Rings with Property (A) and their extensions

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Received 21 September 2006
Available online 12 February 2007
Communicated by Kent R. Fuller

Abstract

A commutative ring \( R \) has Property (A) if every finitely generated ideal of \( R \) consisting entirely of zero-divisors has a nonzero annihilator. We continue in this paper the study of rings with Property (A). We extend Property (A) to noncommutative rings, and study such rings. Moreover, we study several extensions of rings with Property (A) including matrix rings, polynomial rings, power series rings and classical quotient rings. Finally, we characterize when the space of minimal prime ideals of rings with Property (A) is compact.

Keywords: Ring with Property (A); Biregular ring; Matrix ring; Polynomial ring; Compactness

One of important properties of commutative Noetherian rings is that the annihilator of an ideal \( I \) consisting entirely of zero-divisors is nonzero [12, p. 56]. However, this result fails for some non-Noetherian rings, even if the ideal \( I \) is finitely generated [12, p. 63]. Huckaba and Keller [10] introduced the following: a commutative ring \( R \) has Property (A) if every finitely generated ideal of \( R \) consisting entirely of zero-divisors has a nonzero annihilator. Property (A) was originally studied by Quentel [24]. Quentel used the term Condition (C) for Property (A). The class of commutative rings with Property (A) is quite large. For example, Noetherian rings [12, p. 56], rings whose prime ideals are maximal [5], the polynomial ring \( R[x] \) and rings whose

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0021-8693/$ – see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2007.01.042
classical ring of quotients are von Neumann regular [5], are examples of rings with Property (A). Using Property (A), Hinkle and Huckaba [6] extend the concept Kronecker function rings from integral domains to rings with zero divisors. Many authors have studied commutative rings with Property (A) ([2, 5, 9, 10, 19, 20, 24], etc.), and have obtained several results which are useful studying commutative rings with zero-divisors. Property (A) is closely connected with another annihilator condition. Lucas [19] introduced the following: a commutative ring $R$ has the annihilator condition (briefly, (a.c.)) if for each finitely generated ideal $I$ of $R$, there exists an element $b \in R$ with the annihilator of $I$ equals to the annihilator of $b$. Rings with (a.c.) were originally introduced by Henriksen and Jerison [5] for reduced rings. The ring $R[x]$ over a reduced ring $R$ [10], Bezout ring (finitely generated ideals are principal) and many other important commutative rings also have (a.c.) [19]. Property (A) and (a.c.) are equivalent conditions on a reduced ring whose space of minimal prime ideals is compact [10]. However, these two conditions are not equivalent in general [19]. Recently, Lucas [20] also studied the zero-divisor graph of rings with Property (A).

We continue in this paper the study of rings with Property (A). We extend Property (A) to noncommutative rings, and study such rings. Moreover, we study several extensions of rings with Property (A) including matrix rings, polynomial rings and classical quotient rings. Finally, we characterize when the space of minimal prime ideals of rings with Property (A) is compact.

Throughout this paper, $R$ denotes an associative ring with identity. We write $Z_l(R)$ and $P(R)$ for the set of all left zero-divisors of $R$, the set of all right zero-divisors of $R$ and the prime radical of $R$, respectively. For a nonempty subset $S$ of $R$, $\ell_R(S)$ and $r_R(S)$ denote the left annihilator and the right annihilator of a nonempty subset $S$ of $R$, respectively.

1. Rings with Property (A)

We begin with the following definition.

**Definition 1.1.** We say that a ring $R$ has right (left) Property (A) if for every finitely generated two-sided ideal $I \subseteq Z_l(R)$ ($Z_r(R)$), there exists nonzero $a \in R$ ($b \in R$) such that $Ia = 0$ ($bI = 0$). A ring $R$ is said to have Property (A) if $R$ has right and left Property (A).

We note that if a prime ring $R$ has a nonzero ideal $I$ such that $I \subseteq Z_l(R)$ ($Z_r(R)$) then $R$ does not have Property (A) because $Ir_R(I) = 0$ ($\ell_R(I)I = 0$) and so $r_R(I) = 0$ ($\ell_R(I) = 0$). Note that there exists a prime ring with a nonzero ideal $I$ such that $0 \neq I \subseteq Z_l(R)$ [16, p. 178]. However, if an arbitrary ring (in particular, a prime ring) has no nonzero ideals contained in $Z_l(R)$ (or $Z_r(R)$) then $R$ has Property (A). For example, the $n$-by-$n$ full matrix ring over a field $F$ is a simple prime ring, and so it has Property (A).

The following example shows that Property (A) is not left–right symmetric.

**Example 1.2.** Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and let $L = \mathbb{Z}_2[x]/(x^2)$. If $\delta$ denotes the image of $x$ in $L$, then $L = \mathbb{Z}_2 \oplus \mathbb{Z}_2\delta$ with $\delta^2 = 0$. Now consider the ring $R = (L/\mathbb{Z}_2\delta, L/\mathbb{Z}_2\delta)_L$. Let $e_{ij}$ be the usual matrix units, and let $I$ be a nontrivial two-sided ideal. If $I(\delta e_{22}) \neq 0$, then $I$ contains an element of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} a & b \\ 0 & 1+\delta \end{pmatrix}$. Multiplying this element on the left by $e_{22}$ or $(1+\delta)e_{22}$ (respectively) shows that $e_{22} \in I$. But then $e_{12} = e_{12}e_{22} \in I$. Since $I$ is nontrivial, we have that $I = (0_L/\mathbb{Z}_2\delta)_L$. Thus $e_{12}$ annihilates $I$ on the right. This shows $R$ has right Property (A).

We claim that $R$ does not have left Property (A). Consider the ideal $\begin{pmatrix} 0_L/\mathbb{Z}_2\delta \\ 0 \end{pmatrix}$ of $R$ contained in $Z_r(R)$. Suppose that $\begin{pmatrix} a & b \\ 0 & c+d\delta \end{pmatrix} \begin{pmatrix} 0_L/\mathbb{Z}_2\delta \\ 0 \end{pmatrix} = O$. Then $\begin{pmatrix} a & b \\ 0 & c+d\delta \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = O$ and so $b = 0,$
Lemma 1.5. One can prove the following lemma by adapting the proof of [12, Theorem 81]. Note that in reduced rings all minimal prime ideals are completely prime [16, Lemma 12.6].

Property (A).

Example 1.4. Consider the ring \( R = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} \), where \( F \) is an arbitrary field. Then \( R \) is left and right Noetherian. But \( R \) has neither left nor right Property (A). In fact, consider the two finitely generated ideals \( I = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} \) and \( J = \begin{pmatrix} 0 & F \\ F & 0 \end{pmatrix} \) of \( R \). Then \( I \subseteq Z_l(R) \) and \( J \subseteq Z_r(R) \). But there do not exist nonzero \( a, b \in R \) such that \( Ia = 0 \) and \( bJ = 0 \).

We now observe conditions when reduced rings or Noetherian rings have Property (A). A prime ideal \( P \) of \( R \) is called completely prime if for \( a, b \in R \), \( ab \in P \) implies \( a \in P \) or \( b \in P \). Note that in reduced rings all minimal prime ideals are completely prime [16, Lemma 12.6]. We can prove the following lemma by adapting the proof of [12, Theorem 81].

Lemma 1.5. Let \( R \) be a reduced ring and \( I \) an ideal of \( R \). If \( I \subseteq P_1 \cup \cdots \cup P_n \), where \( P_1, \ldots, P_n \) are minimal prime ideals of \( R \), then \( I \subseteq P_i \) for some \( i \).

Proof. Suppose that \( I \notin P_i \) for every \( 1 \leq i \leq n \). It then suffices to assume that \( n > 1 \) and \( n \) is minimal; that is, for each \( i \), \( I \notin \bigcup_{j \neq i} P_j \). Then there exists \( a_i \in I \setminus \bigcup_{j \neq i} P_j \) for each \( i \). Since \( I \subseteq \bigcup_{i=1}^n P_i \), each \( a_i \in P_i \), where \( 1 = i, \ldots, n \). Since \( I \) is an ideal of \( R \), \( a_1 + a_2 \cdots a_n \in I \) and so \( a_1 + a_2 \cdots a_n \in \bigcup_{i=1}^n P_i \). Hence \( a_1 + a_2 \cdots a_n \in P_j \) for some \( j \). Since \( P_1 \) is completely prime, \( a_k \in P_1 \) for some \( k > 1 \), which is a contradiction. If \( j > 1 \), then \( a_1 \in P_j \) since \( a_2 \cdots a_n \in P_j \), which is also a contradiction. Therefore \( I \subseteq P_i \) for some \( i \).

Theorem 1.6. If \( R \) is a reduced ring with finitely many minimal prime ideals, then \( R \) has Property (A).
Proof. Let $a$ be a zero-divisor in $R$. Then there exists a minimal prime ideal containing $a$. For, suppose that $a \notin P$ for all minimal prime ideals $P$ of $R$. Then $r_R(a) \subseteq P$ and so $r_R(a) \subseteq P(R)$. Since $R$ is reduced, $P(R) = 0$ and so $r_R(a) = 0$, which is a contradiction. Let $I = \sum_{i=1}^n Ra_i R \subseteq Z_l(R)$. Then by hypothesis and the preceding argument, $I$ is contained in the union of finitely many minimal prime ideals of $R$. By Lemma 1.5, there exists a minimal prime ideal $P$ of $R$ such that $I \subseteq P$. For each $a_i \in I \subseteq P$, there exists $x_i \in R \setminus P$ such that $a_i x_i = 0$ by [15, Corollary 2.12]. Note that $x_1 \cdots x_n \neq 0$ because $P$ is completely prime by [16, Lemma 12.6]. Since $R$ is reduced and $x_1 x_2 \cdots a_1 x_1 \cdots x_n = 0, a_1 x_1 \cdots x_n = 0$ by [15, Lemma 1.2] and also $Ra_i Rx_1 \cdots x_n = 0$ for each $i$. Thus $I x_1 \cdots x_n = 0$, and therefore $R$ has right Property (A). By symmetry, $R$ also has left Property (A).  

Corollary 1.7. Let $R$ be a reduced ring with a.c.c. on right annihilators. Then $R$ has Property (A).

Proof. By [3, Lemma 1.16], $R$ has only finitely many minimal prime ideals. Hence this result is completed.  

We point out here that in a semiprime ring there are multiple characterizations of finitely many minimal prime ideals [17, Theorem 11.43].

For a commutative ring $R$, if every prime ideal of $R$ is maximal then $R$ has Property (A) [9, Corollary 2.12]. However, it is not true in noncommutative ring in general. In Example 1.4, every prime ideal of the ring $R$ is maximal but $R$ does not have Property (A). A ring $R$ is called reversible if for $r, s \in R$, $rs = 0$ implies $sr = 0$ [23].

Proposition 1.8. Let $R$ be a reversible ring. If every prime ideal of $R$ is maximal, then $R$ has Property (A).

Proof. Let $I = \sum_{i=1}^n Ra_i R \subseteq Z_l(R)$. Then $I \subseteq P$ for some maximal ideal $P$ of $R$. Note that $P$ is a minimal prime ideal. If $I \subseteq P(R)$, then $I$ is nilpotent. For, let $a_i^{k_i} = 0$ for some positive integer $k_i$, where $i = 1, 2, \ldots, n$. Since $R$ is reversible, $(Ra_i R)^{k_i} = 0$ and so $I^{k_i} = 0$, where $k = \sum_{i=1}^n k_i$. Let $s$ be the minimal positive integer such that $I^s = 0$. Thus $0 \neq I^{s-1} \subseteq r_R(I) = \ell_R(I)$. Suppose $I \notin P(R)$. Let $\bar{R} = R/P(R)$. Note that $\bar{R}$ is reduced since $P(R)$ is the set of all nilpotent elements in a reversible ring $R$. Let $\bar{I} = (I + P(R))/P(R)$ and $\bar{P} = P/P(R)$. Reordering the $a_i$ if necessary, let $a_1, \ldots, a_t \notin P(R)$ and $a_i \in P(R)$ for $i > t$, where $1 \leq t \leq n$. Then for each $\bar{a}_j \in \bar{I} \subseteq \bar{P}$ where $1 \leq j \leq t$, there exists $\bar{x}_j \in \bar{R} \setminus \bar{P}$ such that $\bar{a}_j \bar{x}_j = 0$ by [15, Corollary 1.4] because $\bar{R}$ is reduced and $\bar{P}$ is minimal prime. Note that $\bar{x}_1 \cdots \bar{x}_t \neq 0$ because $\bar{P}$ is completely prime. Since $\bar{R}$ is reduced and $\bar{a}_j \bar{x}_j \cdots \bar{x}_t = 0, \bar{a}_j \bar{x}_1 \cdots \bar{x}_t = 0$ by [15, Lemma 1.2] and also $\bar{R} \bar{a}_j \bar{R} \bar{x}_j \cdots \bar{x}_t = 0$ for each $1 \leq j \leq t$. Let $\bar{b} = \bar{x}_1 \cdots \bar{x}_t$. Then $\bar{I} \bar{b} \subseteq P(R)$ and so $(\bar{I} \bar{b})^m = 0$ for some minimal integer $m \geq 1$. If $m = 1$, then $\bar{I} \bar{b} = 0$ and so we are done. Suppose that $m \geq 2$. If $b(\bar{I} \bar{b})^{m-1} \neq 0$, then we are done. If $b(\bar{I} \bar{b})^{m-1} = 0$, then $(\bar{I} \bar{b})^{m-1} = 0$ because $R$ is reversible. If $b(\bar{I} \bar{b})^{m-2} b \neq 0$, then we are done. Continuing this fashion, we have $\bar{I} \bar{b}^m = 0$. Since $b \notin P(R)$, $b$ is not nilpotent since $R$ is reversible, and so $b^m \neq 0$. Therefore $R$ has right Property (A). By symmetry, $R$ also has left Property (A).  

Corollary 1.9. If $R$ is a reduced ring whose prime ideals are maximal, then $R$ has Property (A).
A left Kasch ring (i.e., $r_R(M) \neq 0$ for all maximal left ideals $M$ of $R$) has right Property (A). Note that a ring $R$ is quasi-Frobenius if and only if $R$ is right Noetherian and satisfies the double annihilator conditions: $\ell_R(r_R(I)) = I$ and $r_R(\ell_R(J)) = J$ for a left ideal $I$ and a right ideal $J$ of $R$. This implies quasi-Frobenius rings are left and right Kasch, and hence they have Property (A).

We now note that if a ring $R$ is right self-injective then $\ell_R(r_R(A)) = A$ for a finitely generated left ideal $A$ of $R$. Since a quasi-Frobenius ring is right Noetherian and right self-injective, it is natural to conjecture that a right self-injective ring has right Property (A). But the following example erases the possibility, even if $R$ is right self-injective von Neumann regular.

**Example 1.10.** Let $V$ be a vector space over a field $F$ with countably infinite basis. Let $R = \text{End}_F(V)$ be the endomorphism ring of $V$, thinking of $V$ as a right $R$-module. Then $R$ is a right self-injective von Neumann regular ring. We now claim that $R$ does not have right Property (A). We first note that the ideals of $R$ are only $\{0\}$, $I = \{f \in R: \text{rank}(\text{Im} f) < \infty\}$ and $R$. If we take $0 \neq a \in I$ then $I = RaR$. Since $R$ is von Neumann regular, for each $x \in I$, $Rx = Re$ for some $e^2 = e \in R$. Thus $x = xe$ and hence $x(1-e) = 0$. This implies $I \subseteq Z_l(R)$. Moreover, $xR = fR$ for some idempotent $f$, and so $(1-f)x = 0$. This implies $I \subseteq Z_f(R)$. But there are no nonzero elements $b, c \in R$ such that $Ib = cI = 0$. Therefore $R$ does not have Property (A).

A ring $R$ is called biregular if every principal ideal of $R$ is generated by central idempotent of $R$ [4, p. 89].

**Proposition 1.11.** If $R$ is a biregular ring then $R$ has Property (A).

**Proof.** Let $I = \sum_{i=1}^{n} Ra_i R \subseteq Z_l(R)$. By hypothesis, $Ra_i R = e_i R$ for some central idempotent $e_i$ of $R$. Note that $I = Re_1 \oplus I(1-e_1)$. Then $I(1-e_1) = \sum_{i=2}^{n} Ra_i (1-e_1) R$. Note that $I(1-e_1) = Re_2 \oplus I(1-e_1)(1-e_2)$ since $Ra_2(1-e_1)R = e_2 R$ for some central idempotent $e_2$. If we continue in this fashion, we have $I = \bigoplus_{i=1}^{n} Re_i$. Let $f = \sum_{i=1}^{n} e_i$. Then $f^2 = f \neq 1$ and moreover $I = Rf$. Thus $I(1-f) = 0$, where $1-f \neq 0$. Therefore $R$ has right Property (A). Similarly we obtain $R$ has left Property (A). \[\square\]

From Proposition 1.11, we have the following. Let $A$ and $B$ be right $R$-modules. $A \preceq B$ means that $A$ is isomorphic to a submodule of $B$. A von Neumann regular ring $R$ satisfies general comparability provided that for any $x, y \in R$, there exists a central idempotent $e \in R$ such that $ex R \preceq ey R$ and $(1-e)y R \preceq (1-e)x R$ [4, p. 83].

**Corollary 1.12.** If $R$ satisfies the any of the following conditions (1)–(4), then $R$ has Property (A).

1. $R$ is a strongly regular ring (i.e., reduced von Neumann regular ring).
2. $R$ is a left weakly regular ring (i.e., $Ra = RaRa$ for any $a \in R$) with (a.c.c.) on right annihilators.
3. $R$ is a self-injective von Neumann regular ring with polynomial identity.
4. $R$ is a von Neumann regular ring with general comparability and every prime ideal of $R$ is maximal.
**Proof.** (1) is obvious. (2) follows from [8, Lemma 2.4]. (3) follows from [1, Corollary 3.2]. (4) follows from [4, Corollary 8.24]. □

Example 1.10 shows that the condition “polynomial identity” in Corollary 1.12(3) is essential. Moreover, Example 1.13 shows that the condition “$R$ is self-injective” is essential in Corollary 1.12(3).

**Example 1.13.** Let $F$ be a field. Set $F_n = F$ for $n = 1, 2, \ldots$, let $K = \bigoplus_{i=1}^\infty F_i$ and let $S = \langle \bigoplus_{i=1}^\infty F_i, \langle 1 \rangle \rangle$ be the $F$-subalgebra of $\prod_{i=1}^\infty F_i$ generated by $\bigoplus_{i=1}^\infty F_i$ and $\langle 1 \rangle$, where $\langle 1 \rangle$ is the identity of $\prod_{i=1}^\infty F_i$. Consider the ring $R = \langle S \prod K \rangle$. Then $R$ is a von Neumann regular ring [4, Example 1.8] with polynomial identity and note that $R$ is not a prime ring. For, let

\[
a = \begin{pmatrix}
1 & 0 & 0 & \ldots \\
0 & 0 & 0 & \ldots
\end{pmatrix}
\quad \text{and} \quad
b = \begin{pmatrix}
0 & 1 & 0 & \ldots \\
0 & 0 & 0 & \ldots
\end{pmatrix}.
\]

We have $a, b \neq 0$ but $aRb = O$, where $O$ is the zero matrix in $R$.

We now claim that $R$ does not have right Property (A). Let $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R$. Note that $R\alpha R = \langle S \prod K \rangle$. Let $\beta = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ be a nonzero element in $R\alpha R$. Then $\langle b_i \rangle, \langle c_i \rangle, \langle d_i \rangle \in K$ and so there exists a positive integer $t \geq 1$ such that $b_j = c_j = d_j = 0$ for all $j \geq t$. Let $\gamma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\beta\gamma = O = \gamma\beta$ and so $R\alpha R \subseteq Z_l(R)$. Suppose $(R\alpha R)\delta = O$ for some nonzero $\delta \in R$. Let $\delta = \begin{pmatrix} x_i & y_i \\ z_i & w_i \end{pmatrix} \in R$. Then

\[
(R\alpha R)\delta = \begin{pmatrix} S & K \\ K & K \end{pmatrix} \begin{pmatrix} x_i & y_i \\ z_i & w_i \end{pmatrix} = O.
\]

Since $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R\alpha R$,

\[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_i & y_i \\ z_i & w_i \end{pmatrix} = \begin{pmatrix} x_i & y_i \\ 0 & 0 \end{pmatrix} = O.
\]

So $\langle x_i \rangle = \langle y_i \rangle = \langle 0 \rangle$. Since $\langle z_i \rangle \in K$, there exists a positive integer $t_0 \geq 1$ such that $z_i = 0$ for all $i \geq t_0$. Now $\langle w_i \rangle \in S$, so $\langle w_i \rangle = \langle w_0, w_1, \ldots, w_n, w, w, \ldots \rangle$ for some positive integer $n \geq 1$. Let $\langle u_j \rangle = \langle 0, 0, 1, 0, \ldots \rangle$ with $(n+1)$th entry 1. Then $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in R\alpha R$ and so $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$. Hence $w = 0$. Now, take $m = \max\{t_0, n\}$. Let $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in R\alpha R$, where $\langle v_i \rangle = \langle 0, 0, 1, 0, 0, \ldots \rangle$ with $v_i = 0$ for all $i \geq m + 1$. Then

\[
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O.
\]

So $\langle z_i \rangle = \langle w_i \rangle = \langle 0 \rangle$ and hence $\delta = O$, which is a contradiction. Therefore $R$ does not have right Property (A). Moreover, we note that the ideal $R\alpha R$ of $R$ cannot be generated by a central idempotent. Thus $R$ is not biregular, and so not self-injective by [1, Corollary 3.2].
2. Extensions of rings with Property (A)

In this section we study the extensions of rings with Property (A) which provide several examples of rings with Property (A). We first observe the matrix ring over rings with Property (A).

**Theorem 2.1.** If a ring $R$ has right Property (A), then the full matrix ring $\mathbb{M}_n(R)$ over $R$ has right Property (A) for any $n \geq 1$.

**Proof.** Suppose that $R$ has right Property (A). Let $e_{ij}$ denote the usual matrix units. Fix matrices $A_k \in \mathbb{M}_n(R)$ and suppose

$$X = \sum_{k=1}^{m} \mathbb{M}_n(R) A_k \mathbb{M}_n(R) \subseteq Z_l(\mathbb{M}_n(R)).$$

Write the $(i, j)$th entry of $A_k$ as $a_{ij}^k$. We show that $J = \sum_{i,j,k} R a_{ij}^k R \subseteq Z_l(R)$. Fix $a \in J$, and write $a = \sum_{s=1}^{s} r_i a_{ij}^k s_t$ for some $r_i, s_t \in R$, for $1 \leq t \leq s$. Define the matrix

$$A = \sum_{t=1}^{n} \sum_{d=1}^{s} r_t e_{di} A_k e_{jd} s_t.$$

Then $A \in X \subseteq Z_l(\mathbb{M}_n(R))$. Computing, we find

$$A = \sum_{d=1}^{n} \sum_{t=1}^{s} r_t e_{di} a_{ij}^k e_{jd} s_t = \sum_{d=1}^{n} e_{dd} \sum_{t=1}^{s} r_t a_{ij}^k s_t = a I_n$$

where $I_n$ is the identity matrix. Then $a I_n = A \in Z_l(\mathbb{M}_n(R))$ implies $a \in Z_l(R)$. Therefore $J \subseteq Z_l(R)$. Since $R$ has right Property (A), $J u = 0$ for some nonzero $u \in R$. Note that $X(uI_n) = O$. Therefore $\mathbb{M}_n(R)$ has right Property (A). \qed

The following example shows that Property (A) does not pass to corner rings.

**Example 2.2.** There exists a ring with right Property (A) such that $e Re$ does not have right Property (A) for some idempotent $e^2 = e \in R$.

Let $F$ be a field and

$$R = \left\{ \begin{pmatrix} a & x & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & c & y & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & b & z \\ 0 & 0 & 0 & 0 & c \end{pmatrix} : a, b, c, x, y, z \in F \right\}.$$

Then $R$ is a quasi-Frobenius ring by [14, Example 9] and so $R$ has Property (A) by Corollary 1.9 below. Now let $e = e_{11} + e_{22} + e_{44} + e_{55} \in R$, where $e_{ij}$s are the matrix units. Then $e^2 = e$ and $e Re \cong \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. By Example 1.4, $e Re$ does not have Property (A).
Remark 1. A ring $R$ is called *Baer* if for any nonempty set $X$ of $R$, $r_R(X) = eR$ for some $e^2 = e \in R$. A ring $R$ is called *quasi-Baer* if for any ideal $I$ of $R$, $r_R(I) = eR$ for some $e^2 = e \in R$.

We may ask when Baer or quasi-Baer rings have Property (A). The ring in Example 1.4 is Baer, and so quasi-Baer, which does not have Property (A). Also, rings with right Property (A) are not Baer in general. For example, the polynomial ring $\mathbb{Z}[x]$ over the ring of integers $\mathbb{Z}$ is Baer and has Property (A). By Theorem 2.1, $\mathbb{M}_2(\mathbb{Z}[x])$ has right Property (A), but it is well known that $\mathbb{M}_2(\mathbb{Z}[x])$ is not Baer.

From Theorem 2.1 and Example 1.4, we also note that the subring of rings with right Property (A) needs not to have right Property (A). However we have the following. Let $R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} : a, a_{ij} \in R \right\}$, where $n (\geq 1)$ is a positive integer.

**Theorem 2.3.** Fix $n \geq 1$. A ring $R$ has right Property (A) if and only if the ring $R_n$ has right Property (A).

**Proof.** It suffices to show the case $n = 2$ because the other cases can be proved by the same method. Suppose that $R$ has right Property (A). Let $A_i = \left( \begin{array}{c} a_i \\ b_i \\ \end{array} \right) \in \mathbb{Z}^2$ and $X = \sum_{i=1}^n R_2 A_i R_2 \subseteq \mathbb{Z}_l(R_2)$. For each $i$ and for any $r_{ij}, s_{ij} \in R$,

$$\sum_{j=1}^{k_i} \begin{pmatrix} r_{ij} & 0 \\ 0 & r_{ij} \end{pmatrix} \begin{pmatrix} a_i & b_i \\ 0 & a_i \end{pmatrix} \begin{pmatrix} s_{ij} & 0 \\ 0 & s_{ij} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{k_i} r_{ij} a_i s_{ij} & \sum_{j=1}^{k_i} r_{ij} b_i s_{ij} \\ 0 & \sum_{j=1}^{k_i} r_{ij} a_i s_{ij} \end{pmatrix} \in R_2 A_i R_2.$$  

Let $X_i = \sum_{j=1}^{k_i} r_{ij} a_i s_{ij}$ and $Y_i = \sum_{j=1}^{k_i} r_{ij} b_i s_{ij}$. Then

$$X' = \begin{pmatrix} \sum_{i=1}^m X_i \\ \sum_{i=1}^m Y_i \end{pmatrix} \in X \subseteq \mathbb{Z}_l(R_2).$$

Thus there exists a nonzero $(a b \choose 0 a)$ such that $X' (a b \choose 0 a) = O$. This implies that $Y = \sum_{i=1}^m R a_i R \subseteq \mathbb{Z}_l(R)$. Since $R$ has right Property (A), $Yu = 0$ for some nonzero $u \in R$. Notice that $X (0 u \choose 0 0) = O$. Therefore $R_2$ has right Property (A).

Conversely, let $X_0 = \sum_{i=1}^m R a_i R \subseteq \mathbb{Z}_l(R)$. If we set $X = \sum_{i=1}^m R_2 A_i R_2$ where $A_i = \left( \begin{array}{c} a_i \\ 0 \end{array} \right)$, then

$$X = \left\{ \begin{pmatrix} b & c \\ 0 & b \end{pmatrix} : b, c \in X_0 \right\}.$$  

Let $A = (b \choose 0 c)$ be a nonzero element in $X$. Assume $b \neq 0$. Since $b \in X_0$, there exists nonzero $u \in R$ such that $bu = 0$. Then $A (0 u \choose 0 0) = O$. Thus $A \in \mathbb{Z}_l(R_2)$. Assume $b = 0$ and $c \neq 0$. Since $c \in X_0$, there exists nonzero $v \in R$ such that $cv = 0$. Then $A (0 v \choose 0 0) = O$. Thus $A \in \mathbb{Z}_l(R_2)$, and hence $X \subseteq \mathbb{Z}_l(R_2)$. Since $R_2$ has right Property (A), there exists a nonzero $B = \left( \begin{array}{c} a \choose b \end{array} \right) \in R_2$ such
that \( XB = O \). If \( \alpha \neq 0 \), then for any \( b \in X_0, (b \alpha)(\alpha \beta) = O \) since \( (b \alpha) \in X \). Thus \( b\alpha = 0 \) and hence \( X_0\alpha = 0 \). If \( \alpha = 0 \) and \( \beta \neq 0 \), then \( (b \alpha)(0 \beta) = O \). Thus \( b\beta = 0 \) and hence \( X_0\beta = 0 \). Therefore \( R \) has right Property (A). \( \square \)

Given a ring \( R \) and a \((R,R)\)-bimodule \( M \), the trivial extension of \( R \) by \( M \) is the ring \( T(R, M) = R \oplus M \) with the usual addition and the following multiplication:

\[
(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).
\]

This is isomorphic to the ring of all matrices \( \left( \begin{array}{c} r_m \end{array} \right) \), where \( r \in R \) and \( m \in M \) and the usual matrix operations are used.

By Theorem 2.3, a ring \( R \) has right Property (A) if and only if the trivial extension \( T(R, R) \) of \( R \) has right Property (A). But for a ring \( R \) with right Property (A) and a \((R,R)\)-bimodule \( M \), the trivial extension \( T(R, M) \) of \( R \) by \( M \) does not have right Property (A) in general.

**Example 2.4.** We refer an example in [12, p. 63]. Let \( F \) be a field and let \( R = F[x, y] \) the polynomial ring over \( F \) with commuting indeterminates \( x \) and \( y \). Let \( A = \bigoplus_p R/(p) \), where \( p \) ranges over the primes of \( R \) and \( (p) \) is the principal ideal of \( R \) generated by \( p \). Consider the trivial extension of \( R \) by \( A \)

\[
T = T(R, A) = \left\{ \left( \begin{array}{c} r_m \end{array} \right) : r \in R \right\}.
\]

where \( \bar{a}_p = a_p + (p) \). Note that \( x \) and \( y \) are primes in \( R \). Let \( I = (x, y) \) be the ideal of \( R \) generated by \( x \) and \( y \). Then \( I \neq R \) and note that

\[
I \subseteq \ell_R(A) = \{ r \in R : rc = 0 \text{ for some } c \in A \}.
\]

For, let \( 0 \neq \alpha \in I \). Then \( \alpha = c_1x + c_2y = p_1p_2\cdots p_n \) because \( R \) is a UFD. Thus \( \langle \ldots, \bar{0}, 1 + (p_1), \ldots, 1 + (p_n), \bar{0}, \ldots \rangle = \langle \bar{0} \rangle \), and hence \( \alpha \in \ell_R(A) \).

Let

\[
J = IT = \left\{ \sum_{\text{finite}} \left( \begin{array}{c} \alpha \bar{0} \\ \alpha \alpha \end{array} \right) \left( \begin{array}{c} r \bar{0} \\ 0 \end{array} \right) : \alpha \in I, r \in R \right\}.
\]

Then \( J \) is a finitely generated ideal and \( J \subseteq Z_l(T) \). For, let \( \left( \begin{array}{c} \gamma \bar{0} \\ \gamma \gamma \end{array} \right) \in J \). Then \( \gamma \in I \subseteq \ell_R(A) \).

If \( \gamma \neq 0 \), then \( \gamma \langle \bar{a}_p \rangle = \langle \bar{0} \rangle \) for some nonzero \( \langle \bar{a}_p \rangle \in A \). Thus \( \left( \begin{array}{c} \gamma \bar{0} \\ \gamma \gamma \end{array} \right) \left( \begin{array}{c} \bar{0} \bar{0} \\ 0 \bar{0} \end{array} \right) = O \), where \( O \) denotes the zero element of \( T \). If \( \gamma = 0 \), then \( \left( \begin{array}{c} \gamma \bar{0} \\ \gamma \gamma \end{array} \right) \left( \begin{array}{c} \bar{0} \bar{0} \\ 0 \bar{0} \end{array} \right) = O \), and therefore \( J \subseteq Z_l(T) \).

We now show that \( r_T(J) = O \). If \( \left( \begin{array}{c} \beta \bar{0} \\ \beta \beta \end{array} \right) \in r_T(J) \), then \( \left( \begin{array}{c} x \bar{0} \bar{0} \\ x \alpha \bar{a}_p \end{array} \right) \left( \begin{array}{c} \beta \bar{0} \\ 0 \bar{0} \end{array} \right) = O \). So \( \beta x = 0 \) and \( x(\bar{b}_p) + x(\bar{a}_p) = \langle \bar{0} \rangle \). Thus we have \( \beta = 0 \) and hence \( x(\bar{b}_p) = \langle \bar{0} \rangle \). This implies that \( xb_p \in (p) \) for each \( p \). If \( x \neq p \), then \( b_p \in (p) \) and so \( \langle \bar{b}_p \rangle = \langle \bar{0}, b_x + (x), \bar{0}, \ldots \rangle \). Also \( \left( \begin{array}{c} 0 \bar{0} \\ y \bar{0} \end{array} \right) \left( \begin{array}{c} \bar{0} \\ 0 \bar{0} \end{array} \right) = \left( \begin{array}{c} \bar{0} \\ \bar{0} \bar{0} \end{array} \right) \). So \( \langle \bar{0} \rangle = y(\bar{b}_p) = y(\ldots, \bar{0}, b_x + (x), \bar{0}, \ldots) \), and hence \( yb_x \in (x) \).
Since \( y \notin (x) \), we have \( b_x \in (x) \) and so \( \langle \bar{b}_p \rangle = \langle 0 \rangle \). Hence \( \beta^T (\bar{b}_p) = O \), and therefore \( T \) does not have Property (A).

Let

\[
V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_2 \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} : a_1, a_2, \ldots, a_n \in R \right\},
\]

where \( n(\geq 1) \) is a positive integer. The ring \( V_n \) is a subring of \( R_n \), and by the same method of proof as in Theorem 2.3 we have:

**Proposition 2.5.** Fix \( n \geq 1 \). A ring \( R \) has right Property (A) if and only if \( V_n(R) \) has right Property (A).

Following [18], let \( RA = \{ rA : r \in R \} \) for any \( A \in \mathbb{M}_n(R) \), and for \( n \geq 0 \) let \( V = \sum_{i=1}^{n-1} e_{i(i+1)} \), where the \( e_{i,j} \)s are the matrix units. Then note that for any integer \( n \geq 1 \),

\[
V_n(R) = RI_n + RV + \cdots + RV^{n-1}.
\]

Define \( \rho : V_n(R) \rightarrow R[x]/(x^n) \) by \( \rho(a_0 I_n + a_1 V + \cdots + a_{n-1} V^{n-1}) = \sum_{i=0}^{n-1} a_i x^i + (x^n) \). Then \( \rho \) is a ring isomorphism. So we have the following.

**Corollary 2.6.** Fix \( n \geq 1 \). A ring \( R \) has right Property (A) if and only if \( R[x]/(x^n) \) has right Property (A).

We now consider the polynomial rings with right Property (A). Huckaba and Keller [10, Theorem 1] proved that if \( R \) is a commutative nontrivial graded ring then \( R \) has Property (A). As a corollary, the polynomial ring \( R[x] \) over any commutative ring \( R \) has Property (A). However it is not true for a noncommutative ring even if it is a semiprime ring with polynomial identity as follows.

**Example 2.7.** Let \( \mathbb{Z}_2 \) be the ring of integers modulo 2. Consider the ring

\[
R = \left\{ \langle a_i \rangle : a_i \in \mathbb{M}_2(\mathbb{Z}_2), \ a_i \text{ is eventually in } \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix} \right\}.
\]

Then \( R \) is a semiprime ring with polynomial identity. Now we consider the polynomial ring \( R[x] \) over the ring \( R \). Note that

\[
R[x] = \left\{ \langle f_i \rangle : f_i \in \mathbb{M}_2(\mathbb{Z}_2[x]), \ f_i \text{ is eventually in } \begin{pmatrix} \mathbb{Z}_2[x] & \mathbb{Z}_2[x] \\ 0 & \mathbb{Z}_2[x] \end{pmatrix} \right\}.
\]
Let \( \alpha = \left\{ \left( \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right), \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right), \ldots \right\} \subseteq R \). Note that
\[
R[x] \alpha R[x] = \left\{ \langle f_i \rangle : f_i \in \mathbb{M}_2(\mathbb{Z}_2[x]), f_i \text{ is eventually in } \left( \begin{smallmatrix} 0 & \mathbb{Z}_2[x] \\ 0 & 0 \end{smallmatrix} \right) \right\}.
\]

Given \( \langle f_i \rangle \in R[x] \alpha R[x] \), there exists a positive integer \( n \geq 1 \) such that \( f_i \in \left( \begin{smallmatrix} 0 & \mathbb{Z}_2[x] \\ 0 & 0 \end{smallmatrix} \right) \) for any \( i \geq n \). Consider \( \langle g_i \rangle \in R[x] \), where \( g_1 = \ldots = g_n = O \), \( g_k = \left( \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right) \) for all \( k \geq n + 1 \). Then \( \langle f_i \rangle \langle g_i \rangle = (O) \), where \( (O) \) is the zero element of \( R[x] \). Thus \( \langle f_i \rangle \in Z_l(R[x]) \) and hence \( R[x] \alpha R[x] \subseteq Z_l(R[x]) \). We claim that there does not exist nonzero \( h \in R[x] \) such that \( (R[x] \alpha R[x])h = (O) \). Suppose \( (R[x] \alpha R[x])h = (O) \) for some nonzero \( h \in R[x] \). Let \( h = \langle h_k \rangle \in R[x] \). Then there exists a positive integer \( t \geq 1 \) such that \( h_k \in \left( \begin{smallmatrix} \mathbb{Z}_2[x] & \mathbb{Z}_2[x] \\ 0 & 0 \end{smallmatrix} \right) \) for any \( k \geq t \). Put \( E_i = \langle O, \ldots, O, e_i, O, \ldots \rangle \), where \( e_i = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \). For each \( i \geq 1 \), \( E_i \in R[x] \alpha R[x] \) and \( E_i h = \langle O, \ldots, O, h_i, O, \ldots \rangle = (O) \). So \( h_i = O \) and hence \( h = 0 \), which is a contradiction. Therefore \( R[x] \) does not have right Property (A). Moreover it does not have left Property (A) by a similar method.

By the same method as above, \( R \) does not have Property (A).

From the preceding example, it is natural to raise the following question.

**Question 1.**

(1) If \( R \) is a right duo ring, then does the polynomial ring \( R[x] \) have right Property (A)?

(2) If a ring \( R \) has right Property (A), then does the polynomial ring \( R[x] \) have right Property (A)?

**Remark 2.** There exists a ring \( R \) which does not have Property (A) whose the polynomial ring \( R[x] \) has Property (A). For the polynomial ring \( R[x] \) over any commutative ring \( R \) has Property (A) [10, Theorem 1] and there is a commutative ring \( R \) which does not have Property (A).

**Lemma 2.8.** For a ring \( R \), \( R[x] \) has right Property (A) if and only if whenever \( f(x)R[x] \subseteq Z_l(R[x]), r_{R[x]}(f(x)R[x]) \neq 0 \).

**Proof.** Let \( I = \sum_{i=1}^{k} R[x] f_i(x) R[x] \subseteq Z_l(R[x]) \), where \( f_i(x) = a_{i_0} + a_{i_1} x + \cdots + a_{i_n} x^{n_i} \).

Then \( g(x) = \sum_{i=1}^{k} f_i(x) x^{n_i+\cdots+n_{i-1}+1} \in I \), where \( n_0 = 0 \). Thus \( g(x)R[x] \subseteq I \). By hypothesis, \( r_{R[x]}(g(x)R[x]) = r_{R[x]}(R[x] g(x) R[x]) \neq 0 \). By the main theorem of [22] (or [7, Theorem 2.2]), \( r_{R[x]}(R[x] g(x) R[x]) \cap R \neq 0 \). Hence \( (R[x] g(x) R[x]) r = 0 \) for some nonzero \( r \in R \).

Since \( R g(x) R \subseteq R[x] g(x) R[x] \) and \( f_i(x) = a_{i_0} + a_{i_1} x + \cdots + a_{i_n} x^{n_i} \), \( (R a_i r) r = 0 \). Thus \( I r = \sum_{i=1}^{k} R[x] f_i(x) R[x] ) r = 0 \). Therefore \( R[x] \) has right Property (A). The converse is clear. \( \square \)

We now observe rings whose polynomial rings have Property (A). A ring \( R \) is called semicommutative if for \( a, b \in R \), \( ab = 0 \) implies \( a R b = 0 \). Reduced rings and reversible rings are semicommutative rings but the converse is not true in each case. It is well known that a ring \( R \) is reduced if and only if \( R[x] \) is reduced. But by [11, Example 2], there exists a semicommutative ring whose polynomial rings are not semicommutative. Recently Nielsen [23] defined the following: a ring \( R \) is right McCoy when the equation \( f(x)g(x) = 0 \) over \( R[x] \), where \( f(x), g(x) \neq 0 \),
implies there exists a nonzero \( r \in R \) with \( f(x)r = 0 \). Left McCoy rings are defined similarly. McCoy rings are the left and right McCoy rings. It is well known that commutative rings are always McCoy rings [21, Theorem 2], and moreover it is not true for noncommutative rings [27]. Nielsen [23, Theorem 2] proved that if \( R \) is a reversible ring then \( R \) is right McCoy ring. But there exists a semicommutative ring which is not right McCoy [23, Claims 6 and 7]. Following Rege and Chhawchharia [25], a ring \( R \) is called Armendariz if \( a_i b_j = 0 \) for all \( i \) and \( j \) whenever polynomials \( f(x) = \sum_{i=0}^{m} a_i x^i \) and \( g(x) = \sum_{j=0}^{n} b_j x^j \) in \( R[x] \) satisfy \( f(x)g(x) = 0 \) (for more detail, see [13]).

**Proposition 2.9.** Let \( R \) be a right McCoy ring. If \( R \) has right Property (A), then the polynomial ring \( R[x] \) has right Property (A).

**Proof.** Let \( f(x) = \sum_{i=0}^{k} a_i x^i \in R[x] \) be such that \( X = R[x]f(x)R[x] \subseteq Z_l(R[x]) \). Set the ideal \( I = \sum_{i=0}^{k} R a_i R \). Let \( \sum_{j=0}^{N} f(x)g_j(x)l \in I \) where \( a_i \in \{ a_0, a_1, \ldots, a_k \} \) and \( r_j, s_j \in R \), for each \( j \). Set \( (x) = \sum_{j=0}^{N} r_j f(x)g_j x^{j(k+1)} \), which has coefficients consisting of \( r_j a_i s_j \) for every \( j \); say they occur in degrees \( l_0, l_1, \ldots, l_N \). We multiply \( g(x) \) on the right by \( h(x) = \sum_{j=0}^{N} x^{l_j} \) and then the degree \( l_N \) coefficient of \( g(x)h(x) \) is exactly \( a \). But \( g(x)h(x) \in X \subseteq Z_l(R[x]) \), and is nonzero, so is annihilated on the right by some \( 0 \neq r \in R \) since \( R \) is right McCoy. Hence \( ar = 0 \). Since \( a \) was arbitrary, \( I \subseteq Z_l(R) \). Since \( R \) has right Property (A), there exists nonzero \( u \in R \) such that \( IU = 0 \). Note that \( Xu = 0 \). By Lemma 2.8, \( R[x] \) has right Property (A). \( \square \)

**Remark 3.**

1. Armendariz rings are McCoy rings. For an Armendariz ring with right Property (A), \( R[x] \) has right Property (A). In general, McCoy rings need not be Armendariz. For example, there exists a commutative ring which is not Armendariz [25].

2. Rings with right Property (A) are, in general, neither right McCoy nor Armendariz. For example, let \( Z_4 \) be the ring of integers modulo 4. Then \( \mathbb{M}_2(Z_4) \) has Property (A) by Theorem 2.1. But it is not right McCoy by [27] and moreover it is not Armendariz.

3. Right McCoy rings do not have Property (A) in general. Also Armendariz rings do not have Property (A) in general. For example, there exists a reduced ring which does not have Property (A).

**Proposition 2.10.** If \( R \) is a semicommutative and right McCoy ring then \( R[x] \) has right Property (A).

**Proof.** Let \( X = f(x)R[x] \subseteq Z_l(R[x]) \), where \( f(x) = a_0 + a_1 x + \cdots + a_k x^k \). Then \( f(x)h(x) = 0 \) for some nonzero \( h(x) \in R[x] \). If \( R \) is right McCoy, there exists nonzero \( c \in R \) such that \( f(x)c = 0 \). This implies \( a_i c = 0 \) for all \( i \), and then semicommutativity implies \( a_i Rc = 0 \) for all \( i \). But this clearly implies \( f(x)R[x]c = 0 \). Therefore \( R[x] \) has right Property (A) by Lemma 2.8. \( \square \)

**Corollary 2.11.** If \( R \) is a reversible ring then \( R[x] \) has Property (A).

**Corollary 2.12.** For a ring \( R \), if \( R[x] \) is semicommutative, then \( R[x] \) has Property (A).
Proof. Suppose $R[x]$ is a semicommutative ring. Then $R$ is also semicommutative. Moreover, by the main theorem of [22], $R$ is McCoy. Therefore $R[x]$ has Property (A). $\square$

In Example 2.7, the polynomial ring $R[x]$ is not semicommutative. So the condition in Corollary 2.12 is essential.

Corollary 2.13. For any commutative or reduced ring $R$, $R[x]$ has Property (A).

Proposition 2.14. Let $R$ be a ring and $\Delta$ a multiplicatively closed subset of $R$ consisting of central regular elements. Then $R$ has right Property (A) if and only if $\Delta^{-1} R$ has right Property (A).

Proof. Suppose that $R$ has right Property (A). Let $S = \Delta^{-1} R$ and $X$ a finitely generated ideal in $Z_l(S)$. Then we may assume that $X = \sum_{i=1}^n S a_i b^{-1} S \subseteq Z_l(S)$. Set

$$Y = \sum_{i=1}^n R a_i R \subseteq X \subseteq Z_l(S).$$

Thus, given an element $y \in Y$ there exists a nonzero $cd^{-1} \in S$ with $ycd^{-1} = 0$, hence $yc = 0$. Therefore $Y \subseteq Z_l(R)$. Since $R$ has right Property (A), there exists a nonzero $u \in R$ such that $Yu = 0$. Then note that $Xu = 0$. Therefore $S$ has right Property (A).

Conversely, suppose that $S$ has right Property (A). Let $\sum_{i=1}^n R a_i R \subseteq Z_l(R)$. Then note that $X = \sum_{i=1}^n S a_i S \subseteq Z_l(S)$. Since $S$ has right Property (A), there exists nonzero $ab^{-1} \in S$ such that $Xab^{-1} = 0$. Note that $(\sum_{i=1}^n R a_i R)a = 0$. Therefore $R$ has right Property (A). $\square$

The ring of Laurent polynomials in $x$, coefficients in a ring $R$, consists of all formal sums $\sum_{i=k}^n m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and $k, n$ are (possibly negative) integers; denote it by $R[x; x^{-1}]$.

Corollary 2.15. For a ring $R$, the following statements are equivalent:

1. $R[x]$ has right Property (A);
2. $R[x; x^{-1}]$ has right Property (A).

Proof. Let $\Delta = \{1, x, x^2, \ldots \}$. Then $\Delta$ is a multiplicatively closed subset of $R[x]$ consisting of central regular elements. Note that $R[x; x^{-1}] = \Delta^{-1} R[x]$. Thus, by Proposition 2.14, $R[x]$ has right Property (A) if and only if $R[x; x^{-1}]$ has right Property (A). $\square$

We consider the ring $R$ in Example 2.7. By the same method, we can check that the power series ring $R[[x]]$ over $R$ does not have Property (A). Thus we may raise the following questions.

Question 2.

1. Does the power series ring $R[[x]]$ over a commutative ring $R$ have Property (A)?
2. If a ring $R$ has right Property (A), then does the power series ring $R[[x]]$ over $R$ have right Property (A)?
We now observe a case when the classical right quotient ring of a ring $R$ with right Property (A) also has right Property (A).

**Proposition 2.16.** Let $R$ be a reduced ring. If $R$ has right Property (A), then the classical right quotient ring $Q_r(R)$ of $R$ has right Property (A).

**Proof.** Let $Q = Q_r(R)$ and $J = \sum_{i=1}^{n} Qa_i Q \subseteq Z_l(Q)$, where we may assume after scaling on the right, that $a_i \in R$. Let $I = \sum_{i=1}^{n} Ra_i R$. Then $I \subseteq J \subseteq Z_l(Q)$. Thus for each $a \in I$ there exists nonzero $st^{-1} \in Q$ such that $ast^{-1} = 0$. Thus $as = 0$ and so $I \subseteq Z_l(R)$. Since $R$ has right Property (A), there exists nonzero $c \in R$ such that $Ic = 0$. Since $R$ is a reduced ring, $Q$ is also reduced by [13, Theorem 16]. Now $a_i c = 0$ and so $Qa_i Qc = 0$ for each $i$. Hence $Jc = 0$ and therefore $Q$ has right Property (A). \(\square\)

**Remark 4.** Let $R$ be a reduced ring with the classical quotient ring $Q = Q(R)$ (i.e., $R$ is a two-sided order in $Q$). Then $R$ has right Property (A) if and only if $Q$ has right Property (A). For, let $I = \sum_{i=1}^{n} Ra_i R \subseteq Z_l(R)$. Suppose $J = \sum_{i=1}^{n} Qa_i Q \not\subseteq Z_l(Q)$. Then $J$ contains a regular element and so $J = Q$. Thus $I = \sum p_i a_i q_i$, where $p_i, q_i \in Q$ and $i$ is finite. Let $p_i = c_i d^{-1}$ and $q_i = r_i s^{-1}$ with common denominator $d, s$, where $d, s$ are regular elements in $R$, and moreover $c_i d^{-1} = v^{-1} u_i$ for a regular element $v$ because $R$ is a two-sided order in $Q$. Then $v s = \sum_{i=1}^{n} u_i a_i r_i \in I$, which is a contradiction to $I \subseteq Z_l(R)$. Hence $J \subseteq Z_l(Q)$. By hypothesis, there exists a nonzero $x = st^{-1} \in Q$ such that $Jx = 0$. Note that $Ix \subseteq Jx = 0$. Therefore $R$ has right Property (A).

**Proposition 2.17.** If $R$ is a semiprime right Goldie ring, then the classical right quotient ring $Q_r(R)$ of $R$ has Property (A).

**Proof.** Suppose that $R$ is semiprime right Goldie. Then by the Goldie Theorem, $Q_r(R)$ is semisimple Artinian. Thus $Q_r(R)$ has Property (A). \(\square\)

### 3. Compactness of $\text{Min}(R)$

For a ring $R$, $\text{Spec}(R)$ (Min$(R)$) denotes the set of all (minimal) prime ideals of $R$. For any subset $A$ of $R$, define $\text{supp}(A) = \{P \in \text{Spec}(R): A \not\subseteq P\}$ and $\text{hull}(A) = \text{Spec}(R) \setminus \text{supp}(A)$. In case $A = \{a\}$, we write $\text{supp}(a)$ and $\text{hull}(a)$. Shin [26, Lemma 3.1] proved that for any ring $R$, $\{\text{supp}(a): a \in R\}$ forms a basis (for open sets) on $\text{Spec}(R)$. This topology is called the hull–kernel topology. Then Min$(R)$ is regarded as a subspace of $\text{Spec}(R)$. Let $T = Q_{\max}^r(R)$ be a right maximal quotient ring of a ring $R$. We shall use the notation: $S(a) = \{M \in \text{Spec}(T): a \not\in M\}$, $H(a) = \text{Spec}(T) \setminus S(a)$ for each $a \in T$. Moreover, we shall use $s(a) = \text{supp}(a) \cap \text{Min}(R)$ and $h(a) = \text{Min}(R) \setminus s(a)$ for each $a \in R$.

For a commutative reduced ring $R$, the total quotient ring $T$ of $R$ is von Neumann regular if and only if $R$ has Property (A) and Min$(R)$ is compact [10, Theorem B]. We now observe a similar result in noncommutative rings.

**Lemma 3.1.** Let $I$ be a finitely generated ideal of a reduced ring $R$. Then $I$ is contained in a minimal prime ideal of $R$ if and only if $r_R(I) \neq 0$. 
Proof. Let $R$ be reduced and $I = \sum_{i=1}^{n} Ra_i R$. Assume that $I$ is contained in a minimal prime ideal $P$ of $R$. Then there exists $x \in X$ such that $x \subseteq B = 0$ by [15, Lemma 1.3] because $B \subseteq P$, where $X = R \setminus P$ and $B = \{a_1, \ldots, a_n\}$. Since $R$ is reduced, we get $BRx = 0$ and hence $Ix = 0$, concluding $r_R(I) \neq 0$. Conversely, suppose that $r_R(I) \neq 0$ and $I$ is not contained in any minimal prime ideal of $R$. Then $r_R(I) \subseteq P(R)$ since $Ir_R(I) = 0$. Since $R$ is reduced, $P(R) = 0$, a contradiction. □

Lemma 3.2. Suppose a ring $R$ has a right maximal quotient ring $T$ which is reduced. Then the following statements are equivalent:

1. $\text{Min}(R)$ is compact.
2. $\text{Min}(R) = \{M \cap R: M \in \text{Spec}(T)\}$.
3. For each $a \in R$, $S(J) = \text{Spec}(T) \setminus S(a)$ for a finitely generated ideal $J$ of $R$.
4. For each $a \in R$, there exists a finitely generated ideal $J$ of $R$ such that $J \subseteq r_R(a)$ and $r_R((J,a)) = 0$, where $(J,a)$ is the ideal of $R$ generated by $J$ and $a$.

Proof. Note that $\text{Spec}(T)$ is a compact $T_1$-space. For, since $T$ is reduced, $T$ is von Neumann regular (see [26, Remarks 4.18(b)]), and so it is strongly regular. Thus $\text{Spec}(T)$ is a compact $T_1$-space by [26, Proposition 4.3 and Lemma 3.2]. Therefore (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by [26, Theorem 4.13].

(1) $\Rightarrow$ (4). By [15, Corollary 1.4], $(r_R(a), a) \not\subseteq P$ for any minimal prime ideal $P$ of $R$. Thus

$$\text{Min}(R) = s(a) \cup \left(\bigcup \{s(b_i): b_i \in r_R(a)\}\right).$$

Since $\text{Min}(R)$ is compact, $\text{Min}(R) = s(a) \cup s(b_1) \cup \cdots \cup s(b_n)$ for some $b_1, \ldots, b_n \in r_R(a)$. Let $J = (b_1, \ldots, b_n)$ and $P \in \text{Min}(R)$. Then note that $(J, a) \not\subseteq P$. If $c \in r_R((J,a))$, then $(J,a)c = 0 \in P$ and so $c \in P(R) = 0$ because $R$ is reduced. Therefore $r_R((J,a)) = 0$.

(4) $\Rightarrow$ (3). It suffices to show that if $J$ is the finitely generated ideal of the hypothesis, then $S(J) = \text{Spec}(T) \setminus S(a)$. Suppose $M \in \text{Spec}(T) \setminus S(a)$. Then $a \in M \cap R$. But $(J,a)$ is dense in $R$ implies $(J,a)$ is dense in $T$ and so $r_T((J,a)) = 0$. This implies that $(J,a)$ is not contained in any minimal prime ideal of $T$. Thus $(J,a) \not\subseteq M$ and so $(J,a) \not\subseteq M \cap R$. Hence $J \not\subseteq M \cap R$ and so $\text{Spec}(T) \setminus S(a) \subseteq S(J)$. Now take $M \in S(J)$. Then $J \not\subseteq M \cap R$. Since $aJ = 0$, $a \in M \cap R$. Therefore $M \in \text{Spec}(T) \setminus S(a)$. □

Theorem 3.3. Suppose a ring $R$ has a right maximal quotient ring which is reduced. If $R$ has the two-sided classical quotient ring $Q = Q(R)$, then the following statements are equivalent:

1. $Q$ is a biregular ring;
2. $R$ has Property (A) and $\text{Min}(R)$ is compact.

Proof. (1) $\Rightarrow$ (2). Assume that $Q$ is biregular. Then $Q$ has Property (A) by Proposition 1.11. By Remark 4, $R$ has Property (A). We next show that $\text{Min}(R)$ is compact. Since every prime ideal of $Q$ is maximal, $\text{Min}(Q) = \text{Spec}(Q)$. Let $Y = \{M \cap R: M \in \text{Spec}(R)\}$. Then we claim that $\text{Min}(R) = Y$. By [26, Lemma 4.12], for each $P \in \text{Min}(R)$ there exists $M \in \text{Min}(Q)$ such that $M \cap R \subseteq P$. Since $P$ is minimal and $M \cap R \neq 0$, $M \cap R = P$. So $\text{Min}(R) \subseteq Y$. Now, let $M \cap R \in Y$. In fact, $M \cap R$ is a minimal prime ideal of $R$. Let $P \subseteq M \cap R$ be a minimal prime ideal of $R$. Given $a \in M \cap R \subseteq M$, since $M$ is a minimal prime of $Q$ and $Q$ is reduced, there
exists $bc^{-1}Q \setminus M$ with $a(bc^{-1}) = 0$. Hence $ab = 0$, and since $b \in (Q \setminus M) \cap R \subseteq R \setminus P$, we have $b \notin P$. But $P$ is completely prime, so $a \in P$. Since $a$ is arbitrary, $M \cap R \subseteq P$. Therefore $\text{Min}(R) = Y$. Now we define a map $f : \text{Spec}(Q) \to \text{Spec}(R)$ by $f(M) = M \cap R$. Note that $f$ is continuous and $\text{Min}(R)$ is a continuous image of $\text{Spec}(Q)$. Since $\text{Spec}(Q)$ is compact, $\text{Min}(R)$ is compact.

For the case $(2) \Rightarrow (1)$, we need Remark 4, Lemma 3.2 and [9, Theorem 4.5]. So we omit the proof because the proof in [9, Theorem 4.5] needs only minor modifications to apply to our case by using Lemma 3.2. □

Acknowledgments

We thank the referee for a very careful reading of the paper, and many valuable comments, which have greatly improved the paper. We also thank Professor Y. Hirano for his useful suggestions. The first named author was supported by the Kyung Hee University in 2007, the second named author was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2005-003-C00011), while the third named author was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2005-015-C00011).

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