The joint distribution of sojourn times in finite semi-Markov processes

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Let \( Y = \{ Y_t : t \geq 0 \} \) be a semi-Markov process whose state space \( S \) is finite. Assume that \( Y \) is either irreducible and \( S \) is then partitioned into two classes \( A_1, A_2 \), or that \( Y \) is absorbing and \( S \) is partitioned into \( A_1, A_2, A_3 \), where \( A_3 \) is the set of all absorbing states of \( Y \). Denoting by \( T_{A_i,j} \) the \( j \)th sojourn of \( Y \) in \( A_i, i = 1, 2 \), we determine the Laplace transform of the joint distribution of \( T = \{ T_{A_i,j} : i = 1, 2; j = 1, \ldots, m \} \). This result is derived from a recurrence relation for the Laplace transform of \( T \). The proof of the recurrence relation itself is based on what could be called a 'generalized renewal argument'. Some known results on sojourn times in Markov and semi-Markov processes are also rederived using our main theorem. A procedure for obtaining the Laplace transform of the vector of sojourn times in special cases if \( S \) is partitioned into more than two non-absorbing classes is also considered.

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sojourn time * semi-Markov process * Markov process * Laplace transform * renewal theory * reliability modelling

1. Introduction

Markov and semi-Markov processes are a useful tool for reliability modelling (see, e.g., Singh and Billinton, 1977). In a series of papers formulated in the reliability setting, Rubino and Sericola (1989a,b,c, 1991) initiated the study of the distribution of sojourn times of finite Markov and semi-Markov processes. This paper provides a unified approach to the questions considered by Rubino and Sericola (1989a,b,c, 1991), as well as it continues some recent work by the author on sojourn times (see Csenki, 1990, 1991, 1992).

Let \( Y = \{ Y_t : t \in [0, \infty) \} \) be a homogeneous semi-Markov process which is either irreducible or absorbing. We assume that the state space \( S \) of \( Y \) is finite. \( S \) will be partitioned into two disjoint sets \( A_1, A_2 \) if \( Y \) is irreducible and it will be partitioned into three sets if \( Y \) is absorbing, with \( A_3 \) standing for the set of all absorbing states of \( Y \). Thus, \( S = A_1 \cup A_2 \) or \( S = A_1 \cup A_2 \cup A_3 \) depending on whether \( Y \) is irreducible or absorbing; it is \( A_i \cap A_j = \emptyset \) for \( i \neq j \). In reliability applications (see Singh and Billinton, 1977), \( A_1 \) and \( A_2 \) may stand for the failed (but repairable) set of states and the set of working states respectively. The variables to be studied in this paper...
Sojourn times are the system’s sojourn times: let for $i = 1, 2$ and $j = 1, 2, \ldots$, $T_{A_i,j}$ be the $j$th sojourn of $Y$ in $A_i$. As usual, in the absorbing case we put $T_{A_i,j} = 0$ if $A_i$ is visited by $Y$ less than $j$ times until absorption.

Let us give a brief review of some of the related results available in the literature. Rubino and Sericola (1989a) obtained the distribution of individual sojourn times under the assumption that $Y$ is an absorbing semi-Markov process with $S$ partitioned as $S = A_1 \cup A_2 \cup \{\omega\}$ where $\omega$ is an absorbing state. In Rubino and Sericola (1991), absorbing Markov processes are considered with $S$ partitioned in the same manner. In Rubino and Sericola (1989b,c), results are established for random variables related to sojourn times in absorbing and irreducible finite Markov processes respectively when $S$ is partitioned into two subsets (with $\{\omega\}$ added in the absorbing case). Some recent results by the author have extended those by Rubino and Sericola: in Csenki (1990, 1992) the joint distribution of the sojourn times $\{T_{A_i,j} : i = 1, 2; j = 1, \ldots, m\}$ is evaluated under the Markov assumption in both the irreducible and absorbing cases. (In the latter case $A_3 = \{\omega\}$). There, the method of proof was based on a time-discretization technique which gives the continuous-time result as the limit of the corresponding discrete-parameter result which itself is established by an elementary probabilistic reasoning. In a related paper by the author on irreducible finite semi-Markov processes (see Csenki, 1991), a renewal-theoretical reasoning is used to obtain the Laplace transform of what can be considered as the analogue of the renewal density. If $N_{A_i}(t)$ stands for the number of visits by $Y$ to $A_i$ in $[0, t]$ in the irreducible case, then for any $k = 1, 2, \ldots$, the mapping

$$t \rightarrow E(N_{A_i}(t))^k, \quad t \geq 0,$$

is the distribution function of a measure on $[0, \infty)$. In Csenki (1991), the Laplace transform of this measure was evaluated by a renewal-theoretical reasoning.

In the present paper, the renewal-theoretical approach is further developed to arrive at a theorem which allows a closed form expression for the Laplace transform of

$$T = \{T_{A_i,j} : i = 1, 2; j = 1, \ldots, m\}$$

to be determined for any given $m$. The key tool is a recurrence relation for the Laplace transform of $T$ and it will be presented in Section 2. The Laplace transform of $T$ itself will be deduced in Section 3 from which then most known results on sojourn times in Markov and semi-Markov processes easily follow; a few examples of this are discussed in Section 3 too. Furthermore, Section 3 also contains an indication of how the key theorem (which will be formulated for semi-Markov processes whose state space is partitioned into any finite number of classes) can be used to arrive at the Laplace transform of the vector of sojourn times if $S$ is partitioned into three non-absorbing classes. Section 4 is devoted to the proof of the key tool.

All the results discussed here easily carry over to semi-Markov reward processes. (See also Csenki, 1992; and Rubino and Sericola, 1989a.) The holding time
distributions are then interpreted as the (random) rewards generated by the process when visiting the individual states. The sequence of visits is defined by the underlying (discrete-time) embedded Markov chain.

2. Recurrence relation for the Laplace transform of the vector of sojourn times

The result here will be formulated for a semi-Markov process \( Y = \{ Y_t : t \in [0, \infty) \} \) which is absorbing and whose state space \( S \) is partitioned into \( n+1 \) \((\geq 3)\) disjoint classes: \( S = A_1 \cup \cdots \cup A_n \cup A_{n+1}, \) where \( A_{n+1} \) is the set of all absorbing states of \( Y. \) Assuming \( Y \) to be absorbing does not entail any restrictions since, as was already justified in Csenki (1992), the corresponding formula for the irreducible case is obtained by substituting zero for the transition probabilities to \( A_{n+1} \) for the underlying embedded Markov chain. Assuming \( n \geq 3 \) (rather than \( n = 3 \) which would be enough to deduce Theorem 2 below) gives us the added bonus of being able to obtain the Laplace transform of the vector of sojourn times in special cases if \( S \) is partitioned into more than three classes.

We start with some remarks on the notation. \( I \) and \( 0 \) stand for the identity and zero matrix respectively; \( 1 \) is the column-vector of all ones. As usual, submatrices of a matrix will be indicated by appropriate subscripts by subsets of the index set. The transition probability matrix of the embedded Markov chain \( X = \{ X_n : n = 0, 1, \ldots \} \) will be denoted by \( R. \) It is such that \( r_{s,s} = 0 \) for all \( s \in S. \) \( \alpha^T = (\alpha_A^T, \ldots, \alpha_{A_n}^T, 0) \) is the initial probability vector. \( F_{s,s}(t) \) stands for the distribution function of the holding time in \( s \in S \setminus A_{n+1} \) given that the next state to be visited is \( s' \in S, s' \neq s. \) Define the matrix \( Q^*(\rho) = \{ q^*_s(\rho) : s \in S \setminus A_{n+1}, s' \in S \} \) for \( \rho \geq 0 \) by

\[
q^*_s(\rho) = \begin{cases} 
 0, & \text{if } s \neq s', \\
 0, & \text{if } s = s'.
\end{cases}
\]

Let \( m \geq 1 \) be a fixed integer and define the \( m \times m \) matrix \( U \) by

\[
u_{i,j} = \begin{cases} 
 1, & \text{for } j - i = 1, \\
 0, & \text{otherwise}.
\end{cases}
\]

For \( s \in S \setminus A_{n+1} \) and \( \tau_i = (\tau_{1,i}, \ldots, \tau_{m,i})^T \in [0, \infty)^m \quad (i = 1, \ldots, n) \) put

\[
\psi_s(\tau_1, \ldots, \tau_n) = E \left\{ \exp \left[ - \sum_{i=1}^n (T_{A,s}, \ldots, T_{A_{n},s}) \right] \right\} | Y_0 = s \}.
\]

Define the column-vectors \( \psi_{A_k}(\tau_1, \ldots, \tau_n) \) by

\[
\psi_{A_k}(\tau_1, \ldots, \tau_n) = \{ \psi_s(\tau_1, \ldots, \tau_n) : s \in A_k \}, \quad k = 1, \ldots, n.
\]

In Theorem 1, a recurrence relation is established for these vectors of Laplace transforms. It is as follows.
Theorem 1. For all $i \in \{1, \ldots, n\}$, the matrix $I - Q_{A_i}^* (\rho) A_i$ is invertible and we have

$$
\psi_{A_i} (\tau_1, \ldots, \tau_n) = \sum_{k=1}^{n} \mathbf{K}_{i}^{A_i} (1^T (I - U) \tau_1) \psi_{A_i} (\tau_1, \ldots, \tau_{i-1}, U\tau_i, \tau_{i+1}, \ldots, \tau_n) + \mathbf{K}_{i}^{A_i} (1^T (I - U) \tau_i) 1,
$$

(2.1)

where the matrices $\mathbf{K}_{k}^{A_i} (\rho)$, $k \in \{1, \ldots, n+1\} \setminus \{i\}$, are defined for $\rho \geq 0$ by

$$
\mathbf{K}_{k}^{A_i} (\rho) = (I - Q_{A_i}^* (\rho))^{-1} Q_{A_i}^* (\rho).
$$

(2.2)

Note. The last term on the r.h.s. of (2.1) is zero if $Y$ is irreducible since $Q_{A_i}^* (\rho) = 0$.

(This follows from $R_{(S \setminus A_{n+1}) \setminus A_{n+1}} = 0$.)

3. Laplace transforms of vectors of sojourn times

3.1. $S$ is partitioned into three subsets

This is the case examined hereto in the literature for special cases only, e.g., under the Markov assumption or for the semi-Markov case but only for the marginal distributions of the sojourn time vector. The following holds true in general.

Theorem 2. Let $Y$ be an irreducible (or absorbing) semi-Markov process with $S = A_1 \cup A_2$ (or $S = A_1 \cup A_2 \cup A_3$). Then, the Laplace transform of $\{T_{A_i, j}: i = 1, 2; j = 1, \ldots, m\}$, $\psi (\tau_1, \tau_2)$, is given by

$$
\psi (\tau_1, \tau_2) = - \alpha_{A_1} \left[ \prod_{k=0}^{m-1} \mathbf{K}_{A_1 A_2}^* (1^T (I - U) U^k \tau_1) \mathbf{K}_{A_1 A_2}^* (1^T (I - U) U^k \tau_2) \right] 1 + \alpha_{A_2} \left[ \prod_{k=0}^{m-1} \mathbf{K}_{A_2 A_1}^* (1^T (I - U) U^k \tau_2) \mathbf{K}_{A_2 A_1}^* (1^T (I - U) U^k \tau_1) \right] 1
$$

$$
+ \alpha_{A_1} \sum_{l=0}^{m-1} \left[ \prod_{k=0}^{l-1} \mathbf{K}_{A_1 A_2}^* (1^T (I - U) U^k \tau_1) \mathbf{K}_{A_1 A_2}^* (1^T (I - U) U^k \tau_2) \right] \times \left[ \mathbf{K}_{A_1 A_2}^* (1^T (I - U) U^l \tau_1) \mathbf{K}_{A_1 A_3}^* (1^T (I - U) U^l \tau_2) \right] 1
$$

$$
+ \mathbf{K}_{A_1 A_3}^* (1^T (I - U) U^l \tau_1) \mathbf{K}_{A_1 A_3}^* (1^T (I - U) U^l \tau_2) \right] 1,
$$

(3.1)

where the matrices $\mathbf{K}_{A_i A_k}^* (i \neq k)$ are defined by (2.2).
Proof. Applying (2.1) twice, gives

\[
\psi_{A_i}(\tau_1, \tau_2) = K_{A_i}^*(I^T(I-U)\tau_1)K_{A_i}^*(I^T(I-U)\tau_2)\psi_{A_i}(U^T\tau_1, U^T\tau_2) \\
+ (K_{A_i}^*(I^T(I-U)\tau_1)K_{A_i}^*(I^T(I-U)\tau_2) + K_{A_i}^*(I^T(I-U)\tau_1))1.
\]

From (3.2), we have for \( j = 1, 2, \ldots \) by induction that

\[
\psi_{A_i}(\tau_1, \tau_2) = \prod_{k=0}^{j-1} \{K_{A_i}^*(I^T(I-U)U^k\tau_1)K_{A_i}^*(I^T(I-U)U^k\tau_2)\} \psi_{A_i}(U^T\tau_1, U^T\tau_2) \\
+ \sum_{l=0}^{j-1} \left[ \prod_{k=0}^{l-1} \{K_{A_i}^*(I^T(I-U)U^k\tau_1)K_{A_i}^*(I^T(I-U)U^k\tau_2)\} \right] \\
\times \{K_{A_i}^*(I^T(I-U)U^l\tau_1)K_{A_i}^*(I^T(I-U)U^l\tau_2) \}
\]

\[
\times \{K_{A_i}^*(I^T(I-U)U^l\tau_1)\}1. \tag{3.3}
\]

Using \( U^m = 0 \) and \( \psi_{A_i}(0, 0) = 1 \), (3.1) follows from (3.3) with \( j = m \) since

\[
\psi(\tau_1, \tau_2) = \alpha_{A_i}^T \psi_{A_i}(\tau_1, \tau_2) + \alpha_{A_i}^T \psi_{A_i}(\tau_1, \tau_2). \quad \square
\]

Some special cases will now be examined. Notice first that the arguments of the matrices \( K^* \) on the right-hand side of (3.1) are given by

\[
I^T(I-U)U^k\tau_i = \tau_{k+1,i}, \quad k = 0, 1, \ldots, m-1.
\]

It was shown in Csenki (1991) that if \( Y \) is a continuous-parameter Markov process with transition rate matrix \( A \), then

\[
K_{A_i}^*(\rho) = -(A_{A_i} - \rho I)^{-1}A_{A_i}, \quad i \in \{1, 2\}, \quad j \in \{1, 2, 3\}\setminus\{i\}.
\]

Assume (for simplicity) that \( Y \) is irreducible. (3.1) then becomes

\[
\psi(\tau_1, \tau_2) = \alpha_{A_1}^T \prod_{k=0}^{m-1} \{(A_{A_1} - \tau_{k+1,1}I)^{-1}A_{A_2} - \alpha_{A_2}^T \prod_{k=0}^{m-1} \{(A_{A_2} - \tau_{k+1,2}I)^{-1}A_{A_3}\}1
\]

This is, of course, a known result (cf. Csenki, 1990).

As a second application of Theorem 2, under the general semi-Markov assumption the distribution of the \( m \)th sojourn time in \( A_1 \) will be represented in terms of the first sojourn time in \( A_1 \). This result can be found in Rubino and Sericola (1989a) and it is as follows.

**Theorem 3.** Define the vector of Laplace transforms \( \phi_{A_i}(\rho) = \{\phi_a(\rho) : a \in A_i\} \) by

\[
\phi_a(\rho) = E\{\exp[-\rho T_{A_i,1}] | Y_0 = a\}, \quad a \in A_1.
\]
Then, under the assumptions of Theorem 1, the Laplace transform of the mth sojourn in $A_1$ is given by

$$E\{\exp[-\rho T_{A_1,m}]\} = 1 - \{\alpha_{A_1}^T + \alpha_{A_2}^T H\}G^{-1}(1 - \phi_{A_1}(\rho)), \quad m = 1, 2, \ldots$$

(3.4)

with $G$ and $H$ respectively defined by

$$G = (I - R_{A_1A_1})^{-1}R_{A_1A_2}(I - R_{A_2A_2})^{-1}R_{A_2A_1},$$

$$H = (I - R_{A_2A_2})^{-1}R_{A_2A_1}.$$

Note. In Rubino and Sericola (1989a), the above result was presented in terms of distribution functions rather than Laplace transforms; in fact, there was no use made of Laplace transforms there. (3.4) implies the result for distribution functions by first dividing both sides of (3.4) by $\rho > 0$ (which then gives the corresponding equation for the Laplace transforms of the distribution functions). This implies the desired form:

$$P(T_{A_1,m} \leq t) = 1 - \{\alpha_{A_1}^T + \alpha_{A_2}^T H\}G^{-1}(1 - \phi_{A_1}(\rho)), \quad t > 0.$$

This result shows that once the distribution functions $P(T_{A_1,m} \leq t | Y_0 = a)$, $a \in A_1$, are available (as closed form expressions or in a numerical form) the knowledge of the transition probability matrix of the embedded Markov chain suffices to evaluate the distribution function of the mth sojourn time in $A_1$.

**Proof of Theorem 3.** For $\rho > 0$ put $\tau_1 = (0, 0, \ldots, 0, \rho)^T$, $\tau_1 = 0 \in [0, \infty)^m$ and notice that

$$1^T(I - U)U^k\tau_1 = \begin{cases} 0 & \text{for } k = 0, 1, \ldots, m - 2, \\ \rho & \text{for } k = m - 1, \end{cases}$$

and

$$1^T(I - U)U^k\tau_2 = 0 \quad \text{for } k = 0, \ldots, m - 1.$$
Using now
\[ K_{A_1,A_2}^*(0)K_{A_2,A_1}^*(0) = G, \quad K_{A_1,A_1}^*(0) = H, \]
and
\[ K_{A_i,A_j}^*(0)1 + K_{A_j,A_i}^*(0)1 = 1, \quad i, j \in \{1, 2\}, \quad i \neq j, \]
(with \(i = 2, j = 1\)) we get from (3.5),
\[ E\{\exp[-\rho T_{A_1,m}]\} = \{\alpha_{A_1}^T + \alpha_{A_2}^T H\} G^{-1} \{K_{A_1,A_2}^*(\rho)1 + K_{A_1,A_1}^*(\rho)1\} + c, \]
with some constant \(c\) not depending on \(\rho\). From (3.7) (with \(\rho = 0\)) and (3.6) (with \(i = 1, j = 2\)) it follows that
\[ c = 1 - \{\alpha_{A_1}^T + \alpha_{A_2}^T H\} G^{-1}1. \]
It is therefore
\[ E\{\exp[-\rho T_{A_1,m}]\} = 1 - \{\alpha_{A_1}^T + \alpha_{A_2}^T H\} G^{-1}\{1 - \{K_{A_1,A_2}^*(\rho)1 + K_{A_1,A_1}^*(\rho)1\}\}. \]
Putting \(m = 1\) and \(\alpha_{A_1} = 0\) in (3.8), gives
\[ \phi_{A_1}(\rho) - 1 - \{1 - \{K_{A_1,A_2}^*(\rho)1 + K_{A_1,A_1}^*(\rho)1\}\} = K_{A_1,A_2}^*(\rho)1 + K_{A_1,A_1}^*(\rho)1. \]
(3.4) now follows from (3.8) and (3.9).

3.2. \(S\) is partitioned into four subsets

Theorem 1 is a suitable tool for determining the Laplace transform of sojourn times with a reasonable effort for \(n \geq 3\) in special cases only. We shall restrict ourselves to the discussion of the case \(n = 3\) which already gives an insight into the difficulties arising in general for \(n \geq 3\).

Theorem 1, applied consecutively three times, gives of what could be thought of as an analogue of (3.2) for \(n = 3\):
\[ \psi_{A_1}(\tau_1, \tau_2, \tau_3) = K_{A_1,A_2}^*(1^T(I - U)\tau_1)\{K_{A_2,A_1}^*(1^T(I - U)\tau_2)\psi_{A_1}(U\tau_1, U\tau_2, \tau_3) \]
\[ + K_{A_2,A_1}^*(1^T(I - U)\tau_2)\{K_{A_2,A_1}^*(1^T(I - U)\tau_3) \]
\[ \times \psi_{A_1}(U - \tau_1, U\tau_2, U\tau_3) + K_{A_1,A_2}^*(1^T(I - U)\tau_3) \]
\[ \times \psi_{A_1}(U\tau_1, U\tau_2, U\tau_3) + K_{A_1,A_2}^*(1^T(I - U)\tau_2)1 \}
\[ + K_{A_2,A_1}^*(1^T(I - U)\tau_2)1 \} \]
Unfortunately, unlike in the case \( n = 2 \), the right-hand side of (3.10) involves terms which cannot be reduced to an already known quantity by a repeated application of (3.10); this will namely still contain the terms

\[
\psi_{A_i}(U_{\tau_1}, U_{\tau_2}, U_{\tau_3}) \quad \text{and} \quad \psi_{A_i}(U_{\tau_1}, \tau_2, U_{\tau_3})
\]

after applying it \( l \) times. Furthermore, additional difficulty arises from the \( \psi_{A_i} \) - and \( \psi_{A_i} \) -terms on the right-hand side of (3.10).

The latter problem is easily avoided, however, by considering only those semi-Markov processes for which no transitions can occur between \( A_2 \) and \( A_3 \), i.e.,

\[
R_{A_2A_3} = 0, \quad R_{A_3A_2} = 0.
\]  

(3.11)

This class of processes is still interesting enough as far as reliability applications are concerned: \( A_1 \) will then stand for the system’s working states; \( A_2 \) and \( A_3 \) are two distinct sets of states from which error recovery is possible and which do not communicate with each other; \( A_4 \) stands for ultimate system breakdown. The initial probability vector will be concentrated on \( A_1 \) if at time \( t = 0 \) the system is operational—in this case it suffices to evaluate \( \psi_{A_1} \) to characterize fully the joint distribution of the system’s sojourn times. (A software-hardware system along somewhat similar lines was described and analyzed by Sumita and Masuda (1986).)

We shall describe a procedure by means of which \( \psi_{A_1}(\tau_1, \tau_2, \tau_3) \) can be evaluated for any given \( m \) if (3.11) holds. It should be noted, however, that it is a recursive scheme and no closed form expression is available for \( \psi_{A_1} \) in any other case than \( n = 2 \). Assuming thus (3.11), (3.10) can be rewritten as

\[
+ K^*_{A_1A_2}(1^T(I-U)\tau_1)K^*_{A_2A_1}(1^T(I-U)\tau_2)\psi_{A_1}(U_{\tau_1}, U_{\tau_2}, U_{\tau_3})
+ K^*_{A_1A_3}(1^T(I-U)\tau_2)\psi_{A_1}(U_{\tau_1}, U_{\tau_2}, U_{\tau_3})
\times \psi_{A_1}(U_{\tau_1}, U_{\tau_2}, U_{\tau_3})
+ K^*_{A_2A_1}(1^T(I-U)\tau_2)\psi_{A_1}(U_{\tau_1}, U_{\tau_2}, U_{\tau_3})
+ K^*_{A_2A_3}(1^T(I-U)\tau_3)\psi_{A_1}(U_{\tau_1}, U_{\tau_2}, U_{\tau_3})
+ K^*_{A_3A_1}(1^T(I-U)\tau_3)\psi_{A_1}(U_{\tau_1}, U_{\tau_2}, U_{\tau_3})
+ K^*_{A_3A_2}(1^T(I-U)\tau_3)\psi_{A_1}(U_{\tau_1}, U_{\tau_2}, U_{\tau_3})
\]

(3.12)
Due to $U^n = 0$, a repeated application of (3.12) will eventually result in an equation expressing $\psi_{A_1}(\tau_1, \tau_2, \tau_3)$ in terms of $\psi_{A_1}(0, \tau_2, 0)$ and $\psi_{A_1}(0, 0, \tau_3)$. But, these two quantities are easily obtainable from (3.12). To obtain $\psi_{A_1}(0, 0, \tau_3)$, for example, put in (3.12) $\tau_1 = 0$ and $\tau_2 = 0$,

$$\psi_{A_1}(0, 0, \tau_3)$$

$$= K^*_{A_1,A_2}(0)K^*_{A_2,A_1}(0)\psi_{A_1}(0, 0, \tau_3)$$

$$+ K^*_{A_1,A_2}(0)K^*_{A_2,A_1}(1^T(I - U)\tau_3)\psi_{A_1}(0, 0, U\tau_3)$$

$$+ \{K^*_{A_1,A_2}(0)K^*_{A_2,A_1}(0)\}1 + K^*_{A_1,A_2}(0)1.$$  

(3.13)

(3.13) allows $\psi_{A_1}(0, 0, \tau_3)$ to be expressed in terms of $\psi_{A_1}(0, 0, U\tau_3)$ as follows:

$$\psi_{A_1}(0, 0, \tau_3)$$

$$= (I - G)^{-1}\{K^*_{A_1,A_2}(0)K^*_{A_2,A_1}(1^T(I - U)\tau_3)\psi_{A_1}(0, 0, U\tau_3)$$

$$+ \{K^*_{A_1,A_2}(0)K^*_{A_2,A_1}(0)\}1 + K^*_{A_1,A_2}(0)1\}, \quad (3.14)$$

where $G$ is defined by

$$G = (I - R_{A_1,A_2})^{-1}R_{A_1,A_2}(I - R_{A_2,A_1})^{-1}R_{A_1,A_1}.$$

$I - G$ is invertible since $G$ is (obviously) element-wise non-negative and

$$G^{\prime} \to 0 \quad \text{as} \quad j \to +\infty. \quad (3.15)$$

Repeated application of (3.14) gives of course $\psi_{A_1}(0, 0, \tau_3)$ by virtue of $U^n = 0$.

A simple probabilistic justification of (3.15) is as follows. The matrix $R_{A_1, A_2, A_1, A_2, \omega}$ can be supplemented by one row and one column such that it becomes the transition probability matrix of an absorbing Markov chain on $A_1 \cup A_2 \cup \{\omega\}$. In Csenki (1992) it was shown that for this chain the number of visits to $A_1$ until absorption, $N$, say, satisfies

$$P(N \geq j) = \beta^T G^{j-1}1, \quad j = 1, 2, \ldots, \quad (3.16)$$

if the chain is started in $A_1$ according to some initial probability vector $\beta$. $j \to +\infty$ in (3.16) implies (3.15) since $\beta$ is arbitrary.

4. Proof of Theorem 1

The proof is based on a repeated application of what could be termed a `modified renewal argument'. In classical renewal theory, (see, e.g., Bhat, 1984; or Karlin and Taylor, 1975), an event is sought at which the process under consideration is regenerated, i.e., at which a probabilistic replica of the process is started again. There is no such unique event in our case; but, it is possible to identify a sequence of events iterrelated by a system of equations in the Laplace transform domain. This system of equations will turn out to be (2.1) and (2.2).
Without loss of generality we may restrict ourselves to \(i = 1\) in (2.1). We start with an auxiliary result for which some definitions are needed. Define for \(a \in A_1, a' \in S \setminus A_1\), and \(\rho \geq 0\) measures \(\mu_{a,a';; \rho}\) on \([0, \infty)\) by
\[
\mu_{a,a';; \rho}(D) = P(\{\rho T_{A_1,1} \in D\} \cap E(a') \mid Y_0 = a),
\]
where the event \(E(a')\) is defined by
\[
E(a') = \{Y \text{ visits } a' \text{ immediately after leaving } A_1\}.
\]
Put for \(a \in A_1\) and \(a' \in S \setminus A_1\),
\[
\kappa_{a,a'} = \mu_{a,a';; 1}.
\]
We shall need the following lemma.

**Lemma 1.** For any \(k \in \{2, 3, \ldots, n + 1\}\), the Laplace transforms \(\{\kappa_{a,a'}\}; a \in A_1, a' \in A_k\) of the measures defined by (4.1) are given in matrix form by (2.2), i.e., written element-wise,
\[
\kappa_{a,a'}(\rho) = \kappa_{a,a'}^*(\rho), \quad \rho \geq 0, \quad a \in A_1, \quad a' \in A_k.
\]
Furthermore, the Laplace transform of \(\mu_{a,a';; \rho}\) is given by
\[
\mu_{a,a';; \rho}(\tau) = \kappa_{a,a'}^*(\rho \tau), \quad \tau \geq 0.
\]

**Proof.** Assume \(\sigma > 0\) arbitrary but fixed. An analogue of the renewal argument gives for \(t \geq 0\),
\[
\mu_{a,a'; \sigma}([0, t]) = \sum_{i=1}^{n+1} \sum_{a \in A_1} P(\{\sigma T_{A_i,1} \leq t\} \cap E(a') \mid X_0 = a, X_1 = a'' \} P(X_1 = a'' \mid X_0 = a)
\]
\[
= \sum_{a' \in A_1} P(\{\sigma T_{A_1,1} \leq t\} \cap E(a') \mid X_0 = a, X_1 = a'' \} P(X_1 = a'' \mid X_0 = a)
\]
\[
+ P(\{\sigma T_{A_1,1} \leq t\} \cap E(a') \mid X_0 = a, X_1 = a'' \} P(X_1 = a'' \mid X_0 = a)
\]
\[
+ \sum_{a' \in A_1} \int_{(0, \infty)} P\left(\left\{T_{A_i,1} \leq \frac{t}{\sigma} - v\right\} \cap E(a') \mid X_0 = a'' \right) F_{a', a''}(dv)
\]
\[
+ r_{a,a'} F_{a,a}(t/\sigma).
\]
The above can be written in terms of \(\kappa_{a,a'}\) (defined in (4.1)),
\[
\kappa_{a,a'}([0, t/\sigma]) = \sum_{a' \in A_1} r_{a,a'} \int_{(0, \infty)} \kappa_{a', a''} \left(\left[0, \frac{t}{\sigma} - v\right]\right) F_{a'', a''}(dv)
\]
\[
+ r_{a,a'} F_{a,a}(t/\sigma), \quad t \geq 0.
\]
Taking Laplace transforms in (4.4), we get for \(\rho \geq 0\),
\[
\kappa_{a,a'}^*(\rho) = \sum_{a' \in A_1} q_{a,a'}^*(\rho) \kappa_{a', a''}^*(\rho) + q_{a,a}^*(\rho),
\]
which in matrix form is written as

\[
K_{A_i, A_k}(\rho) = Q_{A_i, A_k}^*(\rho) K_{A_i, A_k}(\rho) + Q_{A_i, A_k}^*(\rho).
\]  

(4.5) implies

\[
K_{A_i, A_k}(\rho) = (I - Q_{A_i, A_k}^*(\rho))^{-1} Q_{A_i, A_k}^*(\rho)
\]

and thus (4.2) since \( I - Q_{A_i, A_k}^*(\rho) \) is invertible. (This follows from the fact that, for fixed \( \rho \), \( Q_{A_i, A_k}^*(\rho) \) can be thought of as the \((A_i, A_k)\)-submatrix of the transition probability matrix of a suitably defined absorbing Markov chain, from which we have the invertibility of \( I - Q_{A_i, A_k}^*(\rho) \) by Csenki (1992).) To see (4.3), notice that for \( \rho > 0 \),

\[
\mu_{a, a'}([0, t]) = \kappa_{a, a'}([0, t/\rho]),
\]

from which (4.3) follows. For \( \rho = 0 \), (4.3) follows from

\[
\mu_{a, a'}(0) = P(E(a') \mid Y_0 = a) = \kappa_{a, a}([0, \infty)) = \kappa_{a, a'}(0).
\]

**Proof of Theorem 1.** We are going to examine the distribution of the random variable \( W \) defined for a fixed set of vectors \( \tau_1, \ldots, \tau_n \in [0, \infty)^m \) by

\[
W(\tau_1, \ldots, \tau_n) = \sum_{i=1}^{n} (T_{a_i,1}, \ldots, T_{a_i,m}) \tau_i.
\]

It is for \( a \in A_1 \),

\[
P(W(\tau_1, \ldots, \tau_n) \leq t \mid Y_0 = a) = \sum_{k=2}^{n+1} \sum_{a' \in A_k} P(W(\tau_1, \ldots, \tau_n) \leq t) \cap E(a') \mid Y_0 = a).
\]  

(4.6)

If \( k \in \{2, \ldots, n\} \) and \( a' \in A_k \) then

\[
P(W(\tau_1, \ldots, \tau_n) \leq t) \cap E(a') \mid Y_0 = a) = P\left\{ \tau_{1,1} T_{A_1,1} + \sum_{j=2}^{m} \tau_{1,j} T_{A_1,j} + \sum_{i=2}^{n} (T_{A_i,1}, \ldots, T_{A_i,m}) \tau_i \leq t \right\} \cap E(a') \mid Y_0 = a
\]

\[
= \int_{[0, t]} P\left( \sum_{j=2}^{m-1} \tau_{1,j+1} T_{A_1,j} + \sum_{i=2}^{n} (T_{A_i,1}, \ldots, T_{A_i,m}) \tau_i \leq t - v \mid Y_0 = a' \right) \mu_{a', \tau_{1,1}}(dv)
\]

\[
= \int_{[0, t]} P(W(U\tau_1, \tau_2, \ldots, \tau_n) \leq t-v \mid Y_0 = a') \mu_{a', \tau_{1,1}}(dv).
\]

(4.7)

If \( a' \in A_{n+1} \), we have

\[
P(W(\tau_1, \ldots, \tau_n) \leq t) \cap E(a') \mid Y_0 = a) = P(\{\tau_{1,1} T_{A_1,1} \cap E(a') \mid Y_0 = a) = \mu_{a, a'; \tau_{1,1}}([0, t])
\]

(4.8)
Substituting (4.7) and (4.8) into (4.6) gives

\[ P(W(\tau_1, \ldots, \tau_n) \leq t \mid Y_0 = a) = \sum_{k=2}^{n} \sum_{a' \in A_k} \int_{[0,t]} P(W(U\tau_1, \tau_2, \ldots, \tau_n) \leq t - v \mid Y_0 = a') \mu_{a',a':\tau_1,t}(dv) \]

\[ + \sum_{a' \in A_{n+1}} \mu_{a,a':\tau_1,t}([0,t]) \text{.} \]  

(4.9)

Taking Laplace transforms in (4.9), we get for \( \rho \geq 0 \),

\[ E\{\exp[-W(\tau_1, \ldots, \tau_n)\rho] \mid Y_0 = a\} = \sum_{k=2}^{n} \sum_{a' \in A_k} E\{\exp[-W(U\tau_1, \tau_2, \ldots, \tau_n)\rho] \mid Y_0 = a'\} \mu_{a,a':\tau_1,t}^*(\rho) \]

\[ + \sum_{a' \in A_{n+1}} \mu_{a,a':\tau_1,t}^*(\rho) \text{.} \]  

(4.10)

Putting \( \rho = 1 \) in (4.10), we see by (4.2), (4.3) and

\[ \tau_{1,1} = (I - U) \tau_1, \]

that

\[ \psi_a(\tau_1, \ldots, \tau_n) = \sum_{k=2}^{n} \sum_{a' \in A_k} \psi_a(U\tau_1, \tau_2, \ldots, \tau_n) \kappa_{a,a}^*(1^T(I - U)\tau_1) \]

\[ + \sum_{a' \in A_{n+1}} \kappa_{a,a}^*(1^T(I - U)\tau_1) \text{.} \]  

(4.11)

The matrix form of (4.11) is (2.1). \( \square \)

References


