Integral Representations of Cyclic Groups of Order $p^2$

Ming-chang Kang*

Department of Mathematics, National Taiwan University, Taipei, Taiwan, Republic of China

E-mail address: kang@math.ntu.edu.tw

Communicated by Kent R. Fuller

Received October 30, 1996

Let $p$ be any prime number, $G$ be the cyclic group of order $p^2$, and $\Lambda := RG$ be the group algebra of $G$ over a Dedekind domain $R$ such that $pR$ is a maximal ideal in $R$ and both $R[T]/\Phi_p(T)$ and $R[T]/\Phi_p(T)$ are Dedekind domains also, where $\Phi_p(T)$ is the $n$th cyclotomic polynomial. We shall provide a full list of indecomposable $\Lambda$ lattices constructed in an explicit way. A complete set of invariants of these indecomposable $\Lambda$ lattices will be determined also, under an additional assumption on $R$. The above results were obtained by Reiner [Pacific J. Math. 78 (1978), 467–501] when $R = \mathbb{Z}$. The problem of torsion-free cancellation was studied by Reiner [17, 18]. The key words: integral representations, lattices, projective modules.

1. INTRODUCTION

Let $G$ be a finite group, $R$ be a Dedekind domain, and $\Lambda := RG$ be the integral group ring. A $\Lambda$ lattice $M$ is a finitely generated $\Lambda$ module which is torsion-free over $R$. When $G$ is a cyclic group of prime order $p$ or square-free order $n$, the structure of $\Lambda$ lattices is well understood [5; 11; 12; 16; 19, Theorem 4.19, p. 74]. When $G$ is a cyclic group of order $p^2$, where $p$ is a prime number, the situation is more complicated. Heller and Reiner [7] and Berman and Gudivok [2] independently showed that there are only finitely many indecomposable $\Lambda$ lattices in this case; moreover, when $R$ is a discrete valuation ring, all these indecomposable $\Lambda$ lattices were exhibited in terms of group extensions. A full list of indecomposable $\Lambda$ lattices and the invariants of these $\Lambda$ lattices in the case $R = \mathbb{Z}$ were studied by Reiner [17, 18]. The problem of torsion-free cancellation was

* Work partially supported by National Science Council, Republic of China.

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0021-8693/98 $25.00
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studied by Wiegand [25] and Swan [20]. The multiplicative structure of the Grothendieck ring of \( L \) lattices was studied by Rudko and Jones and Michler [8]. For more details of the history of integral representations, see [3, Chap. 4].

The purpose of this paper is to construct integral representations of cyclic groups of order \( p^2 \) over a Dedekind domain satisfying the conditions (R1) and (R2) (see Definition 1.2). As mentioned before, the case when the Dedekind domain is \( \mathbb{Z} \) was considered by Reiner [18]. However, this paper is independent of previous results of Reiner, because we study \( \Lambda \) lattices via pull-back diagrams instead of the extension groups Ext. Even in the case when the Dedekind domain \( R \) is \( \mathbb{Z} \), it seems that our formulation is more convenient for applications and provides another perspective of integral representations. A similar approach to the case of cyclic groups of prime order and of square-free order may be carried out also. As a sample, let us state the formulation of Reiner's theorem [16], which was recast in [19, Theorem 4.19, p. 74]:

**Theorem 1.1.** Let \( p \) be any prime number and \( R \) be any Dedekind domain such that the principal ideal \( pR \) is a maximal ideal and \( R[T]/\Phi_p(T) \) is a Dedekind domain, where \( \Phi_p(T) \) is the \( p \)th cyclotomic polynomial. Then every lattice over \( A := R[T]/(T^p - 1) \) is a direct sum of rank 1 projective modules over \( A \), \( R[T]/\Phi_p(T) \), or \( R = R[T]/(T - 1) \).

(The reader should keep it in mind that a module over \( R[T]/\Phi_p(T) \) or \( R[T]/(T - 1) \) is regarded as an \( A \) module by the canonical projection \( A \rightarrow R[T]/\Phi_p(T) \) or \( A \rightarrow R[T]/(T - 1) \). We shall adopt this practice in various similar cases in this paper without further explanation.)

**Definition 1.2.** From now on we shall consider integral representations of cyclic groups of order \( p^2 \). For emphasis, we recall that \( p \) is a prime number, \( \Lambda := R[T]/(T^p - 1) \), where \( R \) is a Dedekind domain satisfying the following conditions:

(R1) The principal ideal \( pR \) (\( \neq 0 \)) is a maximal ideal in \( R \).

(R2) Both the rings \( R[T]/\Phi_p(T) \) and \( R[T]/\Phi_{p^2}(T) \) are Dedekind domains.

Note that the condition (R1) ensures that \( \text{char } R = 0 \). Besides the ring of integers or the ring of \( p \)-adic integers, the ring of integers \( \mathcal{O}_K \) of the following algebraic number field \( K \) also satisfies the conditions (R1) and (R2), where

(i) \( K = \mathbb{Q}(\exp(2\pi\sqrt{-1}/n)) \) with \( p \nmid n \) and \( \Phi_p(X) \) being irreducible in \( \mathbb{Z}/p\mathbb{Z}[X] \) or
(ii) $K = \mathbb{Q}(\sqrt{n})$, where $p$ is an odd prime number, $p \not| n$, and $(n/p) = -1$.

(See Examples 2.5–2.7 for more details.)

Before stating our results, let us fix some terminology and notation.

**Definition 1.3.** Define

$$
\Lambda := R[T]/(T^{p^i} - 1), \quad A_1 := R[T]/\Phi_p(T),
$$

$$
A_2 := R[T]/(T^p - 1), \quad A_0 := R[T]/\langle p, (T - 1)^p \rangle.
$$

The following maps are canonical projections:

$$
\pi_i: \Lambda \to A_i \quad \text{for} \ 1 \leq i \leq 2,
$$

$$
p_i: A_i \to A_0 \quad \text{for} \ 1 \leq i \leq 2,
$$

$$
p_i(j): A_i \to A_0 \to R[T]/\langle p, (T - 1)^i \rangle
$$

for $1 \leq i \leq 2, 1 \leq j \leq p$,

$$
p_2'(j): R[T]/\Phi_p(T) \to R[T]/\langle p, (T - 1)^{p-1} \rangle \to R[T]/\langle p, (T - 1)^j \rangle
$$

for $1 \leq j \leq p - 1$,

$$
p^n: R[T]/(T - 1) \to R[T]/\langle p, T - 1 \rangle.
$$

**Definition 1.4.** For $1 \leq j \leq p$, the ring $\Lambda(j)$ is defined by the pull-back diagram

$$
\begin{array}{ccc}
\Lambda(j) & \longrightarrow & A_1 \\
\downarrow & & \downarrow_{p_1(j)} \\
A_2 & \longrightarrow & R[T]/\langle p, (T - 1)^j \rangle
\end{array}
$$

i.e., $\Lambda(j) := \{(a_1, a_2) \in A_1 \times A_2: p_1(j)(a_1) = p_1(j)(a_2)\}$.

Note that if $j_1 < j_2$, there is a natural inclusion from $\Lambda(j_2)$ into $\Lambda(j_1)$. $\Lambda(p)$ is nothing but $\Lambda$ itself.

**Definition 1.5.** For $1 \leq j \leq p - 1$, the ring $\Lambda(j)$ is defined by the pull-back diagram

$$
\begin{array}{ccc}
\Lambda(j) & \longrightarrow & A_1 \\
\downarrow & & \downarrow_{p_1(j)} \\
R[T]/\Phi_p(T) & \longrightarrow & R[T]/\langle p, (T - 1)^j \rangle
\end{array}
$$

Note that $\Lambda(p - 1) = R[T]/\Phi_p(T)$.
Definition 1.6. The ring $\Lambda'$ is defined by the pull-back diagram

\[
\begin{array}{c}
\Lambda' \downarrow \\
\begin{array}{c}
\longrightarrow \\
\text{R}(T)/(T-1) \rightarrow \text{R}(T)/(p, T-1)
\end{array}
\end{array}
\]

Note that $\Lambda' \cong \text{R}(T)/(T-1)\Phi_p(T)$.

Theorem 1.7. A full list of indecomposable $\Lambda$ lattices is as follows:

Type I. Any rank 1 projective module over the various rings

$A_1, A_2, R[T]/\Phi_p(T), R[T]/(T-1), \Lambda(j)$ for $1 \leq j \leq p$,

$\Lambda(j)$ for $1 \leq j \leq p - 1$, and $\Lambda'$.

Type (II; j) for $2 \leq j \leq p - 1$. (If $p = 2$, there is no $\Lambda$ lattice of Type (II; j).)

Type (III; j) for $1 \leq j \leq p - 1$.

(See Theorem 5.1 for more details.)

The description of lattices of Types (I, j) and (III; j) will be given below in Definitions 1.9 and 1.10.

Definition 1.8. Let $M, P, Q$, and $S$ be rank 1 projective modules over $A_1, A_2, R[T]/\Phi_p(T)$, and $R (= R[T]/(T-1))$, respectively. Note that both $M \otimes_A A_0$ and $P \otimes_A A_0$ are free $A_0$ modules of rank 1. Moreover, $Q \otimes_A A_0$ can be regarded as a free module of rank 1 over $R[T]/(p, (T-1)^{p-1})$ because

\[
Q \otimes A_0 = \left( \begin{array}{c} Q \\ R[T]/\Phi_p(T) \end{array} \right) \otimes A_0
\]

\[
\cong Q \otimes \left( \begin{array}{c} R[T]/\Phi_p(T) \\ A_0 \end{array} \right)
\]

\[
\cong Q \otimes \left( \begin{array}{c} R[T]/(p, (T-1)^{p-1}) \end{array} \right).
\]

Similarly, $S \otimes A_2 R$ is a rank 1 free module over $R/pR (\cong R[T]/(p, T-1))$.

Therefore, all these modules are cyclic modules when regarded as $A_0$ modules. We choose generators $w, x, y, z$ for them, i.e., $M \otimes_A A_0 = A_0 w, P \otimes_A A_0 = A_0 x, Q \otimes_A A_0 = A_0 y,$ and $S \otimes_A A_0 = A_0 z$. When the maps $M \rightarrow A_0 w, P \rightarrow A_0 x, Q \rightarrow A_0 y$, and $S \rightarrow A_0 z$ are unspecified, they are understood to be the canonical projections.
DEFINITION 1.9. For $2 \leq j \leq p - 1$, a $\Lambda$ lattice of Type (II; $j$) is the pull-back of the diagram

$$
\begin{array}{c}
\vdots \\
M_1 \\
\downarrow q_1 \\
P \oplus S \xrightarrow{q_2} R[T]/\langle p, (T-1)^j \rangle,
\end{array}
$$

where $M_1$, $P$, and $S$ are rank 1 projective modules over $A_1$, $A_2$, and $R$, respectively, with the maps $q_1$ and $q_2$ defined by

$$
q_1: M_1 \rightarrow A_0 w \xrightarrow{\delta_1} R[T]/\langle p, (T-1)^j \rangle, \quad w \mapsto \lambda_1,
$$

$$
q_2: P \oplus S \rightarrow A_0 x \oplus A_0 z \xrightarrow{\delta_2} R[T]/\langle p, (T-1)^j \rangle,
$$

$$
x \mapsto \lambda_2, \quad z \mapsto a(T-1)^{j-1},
$$

where $\lambda_1, \lambda_2 \in U(R[T]/\langle p, (T-1)^j \rangle)$ and $a \in U(R/pR)$. (Note that for a commutative ring $B$, $U(B)$ denotes its group of units.)

We shall say that $M$ is defined by $M_1$ and $P \oplus S$ with transition matrix

$$
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & a(T-1)^{j-1}
\end{pmatrix}.
$$

DEFINITION 1.10. For $1 \leq j \leq p - 1$, a lattice of Type (III; $j$) is the pull-back of the diagram

$$
\begin{array}{c}
\vdots \\
M_1 \\
\downarrow q_1 \\
Q \oplus S \xrightarrow{q_2} R[T]/\langle p, (T-1)^j \rangle,
\end{array}
$$

where $M_1$, $Q$, and $S$ are rank 1 projective modules over $A_1$, $R[T]/\Phi_p(T)$, and $R$, respectively, with the maps $q_1$ and $q_2$ defined by

$$
q_1: M_1 \rightarrow A_0 w \xrightarrow{\delta_1} R[T]/\langle p, (T-1)^j \rangle, \quad w \mapsto \lambda_1,
$$

$$
q_2: Q \oplus S \rightarrow A_0 y \oplus A_0 z \xrightarrow{\delta_2} R[T]/\langle p, (T-1)^j \rangle,
$$

$$
y \mapsto \lambda_2, \quad z \mapsto a(T-1)^{j-1},
$$

where $\lambda_1, \lambda_2 \in U(R[T]/\langle p, (T-1)^j \rangle)$ and $a \in U(R/pR)$.12
We shall say that $M$ is defined by $M_1$ and $Q \oplus S$ with transition matrix
\[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & a(T - 1)^{j-1}
\end{pmatrix}.
\]

The invariants of $\Lambda$ lattices of Type I can be studied via Picard groups over various rings and the associated Mayer–Vietoris sequences [15, Theorem 3.3, p. 28; 9, Sect. 4].

**Definition 1.11.** For $2 \leq j \leq p - 1$, denote by $\Sigma(\Pi; j)$ the set of all isomorphism classes of $\Lambda$ lattices of Type $(\Pi; j)$. For $1 \leq j \leq p - 1$, denote by $\Sigma(\Pi; j)$ the set of all isomorphism classes of $\Lambda$ lattices of Type $(\Pi; j)$.

**Definition 1.12.** For $1 \leq j \leq p - 1$, let $W_j$ be the cokernel of the group homomorphism $U(R[T]/\Phi_p(T)) \to U(R[T]/\langle p, (T - 1)^i \rangle)$, which is induced by the canonical projection $p_i^j: R[T]/\Phi_p(T) \to R[T]/\langle p, (T - 1)^i \rangle$.

For $1 \leq j \leq p - 1$, let $V_j$ be the cokernel of the group homomorphism
\[
\begin{align*}
U(A_1) \times U(R[T]/\Phi_p(R)) & \times U(R) \\
\to U(R[T]/\langle p, (T - 1)^i \rangle) \times U(R/pR),
\end{align*}
\]
where $\mu, \eta, \pi$ are the images of $\mu, \eta, \pi$ in the corresponding target groups, respectively.

Finally, $X$ is the cokernel of the map
\[
U(T) \times U(R/pR)^2 \to U(R/pR)
\]
where $U(R/pR)^2 := \{a^2: a \in U(R/pR)\}$, the subgroup of square classes in $U(R/pR)$.

**Theorem 1.13.** For $1 \leq j \leq p - 1$, there is a set-theoretic bijection $g_j$ defined by
\[
g_j: \Sigma(\Pi; j) \to \text{Pic}(A_1) \times \text{Pic}(R[T]/\Phi_p(T)) \times \text{Pic}(R) \times V_j,
\]
\[
[M] \mapsto ([M_1], [Q], [S], (\lambda_1^{-1}, \lambda_2, \lambda_3)),
\]
where $M$ is the $\Lambda$ lattice defined by $M_1$ and $Q \oplus S$ with transition matrix
\[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & (T - 1)^{j-1}
\end{pmatrix}.
\]

(The transition matrix may be simplified as above by the arguments in Case 3 of Section 5. See Theorem 5.1 also.)

**Theorem 1.14.** Assume that (i) $p \neq 2$, $\text{Pic}(R) = 0$, and $U(A_1) \to U(R/pR)$ is surjective or (ii) $p = 2$ and $R = \mathbb{Z}$ or a discrete valuation ring of rank $1$ (DVR). For $2 \leq j \leq p - 1$, there is a set-theoretic bijection $f_j$ defined by
\[
f_j: \sum (l; j) \to \text{Pic}(A_1) \times \text{Pic}(A_2) \times W_{j-1} \times X,
\]
\[
[M] \to ([M_1], [P], \lambda, \lambda),
\]
where $M$ is the $\Lambda$ lattice defined by $M_1$ and $P \oplus R$ with transition matrix
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda & (T - 1)^{j-1}
\end{pmatrix}.
\]

(The transition matrix may be simplified as above by Lemma 7.3.)

**Definition 1.15.** For any maximal ideal $\mathfrak{M}$ in $R$, $R_{\mathfrak{M}}$ denotes the localization of $R$ at $\mathfrak{M}$ and $R_{\hat{\mathfrak{M}}}$ denotes the completion of $R_{\mathfrak{M}}$. We shall define
\[
\Lambda_{\mathfrak{M}} := \Lambda \otimes_R R_{\mathfrak{M}}, \quad \hat{\Lambda}_{\mathfrak{M}} := \Lambda \otimes_R \hat{R}_{\mathfrak{M}}.
\]

If $M$ and $M'$ are two $\Lambda$ lattices, we shall say that $M$ and $M'$ belong to the same genus if
\[
M \otimes_{\Lambda} \hat{\Lambda}_{\mathfrak{M}} = M' \otimes_{\Lambda} \hat{\Lambda}_{\mathfrak{M}}
\]
for any maximal ideal $\mathfrak{M}$ in $R$.

**Theorem 1.16.** Let $M$ and $M'$ be two indecomposable $\Lambda$ lattices. Then $M$ and $M'$ belong to the same genus if and only if one of the following three statements holds: (i) both $M$ and $M'$ are rank $1$ projective modules over the same ring considered in Theorem 1.7, (ii) both $M$ and $M'$ are of Type $(II; j)$ for the same $j$, or (iii) both $M$ and $M'$ are of Type $(III; j)$ for the same $j$.

We remark that, besides $\mathbb{Z}$ or DVRs, there are lots of Dedekind domains and prime numbers $p$ satisfying conditions (R1), (R2), and the surjectivity for the map $U(A_1) \to U(R/pR)$ i.e., the extra assumption in
Theorem 1.14; for example let \( R = \mathbb{Z} + \mathbb{Z}(1 + \sqrt{5})/2 \) and \( p = 59 \). In fact, if the order of the image of some fundamental unit of a real quadratic field satisfying (R1) and (R2) is \( p^2 - 1 \) in \( R/pR \), then the pair \( R \) and \( p \) is a candidate. Moreover, in case \( R = \mathbb{Z} \), Theorems 1.7, 1.13, 1.14, and 1.16 correspond to [18, Theorem 7.3; 3, (34.35) Theorem, p. 739].

It remains to determine the structure of the groups \( W' \), \( V' \), and \( X \). When \( R = \mathbb{Z}, X = 0 \) if \( p \equiv 2 \) or \( p \equiv 3 \pmod{4} \) and \( X = \mathbb{Z}_2 \) if \( p \equiv 1 \pmod{4} \). On the other hand, the structure of \( W' \) has been studied by Galovich [4], Kervaire and Murthy [10, Sect. 6] when \( R = \mathbb{Z} \) and the prime number \( p \) is a semiregular prime number. (Sometimes a semiregular prime number is called a properly irregular prime number. See Definition 6.3 for more details.) Since Vandiver’s conjecture suggests that every prime number is semiregular [24, Remark, p. 159] and this conjecture has been verified for prime numbers less than 4,000,000 [23, 1], we cannot help but persuade ourselves that the problem has been solved for most (or many?) prime numbers.

Finally we should like to point out that in deducing our results, no previous results or knowledge of the theory of orders and integral representations is assumed. A good understanding of commutative algebra and the theory of projective modules [14; 15, pp. 19–32] will be enough to solve the problem considered in this paper. In fact we regard the group ring \( \Lambda = R[T]/(T^{p^2} - 1) \) as a one-dimensional noetherian ring whose spectrum is patched by two proper closed subspaces along a nonreduced closed subscheme. Such a viewpoint is different from the usual technique of extensions of lattices [18, Sects. 1–3]. This approach of studying \( \Lambda \) lattices was adopted by Swan [19, pp. 77–83], Haefner and Klinger [6], and others. An advantage of this approach is that it ensures the construction of indecomposable lattices more explicitly. We remark that by our method it is rather straightforward and functorial to determine the isomorphism classes.

The paper is organized as follows. We list some lemmas in Section 2 and the basic facts of pull-back diagrams are recalled in Section 3. The companion matrix is introduced in Section 4. It will be shown in Section 5 that in order to decompose a lattice into a direct sum of indecomposable lattices, it will suffice to simplify the companion matrix associated to the lattice. Thus the proof of Theorem 1.7 is finished. In Section 6, Theorem 1.13 is proved while Theorem 1.14 is proved in Section 7. The determination of \( W' \) is discussed in the last section and Theorem 1.16 is proved.

Standing Notation

All the notation introduced in this section will remain in force for subsequent sections unless otherwise specified. In particular, \( \Lambda := \mathbb{Z} + \mathbb{Z}(1 + \sqrt{5})/2 \) and \( p = 59 \). In fact, if the order of the image of some fundamental unit of a real quadratic field satisfying (R1) and (R2) is \( p^2 - 1 \) in \( R/pR \), then the pair \( R \) and \( p \) is a candidate. Moreover, in case \( R = \mathbb{Z} \), Theorems 1.7, 1.13, 1.14, and 1.16 correspond to [18, Theorem 7.3; 3, (34.35) Theorem, p. 739].

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2. SOME LEMMAS

LEMMA 2.1. Let B be a commutative ring, I be an ideal in B, M be a finitely generated B module, and N be a finitely generated B/I module. Assume that

(i) B/I is a zero-dimensional principal ideal ring, i.e., dim B/I = 0 and each ideal in B/I is generated by one element;

(ii) M = M₁ ⊕ M₂ ⊕ ⋯ ⊕ Mₘ, where each Mᵢ is a rank 1 projective B module;

(iii) The minimal number of generators for N is l and N = N₁ ⊕ N₂ ⊕ ⋯ ⊕ Nₗ, where each Nᵢ is a cyclic B/I module.

If ϕ: M → N is any surjection (of B modules), then there is an automorphism ψ: M → M such that ϕ ◦ ψ(Mᵢ) = Nᵢ for 1 ≤ i ≤ l and ϕ ◦ ψ(Mᵢ) = 0 for l + 1 ≤ i ≤ m. In particular, m ≥ l.

Remark. Compare [19, Corollary 4.24, p. 77] and the above lemma.

Proof. Since B/I is a zero-dimensional principal ideal ring, write

\[ \frac{B}{I} = B₁ ⊕ ⋯ ⊕ Bₙ, \]

where each Bᵢ is a zero-dimensional local ring whose maximal ideal is generated by uᵢ [26, Theorem 33, p. 245]. Let αᵢ be the positive integer so that uᵢ^αᵢ = 0 and uᵢ^αᵢ−1 ≠ 0. Since \( B₁/I, B₂/I, \ldots, Bₙ/I \) are principal ideal rings, finitely generated modules are direct sums of cyclic modules. In fact, all the cyclic modules, including the zero module, of \( Bᵢ \) are of the form

\[ Bᵢ/uᵢ^λ Bᵢ, \quad 0 ≤ λ ≤ αᵢ. \]
Consider the factorization of \( \varphi = \varphi_2 \circ \varphi_1 \).

\[
M \xrightarrow{\varphi_1} M/IM \xrightarrow{\varphi_2} N,
\]

where \( \varphi_1 \) is the natural projection.

Note that \( M/IM \) is a projective module of constant rank \( m \) over \( B/I \).

Hence we have

\[
M/IM \simeq B_1^m \oplus \cdots \oplus B_n^m. \tag{2}
\]

Consider the decomposition of \( N_j, 1 \leq j \leq l \), relative to the decomposition of \( B/I \) in (1). Write

\[
N_j = N_{j,1} \oplus \cdots \oplus N_{j,n},
\]

where each \( N_{j,i} \) is a cyclic module over \( B_j \). Note that it is allowed that \( N_{j,i} = 0 \) for some \( i \).

Note that \( \varphi_2(B_i^m) = N_{1,i} \oplus N_{2,i} \oplus \cdots \oplus N_{l,i} \); in particular, \( m \geq l \) since \( l \) is the minimal number of generators of \( N \). Furthermore, we get a surjection \( \varphi_2: B_i^m \to N_{1,i} \oplus N_{2,i} \oplus \cdots \oplus N_{l,i} \).

Let \( z_i \) be the generator of \( N_{i,i} \), i.e., \( N_{i,i} = B_i z_i, 1 \leq i \leq l \). When \( N_{i,i} = 0 \), we understand \( z_i = 0 \). Thus we have the surjection

\[
\varphi_2: B_i^m \to \bigoplus_{1 \leq t \leq l} B_i \cdot z_t \tag{3}
\]

In (2), \( B_i^m = (M_j \oplus \cdots \oplus M_m) \otimes_B B_i \). Choose \( x_1 \in M_1, \ldots, x_m \in M_m \) so that \( B_i^m = \bigoplus_{1 \leq j \leq m} B_i \cdot \bar{x}_j \), where \( \bar{x}_j \) is the image of \( x_j \) in \( M_j \otimes_B B_i \). (For different \( i \), these \( m \) elements \( x_1, \ldots, x_m \) might be chosen in different ways.)

In (3), take modulo the maximal ideal \( u_i B_i \). We get a surjection of vector spaces over the field \( B_i/u_i B_i \):

\[
\varphi_2: \bigoplus_{1 \leq j \leq m} (B_i/u_i B_i) \bar{x}_j \to \bigoplus_{1 \leq t \leq l} (B_i/u_i B_i) \bar{z}_t.
\]

Choose a basis \( \bar{y}_1, \ldots, \bar{y}_m \) in \( \bigoplus_{1 \leq j \leq m} (B_i/u_i B_i) \bar{x}_j \), so that \( \varphi(\bar{y}_1) = \bar{z}_1 \) for \( 1 \leq t \leq l \) and \( \varphi \) maps the remaining \( \bar{y}_s \) into zero. Lift \( \bar{y}_1, \ldots, \bar{y}_m \) to \( y_1, \ldots, y_m \) in \( \bigoplus_{1 \leq j \leq m} B_i \cdot \bar{x}_j \). By Nakayama's lemma, \( \{y_1, \ldots, y_m\} \) is also a free basis of the module \( \bigoplus_{1 \leq j \leq m} B_i \cdot \bar{x}_j \). Find an \( \alpha_i \in \text{SL}_m(B_i) \) and an invertible element \( c \) in \( B_i \) so that \( \alpha_i \) maps \( \{\bar{x}_1, \ldots, \bar{x}_m\} \) onto \( \{c y_1, \ldots, y_m\} \) with \( \alpha_i(\bar{x}_j) = c y_j \), \( \alpha_i(\bar{x}_j) = y_j \) for \( 2 \leq t \leq m \).

For each \( i, 1 \leq i \leq n \), construct \( \alpha_i \in \text{SL}_m(B_i) \) as above. Let \( \alpha \) be the automorphism of \( M/IM = B_1^m \oplus \cdots \oplus B_n^m \) whose \( i \)-th component is \( \alpha_i \).

Since \( \det(\alpha) = 1 \) and \( B/I \) is a zero-dimensional noetherian ring, by
Lemma 2.2. below, $\alpha$ can be written as a finite product of elementary matrices. Now $\alpha$ can be lifted to an automorphism $\psi$ of $M$ by Lemma 2.3 below. Moreover, $\psi$ has the required property.

For the sake of completeness, we record the following two lemmas.

**Lemma 2.2.** If $\Lambda$ is a commutative ring with Jacobson radical $J$, suppose that, for any $\psi \in \text{SL}_n(\Lambda/J)$, $\psi$ can be written as a finite product of elementary matrices over $\Lambda/J$. Then, for any $\varphi \in \text{SL}_n(\Lambda)$, $\varphi$ can be written as a finite product of elementary matrices over $\Lambda$.

**Proof.** Suppose $\varphi \in \text{SL}_n(\Lambda)$. Let $\Phi: \text{SL}_n(\Lambda) \to \text{SL}_n(\Lambda/J)$ be the group homomorphism induced by the natural projection $\Lambda \to \Lambda/J$. Denote $\Phi(\varphi) = \psi \in \text{SL}_n(\Lambda/J)$. By the assumption, $\psi = \psi_1\psi_2 \cdots \psi_n$ for some integer $n$ so that each $\psi_i$ is an elementary matrix. Choose any elementary matrix $\varphi_i$ over $\Lambda$ for each $1 \leq i \leq n$ so that $\Phi(\varphi_i) = \psi_i$. Then

$$\varphi = \varphi_0\varphi_1\varphi_2 \cdots \varphi_n$$

for some $\varphi_0 \in \ker(\Phi)$. Write

$$\varphi_0 = (b_{ij})_{1 \leq i, j \leq m},$$

where $b_{11}, b_{22}, \ldots, b_{nn}, b_{1n} \in 1 + J$ and $b_{ij} \in J$ if $i \neq j$. Since $b_{ii}$ is invertible for $1 \leq i \leq m$, it follows that $\varphi_0$ can be transformed into a diagonal matrix $\text{diag}(b_{11}, \ldots, b_{nn})$ by successive elementary row and column operations, where $b_{ii} \in 1 + J$ for $1 \leq i \leq m$. By (4), it is clear that $\det(\varphi_0) = 1$ and thus $b_{11} \cdots b_{nn} = 1$. Now, by similar transformations, the matrix $\text{diag}(b_{11}, \ldots, b_{nn})$ can be transformed to the identity matrix. Hence the result.

**Lemma 2.3** [19, Lemma 4.23, p. 76–76]. Let $B$ be a commutative ring, $I$ an ideal in $B$ and let $M$ be a finitely generated $B$ module. Assume that

(i) $M = M_1 \oplus M_2 \oplus \cdots \oplus M_m$, where each $M_i$ is a rank 1 projective $B$ module;

(ii) for $1 \leq i \leq m$, $M_i/IM_i$ is a free $B/I$ module with basis $\bar{x}_i$, where $x_i \in M_i$.

Suppose that $\alpha$ is an automorphism of $M/IM$ such that $\alpha$ is an elementary matrix over $B/I$ with respect to the basis $\bar{x}_1, \ldots, \bar{x}_m$. Then there exists an automorphism $\psi: M \to M$ so that the following diagram commutes,

$$\begin{array}{ccc}
M & \xrightarrow{\varphi} & M \\
\downarrow & & \downarrow \\
M/IM & \xrightarrow{\alpha} & M/IM,
\end{array}$$

where the vertical maps are the canonical projections.
Proof. We may write
\[ \alpha = 1 + ae_{ji} \]
for some \( a \in B \) and some \( i \neq j \). Let
\[ g: M_i/IM_i \to M_j/IM_j \]
be defined by \( g(x_i) = ax_j \). Since \( M_i \) is a projective \( B \) module, we can find \( f: M_i \to M_j \) so that the following diagram commutes:

\[ \begin{array}{ccc}
M_i & \xrightarrow{f} & M_j \\
\downarrow & & \downarrow \\
M_i/IM_i & \xrightarrow{g} & M_j/IM_j.
\end{array} \]

Let \( \epsilon \) be the endomorphism of \( M \) by sending \( M_i \) to \( M_j \) via \( f \) and sending all other \( M_i \) to zero for \( i \neq i \). Since \( i \neq j \), we find \( \epsilon \circ \epsilon = 0 \) and hence \( 1 + \epsilon \) is the desired automorphism on \( M \) lifting \( \alpha \).

For the convenience of the reader, we record one more lemma of elementary nature, whose proof is omitted.

**Lemma 2.4.** If \( B \) is a Dedekind domain and \( I \) is a nonzero ideal in \( B \), then \( B/I \) is a zero-dimensional principal ideal ring.

**Example 2.5.** Let \( n \) be a positive integer such that \( p \nmid n \) and \( \Phi_t(X) \) is irreducible in \( \mathbb{Z}/p\mathbb{Z}[X] \). Let \( R := \mathbb{Z}[\exp(2\pi \sqrt{-1}/n)] \). Then \( R \) satisfies conditions (R1) and (R2) in Definition 1.2.

Since \( \Phi_t(X) \) is irreducible in \( \mathbb{Z}/p\mathbb{Z}[X] \), it follows that \( pR \) is a maximal ideal in \( R \) by Kummer’s theorem [26, Theorem 34, p. 317].

On the other hand, \( R \) and \( \mathbb{Z}[T]/\Phi_p(T) \) (\( = \mathbb{Z}[\exp(2\pi \sqrt{-1}/p^2)] \)) are linearly disjoint over \( \mathbb{Z} \) because

\[
\left[ \Omega(\exp(2\pi \sqrt{-1}/n)) \otimes \Omega(\exp(2\pi \sqrt{-1}/p^2)) : \mathbb{Q} \right]
= \varphi(n \cdot p^2) = \left[ \Omega(\exp(2\pi \sqrt{-1}/np^2)) : \mathbb{Q} \right]
= \left[ \Omega(\exp(2\pi \sqrt{-1}/n)) \Omega(\exp(2\pi \sqrt{-1}/p^2)) : \mathbb{Q} \right].
\]

Hence the ring of integers in the quotient field of \( R \otimes \mathbb{Z} [T]/\Phi_p(T) \) is just \( R \otimes \mathbb{Z} [T]/\Phi_p(T) \) by [13, Proposition (1.4), p. 91]. In particular, \( R[T]/\Phi_p(T) \) is a Dedekind domain. Similarly, \( R[T]/\Phi_p(T) \) is also a Dedekind domain.
Example 2.6. Let \( n \) be a nonzero square-free integer and \( R \) be the ring of integers on \( \mathbb{Q}(\sqrt[n]{n}) \). Assume that

(i) \( p \nmid n \).

(ii) \( n \equiv 5 \mod 8 \) when \( p = 2 \), and \( (n/p) = -1 \) when \( p \) is an odd prime number.

Then \( R \) satisfies conditions (R1) and (R2) in Definition 1.2.

By [26, Theorem 32, p. 313], \( R \) satisfies the condition (R1). Applying similar arguments as in Example 2.5 and using [13, Proposition (1.4), p. 91], it is easy to verify that \( R \) satisfies (R2).

Example 2.7. Using similar arguments as in Examples 2.5 and 2.6, we find that if \( R \) is a Dedekind domain with quotient field \( K \) such that \( K \) and \( \mathbb{Q}(exp(2\pi\sqrt{-1}/p^2)) \) are linearly disjoint over \( \mathbb{Q} \), e.g., when \( K \) is an algebraic number field with \([K:Q]\) and \( \varphi(p^2) = p^2 - p \) being relatively prime or when \( \mathbb{Q} \) is algebraically closed in \( K \), then \( R \) satisfies condition (R2).

3. Pull-Back Diagrams

Let \( I_1 \) and \( I_2 \) be ideals in \( \Lambda \) generated by \( \Phi_p(T) \) and \( T^p - 1 \), respectively. If \( M \) is a \( \Lambda \) lattice, define

\[
K_i := \{v \in M; I_i \cdot v = 0\} \quad \text{for} \quad 1 \leq i \leq 2;
\]

\[
M_1 := M/K_2, \quad M_2 := M/K_1, \quad M_0 := M/(K_1 + K_2).
\]

It is easy to check that \( M_i \) is an \( A_i \) module for \( 0 \leq i \leq 2 \) and both \( K_i \) and \( M_i \) are \( \Lambda \) lattices for \( 1 \leq i \leq 2 \). Moreover, the following are pull-back diagrams of rings and modules, respectively,

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{\pi_1} & A_1 \\
\downarrow{\pi_1} & & \downarrow{\pi_1} \\
A_2 & \xrightarrow{p_2} & A_0
\end{array}
\quad
\begin{array}{ccc}
M & \xrightarrow{f_1} & M_1 \\
\downarrow{\varphi_1} & & \downarrow{\varphi_1} \\
M_2 & \xrightarrow{q_2} & M_0
\end{array}
\]

i.e., for any \( a_i \in A_i \) (resp. \( a_i \in M_i \)) for \( i = 1, 2 \) with \( p_1(a_1) = p_2(a_2) \) (resp. \( q_1(a_1) = q_2(a_2) \)), there is a unique element \( a \in A \) (resp. \( a \in M \)) so that \( \pi_i(a) = a_i \) (resp. \( \varphi_i(a) = a_i \)), where all the maps are canonical projections.
Conversely, if both \( q_1 \) and \( q_2 \) are surjective, and \( M \) is a pull-back of the diagram

\[
\begin{array}{c}
\text{M}_1 \\
\downarrow \ \phi_1 \\
\text{M}_2 \\
\rightarrow \ \phi_2 \\
\downarrow \\
\text{M}_0
\end{array}
\]

where \( M_i \) is an \( A_i \) module for \( 0 \leq i \leq 2 \), and both \( M_1 \) and \( M_2 \) are torsion-free over \( R \), then \( M \) is a \( \Lambda \) lattice; we shall denote \( M \) by \( \text{pbk}(M_1, M_2, M_0; q_1, q_2) \). Moreover, if we now use \( M \) to construct the pull-back diagram of the preceding paragraph, we see that the two pull-back diagrams are isomorphic.

Suppose that \( M \) and \( M' \) are two \( \Lambda \) lattices and define \( M_1, M_2, M_0, M_1', M_2', M_0' \) as above. It is clear that if \( \varphi: M \rightarrow M' \) is a homomorphism over \( \Lambda \), then \( \varphi \) induces homomorphisms \( \varphi_i: M_i \rightarrow M_i' \) over \( A_i \) for \( 0 \leq i \leq 2 \).

**Lemma 3.1.** Let \( M_i' \) and \( M_i \) be \( A_i \) modules for \( 0 \leq i \leq 2 \), \( M = \text{pbk}(M_1, M_2, M_0; q_1, q_2) \), and \( M' = \text{pbk}(M_1', M_2', M_0'; q_1', q_2') \), where \( q_i \) and \( q_i' \) are epimorphisms for \( 1 \leq i \leq 2 \). If \( \varphi: M \rightarrow M' \) is a homomorphism over \( \Lambda \), then there are induced homomorphisms \( \varphi_i: M_i \rightarrow M_i' \) over \( A_i \) for \( 0 \leq i \leq 2 \) so that the following two diagrams commute:

\[
\begin{array}{c}
\text{M}_1 \\
\downarrow \ \phi_1 \\
\text{M}_2 \\
\rightarrow \ \phi_2 \\
\downarrow \\
\text{M}_0
\end{array}
\]

Conversely, given homomorphisms \( \phi_i: M_i \rightarrow M_i' \) over \( A_i \), making the above diagrams commute, we obtain a unique homomorphism \( \varphi: M \rightarrow M' \) inducing the \( \phi_i \). Moreover, in this situation, \( \phi \) is an isomorphism if and only if each \( \phi_i \) is an isomorphism.

**Proof.** Assume the existence of \( \phi_i \) for \( 0 \leq i \leq 2 \). Define \( \varphi: M \rightarrow M' \) by \( \varphi((v_1, v_2)) = (\varphi_1(v_1), \varphi_2(v_2)) \), where \( v_i \in M \), and \( q_1(v_1) = q_2(v_2) \). Conversely, suppose that \( \varphi: M \rightarrow M' \) is an isomorphism (resp. homomorphism). Since \( M_i \) and \( M_i' \) are uniquely determined by \( M \) and \( M' \), e.g., \( M_1 = \text{Cokernel}(K_2 \rightarrow M) \) and \( K_2 := \{ v \in M : I_2 \cdot v = 0 \} \), the homomorphisms \( \varphi_i \) are also uniquely determined.

**Lemma 3.2.** Let \( B \) be any commutative noetherian ring and \( M \) be a rank 1 projective \( B \) module. Then \( \text{End}_B(M) = B \) and \( \text{Aut}_B(M) = U(B) \).

**Proof.** It suffices to show that \( \text{End}_B(M) = B \). Obviously, \( B \) is a subring of \( \text{End}_B(M) \). It remains to show that the inclusion \( B \hookrightarrow \text{End}_B(M) \) is an
isomorphism as $B$ modules. We may check it at every localization of $B$, but it is certainly the case in the local situation.

Notation 3.3. Suppose $P$ is a rank 1 projective $A_2$ module and $S$ is a rank 1 projective module over $R = R[T]/(T - 1)$. We shall write an endomorphism $\varphi \in \text{End}_{A_2}(P \oplus S)$ as

$$
\varphi = \begin{pmatrix} \nu & \varphi_1 \\ \varphi_2 & \lambda \end{pmatrix},
$$

where $\nu \in \text{End}_A(P) = A_2$, $\varphi_1 \in \text{End}_{A_2}(S, P)$, $\varphi_2 \in \text{End}_{A_2}(P, S)$, and $\lambda \in \text{End}_{A_2}(S) = R$. We shall identify $\varphi_2 \circ \varphi_1$ with an element in $R$. Similar notation works for endomorphisms of $Q \oplus S$ when $Q$ is a rank 1 projective module over $R[T]/\Phi_p(T)$.

Lemma 3.4. Let $P$, $Q$, and $S$ be rank 1 projective modules over $A_2$, $R[T]/\Phi_p(T)$, and $R$, respectively. Then

$$
\text{Aut}_{A_2}(P \oplus S) = \left\{ \begin{pmatrix} \nu & \varphi_1 \\ \varphi_2 & \lambda \end{pmatrix} \in \text{End}_{A_2}(P \oplus S) : \varphi_2 \circ \varphi_1 - \bar{\nu}\lambda \in U(R) \right\},
$$

and $\bar{\nu} \in U(R[T]/\Phi_p(T))$, and $\lambda \in U(R)$.

$$
\text{End}_{A_2}(Q, S) = \text{End}_{A_2}(S, Q) = 0,
$$

$$
\text{Aut}_{A_2}(Q \oplus S) = \left\{ \begin{pmatrix} \eta & 0 \\ 0 & \lambda \end{pmatrix} \in \text{End}_{A_2}(Q \oplus S) : \eta \in U(R[T]/\Phi_p(T)) \right\}.
$$

(Note that in $\text{Aut}_{A_2}(P \oplus S)$, one of $\bar{\nu}$ denotes the image of $\nu$ in $R$, the other denotes the image of $\nu$ in $R[T]/\Phi_p(T)$.)

Proof. Let

$$
\varphi = \begin{pmatrix} \nu & \varphi_1 \\ \varphi_2 & \lambda \end{pmatrix} \in \text{End}_{A_2}(P \oplus S).
$$

We shall show that $\varphi$ is an automorphism if and only if $\varphi_2 \circ \varphi_1 - \bar{\nu}\lambda \in U(R)$ and $\bar{\nu} \in U(R[T]/\Phi_p(T))$.

Since the properties we consider respect the local–global principle (with respect to maximal ideals in $R$), hence we may assume that both $P = A_2$ and $S = R$. In this case, $\varphi_1 \in \text{End}_{A_2}(R, A_2) = \{a \in A_2 : (T - 1) \cdot a = 0\}$.
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\[ \varphi = \begin{pmatrix} \nu & s \Phi_p(T) \\ r & \lambda \end{pmatrix}, \]

where \( r, s \in R \). Note that \( \varphi_2 \circ \varphi_1 \) becomes the multiplication by \( prs \).

Suppose that \( \varphi \) is an automorphism. Then \( \overline{\varphi} : (P \oplus R) \otimes_{A_2} R[T]/(T - 1) \to (P \oplus R) \otimes_{A_2} R[T]/(T - 1) \) is also an automorphism. Since \((P \oplus R) \otimes_{A_2} R[T]/(T - 1) = R^2 \) and \( \overline{\varphi} \) becomes

\[ \begin{pmatrix} \overline{\varphi} & ps \\ r & \lambda \end{pmatrix} \in \text{GL}_2(R), \]

it follows that \( prs - \overline{\varphi} \lambda \in U(R) \). Similarly, \( \overline{\varphi} : (P \oplus S) \otimes_{A_2} R[T]/\Phi_p(T) \to (P \oplus S) \otimes_{A_2} R[T]/\Phi_p(T) \) is an automorphism and \( \overline{\varphi} \) becomes

\[ \begin{pmatrix} \overline{\varphi} & 0 \\ r & \lambda \end{pmatrix}. \]

Thus \( \overline{\varphi} \in U(R[T]/\Phi_p(T)). \)

On the other hand, suppose that \( prs - \overline{\varphi} \lambda \in U(R) \), and \( \overline{\varphi} \in U(R[T]/\Phi_p(T)). \) We shall show that \( \varphi \) is an automorphism.

Note that

\[ A_2 \xrightarrow{(s \Phi_p(T)} R[T]/\Phi_p(T) \]
\[ R \simeq R[T]/(T - 1) \xrightarrow{\langle p, T - 1 \rangle} R[T]/\langle p, T - 1 \rangle \]

is a pull-back diagram, where all the maps are the canonical projections. Hence we may regard \( A_2 \oplus R \) as the pull-back of the diagram

\[ R[T]/\Phi_p(T) \]
\[ R \oplus R \xrightarrow{h} R[T]/\langle p, T - 1 \rangle \cong R/pR, \]

where the vertical map is the canonical projection and the horizontal map \( h \) is defined by \( h((r_1, r_2)) = r_2. \)

By Lemma 3.1, it is easy to see that the endomorphism \( \varphi \) is represented by

\[ \begin{array}{ccc}
R[T]/\Phi_p(T) & \xrightarrow{\varphi_1} & R[T]/\Phi_p(T) \\
\downarrow & & \downarrow \\
R/pR & \xrightarrow{\varphi_0} & R/pR
\end{array} \]

and

\[ \begin{array}{ccc}
R \oplus R & \xrightarrow{\varphi_2} & R \oplus R \\
\downarrow h & & \downarrow h \\
R/pR & \xrightarrow{\varphi_0} & R/pR
\end{array} \]
where $\psi_3$ is the multiplication by $\bar{\nu} \in R[T]/\Phi_p(T)$, $\psi_0$ is the multiplication by $\bar{\nu} \in R/pR$, and $\psi_2$ is given by

$$
\begin{pmatrix}
\bar{\nu} & ps \\
r & \lambda
\end{pmatrix}.
$$

By assumptions, $\psi_i$ is an automorphism for $0 \leq i \leq 2$. Hence $\varphi$ is an automorphism by Lemma 3.1.

To show that $\text{End}_{A_2}(Q, S) = 0$, as before it suffices to consider the case when $Q = R[T]/\Phi_p(T)$ and $S = R$. But $\text{End}_{A_2}(A_2/\Phi_p(T), A_2/(T - 1)) = \{\alpha \in A_2; (T - 1)\alpha = \Phi_p(T)\alpha = 0\} = 0$ since $\langle T - 1, \Phi_p(T) \rangle \cap R = pR$ and $A_2$ is torsion-free over $R$.

We leave it for the reader to determine $\text{End}_{A_2}(S, Q)$ and $\text{Aut}_{A_2}(Q \oplus S)$.

**Notation 3.5.** Suppose that $P$ is a rank 1 projective $A_2$ module such that $P \otimes_{A_2} R$ is a free module. Hence we write $P \otimes_{A_2} R = R \cdot z$ and choose a preimage $v \in P$ of $z$. By the local–global principle, we find that $\{w \in P; (T - 1) \cdot w = 0\} = \Phi_p(T) \cdot P$. It is easy to check that for any $\varphi_1 \in \text{End}_{A_2}(R, P)$, $\varphi_1(1) = s \Phi_p(T)v$ for some $s \in R$. On the other hand, for any $\varphi_2 \in \text{End}_{A_2}(P, R)$, $\varphi_2$ factors through the canonical projection from $P$ onto $P \otimes_{A_2} R = R \cdot z$. If $\varphi_2 \colon P \rightarrow R \cdot z \xrightarrow{\epsilon} R$ and $\varphi_2(v) = r$, i.e., $\bar{\varphi}_2(z) = r$, we shall identify $\varphi_2$ with the “multiplication” by $r$. Hence any $\varphi \in \text{End}_{A_2}(P \otimes R)$ will be written as

$$
\varphi = \begin{pmatrix} v & s \Phi_p(T) \\ r & \lambda \end{pmatrix},
$$

where $v \in A_2$ and $r, s, \lambda \in R$. Since $P \otimes_{A_2} R = R \cdot z$, it follows that $P \otimes_{A_2} R[T]/\langle p, T - 1 \rangle = (R/pR) \cdot \bar{z}$. By Nakayama’s lemma, it follows that

$$
P \otimes_{A_2} A_0 = A_0 z',
$$

where $z'$ is the image of $v \in P$ in $P \otimes_{A_2} A_0$ (Note that $v \mapsto z' \mapsto \bar{z} \in P \otimes_{A_2} R/pR$).

Hence, relative to the basis $(z', 1)$, $\varphi \otimes_{A_2} \text{id}_{A_0}$ can be written as

$$
\begin{pmatrix}
\bar{\nu} & 0 \\
r & \lambda
\end{pmatrix}.
$$

**Lemma 3.6.** All the $A$ lattices of Type I, Type (II; $j$) for $2 \leq j \leq p - 1$, and Type (III; $j$) for $1 \leq j \leq p - 1$ in Theorem 1.7 are indecomposable.
Proof. Let $M$ be any lattice described in Theorem 1.7. Define $M_1, M_2,$ and $M_0$ as the beginning of this section so that each $M_i$ is an $A_i$ module for $0 \leq i \leq 2$ and $M = \text{pbk}(M_1, M_2, M_0; q_1, q_2)$.

Suppose that $M$ decomposes. Then there exists a nontrivial idempotent $e \in \text{End}_A(M)$, i.e., $e^2 = e, \ e \neq 0, 1$. This endomorphism determines idempotent endomorphisms $e_i \in \text{End}_{A_i}(M_i)$ for $0 \leq i \leq 2$ so that the following diagrams commute:

\[
\begin{array}{ccc}
M_1 & \xrightarrow{e_1} & M_1 \\
\downarrow{q_1} & & \downarrow{q_1} \\
M_0 & \xrightarrow{e_0} & M_0
\end{array}
\quad
\begin{array}{ccc}
M_2 & \xrightarrow{e_2} & M_2 \\
\downarrow{q_2} & & \downarrow{q_2} \\
M_0 & \xrightarrow{e_0} & M_0
\end{array}
\]

We shall provide the proof for the case when $M$ is of Type (11; j) and leave the verification of other cases to the reader.

If $M$ is of Type (11; j), then $M$ is defined by $M_1$ and $M_2 := P \oplus S$, where $M_1$, $P$, and $S$ are rank 1 projective modules over $A_1$, $A_2$, and $R$, respectively.

Since $A_1$ is a Dedekind domain and $e_1 \in \text{End}_{A_1}(M_1) = A_1$ by Lemma 3.2, it follows that $e_1 = 1$ or 0 because $A_1$ has no nontrivial idempotent. Hence $e_0 = 0$ or 1 also.

Suppose that $e_1 = 0$. Thus $e_0 = 0$ also. Since $e \neq 0$, it follows that $e_2 \neq 0$. Hence $e_2(P \oplus S)$ is a nonzero direct summand of $P \oplus S$. By the uniqueness of integral representations of the cyclic group of order $p$ [16; 19, Theorem 4.19, p. 74], we find that $e_2(P \oplus S) = P \oplus S$ or $e_2(P \oplus S)$ is isomorphic to $P$ or $S$. In particular, $e_2(P \oplus S)$ contains a direct summand (of $P \oplus S$) which is isomorphic to $P$ or $S$. Thus $q_2(e_2(P \oplus S)) \neq 0$. But this contradicts the fact that $q_2 = e_0 = q_2$ and $e_0 = 0$.

Suppose that $e_1 = 1$, and therefore $e_0 = 1$. We shall show that the only idempotent endomorphism making the second diagram commute is the identity. Hence $e = 1$.

It remains to show that $q_2 = q_2 \circ e_2$ implies that $e_2 = 1$. Localizing at any maximal ideal of $R$, we may assume that $P = A_2$ and $S = R$. Adopting the convention in Notation 3.5, write

\[
e_2 = \begin{pmatrix} \nu & s \Phi_p(T) \\ r & \lambda \end{pmatrix},
\]

where $\nu \in A_2, r, s, \lambda \in R$.

By Definition 1.9, the equality $q_2 \circ e_2 = q_2$ is equivalent to

\[
\begin{pmatrix} \lambda_2 & a(T - 1)^{-1} \\ a(T - 1)^{-1} & \lambda_2 \end{pmatrix} = \begin{pmatrix} \nu & s \Phi_p(T) \\ r & \lambda \end{pmatrix}.
\]
Hence we get that $\lambda_2(1 - \overline{\nu}) = a\overline{\nu}(T - 1)^{j-1}$ in $R[T]/\langle p, (T - 1)^j \rangle$ and $a(\overline{\lambda} - 1) = 0$ in $R/pR$. Since $j \geq 2$ and $\lambda_2 \in U(R[T]/\langle p, (T - 1)^j \rangle)$, $a \in U(R/pR)$, thus $\overline{\nu} = 1 = \overline{\lambda}$ in $R/pR$. Hence $\overline{\nu} + \overline{\lambda} - 1 \neq 0$ in $R/pR$ and therefore $\overline{\nu} + \overline{\lambda} - 1 \neq 0$ in $R$.

Compare the entries of the matrix $e_2^2 = e_2$. It follows that

$$\nu^2 - \nu + rs\Phi_p(T) = s(\nu + \lambda - 1)\Phi_p(T) = 0 \quad \text{in } A_2$$

and

$$\lambda^2 - \lambda + prs = r(\overline{\nu} + \overline{\lambda} - 1) = 0 \quad \text{in } R.$$

Since $\overline{\nu} + \overline{\lambda} - 1 \neq 0$ in $R$, it follows that $r = 0$, $\lambda^2 = \lambda$, and $\nu^2 = \nu$. Since $A_2$ has no nontrivial idempotent and $\overline{\nu} = \overline{\lambda} = 1$ in $R/pR$, it follows that $\nu = 1$, $\lambda = 1$, and thus $s = 0$.

**Lemma 3.7.** Let $M$ and $M'$ be $\Lambda$ lattices of Type $(II; j)$ for $2 \leq j \leq p - 1$, defined by $M_1$ and $P \oplus S$ (for both $M$ and $M'$). If

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & a(T - 1)^{j-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda'_1 & 0 & 0 \\ 0 & \lambda'_2 & a'(T - 1)^{j-1} \end{pmatrix}$$

are the transition matrices of $M$ and $M'$, respectively, then $M \cong M'$ if and only if there exist $\lambda_0 \in U(R[T]/\langle p, (T - 1)^j \rangle)$, $\mu \in U(A_1)$, $\nu \in A_2$, $\overline{\nu} \in R/pR$, and

$$\psi := \begin{pmatrix} \nu \\ \psi_2 \\ \pi \end{pmatrix} \in \text{Aut}_{A_2}(P \oplus S)$$

so that

$$\begin{pmatrix} \lambda_0 \\ 0 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & a(T - 1)^{j-1} \end{pmatrix} = \begin{pmatrix} \lambda'_1 & 0 & 0 \\ 0 & \lambda'_2 & a'(T - 1)^{j-1} \end{pmatrix} \begin{pmatrix} \overline{\mu} & 0 & 0 \\ 0 & \overline{\nu} & 0 \\ 0 & \overline{\pi} & 0 \end{pmatrix} \quad (5)$$

as matrices over $R[T]/\langle p, (T - 1)^j \rangle$.

**Proof.** We shall prove the “only if” direction, leaving the converse to the reader. Choose an isomorphism $\phi: M \to M'$ and let $\lambda_0 \in U(R[T]/\langle p, (T - 1)^j \rangle)$, $\mu \in U(A_1)$, $\psi \in \text{Aut}_{A_2}(P \oplus S)$ stand for the automorphisms $\varphi_0, \varphi_1, \varphi_2$ in Lemma 3.1. Write $\psi$ in the form of Lemma 3.4.
Let $\bar{\psi}_2: P \otimes_{A_2} A_0 = A_0 x \to S \otimes_{A_2} A_0 = A_0 z$ be given by $\bar{\psi}_2(x) = \bar{r}z$ with $\bar{r} \in R/pR$. Note that $\bar{\psi}_1: S \otimes_{A_2} A_0 \to P \otimes_{A_2} A_0$ is the zero map since $\psi_4(S) \subset \Phi_p(T) \cdot P$.

Note that the commutativity of the two diagrams in Lemma 3.1 is equivalent to

$$\lambda_0 \lambda_1 = \lambda'_1 \bar{m}$$

and

$$\lambda_0 (\lambda_2 - \lambda(T - 1)^{j-1}) = (\lambda'_2 - \lambda'(T - 1)^{j-1}) \begin{pmatrix} \bar{p} & 0 \\ \bar{p} & \bar{m} \end{pmatrix},$$

which is equivalent to (5). Hence the result.

The proof of the following lemma is the same as that of Lemma 3.7, so the proof is omitted.

**Lemma 3.8.** Let $M$ and $M'$ be lattices of Type (III; $j$) for $1 \leq i \leq p - 1$, defined by $M_1$ and $Q \oplus S$ (for both $M$ and $M'$). If

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & a(T - 1)^{j-1} \end{pmatrix} \text{ and } \begin{pmatrix} \lambda'_1 & 0 & 0 \\ 0 & \lambda'_2 & \lambda'(T - 1)^{j-1} \end{pmatrix}$$

are the transition matrices of $M$ and $M'$, respectively, then $M = M'$ if and only if there exist $\lambda_0 \in U(R[T]/\langle p, (T - 1)^j \rangle)$, $\mu \in U(A_1)$, $\eta \in U(R[T]/\Phi_p(T))$, and $\pi \in U(R)$ so that

$$\begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & a(T - 1)^{j-1} \end{pmatrix} = \begin{pmatrix} \lambda'_1 & 0 & 0 \\ 0 & \lambda'_2 & \lambda'(T - 1)^{j-1} \end{pmatrix} \begin{pmatrix} \mu & 0 & 0 \\ \eta & 0 & 0 \\ 0 & 0 \end{pmatrix}$$

as matrices over $R[T]/\langle p, (T - 1)^j \rangle$.

4. **Companion Matrices**

Let $M$ be any $A$ lattice. Define $M_1, M_2, M_0$ as in the beginning of Section 3. Note that (i) each $M_i$ is an $A_i$ lattice for $1 \leq i \leq 2$ and $M_0$ is an $A_0$ module; (ii) the diagram

$$M \xrightarrow{\xi_1} M_1 \xrightarrow{\xi_2} M_2 \xrightarrow{q_2} M_0$$
is a pull-back diagram and, for $1 \leq i \leq 2$, $g_i$ is a surjective map over $A_i$ and $q_i$ is a surjective map over $A_j$.

Since $M_i$ is an $A_i$ lattice, it is torsion-free over $A_i$. Hence $M_i$ is a direct sum of rank 1 projective $A_i$ modules [14, Exercise 11.10, p. 86]. So we write

$$M_i = \bigoplus_{1 \leq m \leq e} E_m,$$

where $e$ is a nonnegative integer and each $E_m$ is a rank 1 projective $A_i$ module for $1 \leq m \leq e$.

The structure of $M_2$ has been determined essentially by Reiner [16]. In fact, it is not difficult to obtain the following generalized form of Theorem 1.1: $M_2$ is a direct sum of rank 1 projective modules over $A_2$, $R[T]/\Phi_p(T)$, or $R[T]/(T - 1)$. Hence we may write

$$M_2 = \left( \bigoplus_{1 \leq i \leq a} P_i \right) \oplus \left( \bigoplus_{1 \leq j \leq b} Q_j \right) \oplus \left( \bigoplus_{1 \leq k \leq c} S_k \right),$$

where $a, b, c$ are nonnegative integers, each $P_i$ is a rank 1 projective $A_2$ module, each $Q_j$ is a rank 1 projective module over $R[T]/\Phi_p(T)$, and each $S_k$ is a rank 1 projective module over $R[T]/(T - 1)$.

Since $A_0 = R[T]/\langle p, (T - 1)^p \rangle$ is a zero-dimensional local principal ideal ring because of condition (R1), the $A_0$ module $M_0$ is isomorphic to a direct sum of cyclic $A_0$ modules. So we write

$$M_0 = \bigoplus_{1 \leq l \leq d} A_0 u_l,$$

where $d$ is the number of elements in a minimal generating set of $M_0$ and for each $l$, $1 \leq l \leq d$,

$$A_0 u_l \cong R[T]/\langle p, (T - 1)^{\alpha_l} \rangle$$

for some integer $\alpha_l$ with $1 \leq \alpha_l \leq p$. We may arrange these $\alpha_l$s so that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d$. (Such an ordering is just for the sake of convenience for later discussion.)

Since each map $q_h: M_h \rightarrow M_0$ is a surjective $A_h$ map ($h = 1, 2$), we may factor it as

$$q_h: M_h \xrightarrow{\tilde{p}_h} M_h \otimes_{A_h} A_0 \xrightarrow{\tilde{q}_h} M_0,$$

where $\tilde{p}_h$ is the canonical projection and $\tilde{q}_h$ is a surjective $A_0$ map. To avoid excessive notation we also denote by $\tilde{p}_h$ and $\tilde{q}_h$ the induced maps on $E_m, P_i, Q_j$, and $S_l$, e.g., $P_i \xrightarrow{\tilde{p}_i} P_i \otimes_{A_2} A_0 \xrightarrow{\tilde{q}_i} M_0$, etc.
As in Definition 1.8, all the $A_0$ modules $E_m \otimes_{A_1} A_0$, $P_i \otimes_{A_2} A_0$, $Q_j \otimes_{A_2} A_0$, and $S_k \otimes_{A_2} A_0$ are cyclic modules. So we write

$$E_m \otimes_{A_1} A_0 = A_0 w_m = A_0,$$

$$P_i \otimes_{A_2} A_0 = A_0 x_i = A_0,$$

$$Q_j \otimes_{A_2} A_0 = A_0 y_j = R[T]/\langle p, (T - 1)^{p-1} \rangle,$$

$$S_k \otimes_{A_2} A_0 = A_0 z_k = R[T]/\langle p, T - 1 \rangle.$$

**Definition 4.1.** The companion matrix of $\tilde{q}_2$, denoted by $\Pi$, is the matrix of $\tilde{q}_2$ with respect to the bases $(x_i; 1 \leq i \leq a) \cup (y_j; 1 \leq j \leq b) \cup (z_k; 1 \leq k \leq c)$ and $(u_l; 1 \leq l \leq d)$. Namely,

$$\Pi = (\Pi_1; \Pi_2; \Pi_3),$$

where $\Pi_1$, $\Pi_2$, and $\Pi_3$ are matrices over $A_0$ with sizes $d \times a$, $d \times b$, and $d \times c$, respectively. For example, suppose that

$$\tilde{q}_2(y_j) = \sum_{1 \leq i \leq d} t_{ij} u_i.$$

Then $t_{ij}$ is the $(i, j)$th entry of $\Pi_2$. (Be careful that $(y_j)$, $(z_k)$, and $(u_l)$ are not bases of free modules over $A_0$; all of $A_0 y_j$, $A_0 z_k$, and $A_0 u_l$ are cyclic $A_0$ modules. Thus the entries of the matrix $\Pi$ are not uniquely determined, though this lack of uniqueness will not cause any problems.)

If $\varphi_0: M_0 \rightarrow M_0$ and $\varphi_2: M_2 \rightarrow M_2$ are automorphisms, we shall denote by $\varphi_0 \Pi \varphi_2^{-1}$ the companion matrix of the map

$$M_2 \otimes_{A_2} A_0 \xrightarrow{\varphi_0^{-1}} M_2 \otimes_{A_2} A_0 \xrightarrow{\tilde{q}_2} M_0 \xrightarrow{\varphi_2} M_0$$

with respect to the same bases, where $\varphi_2$ is the induced automorphism on $M_2 \otimes_{A_2} A_0$.

**Lemma 4.2.** Suppose that $a + b + c \geq 2$, $d \geq 1$, and there exist automorphisms $\varphi_0: M_0 \rightarrow M_0$ and $\varphi_2: M_2 \rightarrow M_2$ and integers $\alpha$ and $\beta$ with $1 \leq \alpha \leq d$, $1 \leq \beta \leq a + b + c$, so that

$$\varphi_0 \Pi \varphi_2^{-1} = \begin{pmatrix}
0 & * & \cdots & * \\
* & \cdots & \cdots & * \\
0 & \cdots & \cdots & 0 \\
* & \cdots & \cdots & * 
\end{pmatrix},$$
i.e., all the entries of $\varphi_0 \Pi \varphi_2^{-1}$ in the $\alpha$th row or $\beta$th column are zero except possibly the $(\alpha, \beta)$th. Then $M$ is a direct sum of two nonzero $\Lambda$ lattices.

Proof. Suppose, for example, $\alpha = 1 \leq a$ and $\beta = 2$. Since $q_1 : M_1 \to M_0 = \bigoplus_{1 \leq i \leq d} A_0 \cdot \varphi_1^{-1}(u_i)$ is surjective, by Lemma 2.1 we have $e \geq d$ and there is an automorphism $\varphi_1 : M_1 \to M_1$ so that $\varphi_0 \circ q_1 \circ \varphi_1^{-1}(E_m) = A_0 \cdot u_m$ for $1 \leq m \leq d$ and $\varphi_0 \circ q_1 \circ \varphi_1^{-1}(E_m) = 0$ for $d + 1 \leq m \leq e$. Hence $M$ is isomorphic to $\text{pbk}(M_1, M_2, M_0; \varphi_0 \circ q_1 \circ \varphi_1^{-1}, \varphi_0 \circ q_2 \circ \varphi_2^{-1})$ by Lemma 3.1; the latter pull-back diagram is isomorphic to a direct sum of the following two pull-back diagrams:

\[
\begin{array}{ccc}
P_1 & \xrightarrow{\varphi_0 \circ q_2 \circ \varphi_2^{-1}} & A_0 u_2 \\
\downarrow{\varphi_0 \circ q_1 \circ \varphi_1^{-1}} & & \downarrow{\varphi_0 \circ q_1 \circ \varphi_1^{-1}} \\
M_2/P_1 & \xrightarrow{\varphi_0 \circ q_2 \circ \varphi_2^{-1}} & \bigoplus_{i \neq 2} A_0 u_i
\end{array}
\]

(For details of the latter direct decomposition, see [19, pp. 77–80].)

Lemma 4.3. Let $P$, $Q$, and $S$ be rank 1 projective modules over $A_2$, $R[T]/\Phi_p(T)$, and $R[T]/(T - 1)$, respectively. Choose generators $x$, $y$, and $z$ for the $A_0$ modules $P \otimes A_1 A_2$, $Q \otimes A_1 A_2$, and $S \otimes A_1 A_2$, respectively.

(i) If $\psi_1 \in \text{Aut}_A((P \otimes Q) \otimes A_2 A_0)$ is given by

\[
\begin{pmatrix}
1 & (T - 1)\pi \\
0 & 1
\end{pmatrix}
\]

for some $\pi, \pi' \in A_0$ with respect to the bases $x$ and $y$, then $\psi_1$ can be lifted to $P \otimes Q$, i.e., there exists $\psi'_1 \in \text{Aut}_A((P \otimes Q)$ such that $\psi_1 = \psi'_1 \otimes A_2 \text{id}_{A_0}$.

(ii) If $\psi_2 \in \text{Aut}_A((P \otimes S) \otimes A_2 A_0)$ is given by

\[
\begin{pmatrix}
1 & (T - 1)\pi' \pi \\
0 & 1
\end{pmatrix}
\]

for some $\pi, \pi' \in A_0$ with respect to the bases $x$ and $z$, then $\psi_2$ can be lifted to $P \otimes S$, i.e., there exists $\psi'_2 \in \text{Aut}_A((P \otimes S)$ such that $\psi_2 = \psi'_2 \otimes A_2 \text{id}_{A_0}$.

Proof. (i) We shall consider the first matrix. Choose $v_1 \in P$, $v_2 \in Q$, and $\beta \in A_2$ such that they are mapped to $x$, $y$, and $\pi$ under the canonical projections. Let $N$ be the $A_2$ submodule of $P$ generated by $(T - 1)\beta v_1$. Note that $N$ is, in fact, a module over $R[T]/\Phi_p(T)$. Since $Q$ is a projective module over $R[T]/\Phi_p(T)$, there exists a morphism $\tilde{\omega}$ so that the following
diagram commutes,
\[
\begin{array}{c}
\xymatrix{ Q \ar[d] & N \ar[d] \\
A_0y \ar[r]^\tilde{\omega} & A_0(T - 1)x,}
\end{array}
\]
where \(\omega(y) := (T - 1)x\). Let us denote by \(\tilde{\omega}\) the \(A_2\) morphism \(Q \xrightarrow{\tilde{\omega}} N \subset P\). Then \(1 + \tilde{\omega}\) is the automorphism \(\psi_1\) which we are seeking.

The case for the second matrix is similar and easier. Hence its proof is omitted.

(ii) The proof for the first matrix is similar except that we consider the \(A_2\) submodule of \(P\) generated by \(\Phi_p(T)\beta'v_2\), where \(\beta' \in A_2\) is a preimage of \(\pi\). The details are omitted.

**Lemma 4.4.** If \(P\) and \(P'\) are rank 1 projective \(A_2\) modules, then every automorphism on \((P \oplus P') \otimes_{A_2} A_0\) of the form
\[
\begin{pmatrix}
1 & 0 \\
\pi & 1
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
1 & \pi' \\
0 & 1
\end{pmatrix},
\]
where \(\pi, \pi' \in A_0\) can be lifted to an automorphism on \(P \oplus P'\). The same conclusion holds for \(Q \oplus Q'\) and \(S \oplus S'\) when \(Q\) and \(Q'\) are rank 1 projective modules over \(R[T]/\Phi_p(T)\), and \(S\) and \(S'\) are rank 1 projective \(R\) modules.

**Proof.** Apply Lemma 2.3 to obtain the proof.

### 5. Reduction of the Companion Matrix

The notations in this section are the same as in the preceding section. We shall prove Theorem 1.7 in this section.

Let \(\Pi\) be the companion matrix of \(\widetilde{\eta}_2\) associated to the \(\Lambda\) lattice \(M\). If \(M_0 = 0\), then \(M\) is isomorphic to \(M_1 \oplus M_2\), which is isomorphic to a direct sum of rank 1 projective modules over \(A_1\), \(A_2\), \(R[T]/\Phi_2(T)\), or \(R[T]/(T - 1)\). Hence we may assume \(M_0 \neq 0\). We shall find automorphisms \(\varphi_0: M_0 \rightarrow M_0\) and \(\varphi_2: M_2 \rightarrow M_2\) so that \(\varphi_2 \Pi \varphi_2^{-1}\) will become a “normal form” of Lemma 4.2. We shall solve this problem according to the cases when (i) only one of \(a, b, c\) is not zero, (ii) \(ab \neq 0\) and \(c = 0\), (iii) \(ac \neq 0\) and \(b = 0\), (iv) \(bc \neq 0\) and \(a = 0\), and (v) \(abc \neq 0\).

**Case 1. Only one of \(a, b, c\) is not zero**

We shall apply elementary column operations on \(\Pi\), which amount to elementary matrix transformations on the free module \(M_2 \otimes_{A_2} A_0\). Note
that an elementary matrix transformation is a permissible operation on $M_2 \otimes_A A_0$ because of Lemma 4.4 or Lemma 2.3.

Since $q_2$ is surjective, there exists an invertible element of $A_0$ in the first row. By applying a column operation, we may assume that $\pi_{11} \in U(A_0)$, where $\pi_{11}$ is the entry in the (1,1) place of $\Pi$. By applying successive column operations we get a matrix $\Pi \varphi_2^{-1}$ such that every entry in the first row is zero except that in the (1,1)th place. Write

$$\Pi \varphi_2^{-1} = \begin{pmatrix} \pi_{11} & 0 & \cdots & 0 \\ \lambda_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ \lambda_d & \ddots & \cdots & \ddots \end{pmatrix}.$$ 

Now apply row operations to kill $\lambda_2, \ldots, \lambda_d$ in the above matrix. These row operations amount to an automorphism $\varphi_0: M_0 \to M_0$ satisfying

$$\varphi_0(u_1) = u_1 - \pi_{11}^{-1}(\lambda_2 u_2 + \cdots + \lambda_d u_d)$$

and

$$\varphi_0(u_l) = u_l \text{ for } 2 \leq l \leq d,$$

which is clearly a well-defined automorphism since the only relation for $u_1$ is $(T - 1)^{a_1} u_1 = 0$ and this relation is respected by $u_1 - \pi_{11}^{-1}(\lambda_2 u_2 + \cdots + \lambda_d u_d)$. (Remember that $a_1 \geq a_2 \geq \cdots \geq a_d$ and $(T - 1)^{a_1} u_1 = 0$ for $1 \leq l \leq d$.)

Thus we find

$$\varphi_0 \Pi \varphi_2^{-1} = \begin{pmatrix} \pi_{11} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \ddots & \cdots & \ddots \end{pmatrix}.$$ 

By Lemma 4.2, $M$ is decomposable if $a + b + c \geq 2$. If $M$ is indecomposable, then $a + b + c = 1$. Apply Lemma 2.1. We find that $e = 1$ also. Hence, if $M$ is indecomposable, then $M = \text{pbk}(M_1, M_2, M_0; q_1, q_2)$, where $M_1$ is a rank 1 projective $A_1$ module, $M_2$ is a rank 1 projective module over $A_2$, $R[T]/\Phi_j(T)$, or $R[T]/(T - 1)$, $M_0 = R[T]/\langle p, (T - 1)^{a_1} \rangle$ for some $j$, and both $q_1$ and $q_2$ are surjective. Therefore, $M$ is a rank 1 projective module over $\Lambda(j)$ (for $1 \leq j \leq p$), $\Lambda'(j)$ (for $1 \leq j \leq p - 1$), or $\Lambda'$ by [15, pp. 19 and 20].

Case 2. $ab \neq 0$ and $c = 0$
Subcase 2.1. Suppose that the top row of \( \Pi_2 \) contains a unit of \( A_0 \). Then there is some \( j, 1 \leq j \leq b \), so that \( \tilde{q}_2(y_j) = \lambda u_1 + \Sigma_{2 \leq i < d} (*) u_i \), where \( \lambda \in U(A_0) \) and the coefficients \((*)\) designate some unspecified elements of \( A_0 \). (We will use this notation below as well.)

Apply column operations to kill other entries in the first row of \( \Pi \). These operations can be lifted to automorphisms on \( M_2 \) by Lemma 4.3. Hence

\[
\Pi \varphi_2^{-1} = \begin{pmatrix}
0 & \cdots & 0 & 0 & \cdots & 0 & \lambda & 0 & \cdots & 0 \\
\ast & & & & & & & & & \\
\vdots & & & & & & & & & \\
\vdots & & & & & & & & & \\
\vdots & & & & & & & & & \\
\vdots & & & & & & & & & \\
\end{pmatrix}
\]

for some \( \varphi_2 \in \text{Aut}_{A_0}(M_2) \). By the same arguments as in Case 1, find \( \varphi_0 \in \text{Aut}_{A_0}(M_0) \) so that

\[
\varphi_0 \Pi \varphi_2^{-1} = \begin{pmatrix}
0 & \cdots & 0 & 0 & \cdots & 0 & \lambda & 0 & \cdots & 0 \\
\ast & & & & & & & & & \\
\vdots & & & & & & & & & \\
\vdots & & & & & & & & & \\
\vdots & & & & & & & & & \\
\vdots & & & & & & & & & \\
\end{pmatrix}
\]

Hence \( M \) is decomposable by Lemma 4.2. Note that since \( a \geq 1 \) and \( b \geq 1 \), \( M \) will never be indecomposable in this case.

Subcase 2.2. Suppose that \( \tilde{q}_2(y_j) = (T - 1) \pi_j u_1 + \Sigma_{2 \leq j \leq d} (*) u_i \) for all \( j \), where \( \pi_j \in A_0 \). Since \( \tilde{q}_2 \) is surjective, there exists some \( i, 1 \leq i \leq a \), so that \( \tilde{q}_2(x_i) = \lambda u_1 + \Sigma_{2 \leq j \leq d} (*) u_i \), where \( \lambda \in U(A_0) \).

We may apply column operations to kill other entries in the first row. Note that column operations in \( \Pi_2 \), i.e., among \( \tilde{q}_2(x_t) \) for \( 1 \leq t \leq a \), are permissible operations by Lemma 4.4. On the other hand, to kill \( \pi_j \) in \( \tilde{q}_2(y_j) \) by \( \tilde{q}_2(x_i) \), we shall use the matrix

\[
\begin{pmatrix}
1 & -(T - 1) \pi_j \lambda^{-1} \\
0 & 1
\end{pmatrix}
\]

relative to the basis \( \{x_i, y_j\} \). This automorphism can be lifted to one on \( M_2 \) by Lemma 4.3.
Thus there exist \( \varphi_2 \in \text{Aut}_{A_1}(M_2) \) and \( \varphi_0 \in \text{Aut}_{A_0}(M_0) \) so that
\[
\varphi_0 \varphi_2^{-1} = \begin{pmatrix}
0 & \cdots & 0 & \lambda_0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & * & \cdots & * & \cdots & * & \cdots & \cdots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
0 & & & & & & & & \\
\end{pmatrix}.
\]

Hence \( M \) is decomposable and it will never be an indecomposable \( \Lambda \) lattice in this case.

**Case 3.** \( ac \neq 0 \) and \( b = 0 \)

Since \((T - 1) \cdot z_k = 0\) and \((T - 1)^{a_i} \cdot u_i = 0\), we may write
\[
\tilde{q}_2(z_k) = \sum_{1 \leq l \leq d} (T - 1)^{a_i-1} \pi_{ik} u_i,
\]
where \( \pi_{ik} \in A_0 \) and, for each \( l \), either \( \pi_{ik} = 0 \) or \( \pi_{ik} \in U(A_0) \).

We shall simplify \( \Pi_3 \) by applying permissible column and row operations starting from the lower right corner so that

(i) every row of the new \( \Pi_3 \) has at most a nonzero entry (we do not exclude the possibility that some row of this new \( \Pi_3 \) is identically zero);

(ii) after deleting the rows (in \( \Pi_3 \)) that are identically zero, the remaining part of the new \( \Pi_3 \) has the form
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & \lambda_1 \\
0 & 0 & \cdots & \cdots & \lambda_2 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \cdots & \cdots & \lambda_d \\
\end{pmatrix},
\]

where the left-hand side of the above matrix is the zero matrix and the right-hand side of it is the diagonal matrix of rank \( d' \) (\( \leq d \)).

We shall call the above process the “diagonalization” process applied to the matrix \( \Pi_3 \). (Strictly speaking, when we apply column operations in “diagonalization,” only entries in \( \Pi_3 \) are changed; however, when we apply row operations, entries in \( \Pi_3 \), \( \Pi_2 \), and \( \Pi_1 \) are changed. Thus the name “diagonalization applied to the matrix \( \Pi_3 \)” is abused in some sense.)

The “diagonalization” process for \( \Pi_3 \) will proceed as follows. If the \( d \)th row of \( \Pi_3 \) is identically zero, then consider the \((d - 1)\)st row of \( \Pi_3 \).
Otherwise, applying permissible column operations, we may bring $\Pi_3$ into a matrix of the form

$$
\begin{pmatrix}
(T - 1)^{\alpha_1 - 1} \pi_{1,c} \\
* & \ddots & \\
0 & \cdots & (T - 1)^{\alpha_d - 1} \pi_{d,c}
\end{pmatrix}
$$

for some $\pi_{d,c} \in U(A_q)$. Since $\alpha_1 \geq \cdots \geq \alpha_d$, we can apply row operations to kill the entries lying above $(T - 1)^{\alpha_d - 1} \pi_{d,c}$.

Hence, $\Pi_3$ is transformed into the form

$$
\begin{pmatrix}
* & 0 \\
* & \ddots & \\
0 & \cdots & 0
\end{pmatrix}
$$

for some $\pi_{d,c} \in U(A_q)$.

We may now consider the $(d - 1)$th row of $\Pi_3$ and proceed in the same way. We finally transform $\Pi_3$ into a standard form mentioned before. If some column of this new matrix is identically zero, then there is a direct summand of the lattice $M$ and we have reduced the size of the companion matrix $\Pi_1$ to a matrix of size $(a + (c - 1)) \times d$. Similarly, if the first row of this new matrix is identically zero, apply the reduction process of Case 1 to the matrix $\Pi_1$. Hence we get a matrix of the form

$$
\begin{pmatrix}
0 & \cdots & 0 & \lambda & 0 & \cdots & 0 \\
0 & \ddots & \\
* & \cdots & * & \cdots & * & \ddots & \\
0 & \cdots & \\
\end{pmatrix}
$$

where $\lambda \in U(A_q)$. Note that $\Pi_3$ may not be in the standard form because of the row operations applied to $\Pi$; however, the first row of $\Pi_3$ remains zero, and therefore $M$ is decomposable by Lemma 4.2.

Hence we obtain a new matrix $\Pi$ such that either (a) the first column or the first row of $\Pi_3$ is identically zero or (b) we may assume that

$$
\tilde{q}_2(z_1) = (T - 1)^{\alpha_1 - 1} \pi_1 u_1,
$$
where $\pi_1 \in U(A_0)$. (Note that $d \geq c$ in this case. However, the situation $d > c$ may happen because some rows of the new matrix $\Pi_3$ may be identically zero.)

In case of (a), we get a direct summand of Type I for $M$ and reduce the size of the companion matrix. Hence we shall solve the case of (b).

If $\alpha_1 = 1$, applying permissible column operations to kill all entries of the first row of $\Pi_1$, we get a direct summand again and a smaller companion matrix of size $(a + (c - 1)) \times (d - 1)$.

Thus we may assume that $\alpha_1 \geq 2$. It follows that there exists some $i$, $1 \leq i \leq a$, so that $q_2(x_i) = \lambda u_i + \sum_{2 \leq j \leq d} (\ast) u_j$ for some $\lambda \in U(A_0)$. As before, we may bring $\Pi$ into the form

$$
\begin{pmatrix}
0 & \cdots & 0 & \lambda & 0 & \cdots & 0 & (T - 1)^{\alpha_1 - 1} \pi_1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \ast & \cdots & \cdots & \ast & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\ast & \cdots & \ast & \cdots & \cdots & \cdots & \ast & 0 \\
0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

By row operations, we may kill the entries lying below $\lambda$, but we may create nonzero terms below $(T - 1)^{\alpha_1 - 1} \pi_1$. However, the difficulty can be overcome by “diagonalization” of $\Pi_1$ once again. Note that the “diagonalization” process applied to $\Pi_1$ does not effect the entries of the column $(\lambda, 0, \ldots, 0)$ in $\Pi_1$, but it does create an effect on the first row of $\Pi_1$: it may change from $(0, \ldots, 0, \lambda, 0, \ldots, 0)$ into $(\ast, \ldots, \ast, \ast, \ldots, \ast)$. Thus we should apply column operations to $\Pi_1$ to ensure that the matrix $\Pi$ has the standard form

$$
\begin{pmatrix}
0 & \cdots & 0 & \lambda & 0 & \cdots & 0 & 0 & \cdots & 0 & \mu & 0 & \cdots & 0 \\
0 & \cdots & 0 & \ast & \cdots & \ast & \cdots & \ast & \cdots & \ast & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & \cdots & \ast & \cdots & \ast & \ast & \cdots & 0 & \cdots & 0
\end{pmatrix}
$$

where $\lambda \in U(A_0)$ and $\mu = 0$ or $(T - 1)^{\alpha_1 - 1} \pi_1$ with $\pi_1 \in U(A_0)$. If $\mu = 0$, then $M$ is decomposable as before. If $\mu = (T - 1)^{\alpha_1 - 1} \pi_1$, then $M$
has a direct summand which is a pull-back of $P \oplus S$ and some direct
summand of $M_1$.

We conclude that we get a direct summand which is either of the same

\[ M_1 \]

\[
\begin{pmatrix}
q_1 \\
1
\end{pmatrix}
\]

\[ \xrightarrow{\varphi_2} \]

\[ R[T]/\langle p, (T - 1)^i \rangle, \]

where $1 \leq j \leq p$ and $\varphi_2$ is given by $\varphi_2(x) = \lambda$, $\varphi_2(z) = (T - 1)^{-1} \pi_1$ for

some $\lambda$, and $\pi_1 \in U(A_0)$.

If $j = 1$, apply the column operation to reduce the matrix $(\lambda \quad \pi_1)$ into

$(0 \quad \pi_1)$. Hence this direct summand itself is decomposable. If $j = p$, apply the column operation

\[
\begin{pmatrix}
1 & -(T - 1)^{-1} \pi_1 \lambda^{-1} \\
0 & 1
\end{pmatrix},
\]

which is provided in Lemma 4.3. Again the matrix $(\lambda \quad -(T - 1)^{-1} \pi_1)$ is reduced to $(\lambda \quad 0)$.

In conclusion, besides the lattices of Type 1, we find a new type of

lattice, i.e., a lattice of Type (II; j) for $2 \leq j \leq p - 1$. By Lemma 3.7, we

may assume the transition matrix is of the form

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & (T - 1)^{-1}
\end{pmatrix},
\]

i.e., $\pi_1 = 1 \in R/pR$.

**Case 4.** $bc \neq 0$ and $a = 0$

The proof is very similar to that of Case 3. First we “diagonalize” $\Pi_3$.

Either we may decrease the size of the matrix or we may assume that

\[ q_2(z_1) = (T - 1)^{\alpha_1 - 1} \pi_1 u_1, \tag{6} \]

where $\pi_1 \in U(A_0)$.

Assume formula (6) is valid. If $\alpha_1 \geq 2$, we may proceed as in Case 3. It

remains to solve the case $\alpha_1 = \alpha_2 = \cdots = \alpha_d = 1$. (Note that $\alpha_1 \leq p - 1$

because $(T - 1)^{-1} y_j = 0$ for $1 \leq j \leq b$.) In this case, we may assume that

\[ \tilde{q}_2(y_j) = \sum_{1 \leq l \leq d} \lambda_{ij} u_l, \]

where $\lambda_{ij}$ is either in $U(A_0)$ or zero, because $(T - 1)u_l = 0$ for all $l$. 

If the first row of $\Pi_2$ is identically zero, then (6) provides a direct summand. Otherwise, by column operations and row operations, we may assume that $\Pi$ is of the form

$$
\begin{pmatrix}
0 & \cdots & 0 & \lambda & 0 & \cdots & 0 & \pi_1 & 0 & \cdots & 0 \\
0 & \cdots & * \\
* & \cdots & * & \cdots & * \\
0 & \cdots & * 
\end{pmatrix}.
$$

The above matrix can be simplified as in Case 3 by “diagonalizing” $\Pi_3$ once more and applying column operations to $\Pi_2$ if necessary. We conclude that, in this case, we obtain a new type of lattice, i.e., lattices of Type (III; $j$) for $1 \leq j \leq p - 1$. As before, we may assume the transition matrix is of the form

$$
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & (T - 1)^{j-1} 
\end{pmatrix}.
$$

Note that we cannot exclude the possibility $j = 1$ in this situation because the elementary matrix

$$
\begin{pmatrix}
1 & 0 \\
\pi & 1 
\end{pmatrix}
$$

relative to the basis $(y, z)$ cannot be lifted to an automorphism on $Q \oplus S$ due to the fact that $\text{Hom}_{A_1}(Q, S) = 0$ by Lemma 3.4.

**Case 5.** $abc \neq 0$

Proceed as in Case 4 to “diagonalize” $\Pi_3$. Either we may reduce the size of the matrix or we may assume that formula (6) is valid.

Assume formula (6).

**Subcase 5.1.** $\alpha_1 \geq 2$ and there exists $y_j$, $1 \leq j \leq b$, such that $\tilde{q}_j(y_j) = \pi u_1 + \sum_{i \neq 1} (\pi_1) u_i$ for some $\pi \in U(A_3)$. Apply the process of Subcase 2.1 to the matrix $(\Pi_2; \Pi_3)$. Hence $\Pi$ is transformed into the form

$$
\begin{pmatrix}
0 & \cdots & 0 & \lambda & 0 & \cdots & 0 & (T - 1)^{j-1} \pi_1 & 0 & \cdots & 0 \\
* & \cdots & * & \cdots & * \\
\cdots & \cdots & * & \cdots & * \\
\pi & \cdots & \pi & \cdots & \pi & \cdots & \pi 
\end{pmatrix}.
$$

Now apply row operations and “diagonalize” $\Pi_3$ once again as in Case 4 (or Case 3) and get a direct summand of $M$. 

Subcase 5.2. $\alpha_i \geq 2$ and $q_2(y_j) = (T - 1)(*)u_1 + \sum_{j \neq 1}(*)u_j$ for every $j, 1 \leq j \leq b$. Since $q_2$ is surjective, there exist an $i (1 \leq i \leq a)$ and $x_i$, such that $q_2(x_i) = \pi u_1 + \sum_{j \neq 1}(*)u_j$. Apply the process of Subcase 2.2 and proceed as above. The details are left to the reader.

Subcase 5.3. $\alpha_1 = \alpha_2 = \cdots = \alpha_d = 1$. In this case we can assume that every entry in $\Pi$ is either zero or an invertible element in $A_0$. If all the entries in the first rows of $\Pi_1$ and $\Pi_2$ are zero, then $M$ is decomposable because of formula (6). Otherwise, there are two situations. The first situation is $q_2(y_j) = \pi u_1 + \sum_{j \neq 1}(*)u_j$ for some $j$, where $\pi \in U(A_0)$; the second situation is $q_2(x_i) = \pi u_1 + \sum_{j \neq 1}(*)u_j$ for some $i$, where $\pi \in U(A_0)$ and $q_2(y_j) = \sum_{j \neq 1}(*)u_j$ for all $j$. These two situations are the same as those in Subcases 5.1 and 5.2.

Summarizing the above reduction, we conclude the proof of Theorem 1.7. For emphasis, we record it as follows:

Theorem 5.1. By applying permissible column and row operations on the companion matrix of $q_2$ for any $\Lambda$ lattice $M$, we find a decomposition of $M$ into a direct sum of indecomposable $\Lambda$ lattices. A full list of indecomposable $\Lambda$ lattices is as follows:

Type I. Any rank 1 projective module over any of the following rings

$$A_1, A_2, R[T]/\Phi_p(T), R[T]/(T - 1), \Lambda(j) \quad \text{for } 1 \leq i \leq p,$$

$$\mathcal{N}(j) \quad \text{for } 1 \leq j \leq p - 1 \quad \text{and} \quad \mathcal{N}.$$

Type (II; $j$) for $2 \leq j \leq p - 1$. A $\Lambda$ lattice defined by $M_1$ and $P \oplus S$ with transition matrix

$$\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & (T - 1)^{j-1}
\end{pmatrix},$$

where $M_1$, $P$, and $S$ are rank 1 projective modules over $A_1$, $A_2$, and $R$, respectively.

Type (III; $j$) for $1 \leq j \leq p - 1$. A $\Lambda$ lattice defined by $M_1$ and $Q \oplus S$ with transition matrix

$$\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & (T - 1)^{j-1}
\end{pmatrix},$$

where $M_1$, $Q$, and $S$ are rank 1 projective modules over $A_1$, $R[T]/\Phi_p(t)$, and $R$, respectively.
6. ISOMORPHISM CLASSES OF LATTICES OF TYPE (III; j)

**Lemma 6.1.** Assume that (i) \( p \neq 2 \) or (ii) \( p = 2 \) and \( R = \mathbb{Z} \) or a DVR. For \( 1 \leq j \leq p - 1 \), \( \text{Image}(U(R[T]/\Phi_p(T)) \to U(R[T]/\langle p, (T - 1) \rangle)) \) is contained in \( \text{Image}(U(R[T]/\Phi_p(T)) \to U(R[T]/\langle p, (T - 1) \rangle)) \).

**Proof.** The case when \( p = 2 \) and \( R = \mathbb{Z} \) is easy. For the case when \( R \) is a DVR, use Lemma 8.1. So we may assume that \( p \neq 2 \) and \( R \) is a Dedekind domain satisfying conditions (R1) and (R2). Let \( K_1 \) and \( K_2 \) be the quotient fields of \( R[T]/\Phi_p(T) \) and \( R[T]/\Phi_p(T) \), respectively. Since \( R[T]/\Phi_p(T) \) and \( R[T]/\Phi_p(T) \) are free \( R \) modules of rank \( p - 1 \) and \( p^2 - p \), respectively, it follows that \([K_2:K_1] = p\).

Let \( \omega_2 \) be the image of \( T \) in \( R[T]/\Phi_p(T) \). Then \( R[T]/\Phi_p(T) \) is isomorphic to \( R[\omega_2^2] \) and all the conjugates of \( \omega_2 \) over \( K_1 \) are \( \omega_2, \omega_2^{1+p}, \omega_2^{1+2p}, \ldots, \omega_2^{1+(p-1)p} \). The rest of the proof is the same as that given in the hint of [3, Exercise 4, p. 754].

**Definition 6.2.** We say a Dedekind domain \( R \) satisfies the (CU) condition if the natural map \( U(A_1) \to U(R/pR) \) is surjective for every prime number \( p \).

\( \mathbb{Z} \) satisfies the (CU) condition, because if \( 1 \leq i \leq p - 1 \), the circular unit \( (\zeta^i - 1)/(\zeta - 1) \) is mapped to \( i \), where \( \zeta \) is the image of \( T \) in \( \mathbb{Z}[T]/\Phi_p(T) \). In addition, any DVR satisfies the (CU) condition because of Lemma 8.1.

**Definition 6.3.** A prime number \( p \) is called semiregular (or properly irregular) if \( p \) does not divide the class number of the maximal real subfield \( \mathbb{Q}(\zeta + \zeta^{-1}) \) in \( \mathbb{Q}(\zeta) \), where \( \zeta := \exp(2\pi \sqrt{-1}/p) \). It is an old conjecture, dating back to Kummer and sometimes called Vandiver's conjecture, that every prime number is semiregular [24, Remark, p. 159]. This conjecture has not been proved so far; however, it has been verified by Buhler and others [23; 24, p. 157; 1] that a prime number \( p \) is semiregular if \( p < 4,000,000 \).

Now we start to discuss the isomorphism classes of various indecomposable \( \Lambda \) lattices. For indecomposable \( \Lambda \) lattices of Type I, since they are rank 1 projective modules over various rings, these lattices are classified by the Picard groups of these rings. Since these rings are either Dedekind domains or pull-backs of simpler rings, their Picard groups can be described by the Mayer–Vietoris sequences [15, Theorem 3.3, p. 28; 9, Sect. 4]. We record them as follows.


**Lemma 6.4.** The following sequences are exact:

\[ U(R[T]/\Phi_p(T)) \times U(R[T]/(T - 1)) \]
\[ \to U(R[T]/\langle p, T - 1 \rangle) \to \text{Pic}(A_2) \]
\[ \to \text{Pic}(R[T]/\Phi_p(T)) \times \text{Pic}(R[T]/(T - 1)) \to 0; \]

\[ U(A_1) \times U(A_2) \to U(R[T]/\langle p, (T - 1)^j \rangle) \to \text{Pic}(\Lambda(j)) \]
\[ \to \text{Pic}(A_1) \times \text{Pic}(A_2) \to 0 \quad \text{for} \quad 1 \leq j \leq p; \]

\[ U(A_1) \times U(R[T]/\Phi_p(T)) \to U(R[T]/\langle p, (T - 1)^j \rangle) \]
\[ \to \text{Pic}(\Lambda(j)) \]
\[ \to \text{Pic}(A_1) \times \text{Pic}(R[T]/\Phi_p(T)) \to 0 \]
\[ \quad \text{for} \quad 1 \leq j \leq p - 1; \]

\[ U(A_1) \times U(R[T]/(T - 1)) \to U(R[T]/\langle p, T - 1 \rangle) \to \text{Pic}(\Lambda') \]
\[ \to \text{Pic}(A_1) \times \text{Pic}(R[T]/(T - 1)) \to 0. \]

**Lemma 6.5.**

(i) If \( R = \mathbb{Z} \), then \( \text{Pic}(A_2) = \text{Pic}(R[T]/\Phi_p(T)) \) and \( \text{Pic}(\Lambda') = \text{Pic}(A_1) \).

(ii) If \( p \neq 2 \) or \( R = \mathbb{Z} \), the following is a short exact sequence for \( 1 \leq j \leq p - 1 \):

\[ 0 \to W_j \to \text{Pic}(\Lambda'(j)) \to \text{Pic}(A_1) \times \text{Pic}(R[T]/\Phi_p(T)) \to 0. \]

(iii) If \( R = \mathbb{Z} \) and \( p \) is a semiregular prime number, then the following are exact sequences:

\[ 0 \to W_j \to \text{Pic}(\Lambda(j)) \to \text{Pic}(A_1) \times \text{Pic}(A_2) \to 0 \quad \text{for} \quad 1 \leq j \leq p - 1, \]

\[ U(A_1) \to U(R[T]/\langle p, (T - 1)^p \rangle) \to \text{Pic}(\Lambda(p)) \]
\[ \to \text{Pic}(A_1) \times \text{Pic}(A_2) \to 0. \]

(Recall \( W_j \) and \( V_j \) are defined in Definition 1.12.)

**Proof.** (i) Take the first and fourth exact sequences in Lemma 6.4. Note that both \( U(R[T]/\Phi_p(T)) \to U(R/pR) \) and \( U(A_1) \to U(R/pR) \) are surjective because of the trick of circular units. (See the paragraph after Definition 6.2.)

(ii) Take the third exact sequence in Lemma 6.4. By Lemma 6.1, we can throw away the factor \( U(A_1) \). Hence the result.
(iii) Take the second exact sequence in Lemma 6.4. By [10, Step 1, pp. 444 and 445], \( \text{Image}(U(A_2) \to U(R[T] \langle p, (T - 1)^j \rangle) \) is contained in \( \text{Image}(U(A_1) \to U(R[T] \langle p, (T - 1)^j \rangle) \). Hence the same conclusion is true when the target group is replaced by \( U(R[T] \langle p, (T - 1)^j \rangle) \) for \( 1 \leq j \leq p - 1 \). Thus we can throw away the factor \( U(A_2) \) in this exact sequence. By Lemma 6.1 and [10, line 9, p. 445], \( W_j \) is the cokernel of the first map when \( 1 \leq j \leq p - 1 \).

**Proof of Theorem 1.13.** Let \( M \) and \( M' \) be \( \Lambda \) lattices of Type (III; j) with \( \varphi: M \to M' \). By Lemma 3.1, \( \varphi \) induces isomorphisms \( \varphi_1: M_1 \to M_1' \) and \( \varphi_2: Q \oplus S \to Q' \oplus S' \). Since \( Q = \{ v \in Q \oplus S: \Phi(q) \cdot v = 0 \} \) and \( Q' = \{ v' \in Q' \oplus S': \Phi(q) \cdot v' = 0 \} \), it follows that \( \varphi_2(Q) = Q' \). Similarly \( \varphi_2(S) = S' \). Thus in the definition of \( g_j \), the first three components are well defined. Thus, we can write \( M_1' = M_1 \), \( Q' = Q \), and \( S' = S \).

Let
\[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & (T - 1)^{j-1}
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
\lambda_1' & 0 & 0 \\
0 & \lambda_2' & (T - 1)^{j-1}
\end{pmatrix}
\]
be the transition matrices of \( M \) and \( M' \), respectively.

By Lemma 3.8, there exist \( \lambda_0 \in U(R[T] \langle p, (T - 1)^j \rangle) \), \( \mu \in U(A_1) \), \( \eta \in U(R[T]/\Phi(p)(T)) \), and \( \pi \in U(R) \) so that
\[
\begin{pmatrix}
\lambda_0 & 0 \\
0 & \lambda_0
\end{pmatrix}
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & (T - 1)^{j-1}
\end{pmatrix}
= \begin{pmatrix}
\lambda_1' & 0 & 0 \\
0 & \lambda_2' & (T - 1)^{j-1}
\end{pmatrix}
\begin{pmatrix}
\mu & 0 & 0 \\
0 & \eta & 0 \\
0 & 0 & \pi
\end{pmatrix}.
\]

Comparing the entries of the above matrices, we get
\[
\lambda_0 \lambda_1 = \lambda_1' \mu \quad \text{in} \ U(R[T] \langle p, (T - 1)^j \rangle),
\]
(7)
\[
\lambda_0 \lambda_2 = \lambda_2' \eta \quad \text{in} \ U(R[T] \langle p, (T - 1)^j \rangle),
\]
(8)
\[
\lambda_0 = \pi \quad \text{in} \ U(R/pR).
\]
(9)

Dividing (8) by (7), we find that
\[
\lambda_1^{-1} \lambda_2 = \mu \eta \lambda_1^{-1} \lambda_2.
\]
(10)

Substituting (9) into (7) and taking modulo \( T - 1 \), we find that
\[
\lambda_1' = \mu^{-1} \pi \lambda_1.
\]
(11)

Thus \( g_j \) is well defined.
Clearly \( g_j \) is a surjection. For the injectivity, if \( (\lambda_1^{-1} \lambda_2', \lambda_1') = (\lambda_1^{-1} \lambda_2, \lambda_1) \) in \( V_j \), then there exist \( \mu \in U(A_1) \), \( \eta \in U(R[T]/\Phi_p(T)) \), and \( \pi \in U(R) \) such that (10) and (11) hold. In particular,

\[
\lambda_1^{-1} \lambda_2' \mu = \lambda_2^{-1} \lambda_1 \eta \in U(R[T]/\langle p, (T-1)^j \rangle).
\]

Call this common value \( \lambda_0 \). Then it is routine to check that

\[
\begin{pmatrix}
\lambda_0 & 0 \\
0 & \lambda_0
\end{pmatrix}
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & (T-1)^{j-1}
\end{pmatrix}
= \begin{pmatrix}
\lambda_1' & 0 & 0 \\
0 & \lambda_2' & (T-1)^{j-1}
\end{pmatrix}
\begin{pmatrix}
\mu & 0 & 0 \\
0 & \eta & 0 \\
0 & 0 & \pi
\end{pmatrix}.
\]

By Lemma 3.8, we find that \( M = M' \).

7. ISOMORPHISM CLASSES OF LATTICES OF TYPE (II; j)

**Lemma 7.1.** Assume that (i) \( p \neq 2 \) and \( R \) satisfies the \((CU)\) condition in Definition 6.2 or (ii) \( p = 2 \) and \( R = \mathbb{Z} \) or a DVR. For \( 1 \leq j \leq p - 1 \), every \( \Lambda \) lattice \( M \) of Type (III; j) has a transition matrix of the form

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda & (T-1)^{j-1}
\end{pmatrix},
\]

where \( \lambda \in U(R[T]/\langle p, (T-1)^j \rangle) \).

**Proof.** Suppose that

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & a(T-1)^{j-1}
\end{pmatrix}
\]

is a transition matrix for \( M \). By the \((CU)\) condition, find \( \mu \in U(A_1) \) such that \( \bar{\mu} = a^{-1} \lambda_1 \in U(R/pR) \),

\[
\begin{pmatrix}
\lambda_1^{-1} \bar{\mu} & 0 \\
0 & \lambda_1^{-1} \bar{\mu}
\end{pmatrix}
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & a(T-1)^{j-1}
\end{pmatrix}
\begin{pmatrix}
\bar{\mu}^{-1} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda & (T-1)^{j-1}
\end{pmatrix}
\]

with \( \lambda = \lambda_1^{-1} \lambda_2 \bar{\mu} \in U(R[T]/\langle p, (T-1)^j \rangle) \).
**Theorem 7.2.** Assume that (i) $p \neq 2$ and $R$ satisfies the (CU) condition or (ii) $p = 2$ and $R = \mathbb{Z}$ or a DVR. For $1 \leq j \leq p - 1$, there is a set-theoretic bijection defined by

$$g_j: \Sigma(III; j) \to \text{Pic}(A_1) \times \text{Pic}(R[T]/\Phi_p(T)) \times \text{Pic}(R) \times W_j,$$

where $M$ is the lattice defined by $M_1$ and $Q \oplus S$ with transition matrix

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda & (T - 1)^{j-1}
\end{pmatrix},
$$

which is guaranteed by Lemma 7.1.

**Proof.** In the Proof of Theorem 1.13, taking $(\lambda_1, \lambda_2, 0, 1) = (\lambda, 1) \in V'$, it suffices to show that in the following diagram $\chi$ is an isomorphism:

$$
\begin{array}{c}
U(A_1) \times U(R[T]/\Phi_p(T)) \times U(R) \longrightarrow U(R[T]/\langle p, (T - 1)\rangle) \times U(R/pR) \longrightarrow V' \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \downarrow x \\
U(R[T]/\Phi_p(T)) \quad \longrightarrow \quad U(R[T]/\langle p, (T - 1)\rangle) \quad \longrightarrow \quad W_j
\end{array}
$$

But this is easy by the (CU) condition and Lemma 6.1.

**Lemma 7.3.** Assume that (i) $p \neq 2$, $\text{Pic}(R) = 0$, and $R$ satisfies the (CU) condition or (ii) $p = 2$ and $R = \mathbb{Z}$ or a DVR. For $2 \leq j \leq p - 1$, every $\Lambda$ lattice $M$ of Type (II; $j$) has a transition matrix of the form

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda & (T - 1)^{j-1}
\end{pmatrix},
$$

where $\lambda \in U(R[T]/\langle p, (T - 1)\rangle)$.

**Proof.** Let

$$
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & a(T - 1)^{j-1}
\end{pmatrix}
$$

be any transition matrix for $M$. Consider

$$
\begin{pmatrix}
\lambda_1^{-1} & 0 & 0 \\
0 & \lambda_1^{-1}
\end{pmatrix} \quad \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & a(T - 1)^{j-1}
\end{pmatrix}. 
$$

We may therefore assume that the transition matrix is of the form

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda & a(T - 1)^{j-1}
\end{pmatrix}
$$

from the beginning.
By the (CU) condition, find $\mu \in U(A_2)$ such that $\mu = a$ in $R/pR$. By Lemma 6.1, find $\nu \in U(R[T]/\Phi_p(T))$ so that $\mu = \nu$ in $U(R[T]/(p(T - 1)\nu))$. Let $\nu \in A_2$ be a preimage of $\nu \in U(R[T]/\Phi_p(T))$.

Choose $\nu' \in A_2$ such that $\nu\nu' = 1$ in $R[T]/\Phi_p(T)$. Thus

$$\nu\nu' = 1 + r\Phi_p(T)$$

(12)

for some $r \in R$.

Define $\pi := \nu' \in R = R[T]/(T - 1)$. Let $M$ be defined by $M_1$ and $P \oplus R$. Since Pic($R$) = 0, it follows that $P \oplus R$ is a free module. Thus we may write $P \oplus R = Rz$. Choose $\nu \in P$ so that $\nu$ is mapped onto $z$ by the canonical projection. Define

$$\varphi_1: R \to P,$$

$$1 \mapsto \Phi_p(T)\nu,$$

$$\varphi_2: P \to Rz \to R,$$

$$z \mapsto r,$$

where $r \in R$ is the element found in (12).

By the notation introduced in Definition 3.5,

$$\varphi := \left(\begin{array}{cc}
\nu & \Phi_p(T) \\
r & \pi
\end{array}\right) \in \text{End}_{A_2}(P \oplus R).$$

By Lemma 3.4, $\varphi$ is an automorphism because of (12). Thus we may consider

$$\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & a(T - 1)^{-1} & 0 \\
0 & a & \pi
\end{array}\right)$$

$$\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & \beta & 0 \\
0 & \pi & \pi
\end{array}\right) = \left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & \lambda' & a\pi(T - 1)^{-1} \\
0 & 0 & r
\end{array}\right),$$

where $\lambda'$ is some element in $U(R[T]/(p(T - 1)\nu))$ and $a\pi = \nu\nu' = 1$ in $R/pR$ because of (8) again.

**Lemma 7.4.** Suppose that the natural map $U(R[T]/\Phi_p(T)) \to U(R/pR)$ is surjective. If $P$ and $P'$ are rank 1 projective $A_2$ modules and $S$ is any rank 1 projective $R$ module with $P \oplus S = P' \oplus S$, then $P = P'$.

**Remark.** The condition that $U(R[T]/\Phi_p(T)) \to U(R/pR)$ is surjective is certainly true when $R$ satisfies the (CU) condition, by Lemma 6.1.

**Proof.** Let $P = \text{pbk}(P_0, T, R/pR; q_1, q_2)$ and $P' = \text{pbk}(P'_0, T', R/pR; q'_1, q'_2)$ by the notation in [15, Theorem 2.2, p. 20].

Note that $P \oplus S = \text{pbk}(P_0, T \oplus S, R/pR; q_1, q_2, 0)$ and $P' \oplus S = \text{pbk}(P'_0, T' \oplus S, R/pR; q'_1, q'_2, 0)$. Since $P \oplus S = P' \oplus S$, by Lemma 3.1
we have that \( P_0 = P'_0 \) and \( T \otimes S = T' \otimes S \). Taking the determinant of \( T \otimes S = T' \otimes S \), we get \( T = T' \).

By our assumption and the Mayer–Vietoris sequence [15, Theorem 3.3; 9, Sect. 4], we find an isomorphism

\[
\Phi : \text{Pic}(A_2) \xrightarrow{\sim} \text{Pic}(R[T]/\Phi_p(T)) \times \text{Pic}(R),
\]

where \( \tilde{P} \) is any rank 1 projective \( A_2 \) module. Since \( \Phi([P]) = ([P_0],[T]) = ([P'_0],[T']) = \Phi([P']) \), it follows that \( P \simeq P' \).

**Proof of Theorem 1.14.** By Lemmas 3.1 and 7.4, the first two components of \( f_j \) are well defined. Now suppose that \( M = M' \) and

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda & (T - 1)^{i-1}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda' & (T - 1)^{j-1}
\end{pmatrix}
\]

are the transition functions of \( M \) and \( M' \), respectively.

By Lemma 3.7 and Definition 3.5, there exist \( \lambda_0 \in U(R[T]/\langle p, (T - 1)^i \rangle) \), \( \mu \in U(A_2) \), \( \nu \in A_2 \), and \( r, s \in R \) so that

\[
\varphi := \begin{pmatrix}
\nu \\
r
\end{pmatrix}
\begin{pmatrix}
\Phi_p(T) \\
\pi
\end{pmatrix}
\in \text{Aut}_{A_2}(P \oplus R)
\]

and

\[
\begin{pmatrix}
\lambda_0 & 0 \\
0 & \lambda_0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda & (T - 1)^{i-1}
\end{pmatrix}
\begin{pmatrix}
\overline{\mu} & 0 & 0 \\
0 & \overline{\nu} & 0 \\
0 & \overline{r} & \overline{\pi}
\end{pmatrix}.
\]

Hence we have

\[
\lambda_0 = \overline{\mu} \quad \text{in} \quad U(R[T]/\langle p, (T - 1)^i \rangle),
\]

(13)

\[
\lambda_0 \lambda = \lambda \overline{\nu} + \overline{r}(T - 1)^{i-1} \quad \text{in} \quad U(R[T]/\langle p, (T - 1)^i \rangle),
\]

(14)

\[
\overline{\lambda_0} = \overline{\pi} \quad \text{in} \quad U(R/pR).
\]

(15)

From (14), we get that

\[
\lambda \lambda^{-1} = \lambda_0 \overline{\nu}^{-1} \quad \text{in} \quad U(R[T]/\langle p, (T - 1)^{i-1} \rangle).
\]

Since \( \lambda_0 = \mu \) by (13), we find that \( \lambda_0 \overline{\nu}^{-1} = \mu \overline{\nu}^{-1} \in \text{Image}(U(R[T]/\Phi_p(T)) \to U(R[T]/\langle p, (T - 1)^{i-1} \rangle)) \) because of Lemma 6.1.
On the other hand, in $R/pR$, we have \( \lambda_0 \overline{\nu}^{-1} = \overline{\pi}^{-1} = \overline{\pi} \overline{\nu}^{-2} = \overline{\nu} \overline{\pi} - prs \cdot \overline{\nu}^{-2} \in \text{Image}(U(R) \times U(R/pR)^2 \to U(R/pR)) \), since \( \overline{\nu} \overline{\pi} - prs \in U(R) \) by Lemma 3.4. Hence $f_j$ is well defined. It is easy to see that $f_j$ is a surjection by using the (CU) condition and Lemma 6.1.

It remains to establish the injectivity of $f_j$. Let $M$ and $M'$ be two $\Lambda$ lattices with $f_j([M]) = f_j([M'])$. Let

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda & (T - 1)^{j-1}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda' & (T - 1)^{j-1}
\end{pmatrix}
\]

be the transition matrices of $M$ and $M'$, respectively.

In $R/pR$, since

\[ \overline{\lambda} = \overline{\lambda} \overline{\nu}^{-2} \in U(R/pR), \]

where $u \in U(R)$ and $\nu_0 \in U(R/pR)$, find an element $\pi \in R$ such that $\overline{\pi} \overline{\nu} = \overline{u} \in U(R/pR)$.

By the (CU) condition, find $\mu \in U(A_1)$ such that $\overline{\mu} = \overline{\pi}$ in $U(R/pR)$. By the (CU) condition and Lemma 6.1, find $\nu \in A_2$ such that $\overline{\nu} \in U(R/T)/\Phi_p(T)$ and $\overline{\nu} = \nu_0 \in U(R/pR)$.

In $R$, consider $\overline{\nu} \overline{\pi} - u$. Since $\overline{\nu} \overline{\pi} - \overline{u} = \nu_0 \overline{\pi} - \overline{u} = 0$ in $R/pR$, it follows that $\overline{\nu} \overline{\pi} - u = pr$ for some $r \in R$. Thus

\[ \phi := \begin{pmatrix} \nu & \Phi_p(T) \\ r & \pi \end{pmatrix} \in \text{End}_{A_2}(P \oplus R) \]

is an automorphism by Lemma 3.4. Let $\rho \in U(R[T]/\langle p, (T - 1)^j \rangle)$ and $a \in U(R/pR)$ be defined by the equality

\[ \begin{pmatrix} \overline{\mu} & 0 \\ 0 & \overline{\nu} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & (T - 1)^{j-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & a(T - 1)^{j-1} \end{pmatrix} \begin{pmatrix} \overline{\mu} & 0 & 0 \\ 0 & \overline{\nu} & 0 \end{pmatrix}. \]

Note that $\overline{\mu} = a \overline{\pi} \in U(R/pR)$. Since $\overline{\mu} = \overline{\pi} \in U(R/pR)$, it follows that $a = 1$. Moreover,

\[ \overline{\mu} \lambda = \rho \overline{\nu} + a(T - 1)^{j-1}. \]

Hence, in $R/pR$, we find that

\[ \overline{\rho} \overline{\lambda}^{-1} = \overline{\mu} \overline{\nu}^{-1} = \overline{\pi} \nu_0^{-1} = \overline{\pi} \nu_0^{-2} = \overline{u} \nu_0^{-2} = \overline{\lambda} \overline{\lambda}^{-1}. \]

Thus \( \overline{\rho} = \overline{\lambda} \) in $R/pR$. In other words, we may assume that the transition matrices are chosen so that

\[ \lambda \overline{\lambda}^{-1} \in \text{Image}(U(R[T]/\Phi_p(T)) \to U(R[T]/\langle p, (T - 1)^{j-1} \rangle)) \] (16)
and
\[ \bar{\lambda} = \bar{\lambda} \in U(R/pR). \] (17)

Now choose \( \eta \in U(R[T]/\langle p, (T - 1)^{-1} \rangle) \) such that \( \bar{\lambda} = \bar{\lambda} \eta \) in \( R[T]/\langle p, (T - 1)^{-1} \rangle \). Since \( \bar{\lambda} = \bar{\lambda} \) in \( R/pR \), it follows that \( \bar{\eta} = 1 \) in \( R/pR \). Choose a preimage of \( \eta \) in \( A_2 \). Call it \( \nu_1 \). In \( R/pR \), \( \bar{\nu}_1 - 1 = \bar{\eta} - 1 = 0 \). Hence, in \( R, \bar{\nu}_1 - 1 = p r_1 \) for some \( r_1 \in R \). Then
\[ \varphi' := \begin{pmatrix} \nu_1 & \Phi_p(T) \\ r_1 & 1 \end{pmatrix} \in \text{Aut}_{A_2}(P \oplus R) \]
by Lemma 3.4.

Let \( \rho_1 \in U(R[T]/\langle p, (T - 1)^{-1} \rangle) \) be defined by the equality
\[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho_1 & (T - 1)^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & (T - 1)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \bar{\nu}_1 & 0 \\ 0 & \bar{r}_1 & 1 \end{pmatrix}. \]
Thus we have that
\[ \rho_1 = \lambda \bar{\nu}_1 + \bar{r}_1(T - 1)^{-1}. \]
In \( R[T]/\langle p, (T - 1)^{-1} \rangle \), we get
\[ \bar{\nu}_1 \bar{\lambda}^{-1} = \bar{\nu}_1 = \bar{\eta} = \bar{\lambda} \bar{\lambda}^{-1}. \]
Hence we find that \( \bar{\nu}_1 = \bar{\lambda} \) in \( R[T]/\langle p, (T - 1)^{-1} \rangle \). This means that we can choose the transition matrices for \( M \) and \( M' \) so that they not only satisfy (16) and (17), but also satisfy
\[ \lambda' - \lambda = \bar{a}(T - 1)^{-1} \] (18)
in \( R[T]/\langle p, (T - 1)^{-1} \rangle \) for any \( a \in R \).

Now the simplified relation (18) enables us to conclude \( M = M' \) because, for any \( a \in R \),
\[ \begin{pmatrix} 1 + a \Phi_p(T) & \Phi_p(T) \\ a & 1 \end{pmatrix} \in \text{Aut}_{A_2}(P \oplus R) \]
and
\[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda' & (T - 1)^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & (T - 1)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \bar{a} & 1 \end{pmatrix}. \]
Hence the result.
8. SOME REMARKS ON $W_j$

**Lemma 8.1.** If $R$ is a DVR, then $U(R) \to U(R/pR)$ is surjective. In particular, $R$ satisfies the (CU) condition, $X = 0$ and $W_j = 0$ for $1 \leq j \leq p - 1$.

**Proof.** Only the fact that $W_j = 0$ is not obvious. In order to prove that $U(R[T]/\Phi_p(T)) \to U(R[T]/\langle p, (T - 1)^j \rangle)$ is surjective, since $U(R) \to U(R/pR)$ is surjective, it remains to show that every unit in $R[T]/\langle p, (T - 1)^j \rangle$ of the form

$$1 + \sum_{i=1}^{j-1} a_i (T - 1)^i$$

is in the image of $U(R[T]/\Phi_p(T))$, where $a_i \in R$.

Note that $1 + \sum_{i=1}^{j-1} a_i (T - 1)^i \in R[T]/\Phi_p(T)$ is mapped to the above element. It suffices to show that $1 + \sum_{i=1}^{j-1} a_i (T - 1)^i$ is a unit in $R[T]/\Phi_p(T)$.

We shall show that $T - 1$ belongs to the Jacobson radical of $R[T]/\Phi_p(T)$ and therefore $1 + \sum_{i=1}^{j-1} a_i (T - 1)^i$ is invertible.

Since $R[T]/\Phi_p(T)$ is integral over $R$ and $pR$ is the unique maximal ideal of $R$, hence $p$ belongs to every maximal ideal of $R[T]/\Phi_p(T)$. (In fact, $R[T]/\Phi_p(T)$ is a local domain also.) Let $\mathfrak{M}$ be any maximal ideal of $R[T]/\Phi_p(T)$. In $R[T]/\Phi_p(T)$,

$$0 = \Phi_p(T) = (T - 1)^{p-1} + p \cdot f(T)$$

for some $f(T) \in R[T]/\Phi_p(T)$. Since $p \in \mathfrak{M}$, it follows that $T - 1 \in \mathfrak{M}$ also. Done.

**Theorem 8.2.** If $R$ is a discrete valuation ring, there are precisely $4p + 1$ nonisomorphic indecomposable $\Lambda$ lattices. The following is a full list of indecomposable $\Lambda$ lattices:

- $A_1, A_2, R[T]/\Phi_p(T), R[T]/(T - 1), \Lambda(j)$ for $1 \leq j \leq p$,
- $\Lambda(j)$ for $1 \leq j \leq p - 1, \Lambda'$,
- $\text{pbk}(A_1, A_2 \oplus R, R[T]/\langle p, (T - 1)^j \rangle; q_1, q_2)$ for $2 \leq j \leq p - 1$,
- $\text{pbk}(A_1, R[T]/\Phi_p(T) \oplus R, R[T]/\langle p, (T - 1)^j \rangle; q_1, q_2)$
  for $1 \leq j \leq p - 1$, 

in which $q_1, q_2$ are integers.
where the transition matrix of the last two pull-backs can be chosen to be

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & (T-1)^{i-1}
\end{pmatrix}.
\]

**Proof.** Since \( R \) is a local ring, all the rings appearing in Theorem 5.1 are semilocal. Thus rank 1 projective modules are free modules. Then apply Lemma 8.1.

**Proof of Theorem 1.16.** Let \( \mathfrak{M} \) be any maximal ideal in \( R \) and let \( M \) and \( M' \) be two indecomposable \( \Lambda \) lattices. Suppose that \( M \otimes \Lambda_{\mathfrak{M}} = M' \otimes \Lambda_{\mathfrak{M}} \). If \( M \) is not of Type I, compare the \( A_i \otimes R_{\mathfrak{M}} \) components of \( M \otimes \Lambda_{\mathfrak{M}} \) and \( M' \otimes \Lambda_{\mathfrak{M}} \). Apply Theorem 8.2. Thus both \( M \) and \( M' \) are of Type II; MM or Type III; j. The case when both \( M \) and \( M' \) are of Type I is easy and its proof is omitted.

Conversely, we find that \( M \otimes \Lambda_{\mathfrak{M}} \) is isomorphic to \( M' \otimes \Lambda_{\mathfrak{M}} \) by Theorem 8.2 again. Thus \( M \otimes \Lambda_{\mathfrak{M}} = M' \otimes \Lambda_{\mathfrak{M}} \).

Now we consider the case when \( R = \mathbb{Z} \) and \( p \) is a semiregular prime number. As we mentioned before, the structure of \( W_i \) in this case was studied in [4, 10]. (For related results, see [22, 21].) We shall follow the proof of Kervaire and Murthy. In [10, Sect. 6, Step 2, pp. 445–448], it was shown that the Cokernel(\( \mathcal{U}(A_1) \to \mathcal{U}(R[T]/\langle p, (T-1)^p-1 \rangle) \)) is isomorphic to an elementary abelian \( p \) group of rank \( \frac{j}{2}(p-3) + \delta_p \), where \( \delta_p \) is the number of Bernoulli numbers among \( B_2, B_4, \ldots, B_{p-3} \), whose numerator (in reduced form) is divisible by \( p \). Note that we adopt the convention that \( B_2 = B_4 = B_6 = \cdots = 0 \).

Note that the arguments of Kervaire and Murthy work as well when \( p-1 \) is replaced by any \( j \) with \( 1 \leq j \leq p-1 \). Moreover,

\[
\text{Image}\left\{ \mathcal{U}(A_1) \to \mathcal{U}(R[T]/\langle p, (T-1)^j \rangle) \right\}
= \text{Image}\left\{ \mathcal{U}(R[T]/\Phi_p(T)) \to \mathcal{U}(R[T]/\langle p, (T-1)^j \rangle) \right\}
\]

by Lemma 6.1 and [10, line 9, p. 445]. Hence we get the following:

**Theorem 8.3.** Suppose that \( R = \mathbb{Z} \) and \( p \) is an odd semiregular prime number. For \( 1 \leq j \leq p-1 \), \( W_j \) is isomorphic to an elementary abelian \( p \) group of rank \( \left\lfloor \frac{(j-2)}{2} \right\rfloor + \delta(j) \), where \( \delta(j) \) is the number of Bernoulli numbers among \( B_2, B_4, \ldots, B_{2(j-1)/2} \), whose numerator (in reduced form) is divisible by \( p \).

**Remark 8.4.** When \( R = \mathbb{Z} \) and \( p \) is semiregular, Coker(\( \mathcal{U}(A_1) \to \mathcal{U}(R[T]/\langle p, (T-1)^p \rangle) \)) = \( W_{p-1} \) by [10, lines 1–3, p. 446]. When \( p = 2 \),
Because of Theorem 8.3, the precise number of indecomposable lattices over \( \mathbb{Z}[T]/T^{p^2} - 1 \) when \( p \) is a semiregular prime number can be calculated in terms of class numbers of \( \mathbb{Z}\exp(2\pi i - 1/p) \) and \( \mathbb{Z}\exp(2\pi i - 1/p^2) \), by Theorems 1.7, 1.13, 1.14, 7.2, and Lemma 6.5. In particular, when \( p = 2, 3, 5 \), the number of indecomposable lattices over \( \mathbb{Z}[T]/T^{p^2} - 1 \) is 9, 13, and 40, respectively.

ACKNOWLEDGMENTS

I thank the anonymous referee whose critical and constructive comments greatly improved the presentation of numerous parts of this paper.

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