ELSEVIER
Journal of Pure and Applied Algebra 158 (2001) 309–324
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Bezout domains with stable range 1

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Received 3 August 1998; received in revised form 20 December 1999
Communicated by A.V. Geramita

Abstract

It is shown that certain classes of Bezout domains have stable range 1, and thus are elementary divisor rings. Included is a strengthening of Roquette's principal ideal theorem which states that the holomorphy ring of a family S of valuation rings of a field K, with S having bounded residue fields, is Bezout. A counterpart is also given where a bound is placed on the ramification indices instead of the residue fields, and these results are applied to rings of integer-valued rational functions over these rings. Along the way, characterizations are given of Prüfer domains with torsion class group, Bezout domains, and Bezout domains with stable range 1 in terms of a family $\{\mathscr{B}(t) \mid t \in K\}$ of numerical semigroups associated with the ring R, and a related family $\{\mathscr{D}(t) \mid t \in K\}$ of numerical semigroups. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 13F05; 13F07; 13B22; 13C05; 13F20

0. Introduction

Let R be a commutative ring with identity. Then R is said to be an *elementary divisor ring* if every matrix over R is equivalent to a diagonal matrix [11], or equivalently if every finitely presented R-module is a direct sum of cyclic modules [13]. The classical examples of such rings are the principal ideal domains. In [11] it was shown that an elementary divisor ring R is *Bezout*; that is every finitely generated ideal of R is principal. The main open question on such rings, which has been considered at least as far back as the paper [9], is whether the converse holds for integral domains. Most known constructions of Bezout domains have been shown to always produce elementary divisor rings. See for example [5, Section 4; 3].

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PII: S0022-4049(00)00052-9

By [24, Theorem 2.1] and [11, Theorem 3.2], elementary divisor domains can also be characterized as the Bezout domains R such that each finitely generated R-module M can be generated by n elements whenever, for each maximal ideal P of R, the localization M_P can be generated by n elements over R_P. Thus, the above-mentioned question bears some resemblance to the question of whether each finitely generated ideal of a Prüfer domain is generated by 2 elements. This latter question was raised by Gilmer in 1964 and solved with a counterexample in 1979 [20] (see also [21]). All known counterexamples to Gilmer's question involve real holomorphy rings. Recall that a field F is said to be *formally real* if it has an order, and the *real holomorphy ring* of a formally real field F is $H(F) = \cap\{V \mid V$ is a valuation ring of F with formally real residue field$\}$.

By replacing $\mathbb{R}$, the completion of the rationals $\mathbb{Q}$ with respect to the archimedean valuation of $\mathbb{Q}$, by the completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to one of its non-archimedean valuations, J. Ax, S. Kochen and others defined and studied the analogous classes of formally p-adic holomorphy rings (see [16]).

We recall the definitions from [16]. Let p be a rational prime, let e, f be positive integers and let K be a field of characteristic zero. A valuation ring (V,M) of K is said to be a *p-valuation ring of type* (e,f) if $p \in M$, the residue degree $f' = [V/M : \mathbb{Z}/p\mathbb{Z}]$ divides f, and $M = \pi V$ with $pV = \pi^{e'}V$, $e' \leq e$. A field is said to be *formally p-adic* if there exists a p-valuation on F. The *formally p-adic holomorphy ring of type* (e,f) of a formally p-adic field F is $H(F) = \cap\{V \mid V$ is a p-valuation ring of F of type $(e,f)\}$. In general, if S is a family of valuation rings of the field K, the *holomorphy ring of S* is defined as the ring $R = \cap S$. Unlike real holomorphy rings, the formally p-adic holomorphy rings are always Bezout, as was shown by Roquette [17, 18]. Thus it is natural to ask if such rings are always elementary divisor rings, especially in view of the above-mentioned application of formally real holomorphy rings. We show that such rings always have stable range 1, and thus are elementary divisor rings in a very strong way.

In [7] Gilmer showed that if R is an integrally closed domain for which there exists a monic polynomial f with coefficients in the prime subring of R such that $\{1/f(t) \mid t \in K\} \subseteq R$, then R is Prüfer and if I is a finitely generated ideal of R, then I^{n^k} is principal where n is the degree of f and k is the number of elements in some generating set for I. It follows easily that if there exist such polynomials $f_1, \ldots, f_k \in R[X]$ of relatively prime degrees, R is Bezout. A variation of this was given in [17], where sufficient conditions were also given on the set of residue fields of a family S of valuation rings of a field K for there to exist such polynomials over $R = \cap S$. In particular, the fact that the residue fields of a formally p-adic holomorphy ring of type (e,f) are bounded away from an algebraic closure of their common prime field $\mathbb{Z}/p\mathbb{Z}$ easily implies the existence of monic polynomials $f_1, \ldots, f_k \in R[X]$, $R = H(F)$, of relatively prime degrees such that $\{1/f_i(t) \mid t \in K\} \subseteq R$ for each i. This is in contrast to real holomorphy rings, where there may not exist $f \in R[X]$ of odd degree such that $\{1/f(t) \mid t \in K\} \subseteq R$.

In Section 1 we examine the relationship between the values of monic polynomials and the Prüfer property. It turns out that the hypothesis that there exists one monic polynomial $f \in R[X]$ such that $1/f(t) \in R$ for each $t \in K$ can be weakened to the condition that for each $t \in K - \{0\}$ there exist a monic polynomial $f_t \in R[X]$ such that $1/f_t(t) \in R$. The monic assumption can also be weakened to what we call *monic relative to* t. This means that the leading coefficient of f_t is a unit in each valuation overring of R not containing t. This allows the assignment of a numerical semigroup $\mathscr{B}(t)$ to each $t \in K$. We characterize Prüfer domains with torsion class group and Bezout domains in terms of the family $\{\mathscr{B}(t) \mid t \in K\}$ of numerical semigroups. In Section 2 we compare and generalize the approaches in [7, 17]. In Section 3 we consider, for each $t \in K$, a numerical subsemigroup $\mathscr{D}(t)$ of $\mathscr{B}(t)$ and characterize when R is Bezout with stable range 1 in terms of the semigroups $\mathscr{D}(t)$, $t \in K$. This applies in particular to the Bezout rings produced in [7, 17]. Therefore these rings are Bezout with stable range 1, and thus are elementary divisor rings.

In Section 4 we specialize to the formally p-adic holomorphy rings which were the motivating examples for Roquette's principal ideal theorem (Theorem 2.4), in order to give a complementary result for such rings. We extend the definition of formally p-adic holomorphy ring of type (e, f) by allowing either e or f to be infinite, and show that such rings are Bezout with stable range 1.

In Section 5 we apply the previous results to rings of integer-valued rational functions, to produce further examples of Bezout domains having stable range 1.

1. Prüfer domains with torsion class group

Let R be an integral domain with quotient field K and let $t \in K$. We begin by considering conditions under which $(R + Rt)^n$ is principal. Let $f \in R[X]$ have degree n. If $t \in R$, it is clear that $(R + Rt)^n = f(t)R$ if and only if $1/f(t) \in R$; that is, $f(t)$ is a unit of R. We next consider the equality $(R + Rt)^n = f(t)R$ for $t \in K - R$. We first restrict to the case that R is a valuation ring. By a *valuation ring of* K we mean a valuation ring with quotient field K, and by an *overring* of an integral domain R we mean a ring which contains R as a subring and has the same quotient field as R.

Lemma 1.1. *Let (V, M) be a valuation ring of the field K, let $f \in V[X]$ be of degree $n \geq 1$, and let $t \in K - V$. Then $(V + Vt)^n = f(t)V$ if and only if the leading coefficient of f is a unit of V. If this holds, then $t^i/f(t) \in M^{n-i}$ for $i = 0, \ldots, n$, and $(V + Vt)^n = t^n V = f(t)V$.*

Proof. Let $f(X) = c_0 + c_1X + \cdots + c_{n-1}X^{n-1} + c_nX^n$. Since $t \notin V$, $1/t \in M$. Then

$$f(t)/t^n = c_0(1/t)^n + c_1(1/t)^{n-1} + \cdots + c_{n-1}(1/t) + c_n \in M + c_n.$$

Therefore if c_n is a unit, $f(t)/t^n$ is a unit of V, and thus $f(t)V = t^nV$. It follows that $t^n/f(t) \in V$, and for $0 \leq i < n$, $t^i/f(t) = (1/t^{n-i})(t^n/f(t)) \in M^{n-i}$. Therefore $(V + Vt)^n = t^nV = f(t)V$.

If c_n is not a unit, then $f(t)/t^n \in M + c_n = M$. Therefore $f(t) \in t^nM$, and thus $(V + Vt)^n = t^nV \nsubseteq f(t)V$. □

Lemma 1.2. *Let R be an integrally closed domain with quotient field K, let L be an extension field of K, let $f \in R[X]$ be monic of degree $n \geq 1$ and let $t \in L$. Then $(R + Rt)^n = f(t)R$ if and only if $t \in K$ and $1/f(t) \in R$.*

Proof. ($\Rightarrow$) If $(R + Rt)^n = f(t)R$ then $x = 1/f(t) \in R$ and $y = t/f(t) \in R$. Thus $t = y/x \in K$.

($\Leftarrow$) Clearly $f(t)R \subseteq (R+Rt)^n = (1, t, \ldots, t^n)R$. For the opposite inclusion let (V, M) be a valuation overring of R. Since $1/f(t) \in R \subseteq V$, if $t \in V$ then $t^i/f(t) \in V$ for $i = 0, \ldots, n$. If $t \notin V$, then since f is monic, Lemma 1.1 gives $t^i/f(t) = (1/t^{n-i})(t^n/f(t)) \in M^{n-i}$ for $0 \leq i < n$. In particular $t^i/f(t)$ is contained in each valuation overring of R. Since R is integrally closed, we get $t^i/f(t) \in R$ for $i = 1, \ldots, n$. That is, $(R + Rt)^n \subseteq f(t)R$. □

For an integral domain R with quotient field K let $X(R)$ denote the set of *non-trivial* valuation overrings of R. For $x_1, x_2, \ldots, x_n \in K$, let $E[x_1, \ldots, x_n] = \{V \in X(R) \mid x_1, x_2, \ldots, x_n \in V\}$. Recall that $\{E[x_1, \ldots, x_n] \mid n \in \mathbb{Z}_+, x_1, x_2, \ldots, x_n \in K\}$, is a basis for the Zariski topology on $X(R)$, and $X(R)$ with this topology is called the *Riemann surface of K relative to R* [25]. Using this we give a version of the above lemma which does not require $f \in R[X]$ to be monic. Let $lc(f)$ denote the leading coefficient of f.

Proposition 1.3. *Let R be an integrally closed domain contained in the field K. Let $t \in K$, let n be a positive integer and let $f \in R[X]$ have degree $\leq n$. The following statements are equivalent*:

(i) $(R + Rt)^n = f(t)R$;

(ii) *t is in the quotient field of R*, $1/f(t) \in R$ *and* $E[1/lc(f)] \cup E[t] = X(R)$;

(iii) $\{1/f(t), t/f(t), \ldots, t^n/f(t)\} \subseteq R$.

Proof. The equivalence of (i) and (iii) is immediate.

(i) $\Rightarrow$ (ii) If $(R+Rt)^n = f(t)R$, then $1/f(t) \in R$. To show $E[1/lc(f)] \cup E[t] = X(R)$, let $(V, M) \in X(R)$ and assume $t \notin V$. Then by Lemma 1.1, the equality $(R + Rt)^n = f(t)R$ implies $1/lc(f) \in V$. Thus $E[1/lc(f)] \cup E[t] = X(R)$.

(ii) $\Rightarrow$ (iii) Since R is integrally closed, we have $R = \cap X(R)$. Clearly $f(t)R \subseteq (R + Rt)^n = (1, t, \ldots, t^n)R$. For the opposite inclusion let $(V, M) \in X(R)$. Then $1/f(t) \in V$ by (ii), and thus if $t \in V$ then $1/f(t), t/f(t), \ldots, t^n/f(t) \in V$. If $t \in K - V$; that is, $V \notin E[t]$, then by (ii), $1/lc(f) \in V$. Thus $1/f(t), t/f(t), \ldots, t^n/f(t) \in V$ by Lemma 1.1. Therefore $1/f(t), t/f(t), \ldots, t^n/f(t) \in \cap X(R) = R$. □

We now associate a numerical semigroup $\mathscr{B}(t)$ to $t \in K$. Recall that a *numerical semigroup* is an additive subsemigroup Γ of the non-negative integers $\mathbb{Z}_+$. Our numerical semigroups will not contain 0, and are allowed to be empty. For a non-empty numerical semigroup Γ write $\gcd(\Gamma)$ for the greatest common divisor of the members of Γ. A non-empty numerical semigroup Γ is said to be *primitive* if $\gcd(\Gamma) = 1$. If R is an integral domain with quotient field K, $t \in K$ and $f \in R[X] - R$ with $E[1/lc(f)] \cup E[t] = X(R)$, we say *$f$ is monic relative to t*. For each $t \in K$ let

$$\mathscr{B}(t) = \{\deg(f) \mid f \in R[X] - R \text{ with } E[1/lc(f)] \cup E[t] = X(R) \text{ and } 1/f(t) \in R\}.$$

It is easily seen, either by the previous proposition, or directly, that $\mathscr{B}(t)$ is closed under addition.

We can now give a characterization of when R is Prüfer having quotient field K and torsion class group.

Theorem 1.4. *Let R be an integral domain contained in the field K. The following statements are equivalent*:

(i) *R is Prüfer with quotient field K and with torsion class group.*
(ii) *R is integrally closed and for each $t \in K$, $\mathscr{B}(t) \neq \emptyset$.*
(iii) *For each $t \in K - \{0\}$ there exists a positive integer n_t and $f_t \in R[X]$ with $deg(f_t) \leq n_t$ such that $\{1/f_t(t), t/f_t(t), \ldots, t^{n_t}/f_t(t)\} \subseteq R$.*

Proof. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follow from the corresponding implications of Proposition 1.3.

For (iii) $\Rightarrow$ (i), we first show R is Prüfer. Let $x, y \in R - \{0\}$ and let $t = y/x$. By hypothesis $(R + Rt)^{n_t} = f_t(t)R$ for some positive integer n_t and some $f_t \in R[X]$ of degree $\leq n_t$. Since $(Rx + Ry)^{n_t} = x^{n_t}(R + Rt)^{n_t} = x^{n_t} f_t(t)R$, $Rx + Ry$ is invertible. Thus each ideal generated by two elements is invertible. Since this property passes to the localization R_P for each prime ideal P of R, it follows that R is Prüfer.

To show R has quotient field K let $t \in K - \{0\}$. Since $\{1/f_t(t), t/f_t(t), \ldots, t^{n_t}/f_t(t)\} \subseteq R$, where $n_t \geq \deg(f_t)$, then letting $x = 1/f_t(t) \in R$ and $y = t/f_t(t) \in R$, we see that $t = y/x$ is in the quotient field of R.

Since R is Prüfer, we have $(I + J)^k = I^k + J^k$ for any ideals I, J of R. Indeed this equation is clear if one of I or J contains the other, and locally this is what happens. Then for $x, y \in R$ we have $(Rx + Ry)^{n_t} = x^{n_t}(R + R(y/x))^{n_t} = x^{n_t} f_t(y/x)R$ as noted above. Let I be an ideal of R generated by k elements, $k > 2$. To show I^m is principal for some m write $I = xR + J$ where J is generated by $k - 1$ elements. Using induction we may assume J^{m_1} is principal. Then $I^{m_1} = (xR + J)^{m_1} = (xR)^{m_1} + J^{m_1}$ is generated by 2 elements, and thus for some integer m_2, $(I^{m_1})^{m_2} = I^{m_1 m_2}$ is principal. $\square$

We can now give a characterization of when R is Bezout with quotient field K.

Theorem 1.5. *Let R be an integral domain contained in the field K. Then R is Bezout with quotient field K if and only if R is integrally closed and $\mathscr{B}(t)$ is primitive for each $t \in K$.*

Proof. If R is integrally closed and $\mathscr{B}(t)$ is primitive for each $t \in K$, R is Prüfer with torsion class group by Theorem 1.4. Let $x, y \in R - \{0\}$, and let $t = y/x$. Since the period of the ideal class of $R + Rt$ in the class group of R must divide each $n \in \mathscr{B}(t)$, $R + Rt$ is principal, and thus $Rx + Ry$ is also. Conversely, if R is Bezout with quotient field K then R is integrally closed, and if $t \in K - \{0\}$, then $R + Rt = (a + bt)R$ for some $a, b \in R$. Thus $1 \in \mathscr{B}(t)$ by (i) $\Rightarrow$ (ii) of Proposition 1.3. □

2. Comparisons with constructions of Gilmer and Roquette

In order to clarify the relationship between the results in this paper and the sufficient conditions given in [7, 17] for a domain R to be Prüfer with torsion class group, we begin this section with a proposition which is somewhat intermediate between Theorem 1.4 and the results in [7, 17]. The first two conditions are generalizations of the sufficient conditions considered in [7, 17].

Proposition 2.1. *Let R be a subring of the field K and for each $t \in K$ let $f_t \in R[X]$ be of degree $n_t \geq 1$. The following properties of a ring R are equivalent*:

(1) *f_t is monic relative to t for each $t \in K$ and R is an integrally closed subring of K containing $\{1/f_t(t) \mid t \in K\}$.*

(2) *$R = \cap S$ where S is a set of valuation rings of K, the leading coefficient of f_t is a unit in each $(V, M) \in S$ with $t \notin V$ and $f_t(t) \notin M$ for each $(V, M) \in S$ and $t \in V$.*

(3) *$(R + Rt)^{n_t} = f_t(t)R$ for each $t \in K$.*

Further, if these hold then R is a Prüfer domain with quotient field K and torsion class group.

Proof. (1) $\Rightarrow$ (2) Since R is integrally closed, we can write $R = \cap S$ for a set of valuation overrings. It suffices to show that for each $(V, M) \in S$ and $t \in V$, $f_t(t) \notin M$. But if $t \in V$, then since $1/f_t(t) \in R \subseteq V$, $f_t(t) \notin M$.

(2) $\Rightarrow$ (3) Let $t \in K$. Clearly $f_t(t)R \subseteq (R + Rt)^{n_t} = (1, t, \ldots, t^{n_t})R$. For the opposite inclusion let $(V, M) \in S$. If $t \in V$ then $1/f_t(t) \in V$ by (2), and thus $\{1/f_t(t), t/f_t(t), \ldots, t^{n_t}/f_t(t)\} \subseteq V$. If $t \in K - V$ then since the leading coefficient of f_t is a unit in V by hypothesis, $\{1/f_t(t), t/f_t(t), \ldots, t^{n_t}/f_t(t)\} \subseteq V$ by Lemma 1.1. Therefore $f_t(t)R = (R + Rt)^{n_t}$.

(3) $\Rightarrow$ (1) It follows from (iii) $\Rightarrow$ (i) of Theorem 1.4 that R is Prüfer (with torsion class group) and thus is integrally closed. The rest of this implication follows from (iii) $\Rightarrow$ (ii) of Proposition 1.3.

The last statement follows Theorem 1.4. □

In the above result there was no relationship assumed between f_s, $f_t \in R[X]$ for s, $t \in K$. Instead of letting the f_t vary with t, it was shown in [7, Theorem 2.2; 17, Theorems 1, 2], that if there is a single monic polynomial satisfying the counterparts of (1) and (2), respectively, of the above theorem, then R is Prüfer with torsion class group. In the following result, which will be needed in Section 5, these conditions are conditions (1) and (2), respectively. Condition (3) of Theorem 2.2 was used for a similar purpose in [14, Theorem 2.5]. The following result also shows that the monic assumption in these papers is necessary.

Theorem 2.2. *Let R be a subring of the field K and let $f \in R[X]$ of degree $n \geq 1$. The following properties of a ring R are equivalent*:

(1) *The polynomial f is monic and R is an integrally closed subring of K containing $\{1/f(t) \mid t \in K\}$.*

(2) *f is monic, $R = \cap S$ where S is a set of valuation rings of K, and the image $\overline{f} \in V/M[X]$ has no root in V/M for each $(V,M) \in S$.*

(3) *R is a Prüfer domain, f is monic and $f(r)$ is a unit of R for each $r \in R$.*

(4) $(R + Rt)^n = f(t)R$ for each $t \in K$.

Proof. That (1), (2) and (4) are equivalent follows from Proposition 2.1, and these clearly imply (3). For (3) $\Rightarrow$ (2) observe that if M is a maximal ideal of R, f has no root in $R/M = R_M/MR_M$, and thus since R is Prüfer, (2) holds with $S = \{R_M \mid M$ is a maximal ideal of $R\}$. □

Theorem 2.3 (Gilmer [7, Theorem 2.2]). *Let K be a field and let A be the integral closure of the prime subring of K in K. Let $f \in A[X]$ be monic of degree $n \geq 1$ and have no root in K. If R is an integrally closed subring of K containing $\{1/f(t) \mid t \in K\}$, then R is a Prüfer domain with torsion class group. Further if a fractional ideal I of R is generated by k elements then I^{n^k} is principal.*

The last statement in Theorem 2.3 follows as in the last sentence of the proof of Theorem 1.4.

Theorem 2.4 (Roquette [17, Theorems 1, 2, 18]). *Let S be a set of valuation rings of the field K and let $R = \cap S$*:

(1) *If there exists a monic $f \in R[X]$ of degree $n \geq 1$ such that for each $(V,M) \in S$, the image of f in $V/M[X]$ has no root in V/M, then R is a Prüfer domain and if a fractional ideal I of R is generated by k elements then I^{n^k} is principal.*

(2) *If there exist monic $f_1, \ldots, f_r \in R[X]$ of positive degrees $n_1, \ldots, n_r$, respectively such that for each $(V,M) \in S$, the image of each f_i in $V/M[X]$ has no root in V/M, then R is a Prüfer domain and if a fractional ideal I of R is generated by k elements then I^{d^k} is principal, where d is the greatest common divisor of $\{n_1, \ldots, n_r\}$. In particular, if $d = 1$, R is Bezout.*

The second part of Theorem 2.4 follows immediately from the first. The motivating examples for [17] were the formally p-adic holomorphy rings, and for these rings it is easy to find polynomials as in part (2) of the above theorem. It is natural to ask if the Bezout domains produced by this construction can give examples of Bezout domains which are not elementary divisor rings. We answer this in the following section.

The paper [7] was less concerned with the case where R is Bezout since the motivation there was to produce examples of Prüfer domains, at least partly to test the conjecture that each finitely generated ideal in a Prüfer domain can be generated by two elements. It turns out that this construction, in fact the special case $f = X^2 + 1$ considered in [4], can be used to produce a counterexample to this conjecture, although the first such counterexample, by Schulting, was a different variation of the construction in [4], see [21, Section 6].

3. Stable range 1

In this section we give a characterization of Bezout domains with stable range 1 which applies to the Bezout domains considered [7, 14, 17], and some generalizations considered in the previous section. First we recall some definitions. A sequence $(a_1, a_2, \ldots, a_{s+1})$ of elements of R is said to be *unimodular* if $(a_1, a_2, \ldots, a_{s+1})R = R$, and $(a_1, a_2, \ldots, a_{s+1})$ is said to be *stable* if there exist $b_1, b_2, \ldots, b_s \in R$ such that the sequence $(a_1 + b_1 a_{s+1}, a_2 + b_2 a_{s+1}, \ldots, a_s + b_s a_{s+1})$ is unimodular. Recall that a ring R is said to have *n in the stable range* if every unimodular sequence $(a_1, a_2, \ldots, a_{s+1})$ in R with $s \geq n$ is stable [6]. By a result of Vaserstein [22, Theorem 1], this is equivalent to the property that each unimodular sequence in R of length $n+1$ is stable. If R has n in the stable range and does not have $n-1$ in the stable range, R is sometimes said to have stable range n. We shall only be interested in stable range 1, where we can use the following simple lemma in place of the above-mentioned result of Vaserstein. This lemma is a slight variation of [6, Proposition 5.1]. We reproduce the proof in [6] for the convenience of the reader.

Lemma 3.1. *The following properties of an integral domain R are equivalent*:
(1) *R is Bezout with* 1 *in the stable range.*
(2) *R is Bezout and each unimodular sequence in R of length* 2 *is stable.*
(3) *For each pair of elements $a_1, a_2 \in R$, there exists $d \in R$ such that $(a_1, a_2)R = (a_1 + da_2)R$.*

Proof. Implication $(1) \Rightarrow (2)$ is clear.

For $(2) \Rightarrow (3)$, we may assume $a_1 a_2 \neq 0$. Since R is Bezout, we have $(a_1, a_2)R = aR$ for some $a \in R$. Let $a_1 = aa_1'$, $a_2 = aa_2'$. Then $(a_1', a_2')R = R$, and thus by (2) $a_1' + da_2' = u$ is a unit for some $d \in R$. To show that $(a_1, a_2)R \subseteq (a_1 + da_2)R$, let $b \in (a_1, a_2)R = aR$. Write $b = ab'$, $b' \in R$. Then $ub = uab' = (a_1' + da_2')ab' = (a_1 + da_2)b'$. So $b \in (a_1 + da_2)R$.

For $(3) \Rightarrow (1)$, let $(a_1, a_2, \ldots, a_{n+1}) \in R^{n+1}$ be unimodular. By (3) there exist $b_i \in R$ such that $(a_i, a_{n+1})R = (a_i + b_i a_{n+1})R$ for $i = 1, \ldots, n$. Then $R = (a_1, a_2, \ldots, a_{n+1}) = (a_1 + b_1 a_{n+1}, a_2 + b_2 a_{n+1}, \ldots, a_n + b_n a_{n+1})$. □

The ring of entire functions has stable range 1 [19]. Other examples are given in [6, 10, 23]. For consequences of the stable range 1 condition see [1].

To measure the degree of stability of a unimodular pair (a,b) in R with $a \neq 0$ we associate to $t = b/a$ a subsemigroup $\mathscr{D}(t)$ of $\mathscr{B}(t)$. If R is an integral domain with quotient field K, then for each $t \in K$ let

$$\mathscr{D}(t) = \{\deg(f) \mid f \in R[X] - R \text{ with } E[1/lc(f)] \cup E[t] = X(R) \text{ and } 1/f(t), 1/f(0) \in R\}.$$

Theorem 3.2. *Let R be an integrally closed domain with quotient field K and let $t \in K - \{0\}$. If $\mathscr{D}(t)$ is primitive then $(1,t)R = (1 + at)R$ for some $a \in R$.*

Proof. Let $f_1, f_2, \ldots, f_k \in R[X]$ be monic relative to t of degrees $n_1, n_2, \ldots, n_k$ respectively such that $1/f_i(t)$, $1/f_i(0) \in R$, and let $1 = \sum_{i=1}^{k} r_i n_i$, $r_i \in \mathbb{Z}$. By Proposition 1.3 we have $(R + Rt)^{n_i} = f_i(t)R$ for $i = 1, \ldots, k$. Thus $R + Rt$ is invertible and

$$R + Rt = (R + Rt)^{\sum r_i n_i} = ((R + Rt)^{n_1})^{r_1} \cdots ((R + Rt)^{n_k})^{r_k} = f_1(t)^{r_1} \cdots f_k(t)^{r_k} R.$$

Assume $r_1, r_2, \ldots, r_j$ are positive and the other r_i are negative. Let $r_i = -u_i$ for $i = j+1, \ldots, k$. Since by hypothesis each of the constant terms $f_i(0)$ is a unit of R, after multiplying by a unit of R we may assume that each f_i has constant term 1. Let $f = \prod_{i=1}^{j} f_i(t)^{r_i}$ and $g = \prod_{i=j+1}^{k} f_i(t)^{u_i}$. Then $\deg(f) - \deg(g) = 1$ and

$$f_1(t)^{r_1} \cdots f_k(t)^{r_k} = \frac{f(t)}{g(t)} = \frac{g(t) + f(t) - g(t)}{g(t)} = 1 + \frac{f(t) - g(t)}{g(t)}.$$

Since each of the f_i has constant term 1, we can write this as $1 + th(t)/g(t)$ for $h(X)$, $g(X) \in R[X]$ monic relative to t, and $\deg(h) = \deg(f) - 1 = \deg(g)$.

Since $1/g(t) \in R$ and $E[1/lc(g)] \cup E[t] = X(R)$, by Proposition 1.3 we have $(R + Rt)^{\deg(g)} = g(t)R$. Thus $t^i/g(t) \in R$ for $i = 0, \ldots, \deg(g)$, and therefore $h(t)/g(t) \in R$. Therefore $u(X) = 1 + Xh(t)/g(t) \in R[X]$.

Also $R + Rt = f_1(t)^{r_1} \cdots f_k(t)^{r_k} R = (1 + th(t)/g(t))R = u(t)R$, and thus $1/u(t)$, $1/u(0) \in R$. Therefore the result holds with $a = h(t)/g(t)$. □

Corollary 3.3. *Let R be an integrally closed integral domain with quotient field K and let $t = y/x \in K$, $x, y \in R$. The following statements are equivalent*:

(i) *$\mathscr{D}(t)$ is primitive.*

(ii) *$(1,t)R = (1 + at)R$ for some $a \in R$.*

(iii) *$(x,y)R = (x + ay)R$ for some $a \in R$.*

Proof. That (i) ⇒ (ii) follows from Theorem 3.2, and that (ii) ⇒ (i) follows from Proposition 1.3. The equivalence of (ii) and (iii) follows since $x(1,t)R = (x,y)R$ and $x(1 + at)R = (x + ay)R$. □

We can now give our characterization of Bezout domains with stable range 1.

Theorem 3.4. *Let R be a subring of a field K. Then R is a Bezout domain with stable range* 1 *and quotient field K if and only if R is integrally closed and $\mathscr{D}(t)$ is primitive for each $t \in K$.*

Proof. If R is integrally closed and $\mathscr{D}(t)$ is primitive for each $t \in K$, it follows from Theorem 1.5 that R is Bezout with quotient field K. By Corollary 3.3 R has stable range 1.

Conversely, if R is Bezout with stable range 1 and quotient field K, then by Lemma 3.1, for each $x, y \in R$ there exists $a \in R$ such that $(x, y)R = (x + ay)R$. Thus by Corollary 3.3, $\mathscr{D}(t)$ is primitive for each $t \in K$. □

Corollary 3.5. *Let R be an integrally closed integral domain with quotient field K such that $\mathscr{D}(t)$ is primitive for each $t \in K$. Then R is an elementary divisor domain.*

Proof. By [11, Theorem 5.2] it suffices to show that R is Bezout and if $a, b, c \in R$ with $R = (a, b, c)R$, then $R = (pa, pb + qc)R$ for some $p, q \in R$. But by Theorem 3.4 and its proof, R is Bezout and such p, q exist with $p = 1$. □

The stable range 1 property is much stronger than the property of being an elementary divisor domain. For example the most classical examples of elementary divisor domains, namely the rational integers $\mathbb{Z}$ and the polynomial ring $k[X]$ in one variable over a field, do not have stable range 1. See [6, 23] for example. In fact, whereas stable range 1 requires $\gcd(\mathscr{D}(t)) = 1$ for each $t \in \mathbb{Q}$ by Theorem 3.4, for $\mathbb{Z}$ the set $\{\gcd(\mathscr{D}(t)) \mid t \in \mathbb{Q}\}$ is unbounded. Indeed given $b \in \mathbb{Z}$, $b > 1$, choose $a \in \mathbb{Z}$, $a > 1$ relatively prime to b such that for $i = 1, 2, \ldots, n$, $b^i \not\equiv \pm 1 \pmod{a}$. Consider $(1, a/b, \ldots, (a/b)^n)\mathbb{Z} = (b^n/b^n, ab^{n-1}/b^n, \ldots, a^n/b^n))\mathbb{Z} = (b^n, a)/b^n\mathbb{Z} = 1/b^n\mathbb{Z}$. By Proposition 1.3 we must show that there does not exist $f \in \mathbb{Z}[X]$ such that $\deg(f) \le n$, $f(0) = \pm 1$ and $f(a/b)R = (1, a/b, \ldots, (a/b)^n)\mathbb{Z} = 1/b^n\mathbb{Z}$. But if

$$\pm(1/b^n) = 1 + c_1(a/b) + c_2(a/b)^2 + \cdots + c_n(a/b)^n, \quad c_i \in \mathbb{Z},$$

we would have

$$\pm 1 = b^n + c_1 ab^{n-1} + c_2 a^2 b^{n-2} + \cdots + c_n a^n \in b^n + a\mathbb{Z}.$$

This contradicts $b^n \not\equiv \pm 1 \pmod{a}$. Thus $\gcd(\mathscr{D}(a/b)) > n$.

If we specialize to the situation considered in [7, 17, 14] we get the following result which strengthens, for example, the conclusion in Theorem 2.4.2 from Bezout to Bezout with stable range 1 and quotient field K.

Theorem 3.6. *Let R be an integrally closed subring of the field K, and assume there exist monic $f_1, f_2, \ldots, f_k \in R[X]$ of positive degrees $n_1, n_2, \ldots, n_k$, respectively, which satisfy the equivalent conditions of Theorem* 2.2. If $\mathbb{Z} = (n_1, n_2, \ldots, n_k)\mathbb{Z}$, *then R is a Bezout domain with stable range* 1 *and quotient field K.*

Proof. This follows from Theorem 3.4. □

Several results are given in [17] which give examples of when a ring R satisfies the hypothesis of Theorem 3.6. Of course in each case, by Theorem 3.6, R is a Bezout domain with stable range 1, and thus is an elementary divisor domain. For later reference we state two such results, and recall the proof of the existence of the polynomials as in Theorem 3.6 for the convenience of the reader.

Theorem 3.7. *Let R be the holomorphy ring of a set S of valuation rings of the field K. If V/M is finite for each $(V,M) \in S$ and the set of cardinalities $\{|V/M| \mid (V,M) \in S\}$ is bounded. Then R is a Bezout domain with stable range* 1 *and quotient field K.*

Proof. Let $\{|V/M| \mid (V,M) \in S\} = \{q_i \mid i = 1,\dots,n\}$. Then $g(X) = \prod_{i=1}^{n} X^{q_i} - X$ and $Xg(X)$ vanish on V/M for each $(V,M) \in S$. Thus we may apply Theorem 3.6 with $k = 2$, $f_1 = 1 + g(X)$ and $f_2 = 1 + Xg(X)$ to get the result. □

Following [17] we generalize this as follows. A field F is said to be *residually finite* if there exists a valuation ring (V,M) of F such that V/M is finite. A family $\{F_i \mid i \in I\}$ of fields is said to be *residually bounded* if for each $i \in I$ there is a valuation ring (V_i,M_i) of F_i such that the set of cardinalities $\{|V_i/M_i| \mid i \in I\}$ is bounded.

Theorem 3.8. *Let R be the holomorphy ring of a set S of valuation rings of the field K, and assume that the family of residue fields $\{V/M \mid (V,M) \in S\}$ is residually bounded. Then R is a Bezout domain with stable range* 1 *and quotient field K.*

Proof. Let (V_0,M_0) be the composite of (V,M) and the given valuation ring (V',M') on V/M having finite residue field. That is, V_0 is the inverse image of V' under the canonical map $V \to V/M$. Then each V_0 has quotient field K, and the holomorphy ring $H(S_0)$ of $S_0 = \{(V_0,M_0) \mid (V,M) \in S\}$ is Bezout with stable range 1 and quotient field K by Theorem 3.7, and R is an overring of $H(S_0)$. But an overring of a Bezout ring is clearly Bezout, and by [6, Corollary 5.2], an overring of a Bezout domain having stable range 1 also has stable range 1. □

By Theorem 3.6 R is also Bezout with stable range 1 in the *relative* case considered in [17, Theorem B], where the holomorphy ring R has a subfield F which is *non-exceptional* [17, p. 363] such that, identifying F with a subfield of each V/M, $(V,M) \in S$, there is a uniform bound on the degrees $[V/M\colon F]$. Further, if E is a finite extension field of the p-adic numbers $\mathbb{Q}_p$ for some rational prime p, it also follows from Theorem 3.6 that each *formally p-adic holomorphy ring of type E*, as defined in [15] is a Bezout domain with stable range 1.

It is not difficult to see that the condition in Theorem 3.4 for R to be Bezout with stable range 1 is strictly weaker than the hypothesis in Theorem 2.4.2. For example one could take S to consist of a single valuation domain (V,M) of the form $F + M$ where F is an algebraically closed field. Similarly, one could take R to be a principal ideal domain with infinitely many residue fields isomorphic to $\mathbb{Q}$, and finitely many residue fields isomorphic to the algebraic closure of $\mathbb{Q}$ [8]. Then R is the holomorphy ring of the family $S = \{R_P \mid P$ a maximal ideal of $R\}$, and R is not the holomorphy ring of any proper subfamily of S. Although in this case R is a principal ideal domain by its construction [8], Theorem 3.4 adds that R has stable range 1. In order to give a general condition, which includes these examples, for the existence of monic $f_t \in R[X]$ such that $1/f_t(t), 1/f_t(0) \in R$ for a given $t \in K$, we fix some notation.

Let R be the holomorphy ring of a family S of valuation rings of a field K. For $t \in K$ let $t(V)$ be the image of t under the place corresponding to V. Thus $t(V)$ is the image of t in V/M if $t \in V$ and $t(V) = \infty$ if $t \notin V$. Assume further that R contains a subring A such that each of the $(V,M) \in S$ have the same center P on A. That is $M \cap A = P$ for each $(V,M) \in S$. Then each of the canonical homomorphisms $V \to V/M$, $(V,M) \in S$, can be considered as extensions of the canonical map $A \to A/P$. Let F be the quotient field of A/P, which is canonically embedded in each residue field V/M, $(V,M) \in S$. If $t(V)$ is finite and algebraic over F let $g_{t,V} \in F[X]$ be the minimal polynomial in $F[X]$ of $t(V) \in V/M$. Otherwise let $g_{t,V} = 0$. Then if $f_t(X) \in A[X]$ and $(V,M) \in S$, $f_t(t) \notin M$ if and only if the image $\overline{f_t}(X)$ in $F[X]$ is not in $g_{t,V}F[X]$. Let $\mathrm{Irr}(F,t,S) = \{g_{t,V} \mid V \in S\}$ and let $\mathrm{Irr}(F)$ denote the set of irreducible polynomials in $F[X]$. Then $\mathrm{Irr}(F,t,S) \subseteq \mathrm{Irr}(F)$. We now have the following simple result.

Proposition 3.9. *Let R be the holomorphy ring of a set S of valuation rings of the field K and assume R contains a subring A having a prime ideal P such that $M \cap A = P$ for each $(V,M) \in S$. Let F be the quotient field of A/P.*

(1) If $t \in K$ *is such that* $\mathrm{Irr}(F,t,S) \cup \{X\} \neq \mathrm{Irr}(F)$, *there exists monic* $f_t(X) \in R[X]$ *such that* $1/f_t(t)$, $1/f_t(0) \in R$.
(2) *If $t \in K$ is such that* $\mathrm{Irr}(F) - (\mathrm{Irr}(F,t,S) \cup \{X\})$ *contains polynomials of relatively prime degrees then $\mathscr{D}(t)$ is primitive.*
(3) *If* (1) *holds for every $t \in K$, then R is Prüfer with torsion class group.*
(4) *If* (2) *holds for every $t \in K$, then R is Bezout with stable range* 1.

Proof. For (1) choose a monic preimage $f_t \in A[X]$ of some $f(X) \in \mathrm{Irr}(F) - (\mathrm{Irr}(F,t,S) \cup \{X\})$. Part (2) is similar. Parts (3) and (4) follow from Theorem 1.4 and Theorem 3.4, respectively. □

4. A companion to Roquette's principal ideal theorem

Although Roquette's principal ideal theorem (Theorem 2.4.2) was motivated by the formally p-adic holomorphy rings of type (e,f), his result only used the bound f on the residue fields, and not the bound e on the ramification. This leads one to question if a similar consequence follows from using only the bound e, and in this section we show that indeed it does.

If $(D,M) \subseteq (V,N)$ are valuation rings we say V *dominates* D if $M \subseteq N$. Let (D,M) be a valuation ring of a field K with principal maximal ideal, say $M = aD$. If $(V,\pi V)$

is a valuation domain dominating D with $aV = \pi^j V$, $j \geq 1$, we say $(V, \pi V)$ *has finite ramification index over* (D, aD). We denote the ramification index j by $e(D, V)$.

Lemma 4.1. *Let (D, aD) be a valuation ring of a field K and let $(V, \pi V)$ be a valuation ring of an extension field L of K with finite ramification index $e(D, V) \leq k$. Let $\xi_k(X) = (a + aX^{k+1})/(1 + aX^{k+1})$. Then $\xi_k(V) \subseteq aV$ and $\xi_k(L - V) \subseteq V - \pi V$.*

Proof. Let v be a valuation corresponding to V. If $x \in V$, $a + ax^{k+1} \in aV$ and $1 + ax^{k+1}$ is a unit of V. If $x \in L - V$, the numerator and denominator of $\xi_k(x)$ have the same value. □

Theorem 4.2. *Let (D, aD) be a valuation ring of a field K and let S be a family of valuation rings of an extension field L of K, each having principal maximal ideal. Assume each $(V, \pi V) \in S$ dominates (D, aD) with $e(D, V) \leq k$ for each $(V, \pi V) \in S$. Then the holomorphy ring $H(S) = R$ of S is a Bezout domain with stable range* 1 *and quotient field L.*

Proof. Let $y \in L$. From Lemma 4.1, $\xi_k(y) \in \cap\{V \mid (V, M) \in S\} = R$, and thus $(1, y)R \supseteq (1 + y\xi_k(y))R$. For the opposite inclusion it suffices to see that $1/(1 + y\xi_k(y))$ and $y/(1 + y\xi_k(y)) \in V$ for each $(V, M) \in S$, and this is immediate from Lemma 4.1. The result now follows from Corollary 3.3 and Theorem 3.4. □

We specialize to the formally p-adic holomorphy rings which were the motivating examples for Roquette's principal ideal theorem (Theorem 2.4). We extend the definition of formally p-adic holomorphy ring of type (e, f). Let p be a rational prime and let e, f be positive integers. A valuation ring (V, M) is said to be a *p-valuation ring of type* (∞, f), or (e, ∞), respectively, if the residue degree $f' = [V/M \colon \mathbb{Z}/p\mathbb{Z}]$ divides f, or if $M = \pi V$ with $pV = \pi^{e'} V$, $e' \leq e$, respectively. A field is said to be *formally p-adic* if there exists a p-valuation on F. If $e, f \in \mathbb{N} \cup \{\infty\}$, the *formally p-adic holomorphy ring of type* (e, f) of a formally p-adic field F is $H(F) = \cap\{V \mid V$ is a p-valuation ring of F of type $(e, f)\}$.

Corollary 4.3. *Let R be the formally p-adic holomorphy ring of type (e, f) of a formally p-adic field K, with either e or f finite. Then R is a Bezout domain with stable range* 1 *and quotient field K.*

Proof. This follows from Theorems 3.7 and 4.2. □

5. Integer-valued rational functions

Let D be a domain with quotient field K and let E be a subset of K. The ring of *integer-valued rational functions on E* is defined as $Int^R(E, D) = \{\varphi \in K(X) \mid \varphi(t) \in$

R for each $t \in E\}$ [2]. This ring has been closely associated to formally p-adic holomorphy rings since the latter rings were defined in [12]. Indeed one of the reasons for introducing the formally p-adic holomorphy rings in [12] was to obtain analogues for $\mathbb{Q}_p$ to Artin's Theorem which solves Hilbert's 17th problem: If $r(X) \in \mathbb{R}[X_1,\dots,X_n]$ is such that $r(a) \geq 0$ for each $a \in \mathbb{R}^n$, where $\mathbb{R}$ is the field of real numbers, then $r(X) = \sum_{i=1}^{m} c_i u_i(X)^2$, $c_i \in K$, $c_i > 0$ and $u_i \in \mathbb{R}(X)$. In [12] the rational function $\gamma(X) = (1/p)(X^p - X)/((X^p - X)^2 - 1) \in \mathrm{Int}^R(\mathbb{Q}_p, \mathbb{Z}_p)$, now called "the Kochen operator", takes the place of the squares in Hilbert's 17th problem. Further, integral-valued rational functions also arose naturally in Theorem 3.2 where the coefficient a was obtained as a rational function of $t = b/a$, and similarly in Lemma 4.1 and Theorem 4.2. For integer-valued rational functions we have the following application of Theorem 1.4.

Theorem 5.1. *Let D be an integral domain with quotient field K and let $f \in D[X]$ be monic polynomial such that the equivalent conditions of Theorem* 2.2 *are satisfied. Then f satisfies the equivalent conditions of Theorem* 2.2 *with respect to the integral domain* $\mathrm{Int}^R(K,D)$. *In particular* $\mathrm{Int}^R(K,D)$ *is a Prüfer domain with torsion class group and quotient field $K(X)$.*

Proof. It is immediate that $\mathrm{Int}^R(K,D)$ is integrally closed. Let $\rho \in K(X)$. If $x \in K$ is such that $\rho(x)$ is defined, $1/f(\rho(x)) \in D$ by the hypothesis on f. Thus $1/f(\rho(x)) \in D$ for all but possibly finitely many $x \in K$. Let V be a valuation overring of D. Give K the topology induced by any valuation associated to V, and let $K \cup \{\infty\}$ be the one-point compactification of K. Then $1/f(\rho): K \to K \cup \{\infty\}$ is continuous and thus the preimage B of V under the map $1/f(\rho)$ is a closed set in the V-topology on K. Thus the finite set $K - B$ is open in the V-topology on K. But since $V \neq K$, each non-empty open subset of K must be infinite. Thus $B = K$, and it follows that $1/f(\rho) \in \mathrm{Int}^R(K,V)$. Thus $1/f(\rho) \in \cap\{\mathrm{Int}^R(K,V) \mid V$ is a valuation overring of $D\} = \mathrm{Int}^R(K,D)$. The rest follows from Theorems 1.4 and 2.2. □

Theorem 5.2. *Let D be an integrally closed domain with quotient field K and let $f_1, f_2, \dots, f_k \in D[X]$ be monic polynomials of degree $n_1, n_2, \dots, n_k$, respectively satisfying the equivalent conditions of Theorem* 2.2 *and such that $(n_1,\dots,n_k)\mathbb{Z} = \mathbb{Z}$. Then* $\mathrm{Int}^R(K,D)$ *is an elementary divisor domain with stable range* 1 *and quotient field $K(X)$.*

Proof. This follows immediately from Theorems 5.1 and 3.6. □

Theorem 5.3. *Let (D, aD) be a valuation ring of a field K and let S be a family of valuation rings of an extension field L of K, each having principal maximal ideal. Assume each $(V, \pi V) \in S$ dominates (D, aD) with $e(D,V) \leq k$. Then* $\mathrm{Int}^R(K,D)$ *is a Bezout domain with stable range* 1 *and quotient field $K(X)$.*

Proof. Observe that if $\varphi, \psi \in \mathrm{Int}^R(K,D)$, then letting $\rho = \varphi(X)/\psi(X)$ we have $(1, \rho(y))D = (1 + \xi_k(\rho(y))\rho(y))D$ for all but finitely many $y \in K$ by Lemma 4.1. It follows as in the proof of Theorem 5.1 that this holds for all $y \in K$, and thus $(1, \rho(X))\mathrm{Int}^R(K,D) = (1 + \xi_k(\rho(X))\rho(X))\mathrm{Int}^R(K,D)$. Multiplying both sides by $\psi(X)$ we get the result. □

Corollary 5.4. *If D is the formally p-adic holomorphy ring of type (e, f) of a field K, with either e or f finite, then $\mathrm{Int}^R(K,D)$ is a Bezout domain with stable range* 1 *and quotient field $K(X)$.*

Proof. This follows from Theorems 5.2 and 5.3. □

Observe that $\mathrm{Int}^R(K,D) \subseteq \mathrm{Int}^R(E,D)$ for any subset E of K, and since the properties Prüfer with torsion class group, Bezout, and Bezout with stable range 1, are inherited by overrings, we see that these results also hold for $\mathrm{Int}^R(E,D)$.

In the case that (V, tV) is a valuation ring with principal maximal ideal and quotient field K, it was shown in [2, p. 271] that $\mathrm{Int}^R(K,V)$ is Bezout with stable range 1, and the author greatfully acknowledges the inspiration this result furnished for Theorems 4.2 and 5.3.

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