the empty diagram with no curves. For $n \geqslant 1$, the number of (n, n)-Brauer diagrams is $(2n - 1)!! = (2n - 1)(2n - 3) \cdots (3)(1)$.

For any of these types of diagrams, we call $P = \{\mathbf{1}, \dots, \boldsymbol{n}, \bar{\mathbf{1}}, \dots, \bar{\boldsymbol{n}}\}$ the set of *vertices* of the diagram, $P^+ = \{\mathbf{1}, \dots, \boldsymbol{n}\}$ the set of *top vertices*, and $P^- = \{\bar{\mathbf{1}}, \dots, \bar{\boldsymbol{n}}\}$ the set of *bottom vertices*. A curve or *strand* in the diagram is called a *vertical* or *through* strand if it connects a top vertex and a bottom vertex, and a *horizontal* strand if it connects two top vertices or two bottom vertices.

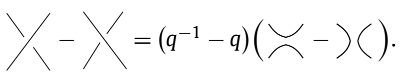
6.2. The BMW algebras

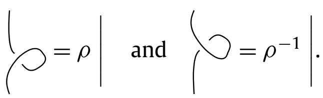
The BMW algebras were first introduced by Birman and Wenzl [5] and independently by Murakami [33] as abstract algebras defined by generators and relations. The version of the presentation given here follows [31] and [32].

Definition 6.1. Let S be a commutative unital ring with invertible elements ρ and q and an element δ satisfying $\rho^{-1} - \rho = (q^{-1} - q)(\delta - 1)$. The *Birman–Wenzl–Murakami algebra* $W_n(S; \rho, q, \delta)$ is the unital S-algebra with generators $g_i^{\pm 1}$ and e_i $(1 \leqslant i \leqslant n-1)$ and relations:

(1) (Inverses) $g_i g_i^{-1} = g_i^{-1} g_i = 1$.
(2) (Essential idempotent relation) $e_i^2 = \delta e_i$.
(3) (Braid relations) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ and $g_i g_j = g_j g_i$ if $|i - j| \geqslant 2$.
(4) (Commutation relations) $g_i e_j = e_j g_i$ and $e_i e_j = e_j e_i$ if $|i - j| \geqslant 2$.
(5) (Tangle relations) $e_i e_{i\pm1} e_i = e_i$, $g_i g_{i\pm1} e_i = e_{i\pm1} e_i$, and $e_i g_{i\pm1} g_i = e_i e_{i\pm1}$.
(6) (Kauffman skein relation) $g_i - g_i^{-1} = (q - q^{-1})(1 - e_i)$.
(7) (Untwisting relations) $g_i e_i = e_i g_i = \rho^{-1} e_i$, and $e_i g_{i\pm1} e_i = \rho e_i$.

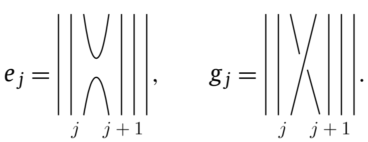
The BMW algebra W_n can also be realized as the algebra of (n, n)-tangle diagrams modulo regular isotopy and the following *Kauffman skein relations:*

(1) Crossing relation: 

(2) Untwisting relation: 

(3) Free loop relation: $T \cup \bigcirc = \delta T$, where $T \cup \bigcirc$ means the union of a tangle diagram T and a closed loop having no crossings with T.

In the tangle picture, e_j and g_j are represented by the following (n, n)-tangle diagrams:



The realization of the BMW algebra as an algebra of tangles is from [32]. See [17, Section 5.4] for more details.

The quotient of the BMW algebra $W_n(S; \rho, q, \delta)$ by the ideal J generated by e_{n-1} is the Hecke algebra $H_n(S; q^2)$. If π_n denotes the quotient map $\pi_n : W_n \to W_n/J$, take $T_i = \pi_n(q\ g_i)$ to obtain an isomorphism with the Hecke algebra as presented in Example 2.13.

The generic ground ring for the BMW algebras is

$$R = \mathbb{Z}\big[\boldsymbol{\rho}^{\pm1}, \boldsymbol{q}^{\pm1}, \boldsymbol{\delta}\big] / \big\langle \boldsymbol{\rho}^{-1} - \boldsymbol{\rho} = \big(\boldsymbol{q}^{-1} - \boldsymbol{q}\big)(\boldsymbol{\delta} - 1)\big\rangle,$$

where $\boldsymbol{\rho}$, $\boldsymbol{q}$, and $\boldsymbol{\delta}$ are indeterminants over $\mathbb{Z}$. R is an integral domain whose field of fractions is $F \cong \mathbb{Q}(\boldsymbol{\rho}, \boldsymbol{q})$ (with $\boldsymbol{\delta} = (\boldsymbol{\rho}^{-1} - \boldsymbol{\rho})/(\boldsymbol{q}^{-1} - \boldsymbol{q}) + 1$ in F). Write W_n for $W_n(R; \boldsymbol{\rho}, \boldsymbol{q}, \boldsymbol{\delta})$ and H_n for $H_n(R; \boldsymbol{q}^2)$. It is shown in [17, Section 5.4], that the pair of towers $(W_n)_{n \geqslant 0}$ and $(H_n)_{n \geqslant 0}$ satisfy the framework axioms of Section 4. In fact, by Example 2.13, the tower of Hecke algebras is strongly coherent, so the pair satisfies the strong version of the framework axioms. Consequently, by Theorem 4.1, the sequence of BMW algebras is a strongly coherent tower of cellular algebras. The partially ordered set Λ_n in the cell datum of W_n is the set of pairs (λ, n), with λ a Young diagram of size $k \leqslant n$ with $n - k$ even. The set of paths $\mathscr{T}((\lambda, n))$ can be identified with up–down tableaux of length n and shape λ, see [11].

The following analogue of Jucys–Murphy elements for the BMW algebras were introduced by Leduc and Ram [27] and Enyang [11]. Define $L_1 = 1$ and $L_{j+1} = g_j L_j g_j$ for $j \geqslant 1$. (Thus, for example, $L_5 = g_4 g_3 g_2 g_1^2 g_2 g_3 g_4$.) The involution on W_n is the unique algebra involution taking $e_i \mapsto e_i$ and $g_i \mapsto g_i$; it leaves each L_j invariant. One can check algebraically that L_n commutes with the generators of W_{n-1}, but this is far easier to see using the geometric realization of W_n. In fact, in the geometric picture, L_n is represented by the braid in which the n-th strand wraps once around the first $(n-1)$ strands.

Let $L_j^{(0)}$ denote the classical JM elements in the Hecke algebras H_n, as defined in Example 3.1. Then we have $\pi_n(L_j) = L_j^{(0)}$ for $1 \leqslant j \leqslant n$; this follows because $\pi_n(L_1) = 1$ and $\pi_n(L_{j+1}) = \boldsymbol{q}^{-2} T_j \pi_n(L_j) T_j$. (This is the correct recursion, because the Hecke algebra parameter q has been replaced by $\boldsymbol{q}^2$.) One can check, using algebraic relations or by using tangle diagrams, that for all $j \geqslant 1$,

$$L_j L_{j+1} e_j = e_j L_j L_{j+1} = \rho^{-2} e_j.$$

(The factor of ρ^{-2} comes from two applications of the untwisting relation (2) above.)

It now follows from Theorem 5.1 that $\{L_j\colon\ j \geqslant 0\}$ is a multiplicative JM–family in $(W_n)_{n \geqslant 0}$, with $L_1 \cdots L_n$ acting by

$$\beta\big((\lambda, n)\big) := \rho^{-(n-k)} \alpha(\lambda)$$

on the cell module $\Delta^{(\lambda,n)}$, if λ is a Young diagram of size k. By Proposition 3.6, the action of the elements L_j on the basis of $\Delta^{(\lambda,n)}$ labelled by up–down tableaux is triangular:

$$L_j a_{\mathfrak{t}}^{\lambda} = \kappa(j, \mathfrak{t})\, a_{\mathfrak{t}}^{\lambda} + \sum_{\mathfrak{s} \succ \mathfrak{t}} r_{\mathfrak{s}} a_{\mathfrak{s}}^{\lambda}, \tag{6.1}$$

with $\kappa(j, \mathfrak{t}) = \frac{\beta(\mathfrak{t}(j))}{\beta(\mathfrak{t}(j-1))}$, for some elements $r_{\mathfrak{s}} \in R$, depending on j and $\mathfrak{t}$. Moreover, if $\mathfrak{t}(j) = (\nu, j)$ and $\mathfrak{t}(j-1) = (\mu, j-1)$, then $|\nu| = |\mu| \pm 1$. If $|\nu| = |\mu| + 1$ and $\nu \setminus \mu = x$, then

$$\kappa(j, \mathfrak{t}) = \frac{\beta((\nu, j))}{\beta((\mu, j-1))} = \frac{\alpha(\nu)}{\alpha(\mu)} = \boldsymbol{q}^{2\kappa(x)},$$

where $\kappa(x)$ is the content of x, namely the column of x minus the row of x. If $|\nu| = |\mu| - 1$ and $\mu \setminus \nu = x$, then

$$\kappa(j, \mathfrak{t}) = \frac{\beta((\nu, j))}{\beta((\mu, j-1))} = \rho^{-2} \frac{\alpha(\nu)}{\alpha(\mu)} = \rho^{-2} \boldsymbol{q}^{-2\kappa(x)}.$$

This recovers Theorem 7.8 of Enyang [11].[2]

[2] The theorem is stated in [11] with dominance order rather than lexicographic order, but it appears that the proof only yields the statement with lexicographic order.

6.3. The Brauer algebras

The Brauer algebras were introduced by Brauer [6] as a device for studying the invariant theory of orthogonal and symplectic groups.

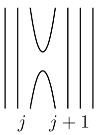
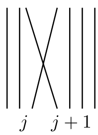
Let S be a commutative ring with identity, with a distinguished element δ. The Brauer algebra $B_n(S,\delta)$ is the free S-module with basis the set of (n,n)-Brauer diagrams, with multiplication defined as follows. The product of two Brauer diagrams is defined to be a certain multiple of another Brauer diagram. Namely, given two Brauer diagrams a, b, first "stack" b over a; the result is a planar tangle that may contain some number of closed curves. Let r denote the number of closed curves, and let c be the Brauer diagram obtained by removing all the closed curves. Then $ab = \delta^r c$.

Definition 6.2. For $n \geqslant 1$, the *Brauer algebra* $B_n(S,\delta)$ over S with parameter δ is the free S-module with basis the set of (n,n)-Brauer diagrams, with the bilinear product determined by the multiplication of Brauer diagrams. In particular, $B_0(S,\delta) = S$.

Note that the Brauer diagrams with only vertical strands are in bijection with permutations of $\{1,\dots,n\}$, and that the multiplication of two such diagrams coincides with the multiplication of permutations. Thus the Brauer algebra contains the group algebra $S\mathfrak{S}_n$ of the permutation group $\mathfrak{S}_n$. The identity element of the Brauer algebra is the diagram corresponding to the trivial permutation. We will note below that $S\mathfrak{S}_n$ is also a quotient of $B_n(S,\delta)$.

The involution i on (n,n)-Brauer diagrams which reflects a diagram in the axis $y = 1/2$ extends linearly to an algebra involution of $B_n(S,\delta)$.

Let e_j and s_j denote the (n,n)-Brauer diagrams:

$e_j =$ , $s_j =$ .

Note that $e_j^2 = \delta e_j$, so e_j is an essential idempotent if $\delta \neq 0$, and nilpotent if $\delta = 0$. We have $i(e_j) = e_j$ and $i(s_j) = s_j$. It is easy to see that $e_1,\dots,e_{n-1}$ and $s_1,\dots,s_{n-1}$ generate $B_n(S,\delta)$ as an algebra.

The products ab and ba of two Brauer diagrams have at most as many through strands as a. Consequently, the span of diagrams with fewer than n through strands is an ideal J in $B_n(S,\delta)$. The ideal J is generated by e_{n-1}. We have $B_n(S,\delta)/J \cong S\mathfrak{S}_n$, as algebras with involutions.

The generic ground ring for the Brauer algebras is $R = \mathbb{Z}[\boldsymbol{\delta}]$, where $\boldsymbol{\delta}$ is an indeterminant. Let $F = \mathbb{Q}(\boldsymbol{\delta})$ denote the field of fractions of R. Write $B_n = B_n(R,\boldsymbol{\delta})$.

It is shown in [17, Section 5.2], that the pair of towers $(B_n)_{n\geqslant 0}$ and $(R\mathfrak{S}_n)_{n\geqslant 0}$ satisfy the framework axioms of Section 4. In fact, since the symmetric group algebra is a specialization of the Hecke algebra, the tower of symmetric group algebras is strongly coherent, so the pair satisfies the strong version of the framework axioms. Consequently, by Theorem 4.1, the sequence of Brauer algebras is a strongly coherent tower of cellular algebras. As for the BMW algebras, the partially ordered set Λ_n in the cell datum of B_n is the set of pairs (λ, n), with λ a Young diagram of size $k \leqslant n$ with $n-k$ even. The set of paths $\mathscr{T}((\lambda,n))$ can be identified with up–down tableaux of length n and shape λ.

We need to recall the Jucys–Murphy elements for the symmetric group algebras, which can be defined inductively by $L_1^{(0)} = 0$, $L_{j+1}^{(0)} = s_j L_j s_j + s_j$. Thus, for example, $L_5^{(0)} = (1,5) + (2,5) + (3,5) + (4,5)$. One has $L_j^{(0)} \in R\mathfrak{S}_j$, and $L_j^{(0)}$ commutes with $R\mathfrak{S}_{j-1}$. $L_1^{(0)} + \cdots + L_n^{(0)}$ is central in $R\mathfrak{S}_n$ and acts as the scalar $\alpha(\lambda) = \sum_{x\in\lambda} \kappa(x)$ on the cell module Δ^λ. Here, λ is a Young diagram of size n and for a cell x of λ, $\kappa(x)$ is the content of x, namely the column co-ordinate minus the row co-ordinate of x. In particular $\{L_j^{(0)}\colon\ j \geqslant 0\}$ is an additive JM–family in the sense of Definition 3.4.

The following analogues of Jucys-Murphy elements for the Brauer algebras were introduced by Nazarov [36]. Let $L_1 = 0$ and $L_{j+1} = s_j L_j s_j + s_j - e_j$. Observe that $\pi_n(L_j) = L_j^{(0)}$ for $1 \leqslant j \leqslant n$, where

$\pi_n : B_n \to R\mathfrak{S}_n$ is the quotient map. Evidently, $L_n \in B_n$. By [36, Proposition 2.3], L_n commutes with B_{n-1}, and for all $j \geqslant 1$,

$$(L_j + L_{j+1})e_j = e_j(L_j + L_{j+1}) = (1-\delta)e_j.$$

It now follows from Theorem 5.3 that $\{L_j\colon\ j \geqslant 0\}$ is an additive JM–family in $(B_n)_{n\geqslant 0}$, with $L_1 + \cdots + L_n$ acting by

$$\beta\big((\lambda, n)\big) := \frac{n-k}{2}(1-\delta) + \alpha(\lambda)$$

on the cell module $\Delta^{(\lambda,n)}$, if λ is a Young diagram of size k.

By Proposition 3.7, the action of the elements L_j on the basis of $\Delta^{(\lambda,n)}$ labelled by up–down tableaux is triangular:

$$L_j a^\lambda_{\mathfrak{t}} = \kappa(j, \mathfrak{t})\, a^\lambda_{\mathfrak{t}} + \sum_{\mathfrak{s} \succ \mathfrak{t}} r_{\mathfrak{s}} a^\lambda_{\mathfrak{s}}, \tag{6.2}$$

with $\kappa(j,\mathfrak{t}) = \beta(\mathfrak{t}(j)) - \beta(\mathfrak{t}(j-1))$, for some elements $r_{\mathfrak{s}} \in R$, depending on j and $\mathfrak{t}$. Moreover, if $\mathfrak{t}(j) = (\nu, j)$ and $\mathfrak{t}(j-1) = (\mu, j-1)$, then $|\nu| = |\mu| \pm 1$. If $|\nu| = |\mu| + 1$ and $\nu \setminus \mu = x$, then

$$\kappa(j,\mathfrak{t}) = \beta\big((\nu, j)\big) - \beta\big((\mu, j-1)\big) = \alpha(\nu) - \alpha(\mu) = \kappa(x).$$

If $|\nu| = |\mu| - 1$ and $\mu \setminus \nu = x$, then

$$\kappa(j,\mathfrak{t}) = \beta\big((\nu, j)\big) - \beta\big((\mu, j-1)\big) = (1-\delta) + \alpha(\nu) - \alpha(\mu) = (1-\delta) - \kappa(x).$$

This recovers Theorem 10.7 of Enyang [11].[3]

6.4. Cyclotomic BMW algebras

The *cyclotomic Birman–Wenzl–Murakami algebras* are BMW analogues of cyclotomic Hecke algebras [2,1]. The cyclotomic BMW algebras were defined by Häring–Oldenburg in [23] and have recently been studied by three groups of mathematicians: Goodman and Hauschild–Mosley [18–20,12,13], Rui, Xu, and Si [41,40], and Wilcox and Yu [45,46,44,47].

6.4.1. Definition of cyclotomic BMW algebras

Definition 6.3. Fix an integer $r \geqslant 1$. A *ground ring* S is a commutative unital ring with parameters ρ, q, δ_j $(j \geqslant 0)$, and $u_1, \ldots, u_r$, with ρ, q, and $u_1, \ldots, u_r$ invertible, and with $\rho^{-1} - \rho = (q^{-1} - q)(\delta_0 - 1)$.

Definition 6.4. Let S be a ground ring with parameters ρ, q, δ_j $(j \geqslant 0)$, and $u_1, \ldots, u_r$. The *cyclotomic BMW algebra* $W_{n,S,r}(u_1, \ldots, u_r)$ is the unital S-algebra with generators $y_1^{\pm 1}$, $g_i^{\pm 1}$ and e_i $(1 \leqslant i \leqslant n-1)$ and relations:

(1) (Inverses) $g_i g_i^{-1} = g_i^{-1} g_i = 1$ and $y_1 y_1^{-1} = y_1^{-1} y_1 = 1$.
(2) (Idempotent relation) $e_i^2 = \delta_0 e_i$.
(3) (Affine braid relations)
  (a) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ and $g_i g_j = g_j g_i$ if $|i-j| \geqslant 2$.
  (b) $y_1 g_1 y_1 g_1 = g_1 y_1 g_1 y_1$ and $y_1 g_j = g_j y_1$ if $j \geqslant 2$.
(4) (Commutation relations)

[3] The same caution about lexicographic order versus dominance order applies here, as in the BMW case.

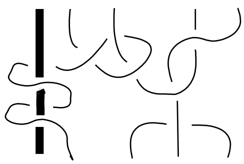
   (a) $g_i e_j = e_j g_i$ and $e_i e_j = e_j e_i$ if $|i - j| \geqslant 2$.
   (b) $y_1 e_j = e_j y_1$ if $j \geqslant 2$.
(5) (Affine tangle relations)
   (a) $e_i e_{i\pm1} e_i = e_i$.
   (b) $g_i g_{i\pm1} e_i = e_{i\pm1} e_i$ and $e_i g_{i\pm1} g_i = e_i e_{i\pm1}$.
   (c) For $j \geqslant 1$, $e_1 y_1^j e_1 = \delta_j e_1$.
(6) (Kauffman skein relation) $g_i - g_i^{-1} = (q - q^{-1})(1 - e_i)$.
(7) (Untwisting relations) $g_i e_i = e_i g_i = \rho^{-1} e_i$ and $e_i g_{i\pm1} e_i = \rho e_i$.
(8) (Unwrapping relation) $e_1 y_1 g_1 y_1 = \rho e_1 = y_1 g_1 y_1 e_1$.
(9) (Cyclotomic relation) $(y_1 - u_1)(y_1 - u_2)\cdots(y_1 - u_r) = 0$.

Thus, a cyclotomic BMW algebra is the quotient of the affine BMW algebra [18], by the cyclotomic relation $(y_1 - u_1)(y_1 - u_2)\cdots(y_1 - u_r) = 0$.

The cyclotomic BMW algebra has a unique algebra involution i fixing each of the generators.

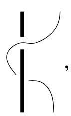
6.4.2. Geometric realization

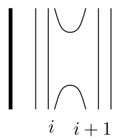
It is shown in [20] and in [44] that the cyclotomic BMW algebra has a geometric realization as the "cyclotomic Kauffman tangle (KT) algebra", assuming admissibility conditions on the ground ring (see below). The cyclotomic KT algebra is described in terms of "affine tangle diagrams", which are just ordinary tangle diagrams with a distinguished vertical strand connecting **1** and $\bar{\mathbf{1}}$, as in the following figure.



The cyclotomic KT algebra is the algebra of affine tangle diagrams, modulo regular isotopy, Kauffman skein relations, and a cyclotomic skein relation, which is a "local" version of the cyclotomic relation of Definition 6.4(9). See [19] for the precise definition.

In the geometric realization, the generators g_i, e_i, and $x_1 = \rho^{-1} y_1$ are represented by the following affine tangle diagrams:

$x_1 =$  , $g_i =$ 

, $e_i =$ 

.

In the geometric picture, the algebra involution i is given on the level of affine tangle diagrams by the map that flips an affine tangle diagram over the horizontal line $y = 1/2$.

6.4.3. Admissibility

The cyclotomic BMW algebras can be defined over an arbitrary ground ring. However, unless the parameters satisfy certain restrictions, the element e_1 is forced to be zero and the algebras collapse to a specialization of the cyclotomic Hecke algebras. In order to understand the algebras and the restrictions on the parameters, it crucial first to focus on the following "optimal" situation:

Definition 6.5. Let S be an integral ground ring with parameters ρ, q, δ_j $(j \geqslant 0)$ and $u_1, \ldots, u_r$, with $q - q^{-1} \neq 0$. One says that S is *admissible* (or that the parameters are *admissible*) if $\{e_1, y_1 e_1, \ldots, y_1^{r-1} e_1\} \subseteq W_{2,S,r}$ is linearly independent over S.

We are also going to restrict our attention to the case that the ground ring is an integral domain, and $q - q^{-1} \neq 0$.

It is shown by Wilcox and Yu in [45] that admissibility is equivalent to finitely many (explicit) polynomial conditions on the parameters. Moreover, these relations give ρ and $(q - q^{-1})\delta_j$ as Laurent polynomials in the remaining parameters $q, u_1, \dots, u_r$; see [45] and [20] for details. An alternative set of explicit conditions on the parameters was proposed by Rui and Xu [41]. It has been shown in [13] that the conditions of Rui and Xu are also equivalent to admissibility (assuming the ground ring is integral and $q - q^{-1} \neq 0$).

Finally, it has been shown that the structure of cyclotomic BMW algebras with non–admissible parameters can be derived from the admissible case [14].

6.4.4. Generic ground ring

There is a universal admissible integral ground ring R for cyclotomic BMW algebras, which is a little more complicated to describe than the generic ground rings for the other algebras we have encountered. We refer to [20], Theorem 3.19 for details. Suffice it to say that the field of fractions F of R is $\mathbb{Q}(\boldsymbol{q}, \boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$, where $\boldsymbol{q}, \boldsymbol{u}_1, \dots, \boldsymbol{u}_r$ are algebraically independent indeterminants over $\mathbb{Q}$; the remaining parameters are given by certain Laurent polynomials in $\boldsymbol{q}$, $\boldsymbol{u}_1, \dots, \boldsymbol{u}_r$, and $(\boldsymbol{q} - \boldsymbol{q}^{-1})^{-1}$, and R is the subring of F generated by all the parameters. Any other admissible integral ground ring S is a module over R, and $W_{n,S,r} \cong W_{n,R,r} \otimes_R S$. We will write W_n for $W_{n,R,r}$.

6.4.5. Cyclotomic BMW algebras and cyclotomic Hecke algebras

We recall the definition of the affine and cyclotomic Hecke algebras, see [1].

Definition 6.6. Let S be a commutative unital ring with an invertible element q. The *affine Hecke algebra* $\widehat{H}_{n,S}(q)$ over S is the S-algebra with generators $T_0, T_1, \dots, T_{n-1}$, with relations:

(1) The generators T_i are invertible, satisfy the braid relations, and the Hecke relations $(T_i - q)(T_i + q) = 0$.
(2) The generator T_0 is invertible, $T_0T_1T_0T_1 = T_1T_0T_1T_0$ and T_0 commutes with T_j for $j \geqslant 2$.

Let $u_1, \dots, u_r$ be additional elements in S. The *cyclotomic Hecke algebra* $H_{n,S,r}(q; u_1, \dots, u_r)$ is the quotient of the affine Hecke algebra $\widehat{H}_{n,S}(q)$ by the polynomial relation $(T_0 - u_1) \cdots (T_0 - u_r) = 0$.

We remark that since the generator T_0 can be rescaled by an arbitrary invertible element of S, only the ratios of the parameters u_i have invariant significance in the definition of the cyclotomic Hecke algebra. The cyclotomic Hecke algebra has a unique algebra involution i leaving each generator invariant. By [2], the cyclotomic Hecke algebras $H_{n,S,r}$ are free S-modules of rank $r^n n!$ and $H_{n,S,r}$ imbeds in $H_{n+1,S,r}$.

The cyclotomic Hecke algebras were shown to be cellular algebras in [21]. In [9], a cellular basis was given generalizing the Murphy basis of the ordinary Hecke algebra. The partially ordered set $\Lambda_n^{(0)}$ in the cell datum for $H_{n,S,r} = H_{n,S,r}(q; u_1, \dots, u_r)$ is the set of r-tuples of Young diagrams with total size n, ordered by dominance. For each $\boldsymbol{\lambda} \in \Lambda_n^{(0)}$, the index set $\mathscr{T}(\boldsymbol{\lambda})$ in the cell datum is the set of standard tableaux of shape $\boldsymbol{\lambda}$; this has the usual meaning: fillings with the numbers $1, \dots, n$, so that the numbers increase in each row and column (separately in each component Young diagram). The cyclotomic Hecke algebras are generically split semisimple; in the semisimple case, the branching diagram has vertices at level n labelled by all r-tuples of Young diagrams of total size n, and $\boldsymbol{\lambda} \nearrow \boldsymbol{\mu}$ if $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by adding one box in one component of $\boldsymbol{\lambda}$. Standard tableaux of shape $\boldsymbol{\lambda}$ can be identified with paths on the generic branching diagram from $\emptyset$ (the r-tuple of empty Young diagrams) to $\boldsymbol{\lambda}$.

By results of Ariki and Mathas [3, Proposition 1.9] and Mathas [30], the sequence of cyclotomic Hecke algebras $(H_{n,S,r})_{n \geqslant 0}$ is a strongly coherent tower of cellular algebras.

Let J be the ideal in $W_n = W_{n,R,r}$ generated by e_{n-1}. It is not hard to show that the quotient W_n/J is isomorphic to $H_{n,R,r}(\boldsymbol{q}^2; \boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$. If π_n denotes the quotient map $\pi_n : W_n \to W_n/J$, take

$T_j = \pi_n(q\, g_i)$ for $j \geqslant 1$, and $T_0 = \pi_n(y_1)$ to obtain an isomorphism with the cyclotomic Hecke algebra as presented above. We will write H_n for $H_{n,R,r}(\boldsymbol{q}^2; \boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$.

It is shown in [17, Section 5.5], that the pair of towers of algebras $(W_n)_{n\geqslant 0}$ and $(H_n)_{n\geqslant 0}$ satisfies the framework axioms of Section 4. Since the sequence of Hecke algebras is strongly coherent, the pairs satisfies the strong version of the framework axioms. Therefore, it follows from Theorem 4.1 that the sequence of cyclotomic BMW algebras is a strongly coherent tower of cellular algebras.

The partially ordered set Λ_n in the cell datum of W_n is the set of pairs $(\boldsymbol{\lambda}, n)$, with $\boldsymbol{\lambda}$ an r-tuple of Young diagrams of total size $k \leqslant n$ with $n - k$ even. The set of paths $\mathscr{T}((\boldsymbol{\lambda}, n))$ can be identified with up–down tableaux of length n and shape $\boldsymbol{\lambda}$, that is sequences of r-tuples of Young diagrams in which each successive r-tuple is obtained from the previous one by either adding or removing one box from one component Young diagram.

6.4.6. JM elements for cyclotomic BMW and Hecke algebras

In the cyclotomic Hecke algebra $H_{n,S,r} = H_{n,S,r}(q; u_1, \dots, u_r)$, define $L_1^{(0)} = T_0$ and $L_{j+1}^{(0)} = q^{-1} T_j L_j^{(0)} T_j$ for $j \geqslant 1$. Then $L_n^{(0)} \in H_{n,S,r}$, $L_n^{(0)}$ is invariant under the involution on $H_{n,S,r}$, and $L_n^{(0)}$ commutes with $H_{n-1,S,r}$. The product $L_1^{(0)} \cdots L_n^{(0)}$ is central in $H_{n,S,r}$.

For an r-tuple of Young diagrams $\boldsymbol{\lambda}$ of total size n and a cell $x \in \boldsymbol{\lambda}$, the multiplicative content of the cell is

$$\kappa(x) = u_j q^{b-a}$$

if x is in row a and column b of the j-th component of $\boldsymbol{\lambda}$. For a standard tableau $\mathfrak{t}$ of shape $\boldsymbol{\lambda}$, and $1 \leqslant j \leqslant n$, let $\kappa(j, \mathfrak{t}) = \kappa(x)$, where x is the cell occupied by j in $\mathfrak{t}$. Let $\{a_\mathfrak{t}^{\boldsymbol{\lambda}}\}$ be the Murphy type basis of the cell module $\Delta^{\boldsymbol{\lambda}}$ indexed by standard tableaux of shape $\boldsymbol{\lambda}$. Then $L_j^{(0)}$ acts by

$$L_j^{(0)} a_\mathfrak{t}^\lambda = \kappa(j, \mathfrak{t}) a_\mathfrak{t}^\lambda + \sum_{\mathfrak{s} \rhd \mathfrak{t}} r_\mathfrak{s} a_\mathfrak{s}^\lambda, \tag{6.3}$$

where the sum is over standard tableaux $\mathfrak{s}$ greater than $\mathfrak{t}$ in dominance order (hence in lexicographic order). These results are from [24, Section 3]. It follows that the product $L_1^{(0)} \cdots L_n^{(0)}$ acts as the scalar $\alpha(\boldsymbol{\lambda}) = \prod_{x \in \boldsymbol{\lambda}} \kappa(x)$ on the cell module $\Delta^{\boldsymbol{\lambda}}$. Thus $\{L_n^{(0)}\colon\ n \geqslant 0\}$ is a multiplicative JM–family in the strongly coherent tower of cellular algebras $(H_{n,S,r})_{n\geqslant 0}$.

Define elements L_j in the cyclotomic BMW algebras $W_n = W_{n,R,r}(\boldsymbol{q}; \boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$ over the generic integral admissible ground ring R by $L_1 = y_1$, $L_{j+1} = g_j L_j g_j$ for $j \geqslant 1$. These are the same as the elements y_j in [20]. We have $L_n \in W_n$ and L_n commutes with W_{n-1}. One can verify that $L_j L_{j+1} e_j = e_j L_j L_{j+1} = e_j$. The computations can be done at the level of the affine BMW algebras,using the algebraic relations or using affine tangle diagrams.

We have $\pi_n(L_1) = T_0 = L_1^{(0)}$, and $\pi_n(L_{j+1}) = \boldsymbol{q}^{-2} T_j \pi_n(L_j) T_j$. Hence $\pi_n(L_j)$ satisfy the recursion for $L_j^{(0)}$ in $H_n = H_{n,R,r}(\boldsymbol{q}^2; \boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$.

It now follows from Theorem 5.1 that $\{L_j\colon\ j \geqslant 0\}$ is a multiplicative JM–family in $(W_n)_{n\geqslant 0}$, with the product $L_1 \cdots L_n$ acting by

$$\beta\big((\boldsymbol{\lambda}, n)\big) := \alpha(\boldsymbol{\lambda})$$

on the cell module $\Delta^{(\boldsymbol{\lambda},n)}$, if $\boldsymbol{\lambda}$ is an r-tuple of Young diagrams of total size k. By Proposition 3.6, the action of the elements L_j on the basis of $\Delta^{(\boldsymbol{\lambda},n)}$ labelled by up–down tableaux is triangular:

$$L_j a_\mathfrak{t}^\lambda = \kappa(j, \mathfrak{t})\, a_\mathfrak{t}^{(\boldsymbol{\lambda},n)} + \sum_{\mathfrak{s} \succ \mathfrak{t}} r_\mathfrak{s} a_\mathfrak{s}^{(\boldsymbol{\lambda},n)}, \tag{6.4}$$

with $\kappa(j,\mathfrak{t}) = \frac{\beta(\mathfrak{t}(j))}{\beta(\mathfrak{t}(j-1))}$, for some elements $r_{\mathfrak{s}} \in R$, depending on j and $\mathfrak{t}$. Moreover, if $\mathfrak{t}(j) = (\boldsymbol{\nu}, j)$ and $\mathfrak{t}(j-1) = (\boldsymbol{\mu}, j-1)$, then $|\boldsymbol{\nu}| = |\boldsymbol{\mu}| \pm 1$. If $|\boldsymbol{\nu}| = |\boldsymbol{\mu}| + 1$ and $\boldsymbol{\nu} \setminus \boldsymbol{\mu} = x$, where x is the cell in row a and column b of the ℓ-th component of $\boldsymbol{\nu}$, then

$$\kappa(j,\mathfrak{t}) = \frac{\alpha(\boldsymbol{\nu})}{\alpha(\boldsymbol{\mu})} = \kappa(x) = \boldsymbol{u}_\ell \boldsymbol{q}^{2(b-a)}.$$

If $|\boldsymbol{\nu}| = |\boldsymbol{\mu}| - 1$ and $\boldsymbol{\mu} \setminus \boldsymbol{\nu} = x$, then

$$\kappa(j,\mathfrak{t}) = \frac{\alpha(\boldsymbol{\nu})}{\alpha(\boldsymbol{\mu})} = \kappa(x)^{-1} = \boldsymbol{u}_\ell^{-1} \boldsymbol{q}^{-2(b-a)}.$$

This recovers Theorem 3.17 of Rui and Si [40].

6.5. *Degenerate cyclotomic BMW algebras (cyclotomic Nazarov–Wenzl algebras)*

Degenerate affine BMW algebras were introduced by Nazarov [36] under the name *affine Wenzl algebras*. The cyclotomic quotients of these algebras were introduced by Ariki, Mathas, and Rui in [4] and studied further by Rui and Si in [39], under the name *cyclotomic Nazarov–Wenzl algebras*. We propose to refer to these algebras as degenerate affine (resp. degenerate cyclotomic) BMW algebras instead, to bring the terminology in line with that used for degenerate affine and cyclotomic Hecke algebras.

6.5.1. *Definition of the degenerate cyclotomic BMW algebras*

Fix a positive integer n and a commutative ring S with multiplicative identity. Let $\Omega = \{\omega_a\colon a \geqslant 0\}$ be a sequence of elements of S.

Definition 6.7. (See Nazarov [36]; Ariki, Mathas, Rui [4].) The *degenerate affine BMW algebra* $W_{n,S}^{\mathrm{aff}} = W_{n,S}^{\mathrm{aff}}(\Omega)$ is the unital associative R-algebra with generators $\{s_i, e_i, x_j\colon 1 \leqslant i < n \text{ and } 1 \leqslant j \leqslant n\}$ and relations:

(1) (Involutions) $s_i^2 = 1$, for $1 \leqslant i < n$.
(2) (Affine braid relations)
  (a) $s_i s_j = s_j s_i$ if $|i-j| > 1$,
  (b) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, for $1 \leqslant i < n-1$,
  (c) $s_i x_j = x_j s_i$ if $j \neq i, i+1$.
(3) (Idempotent relations) $e_i^2 = \omega_0 e_i$, for $1 \leqslant i < n$.
(4) (Commutation relations)
  (a) $s_i e_j = e_j s_i$, if $|i-j| > 1$,
  (b) $e_i e_j = e_j e_i$, if $|i-j| > 1$,
  (c) $e_i x_j = x_j e_i$, if $j \neq i, i+1$,
  (d) $x_i x_j = x_j x_i$, for $1 \leqslant i, j \leqslant n$.
(5) (Skein relations) $s_i x_i - x_{i+1} s_i = e_i - 1$, and $x_i s_i - s_i x_{i+1} = e_i - 1$, for $1 \leqslant i < n$.
(6) (Unwrapping relations) $e_1 x_1^a e_1 = \omega_a e_1$, for $a > 0$.
(7) (Tangle relations)
  (a) $e_i s_i = e_i = s_i e_i$, for $1 \leqslant i \leqslant n-1$,
  (b) $s_i e_{i+1} e_i = s_{i+1} e_i$, and $e_i e_{i+1} s_i = e_i s_{i+1}$, for $1 \leqslant i \leqslant n-2$,
  (c) $e_{i+1} e_i s_{i+1} = e_{i+1} s_i$, and $s_{i+1} e_i e_{i+1} = s_i e_{i+1}$, for $1 \leqslant i \leqslant n-2$.
(8) (Untwisting relations) $e_{i+1} e_i e_{i+1} = e_{i+1}$, and $e_i e_{i+1} e_i = e_i$, for $1 \leqslant i \leqslant n-2$.
(9) (Anti–symmetry relations) $e_i(x_i + x_{i+1}) = 0$, and $(x_i + x_{i+1}) e_i = 0$, for $1 \leqslant i < n$.

Definition 6.8. (See Ariki, Mathas, Rui [4].) Fix an integer $r \geqslant 1$ and elements $u_1, \ldots, u_r$ in S. The *degenerate cyclotomic BMW algebra* $W_{n,S,r} = W_{n,S,r}(u_1, \ldots, u_r)$ is the quotient of the degenerate affine BMW algebra $W_{n,S}^{\mathrm{aff}}(\Omega)$ by the relation $(x_1 - u_1) \cdots (x_1 - u_r) = 0$.

Due to the symmetry of the relations, $W_{n,S}^{\mathrm{aff}}$ has a unique S-linear algebra involution i fixing each of the generators. The involution passes to cyclotomic quotients.

6.5.2. Admissibility

As for the cyclotomic BMW algebras, to understand the degenerate cyclotomic BMW algebras it is crucial to first understand the "optimal" case, namely the case that $W_{r,2}e_1$ is free of rank r. We say that the parameters are *admissible* if this condition holds.

It has been shown in [15] and [14] that admissibility is equivalent to certain polynomial conditions on the parameters that were proposed by Ariki, Mathas and Rui [4], called u-admissibility, and also to an analogue of the admissibility condition of Wilcox and Yu [45] for the cyclotomic BMW algebras. It was shown in [14] that the structure of degenerate cyclotomic BMW algebras with non–admissible parameters can be derived from the admissible case.

In an admissible ground ring, the parameters ω_a are given by specific polynomial functions of $u_1, \dots, u_r$. There is a generic admissible ground ring $R = \mathbb{Z}[\boldsymbol{u}_1, \dots, \boldsymbol{u}_r]$, where the $\boldsymbol{u}_j$ are algebraically independent indeterminants.

Remark 6.9. In previous work on the degenerate cyclotomic BMW algebras, it was always assumed that 2 is invertible in the ground ring. However, it was shown in [14] that this assumption could be eliminated. In particular, there is no need to adjoin $1/2$ to the generic admissible ground ring R.

6.5.3. Some basic properties of degenerate cyclotomic BMW algebras

We establish some elementary properties of degenerate cyclotomic BMW algebras. Several of the properties can be shown for degenerate affine BMW algebras instead. Let S be any appropriate ground ring for the degenerate affine or cyclotomic BMW algebras, and write W_n^{aff} for $W_{n,S}^{\mathrm{aff}}$ and W_n for $W_{n,S,r}$.

Lemma 6.10. *(See [4, Lemma 2.3].) In the affine BMW algebra W_n^{aff}, for $1 \leqslant i < n$ and $a \geqslant 1$, one has*

$$s_i x_i^a = x_{i+1}^a s_i + \sum_{b=1}^{a} x_{i+1}^{b-1}(e_i - 1)x_i^{a-b}. \tag{6.5}$$

Lemma 6.11. *For $n \geqslant 1$, W_n^{aff} is contained in the span of W_{n-1}^{aff} and of elements of the form $a\chi_n b$, where $a, b \in W_{n-1}^{\mathrm{aff}}$ and $\chi_n \in \{e_{n-1}, s_{n-1}, x_n^\alpha \colon \alpha \geqslant 1\}$.*

Proof. We do this by induction on n. The base case $n = 1$ is clear since $W_{1,S,r}$ is generated by x_1. Suppose now that $n > 1$ and make the appropriate induction hypothesis. We have to show that a word in the generators having at least two occurrences of e_{n-1}, s_{n-1}, or a power of x_n can be rewritten as a linear combination of words with fewer occurrences.

Consider a subword $\chi_n y \chi_n'$ with $\chi_n, \chi_n' \in \{e_{n-1}, s_{n-1}, x_n^\alpha \colon \alpha \geqslant 1\}$ and $y \in W_{n-1}^{\mathrm{aff}}$. If one of χ_n, χ_n' is a power of x_n, then it commutes with y; say without loss of generality $\chi_n = x_n^\alpha$. Then $\chi_n y \chi_n' = y x_n^\alpha \chi_n'$. Now consider the cases $\chi_n' = e_{n-1}$, $\chi_n' = s_{n-1}$, and $\chi_n' = x_n^\beta$. We have $y x_n^\alpha e_{n-1} = y(-1)^\alpha x_{n-1}^\alpha e_{n-1}$ and $y x_n^\alpha x_n^\beta = y x_n^{\alpha+\beta}$. Finally $y x_n^\alpha s_{n-1}$ can be dealt with using Lemma 6.10.

Suppose both of χ_n, χ_n' are in $\{e_{n-1}, s_{n-1}\}$. If $y \in W_{n-2,S}^{\mathrm{aff}}$, then $\chi_n y \chi_n' = y \chi_n \chi_n'$. But the product of any two of e_{n-1}, s_{n-1} is either 1 or a multiple of e_{n-1}. If $y \notin W_{n-2,S}^{\mathrm{aff}}$, then we can assume, using the induction hypothesis, that $y = y' \chi y''$, where $y', y'' \in W_{n-2,S}^{\mathrm{aff}}$, and χ is one of e_{n-2}, s_{n-2}, or a power of x_{n-1}. Since χ_n, χ_n' commute with y', y'', we are reduced to considering $\chi_n \chi \chi_n'$. Moreover, if χ is not a power of x_{n-1}, then essentially we are dealing with a computation in the Brauer algebra, which was done in [43, Proposition 2.1]. If one of χ_n, χ_n' is s_{n-1}, then the computation can be done using Lemma 6.10. Thus the only interesting case is $e_{n-1} x_{n-1}^\alpha e_{n-1}$. But by Lemma 4.15 in [4], $e_{n-1} x_{n-1}^\alpha e_{n-1} = \omega e_{n-1}$, where ω is in the center of W_{n-2}^{aff}. $\square$

Lemma 6.12.

(1) *For* $n \geqslant 3$, $e_{n-1} W^{\mathrm{aff}}_{n-1} e_{n-1} = W^{\mathrm{aff}}_{n-2} e_{n-1}$.
(2) $e_1 W^{\mathrm{aff}}_1 e_1 = \langle \omega_j\colon\ j \geqslant 0\rangle e_1$, *where* $\langle \omega_j\colon\ j \geqslant 0\rangle$ *denotes the ideal in* S *generated by all* ω_j.
(3) *For* $n \geqslant 2$, e_{n-1} *commutes with* W^{aff}_{n-2}.

Proof. First we have to show that if $y \in W^{\mathrm{aff}}_{n-1,S}$, then $e_{n-1} y e_{n-1} \in W^{\mathrm{aff}}_{n-2,S} e_{n-1}$. Using Lemma 6.11, we can suppose that either $y \in W^{\mathrm{aff}}_{n-2,S}$ or $y = y'\chi_{n-1} y''$, with $y', y'' \in W^{\mathrm{aff}}_{n-2,S}$, and $\chi_{n-1} \in \{e_{n-2}, s_{n-2}, x^{\alpha}_{n-1}\colon\ \alpha \geqslant 1\}$. For χ_{n-1} a power of x_{n-1}, apply Lemma 4.15 from [4]. In all other cases, the result follows from the defining relations of W^{aff}_n. Thus we have $e_{n-1} W^{\mathrm{aff}}_{n-1,S} e_{n-1} \subseteq W^{\mathrm{aff}}_{n-2,S} e_{n-1}$. For the opposite inclusion, let $x \in W^{\mathrm{aff}}_{n-2,S}$. Then $x e_{n-1} = e_{n-1} x e_{n-2} e_{n-1} \in e_{n-1} W^{\mathrm{aff}}_{n-1,S} e_{n-1}$. Points (2) and (3) are obvious. □

Lemma 6.13. *For* $n \geqslant 2$, $W_n e_{n-1} = W_{n-1} e_{n-1}$.

Proof. The proof is similar to the proof of Lemma 5.3 in [17]. Using Lemma 6.11, if $x \in W_n$ and $x \notin W_{n-1}$, then we can assume that $x = y'\chi_n y''$, with $y', y'' \in W_{n-1}$, and $\chi_n \in \{e_{n-1}, s_{n-1}, x^{\alpha}_n\colon\ \alpha \geqslant 1\}$. Likewise, we can assume that either $y'' \in W_{n-2}$ or that $y'' = z'\chi_{n-1} z''$ with $z', z'' \in W_{n-2}$ and $\chi_{n-1} \in \{e_{n-2}, s_{n-2}, x^{\beta}_{n-1}\colon\ \beta \geqslant 1\}$. The problem reduces to showing that $\chi_n e_{n-1}$ and $\chi_n \chi_{n-1} e_{n-1}$ lie in $W_{n-1} e_{n-1}$ for the various choices of χ_n, χ_{n-1}. Most of the cases follow directly from the defining relations, while $s_{n-1} x^{\beta}_{n-1} e_{n-1}$ must be reduced using Lemma 6.10, and $e_{n-1} x^{\beta}_{n-1} e_{n-1}$ requires the use of Lemma 4.15 in [4]. □

Lemma 6.14. *Let* R *be the universal admissible ring. For* $n \geqslant 1$, *the map* $x \mapsto x e_n$ *is injective from* $W_{n,R,r}$ *to* $W_{n,R,r} e_n$.

Proof. Note that $e_{n+1}(x e_n) e_{n+1} = x e_{n+1}$, so it suffices to show that $x \mapsto x e_{n+1}$ is injective. It follows from Proposition 2.15 and Theorem A in [4] that $W_{n,R,r}$ has a basis of "r-regular monomials". The map $x \mapsto x e_{n+1}$ takes the basis elements of $W_{n,R,r}$ to distinct basis elements of $W_{n+2,R,r}$, so is injective. □

6.5.4. Degenerate cyclotomic Hecke algebras

Definition 6.15. Let S be a commutative ring with identity. The *degenerate affine Hecke algebra* $\widehat{H}_{n,S}$ is the unital associative S-algebra with generators

$$\{s_i, x_j\colon\ 1 \leqslant i < n \text{ and } 1 \leqslant j \leqslant n\},$$

and relations:

(1) (Involutions) $s_i^2 = 1$, for $1 \leqslant i < n$.
(2) (Affine braid relations)
  (a) $s_i s_j = s_j s_i$ if $|i - j| > 1$,
  (b) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, for $1 \leqslant i < n - 1$.
(3) (Commutation relations) $x_i x_j = x_j x_i$, for $1 \leqslant i, j \leqslant n$ and $s_i x_j = x_j s_i$ if $j \neq i, i+1$.
(4) (Skein relations) $s_i x_i - x_{i+1} s_i = -1$, and $x_i s_i - s_i x_{i+1} = -1$, for $1 \leqslant i < n$.

Let $u_1, \dots, u_r$ be elements of S. The *degenerate cyclotomic Hecke algebra* $H_{n,S,r}(u_1, \dots, u_r)$ is the quotient of $\widehat{H}_n$ by the relation $(x_1 - u_1)(x_2 - u_2) \cdots (x_1 - u_r) = 0$.

The degenerate cyclotomic Hecke algebra is a free S-module of rank $r^n n!$, and $H_{n,S,r}(u_1, \dots, u_r) \hookrightarrow H_{n+1,S,r}(u_1, \dots, u_r)$ for all n [26]. $H_{n,S,r}(u_1, \dots, u_r)$ has a unique algebra involution i fixing the generators; the involutions on the tower of degenerate cyclotomic Hecke algebras are consistent.

It is observed in [4, Section 6], that the Murphy type cellular basis of the cyclotomic Hecke algebra from [9] can be easily adapted to the degenerate cyclotomic Hecke algebras. Recall that the partially ordered set $\Lambda_n^{(0)}$ in the cell datum for $H_{n,S,r} = H_{n,S,r}(q; u_1, \dots, u_r)$ is the set of r-tuples of Young diagrams with total size n, ordered by dominance. For each $\boldsymbol{\lambda} \in \Lambda_n^{(0)}$, the index set $\mathscr{T}(\boldsymbol{\lambda})$ in the cell datum is the set of standard tableaux of shape $\boldsymbol{\lambda}$. The proof of strong coherence of the sequence of cyclotomic Hecke algebras in [3, Proposition 1.9], and [30] also applies to the degenerate cyclotomic Hecke algebras.

Let J be the ideal in the degenerate cyclotomic BMW algebra $W_{n,S,r}(u_1, \dots, u_r)$ generated by e_{n-1}. It is straightforward to show that $W_{n,S,r}(u_1, \dots, u_r)/J \cong H_{n,S,r}(u_1, \dots, u_r)$, as algebras with involution.

6.5.5. Verification of the framework axioms for the degenerate cyclotomic BMW algebras

Let R be the generic admissible integral ground ring, $R = \mathbb{Z}[\boldsymbol{u}_1, \dots, \boldsymbol{u}_r]$. In this section, we write W_n for $W_{n,R,r}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$ and H_n for $H_{n,R,r}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$. The field of fractions of R is $F = \mathbb{Q}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$.

Proposition 6.16. *The two sequences of algebras $(W_n)_{n\geqslant 0}$ and $(H_n)_{n\geqslant 0}$ satisfy the strong framework axioms of Section* 4.

Proof. As observed above, $(H_n)_{n\geqslant 0}$ is a strongly coherent tower of cellular algebras, so the strong version of axiom (1) holds. Axioms (2) and (3) are evident. W_n^F is semisimple by [4, Theorem 5.3]. Thus axiom (4) holds.

We observed above that $W_n/W_n e_{n-1} W_n \cong H_n$, as algebras with involutions. Thus axiom (5) holds. Axiom (6) follows from Lemma 6.12 and axiom (7) from Lemmas 6.13 and 6.14. Finally, axiom (8) holds because of the relation $e_{n-1} e_n e_{n-1} = e_{n-1}$. □

Corollary 6.17. *Let S be any admissible ground ring. The sequence of degenerate cyclotomic BMW algebras $(W_{n,S,r})_{n\geqslant 0}$ is a strongly coherent tower of cellular algebras. $W_{n,S,r}$ has cell modules indexed by all pairs $(\boldsymbol{\lambda}, n)$, where $\boldsymbol{\lambda}$ is an r-tuple of Young diagrams of total size $n, n-2, n-4, \dots$. The cell module labeled by $(\boldsymbol{\lambda}, n)$ has a basis labeled by up–down tableaux of length n and shape $\boldsymbol{\lambda}$.*

Cellularity of degenerate cyclotomic BMW algebras was proved in [4, Section 7]. The cell filtration for restricted modules was proved in [39, Theorem 4.15]. The proof of both results here is shorter.

6.5.6. JM elements for degenerate cyclotomic BMW and Hecke algebras

The analogue of Jucys–Murphy elements for the degenerate cyclotomic Hecke algebras $H_{n,S,r} = H_{n,S,r}(u_1, \dots, u_r)$ are just the generators x_k. In order to eventually distinguish between JM elements in the degenerate cyclotomic Hecke algebras and the degenerate cyclotomic BMW algebras, let us introduce the slightly superfluous notation $L_j^{(0)} = x_j$. It follows from the defining relations that $L_1^{(0)} + \cdots + L_n^{(0)}$ is central in $H_{n,S,r}$.

For an r-tuple of Young diagrams $\boldsymbol{\lambda}$ of total size n and a cell $x \in \boldsymbol{\lambda}$, the additive content of the cell is

$$\kappa(x) = u_j + b - a$$

if x is in row a and column b of the j-th component of $\boldsymbol{\lambda}$. For a standard tableau $\mathfrak{t}$ of shape $\boldsymbol{\lambda}$, and $1 \leqslant j \leqslant n$, let $\kappa(j, \mathfrak{t}) = \kappa(x)$, where x is the cell occupied by j in $\mathfrak{t}$. Let $\{a_{\mathfrak{t}}^{\boldsymbol{\lambda}}\}$ be the Murphy type basis of the cell module $\Delta^{\boldsymbol{\lambda}}$ indexed by standard tableaux of shape $\boldsymbol{\lambda}$. Then $L_j^{(0)}$ acts by

$$L_j^{(0)} a_{\mathfrak{t}}^{\boldsymbol{\lambda}} = \kappa(j, \mathfrak{t}) a_{\mathfrak{t}}^{\boldsymbol{\lambda}} + \sum_{\mathfrak{s} \rhd \mathfrak{t}} r_{\mathfrak{s}} a_{\mathfrak{s}}^{\boldsymbol{\lambda}}, \tag{6.6}$$

where the sum is over standard tableaux $\mathfrak{s}$ greater than $\mathfrak{t}$ in dominance order (hence in lexicographic order). It is noted in [4, Lemma 6.6], that this follows by the argument in [24, Section 3]. It follows that the sum $L_1^{(0)} + \cdots + L_n^{(0)}$ acts as the scalar $\alpha(\boldsymbol{\lambda}) = \sum_{x \in \boldsymbol{\lambda}} \kappa(x)$ on the cell module $\Delta^{\boldsymbol{\lambda}}$. Thus $\{L_n^{(0)}\colon\ n \geqslant 0\}$ is an additive JM–family in the strongly coherent tower of cellular algebras $(H_{n,S,r})_{n \geqslant 0}$.

In the degenerate cyclotomic BMW algebras $W_n = W_{n,R,r}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$ over the generic integral admissible ground ring R, we define $L_j = x_j$ for $1 \leqslant j \leqslant n$. We have $L_n \in W_n$ and L_n commutes with W_{n-1}. We have $(L_j + L_{j+1})e_j = e_j(L_j + L_{j+1}) = 0$ by the defining relations. It is clear that $\pi_n(L_j) = L_j^{(0)}$, where $\pi_n : W_n \to H_n = H_{n,R,r}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$ is the quotient map.

It now follows from Theorem 5.3 that $\{L_j\colon\ j \geqslant 0\}$ is an additive JM–family in $(W_n)_{n \geqslant 0}$, with the sum $L_1 + \cdots + L_n$ acting by

$$\beta\big((\boldsymbol{\lambda}, n)\big) := \alpha(\boldsymbol{\lambda})$$

on the cell module $\Delta^{(\boldsymbol{\lambda},n)}$, if $\boldsymbol{\lambda}$ is an r-tuple of Young diagrams of total size k. By Proposition 3.7, the action of the elements L_j on the basis of $\Delta^{(\boldsymbol{\lambda},n)}$ labelled by up–down tableaux is triangular:

$$L_j a_{\mathfrak{t}}^{(\boldsymbol{\lambda},n)} = \kappa(j, \mathfrak{t}) a_{\mathfrak{t}}^{(\boldsymbol{\lambda},n)} + \sum_{\mathfrak{s} \succ \mathfrak{t}} r_{\mathfrak{s}} a_{\mathfrak{s}}^{(\boldsymbol{\lambda},n)}, \tag{6.7}$$

with $\kappa(j, \mathfrak{t}) = \beta(\mathfrak{t}(j)) - \beta(\mathfrak{t}(j-1))$, for some elements $r_{\mathfrak{s}} \in R$, depending on j and $\mathfrak{t}$. Moreover, if $\mathfrak{t}(j) = (\boldsymbol{\nu}, j)$ and $\mathfrak{t}(j-1) = (\boldsymbol{\mu}, j-1)$, then $|\boldsymbol{\nu}| = |\boldsymbol{\mu}| \pm 1$. If $|\boldsymbol{\nu}| = |\boldsymbol{\mu}| + 1$ and $\boldsymbol{\nu} \setminus \boldsymbol{\mu} = x$, where x is the cell in row a and column b of the ℓ-th component of $\boldsymbol{\nu}$, then

$$\kappa(j, \mathfrak{t}) = \alpha(\boldsymbol{\nu}) - \alpha(\boldsymbol{\mu}) = \kappa(x) = \boldsymbol{u}_\ell + (b - a).$$

If $|\boldsymbol{\nu}| = |\boldsymbol{\mu}| - 1$ and $\boldsymbol{\mu} \setminus \boldsymbol{\nu} = x$, then

$$\kappa(j, \mathfrak{t}) = \alpha(\boldsymbol{\nu}) - \alpha(\boldsymbol{\mu}) = -\kappa(x)^{-1} = -\boldsymbol{u}_\ell - (b - a).$$

This recovers Theorem 5.12 of Rui and Si [39].

6.6. The Jones–Temperley–Lieb algebras

Let S be a commutative ring with identity, with distinguished element δ. The Jones–Temperley–Lieb algebra $A_n(S, \delta)$ is the unital S-algebra with generators $e_1, \dots, e_{n-1}$ satisfying the relation:

(1) $e_j^2 = \delta e_j$,
(2) $e_j e_{j\pm1} e_j = e_j$,
(3) $e_j e_k = e_k e_j$, if $|j - k| \geqslant 2$,

whenever all indices involved are in the range from 1 to $n - 1$.

The Jones–Temperley–Lieb algebra can also be realized as the subalgebra of the Brauer algebra, with parameter δ, spanned by Brauer diagrams *without crossings*. If J_n denotes the ideal in $A_n(S, \delta)$ generated by e_{n-1} (or, equivalently, by any e_j), then $A_n(S, \delta)/J_n \cong S$.

The generic ground ring for the Jones–Temperley–Lieb algebras is $R_0 = \mathbb{Z}[\boldsymbol{\delta}]$, where $\boldsymbol{\delta}$ is an indeterminant over $\mathbb{Z}$. It is shown in [17, Section 5.3], that the pair of towers of algebras $(A_n(R_0, \boldsymbol{\delta}))_{n \geqslant 0}$ and $(R_0)_{n \geqslant 0}$ satisfies the framework axioms of Section 4. It follows from Theorem 4.1 that the sequence of Jones–Temperley–Lieb algebras is a strongly coherent tower of cellular algebras. Moreover, the partially ordered set in the cell datum for A_n is naturally realized as

$$\big\{(k, n)\colon\ k \leqslant n \text{ and } n - k \text{ even}\big\}, \quad \text{with}$$
$$(k, n) \leqslant \big(k', n\big) \quad \Leftrightarrow \quad k \geqslant k'. \tag{6.8}$$

Proposition 6.18. *Fix S and δ and write A_n for $A_n(S,\delta)$. For $n \geqslant 0$ and $k \leqslant n$, $A_n^{(k,n)}$ is the ideal in A_n generated by $e_{k+1}e_{k+3}\cdots e_{n-1}$.*

Proof. For $k = n$, we interpret $e_{k+1}e_{k+3}\cdots e_{n-1}$ as 1, so the statement is trivial. In particular, the statement is true for $n = 0, 1$. Let $n \geqslant 2$ and suppose the statement is true for $A_{n'}$ with $n' < n$. By the proof of Theorem 3.2 in [17], in particular Proposition 4.7, for $k < n$ we have $A_n^{(k,n)} = A_n e_{n-1} A_{n-2}^{(k,n-2)} A_n$. Applying the induction hypothesis,

$$\begin{aligned} A_n^{(k,n)} &= A_n e_{n-1} A_{n-2}^{(k,n-2)} A_n \\ &= A_n e_{n-1} A_{n-2}(e_{k+1}e_{k+3}\cdots e_{n-3}) A_{n-2} A_n \\ &= A_n(e_{k+1}e_{k+3}\cdots e_{n-3}e_{n-1}) A_n. \qquad \square \end{aligned}$$

Let R_0 be as above, and let $\boldsymbol{q}^{1/2}$ be a solution to $\boldsymbol{q}^{1/2} + \boldsymbol{q}^{-1/2} = \boldsymbol{\delta}$ in an extension of R_0. Define $R = \mathbb{Z}[\boldsymbol{q}^{\pm 1/2}]$ and let $F = \mathbb{Q}(\boldsymbol{q}^{\pm 1/2})$. Let H_n denote the Hecke algebra $H_{n,R}(\boldsymbol{q})$. Then $\varphi : T_j \mapsto \boldsymbol{q}^{1/2} e_j - 1$ defines a homomorphism from $H_{n,R}(\boldsymbol{q})$ to $A_n(R, \boldsymbol{\delta})$, respecting the algebra involutions. The kernel of φ is the ideal in H_n generated by

$$\xi = T_1T_2T_1 + T_1T_2 + T_2T_1 + T_1 + T_2 + 1, \tag{6.9}$$

see [16, Corollary 2.11.2].

Recall from Example 2.13 that the Hecke algebra H_n has a cell datum whose partially ordered set is the set Y_n of Young diagrams of size n with dominance order. The set Γ_n of Young diagrams with at least three columns is an order ideal in Y_n; let $I_n = H_n(\Gamma_n)$ denote the corresponding i-invariant two sided ideal of H_n.

The proof of the following lemma is straightforward.

Lemma 6.19. *Let A be a cellular algebra. Let Λ denote the partially ordered set in the cell datum for A, let Γ be an order ideal in Λ, and let $A(\Gamma)$ be the corresponding ideal of A. Then $A/A(\Gamma)$ is a cellular algebra, with cellular basis $\{c^\lambda_{\mathfrak{s},\mathfrak{t}} + A(\Gamma)\colon \lambda \in \Lambda \setminus \Gamma;\ \mathfrak{s},\mathfrak{t} \in \mathscr{T}(\lambda)\}$.*

Applying the lemma to the Hecke algebra, we have that H_n/I_n is a cellular algebra, with cellular basis $\{m^\lambda_{\mathfrak{s},\mathfrak{t}} + H_n(\Gamma_n)\colon \lambda \in Y_n \setminus \Gamma_n;\ \mathfrak{s},\mathfrak{t} \in \mathscr{T}(\lambda)\}$. The set $Y_n \setminus \Gamma_n$ is the set of Young diagrams of size n with no more than 2 columns. It is totally ordered by dominance. Write $\lambda(k,n) = (2^{(n-k)/2}, 1^k)$, i.e. the Young diagram with $(n-k)/2$ rows with two boxes and k rows with one box. Then

$$\begin{aligned} Y_n \setminus \Gamma_n = \bigl\{\lambda(k,n)\colon k \leqslant n \text{ and } n-k \text{ even}\bigr\}, \quad \text{with} \\ \lambda(k,n) \trianglelefteq \lambda\bigl(k',n\bigr) \quad \Leftrightarrow \quad k \geqslant k'; \end{aligned} \tag{6.10}$$

compare (6.8).

Lemma 6.20. $H_n/I_n \cong A_n(R, \boldsymbol{\delta})$.

Proof. For $n = 1, 2$, $\Gamma_n = \emptyset$ and $I_n = (0)$. On the other hand, $H_n \cong A_n(R, \boldsymbol{\delta}) \cong R$. For $n \geqslant 3$, let $\mu = (3, 1^{n-3})$. In the notation of [28, Chapter 3], $\xi = m_\mu = m^\mu_{\mathfrak{t}^\mu,\mathfrak{t}^\mu} \in I_n$, where ξ is the element in Eq. (6.9). Hence the ideal $\langle \xi \rangle$ generated by ξ in H_n is contained in I_n. Therefore, we have a surjective homomorphism of involutive algebras $A_n \cong H_n/\langle \xi \rangle \to H_n/I_n$. Both algebras are free of rank $\sum_\lambda (f_\lambda)^2 = \frac{1}{n+1}\binom{2n}{n}$, where the sum is over Young diagrams of size n and no more than two columns, and f_λ is the number of standard Young tableaux of shape λ. Hence, the homomorphism is an isomorphism. $\square$

We identify H_n/I_n with A_n. By slight abuse of notation, we write T_j for the image of T_j in A_n, namely $T_j = \boldsymbol{q}^{1/2}e_j - 1$. Thus $T_j + 1 = \boldsymbol{q}^{1/2}e_j$. We now have potentially two cellular structures on A_n, one inherited from the Hecke algebra and one obtained by the construction of [17, Section 5.3].

By the description of the cellular structure on the Hecke algebra in [28, Chapter 3], we have that $A_n^{\lambda(k,n)}$ is the span of $A_n m_{\lambda(j,n)} A_n$ with $j \leqslant k$, where

$$m_{\lambda(j,n)} = (1 + T_1)(1 + T_3) \cdots (1 + T_{n-j-1}) = \boldsymbol{q}^{(n-j)/2} e_1 e_3 \cdots e_{n-j-1}.$$

Thus, in fact,

$$\begin{aligned} A_n^{\lambda(k,n)} &= A_n(e_1 \cdots e_{n-k-1})A_n \\ &= A_n(e_{k+1} \cdots e_{n-1})A_n = A_n^{(k,n)}. \end{aligned}$$

Moreover, the cell modules from the two cellular structures are explicitly isomorphic:

$$\begin{aligned} \Delta^{\lambda(k,n)} &= A_n(e_1 \cdots e_{n-k-1}) + \breve{A}_n^{\lambda(k,n)} \\ &\cong A_n(e_{k+1} \cdots e_{n-1}) + \breve{A}_n^{(k,n)} = \Delta^{(k,n)}. \end{aligned}$$

We can now import the JM elements from the Hecke algebras (see Example 3.1) to the Jones–Temperley–Lieb algebras. Set $L_1 = 1$ and $L_{j+1} = q^{-1} T_j L_j T_j$ for $j \geqslant 1$. Since the cell modules for the Jones–Temperley–Lieb algebra A_n are in fact cell modules for the Hecke algebra H_n, the triangularity property (3.1) follows, and the product $\prod_{j=1}^{n} L_j$ acts as the scalar

$$\alpha\big(\lambda(k,n)\big) = q^{\sum_{x \in \lambda(k,n)} \kappa(x)}$$

on the cell module $\Delta^{\lambda(k,n)} = \Delta^{(k,n)}$. One can check that

$$\frac{\alpha(\lambda(k,n))}{\alpha(\lambda(k,n-2))} = q^{-n+3},$$

independent of k, for $n \geqslant 2$. It follows from this that $L_n L_{n+1} e_n = e_n L_n L_{n+1} = q^{-n+2} e_n$ for $n \geqslant 1$.

Remark 6.21. The same or similar analogues of Jucys–Murphy elements for the Jones–Temperley–Lieb algebras have been considered in [22] and [10]. Those in [10] are defined over the generic ring $R_0 = \mathbb{Z}[\boldsymbol{\delta}]$, but it is not clear that they have, or can be modified to have, the multiplicative property (resp. additive property) of Definition 3.3 or 3.4.

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