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Tensorial Extensions of Central Simple Algebras

IAN STEWART

Mathematics Institute, University of Warwick, Coventry CV4 7AL, England

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1. INTRODUCTION

Throughout this paper the unqualified term *algebra* will mean a not necessarily associative algebra over a field $\mathfrak{k}$. Given two algebras A and B there is a natural algebra structure on $A \otimes B$ given by

$$(a \otimes b)(c \otimes d) = ac \otimes bd$$

for $a, c \in A$; $b, d \in B$. (All tensor products will be over $\mathfrak{k}$.)

A subalgebra D of an algebra A is an (*r-step*) *subideal* of A if there is a chain

$$D = D_0 \leqslant D_1 \leqslant \cdots \leqslant D_r = A$$

such that each D_i is a two-sided ideal of D_{i+1}, for $i = 0,..., r - 1$. We shall use the notation $D \lhd^r A$ to indicate this situation, writing D si A if the value of r is unimportant. We write $D \lhd A$ to indicate that D is a two-sided ideal of A.

The purpose of this paper is to study the subideal structure of $A \otimes B$, for suitable algebras B. Several authors have looked at the ideal structure of $A \otimes B$ (e.g., Jacobson [5, p. 109], Kaplansky [6, p. 150]), but their results are not immediately applicable to subideals. Their methods, however, can be used. Following them we restrict attention to the case when B is central simple. Our key result, here dubbed the *Sandwich Lemma*, gives conditions under which every subideal of $A \otimes B$ is sandwiched between $I \otimes B$ and $I^\pi \otimes B$ where I is an ideal of A and $\pi > 0$ is an integer.

By suitable choice of A and B we can apply our results to the construction of algebras of various kinds, especially Lie or Jordan algebras. In particular, we obtain an infinite-dimensional residually nilpotent Lie algebra which satisfies the maximal condition for subideals. This appears to be the first nontrivial example of a Lie algebra satisfying this maximal condition, although Lie algebras satisfying the *minimal* condition for subideals are known [14, 17].

The integer π referred to above can be related to a certain combinatorial invariant of B, its *spread*, defined in Section 3. In that section we discuss the calculation of this invariant for certain standard algebras.

I am grateful to Prof. J. A. Green who found a fatal flaw in a preliminary version of the sandwich lemma, and suggested a way of eliminating it.

2. The Sandwich Lemma

If B is an algebra and $b \in B$ we may define the *left* and *right translations* b_L and b_R by

$$\begin{aligned} xb_R &= xb \\ xb_L &= bx \end{aligned} \tag{1}$$

($x \in B$). As b varies throughout B, these translations generate a subalgebra $\mathfrak{M}$ of the associative algebra $\mathfrak{E}$ of all linear transformations of B, called the *multiplication algebra* (Jacobson [5, p. 107], Schafer [10, p. 14]) or *enveloping ring* (Kaplansky [6, p. 147]) of B.

Now B is a right $\mathfrak{M}$-module with action (1). The *centroid* $\mathfrak{d}$ of $\mathfrak{M}$ is the algebra of all $\mathfrak{M}$-endomorphisms of B, these endomorphisms being regarded as left operators on B. If B is simple then $\mathfrak{d}$ is a field (by Jacobson [5, p. 107] and Schur's lemma), and Jacobson's famous *Density Theorem* implies that $\mathfrak{M}$ acts as a dense ring of linear transformations of B, considered as a vector space over $\mathfrak{d}$.

The algebra B is *central simple* if it is simple, and if the centroid $\mathfrak{d}$ is the ground field $\mathfrak{k}$ with its standard action on B. Every simple algebra is central simple over its centroid, and every finite-dimensional simple algebra over an algebraically closed field is central simple (Kaplansky [6, p. 98]).

We now pick a basis $\{p_\alpha\}$ for B. Then every $m \in A \otimes B$ is uniquely expressible in the form

$$m = \sum_\alpha m_\alpha \otimes p_\alpha \tag{2}$$

for $m_\alpha \in A$. Given any subset S of $A \otimes B$ we define $\hat{S}$ to be the subspace of A spanned by all the coefficients m_α of elements $m \in S$ with respect to the given basis.

For any ordinal ρ we define A^ρ by setting $A^1 = A$, $A^{\rho+1} = A^\rho A$, $A^\lambda = \bigcap_{\rho<\lambda} A^\rho$ for limit ordinals λ. The *nucleus* of A is the set of elements $x \in A$ such that $(ab)x = a(bx)$, $(ax)b = a(xb)$, $(xa)b = x(ab)$ for all $a, b \in A$; and the *center* of A is the set of all elements y in the nucleus of A such that $ay = ya$ for all $a \in A$. (See Schafer [10, p. 13, 14]). The algebra A is *ambidextrous* if every left translation is a right translation and every right translation is a left translation.

We may now state:

SANDWICH LEMMA 2.1. *Let A be an algebra, B a central simple algebra. Let S be a subspace of $A \otimes B$ which is idealized by $I \otimes B$ where I is a subalgebra of the nucleus of A. If either*

(1) *I is contained in the center of A,*

(2) *B has an identity,*

or

(3) *B is ambidextrous,*

then there exists an ordinal $\rho \leqslant \omega$ such that

$$I^\rho \hat{S} I^\rho \otimes B \leqslant S.$$

Proof. We deal with the simpler case (1) first. Take any nonzero element

$$m = \sum m_\gamma \otimes p_\gamma$$

in S. Let $W = \{\gamma : m_\gamma \neq 0\}$, and take $\alpha \in W$ such that $m_\alpha \neq 0$. By the density theorem there exists, for any β, an element $t_{\alpha\beta} \in \mathfrak{M}$ such that

$$\begin{aligned} p_\alpha t_{\alpha\beta} &= p_\beta\,, \\ p_\gamma t_{\alpha\beta} &= 0, \qquad \alpha \neq \gamma \in W. \end{aligned}$$

We now modify these $t_{\alpha\beta}$ to obtain elements of the multiplication algebra of $A \otimes B$ having similar properties. Each $t_{\alpha\beta}$ is a sum of terms of the form

$$(b_1)_* \cdots (b_i)_*\,,$$

where $*$ denotes either L or R independently. For given α, β the lengths i of these terms are bounded, say $i \leqslant n$. We select an arbitrary n-tuple $(a_1, \ldots, a_n)$ of elements $a_j \in I$. For each i we split this into i parts, e.g., by taking $(a_1), \ldots, (a_{i-1}), (a_i, \ldots, a_n)$. Consider now the element

$$(a_1 \otimes b_1)_* \cdots (a_{i-1} \otimes b_{i-1})_* (a_i \cdots a_n \otimes b_i)_*$$

of the multiplication algebra of $A \otimes B$, where the $*$'s and b_j are as in $t_{\alpha\beta}$. Let t be the sum of all these elements corresponding to the decomposition of $t_{\alpha\beta}$. Then clearly

$$\begin{aligned} (m_\alpha \otimes p_\alpha) t &= m_\alpha a_1 \cdots a_n \otimes p_\beta\,, \\ (m_\gamma \otimes p_\gamma) t &= 0, \qquad \alpha \neq \gamma \in W. \end{aligned}$$

Since the a_j lie in I the product mt lies in S. But

$$mt = \left(\sum m_\gamma \otimes p_\gamma\right) t$$
$$= m_\alpha a_1 \cdots a_n \otimes p_\beta .$$

If the integers n occurring as W, α, β vary are bounded by ρ then we have

$$\hat{S}I^\rho \otimes B \leqslant S.$$

If the n are unbounded, the same formula holds with ρ replaced by ω. Since $I^\rho \hat{S} I^\rho \leqslant \hat{S} I^\rho$ in case (1) the lemma is proved.

For cases (2) and (3) we need to find some elements to play the part of $a_1, \ldots, a_n$ above, but acting on the left as well as the right. This works provided there are both left and right translations in each summand of $t_{\alpha\beta}$. We must therefore prepare the ground.

In case (2) we simply replace $t_{\alpha\beta}$ by

$$u_{\alpha\beta} = (1)_R (1)_L t_{\alpha\beta} .$$

In case (3) we take $t_{\alpha\beta}$ as before. Now by density again there exists $x \in \mathfrak{M}$ such that $p_\gamma x = p_\gamma$ $(\gamma \in W)$. Then

$$u_{\alpha\beta} = x t_{\alpha\beta}$$

is a sum of terms

$$(b_1)_* \cdots (b_i)_* \tag{3}$$

where each term has at least 2 factors. Since B is ambidextrous we can modify this expression so that it contains at least one left translation and one right translation. We now have

$$p_\alpha u_{\alpha\beta} = p_\beta ,$$
$$p_\gamma u_{\alpha\beta} = 0, \qquad \alpha \neq \gamma \in W,$$

where $u_{\alpha\beta}$ is a sum of terms of the form (3) where $* = R$ or L and *at least one R and one L occur in each term.* Again i is bounded, say by n. We now take n elements $r_1, \ldots, r_n \in I$ and another n elements $l_1, \ldots, l_n \in I$. For a given term we will have j $*$'s equal to R and $i - j$ $*$'s equal to L, with $1 \leqslant j \leqslant i - 1$. We partition $(r_1, \ldots, r_n)$ and $(l_n, \ldots, l_1)$ into j and $i - j$ parts, respectively, keeping them in order. Then each $(b_k)_*$ is replaced by $(q \otimes b_k)_*$, where q is the product in order of the elements in the relevant partition of $(l_n, \ldots, l_1)$ or $(r_1, \ldots, r_n)$, according as $* = L$ or R. The sum t of these modified expressions, corresponding to the decomposition of $u_{\alpha\beta}$, now lies in the multiplication algebra of $A \otimes B$, and t idealizes S. Further,

$$(m_\alpha \otimes p_\alpha)t = l_n \cdots l_1 m_\alpha r_1 \cdots r_n \otimes p_\beta ,$$
$$(m_\gamma \otimes p_\gamma)t = 0, \qquad \alpha \neq \gamma \in W.$$

Hence

$$l_n \cdots l_1 m_\alpha r_1 \cdots r_n \otimes p_\beta \in S.$$

As before, if n is bounded by ρ we have

$$I^\rho \hat{S} I^\rho \otimes B \leqslant S,$$

and if not then

$$I^\omega \hat{S} I^\omega \otimes B \leqslant S.$$

The lemma follows.

We can recover the following result of Jacobson [5, p. 109]:

COROLLARY 2.2. *If B is central simple and A has an identity then every ideal of $A \otimes B$ is of the form $U \otimes B$ where U is an ideal of A.*

Proof. Take $I = \mathfrak{k} \cdot 1$, which lies inside the center.

Remark. B is obviously ambidextrous provided it is either commutative or anticommutative, so that case (3) applies in particular if B is a Lie or (commutative) Jordan algebra.

3. SPREAD

If B is central simple, the *spread* of B is defined to be the least ordinal σ such that every element of the multiplication algebra $\mathfrak{M}$ of B is a sum of terms

$$(b_1)_* \cdots (b_i)_* , \tag{4}$$

each having $\leqslant \sigma$ factors. From the proof of the sandwich lemma it follows that we may take $\rho \leqslant \sigma$ in cases (1) and (3) and $\rho \leqslant \sigma + 1$ in case (2).

Clearly every finite-dimensional central simple algebra has finite spread. We shall show that for infinite-dimensional algebras the spread may be either finite or infinite.

If B is associative then every expression of the form (4) reduces to one of the form $(b_1)_R$, $(b_1)_L$, or $(b_1)_R (b_2)_L$, so the spread is at most 2.

In the case where B is a central simple Lie algebra, we have an improvement on the estimate $\rho \leqslant \sigma$, obtained by making an alternative preparatory move in the sandwich lemma. Thus, starting with

$$t_{\alpha\beta} = \sum (b_1)_* \cdots (b_i)_*$$

as in case (1) (where now each $i \leqslant \sigma$) we use the fact that for all $b \in B$

$$(b)_L = (-b)_R$$

to trade left and right multiplications. For a term where $i > 1$, we make half of the *'s into L's and the rest into R's. For a term where $i = 1$, we use the fact that $B^2 = B$ to express $(b_1)_L$ in the form $\sum x_j y_j$, for suitable $x_j, y_j \in B$. The Lie identities then imply that

$$(b_1)_L = \sum ((x_j)_L(y_j)_R - (y_j)_L(x_j)_R).$$

Now $t_{\alpha\beta}$ is a sum of terms, each with at least one left and one right multiplication, and with at most $[(\sigma + 1)/2]$ of each kind. Thus we may take $\rho \leqslant [(\sigma + 1)/2]$.

The classical Lie algebras of type A_l, B_l, C_l, D_l, G_2, F_4, E_6, E_7, E_8 can be defined over any field of characteristic not 2 or 3 (Seligman [11, 12]) and are then central simple. Calculations using a Chevalley basis (Samelson [9, p. 49], Seligman [12, p. 29]) show that they have spread $\leqslant 8$. Using the improved estimate above we find that the sandwich lemma holds with $\rho \leqslant 5$.

Let G be any additive subgroup of $\mathfrak{k}$ and define $\mathfrak{W}_G$ to have basis $\{w_g : g \in G\}$ and multiplication

$$w_g w_h = (g - h) w_{g+h} . \tag{5}$$

Then $\mathfrak{W}_G$ is a Lie algebra, and if $\operatorname{char}(\mathfrak{k}) \neq 2$ it is central simple. If $\mathfrak{k}$ has characteristic $p > 0$ and G is the additive group of the prime subfield then $\mathfrak{W}_G$ is the *Witt algebra* (Jacobson [4, p. 196]). Taking any element

$$m = \sum m_g w_g$$

of length l (i.e., having l values $m_g \neq 0$), we can multiply it by w_h, where $m_h \neq 0$, and decrease the length by 1. By repeating this process until the length is 1 and then multiplying by one more w_j, it follows that the spread is at most $|G| + 1$.

When G is infinite this bound tells us nothing new. But it is easy to see that $\mathfrak{W}_{\mathbb{Z}}$, for example, has spread ω. If we take A to be a free commutative associative algebra on countably many generators $t_1, t_2, \ldots$ (*without* an identity) then the ideal of $A \otimes \mathfrak{W}_{\mathbb{Z}}$ generated by

$$\begin{aligned}
&t_1 \otimes w_0 \\
&t_2 \otimes w_0 + t_3 \otimes w_1 \\
&t_4 \otimes w_0 + t_5 \otimes w_1 + t_6 \otimes w_2 \\
&\qquad\ldots
\end{aligned}$$

does not contain $A^n \otimes \mathfrak{W}_{\mathbb{Z}}$ for any finite n. Thus the sandwich lemma does not hold for any finite ρ, and the spread must be ω.

On the other hand, the infinite-dimensional analogues of the classical simple Lie algebras have local systems of classical simple Lie algebras and can easily be seen to have spread $\leqslant 8$.

The simple finite-dimensional Jordan algebra of type A_I (Schafer [10, p. 101]) has spread at most 2.

4. Subideals

Our main result on subideals follows from the sandwich lemma.

THEOREM 4.1. *Let A be an associative algebra, B a central simple algebra of finite spread σ. Suppose that either*

(1) *A is commutative,*

(2) *B has an identity,*

or

(3) *B is ambidextrous.*

Suppose that $D \lhd^r A \otimes B$. Let $\hat{D}$ be defined as in Section 2 and let I be the ideal of A generated by $\hat{D}$. Then there exists an integer $\pi = \pi(\sigma, r)$ such that

$$I^\pi \otimes B \leqslant D \leqslant I \otimes B. \tag{6}$$

Conversely, if there is an ideal I of A and an integer π such that D is a subalgebra of $A \otimes B$ and (6) holds, then D is a subideal of $A \otimes B$.

Proof. We have

$$D = D_0 \lhd D_1 \lhd \cdots \lhd D_r = A \otimes B.$$

By the sandwich lemma,

$$A^\rho \hat{D}_{r-1} A^\rho \otimes B \leqslant D_{r-1}$$

so that

$$A^\rho \hat{D} A^\rho \otimes B \leqslant D_{r-1}.$$

The left-hand side idealises D_{r-2}, and we may continue inductively, so that

$$E_r \otimes B \leqslant D,$$

where

$$E_1 = A^\rho \hat{D} A^\rho,$$
$$E_{i+1} = E_i{}^\rho \hat{D} E_i{}^\rho, \qquad \text{for} \quad i \geqslant 1.$$

We claim inductively that there exist integers π_i such that $I^{\pi_i} \leqslant E_i$. Now I is the linear span of

$$\hat{D} + \hat{D}A + A\hat{D} + A\hat{D}A,$$

so that

$$I^3 \leqslant A\hat{D}A.$$

Then

$$\begin{aligned} I^{2\rho+1} &= I^{\rho-1}I^3I^{\rho-1} \\ &\leqslant A^{\rho-1}A\hat{D}AA^{\rho-1} \\ &= E_1. \end{aligned}$$

Suppose now that $I^{\pi_i} \leqslant E_i$. Then

$$\begin{aligned} I^{\rho\pi_i}I^3I^{\rho\pi_i} &\leqslant I^{\rho\pi_i}(A\hat{D}A)\, I^{\rho\pi_i} \\ &\leqslant (I^{\rho\pi_i}A)\, \hat{D}(AI^{\rho\pi_i}) \\ &\leqslant I^{\rho\pi_i}\hat{D}I^{\rho\pi_i} \\ &\leqslant E_i^{\rho}\hat{D}E_i^{\rho} \\ &= E_{i+1}, \end{aligned}$$

so we may take

$$\begin{aligned} \pi_1 &= 2\rho + 1, \\ \pi_{i+1} &= 2\rho\pi_i + 3. \end{aligned}$$

Since we know that we may take $\rho \leqslant \sigma + 1$, this establishes the first part of the theorem.

Conversely, if $I^\pi \otimes B \leqslant D \leqslant I \otimes B$, we have

$$D = (I^\pi \otimes B) + D \lhd (I^{\pi-1} \otimes B) + D \lhd \cdots \lhd I \otimes B \lhd A \otimes B$$

so that D si $A \otimes B$.

From this we can recover a result of Baer [1, p. 41] which we shall need:

LEMMA 4.2 (Baer). *D is a subideal of the associative algebra A if and only if there is an integer π such that*

$$I^\pi \leqslant D \leqslant I$$

when I is the ideal of A generated by D.

Proof. Use 4.1, with $B = \mathfrak{k}$. (Baer's result is in fact more general, and applies to associative *rings*. Baer calls subideals *meta ideals of finite index*). This result does *not* hold for Lie algebras [13, p. 74; 15].

Baer uses this lemma to show that in associative algebras the *join* of (subalgebra generated by) two subideals is a subideal. This fails for Lie algebras (Hartley [2, p. 271], Stewart [13, p. 79] and [15]). We shall say that an algebra *has the join property* if any two subideals generate a subideal.

THEOREM 4.3. *If A and B are algebras which satisfy the hypotheses of* 4.1, *then $A \otimes B$ has the join property.*

Proof. Let D, E si $A \otimes B$. Then there exist ideals H, K of A and integers π, σ such that

$$H^{\pi} \otimes B \leqslant D \leqslant H \otimes B$$

$$K^{\sigma} \otimes B \leqslant E \leqslant K \otimes B.$$

Let F be the join of D, E. Then

$$(H^{\pi} + K^{\sigma}) \otimes B \leqslant F \leqslant (H + K) \otimes B,$$

and

$$(H + K)^{\pi+\sigma} \leqslant H^{\pi} + K^{\sigma},$$

so

$$(H + K)^{\pi+\sigma} \otimes B \leqslant F \leqslant (H + K) \otimes B.$$

Therefore F si $A \otimes B$ by 4.1.

If A is commutative and B is Lie (Jordan) then $A \otimes B$ is Lie (Jordan). Theorem 4.3 provides a wide range of Lie (Jordan) algebras with the join property.

An algebra D is *nilpotent* if $D\mathfrak{M}^n = 0$ where $\mathfrak{M}$ is the multiplication algebra of D. We define $D^{[n]} = D\mathfrak{M}^{n-1}$.

Baer's theorem for rings implies that every nilpotent subideal is contained in a nilpotent ideal. This fails for Lie algebras ([13, 15]), but a class of algebras for which it holds is provided by:

PROPOSITION 4.4. *If A, B are as in* 4.1 *and if D is a nilpotent subideal of $A \otimes B$ then D is contained in a nilpotent ideal.*

Proof. Let I be as in 4.1, so that $I^{\pi} \otimes B \subseteq D$. Since $D^{[n]} = 0$ and B is simple we must have $I^{n\pi} = 0$, whence $I \otimes B$ is a nilpotent ideal $\supseteq D$.

Various generalizations of this are possible: if B is a Lie algebra and A is commutative then $A \otimes B$ is a Lie algebra and every *soluble* subideal is contained in a *nilpotent* ideal (so is in fact nilpotent).

5. Properties Inherited from A

An algebra C is *locally nilpotent* (*locally finite*) if every finite subset is contained in a nilpotent (finite-dimensional) subalgebra. It is *residually nilpotent* (*residually finite*) if it possess a family $\{N_\lambda\}$ of ideals such that $\bigcap N_\lambda = 0$ and C/N_λ is nilpotent (finite-dimensional).

Without difficulty we can prove:

Lemma 5.1. *Let A be an associative algebra, B a finite-dimensional algebra. Then $A \otimes B$ inherits the following properties of A:*

(1) *Residual nilpotence*

(2) *Residual Finiteness*

(3) *Local nilpotence*

(4) *Local finiteness*

(5) *Nilpotence*

(6) *Finiteness of dimension.*

If B is infinite-dimensional, (1), (3) and (5) still carry over to $A \otimes B$, (2) requires residual finiteness of B, (4) local finiteness of B.

In [14, 16] we have defined the chain conditions min-si, min-$\lhd^r$, min-$\lhd$, max-si, max-$\lhd^r$, max-$\lhd$ for Lie algebras. The symbols are self-explanatory, and we extend their use to arbitrary algebras.

Theorem 5.2. *If A, B are as in* 4.1 *and B is finite-dimensional then the following are equivalent:*

(1) *$A \otimes B$ satisfies* max-si,

(2) *$A \otimes B$ satisfies* max-$\lhd^2$,

(3) *Every ideal of $A \otimes B$ satisfies* max-$\lhd$,

(4) *Every ideal of A satisfies* max-$\lhd$,

(5) *A satisfies* max-$\lhd^2$,

(6) *A satisfies* max-si,

(7) *A satisfies* max-$\lhd$ *and if $J \lhd A$, $n > 0$ then J/J^n is finite-dimensional.*

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). We show (4), (5), (6), (7) are equivalent and that (7) $\Rightarrow$ (1).

For any associative A (6) $\Rightarrow$ (5) $\Rightarrow$ (4). Any nilpotent algebra satisfying max-$\lhd$ is finite-dimensional by the obvious argument, so (4) $\Rightarrow$ (7). To show (7) $\Rightarrow$ (6), consider any increasing chain of subideals of A:

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_r \subseteq \cdots.$$

By 4.2 there exist ideals I_r of A and integers j_r such that $I_r^{j_r} \subseteq S_r \subseteq I_r$. Each $I_r^{j_r}$ is an ideal of A, which satisfies max-$\lhd$. Thus the chain

$$I_1 \subseteq I_1 + I_2 \subseteq \cdots \subseteq I_1 + \cdots + I_r \subseteq \cdots$$

stops at $r = R$, say. Then

$$S_i \subseteq J = I_1 + \cdots + I_R$$

for any i. Now,

$$(I_1 + \cdots + I_R)^{j_1+\cdots+j_R} \subseteq I_1^{j_1} + \cdots + I_R^{j_R} \subseteq S_R .$$

If $j = j_1 + \cdots + j_R$ the factor J/J^j is finite-dimensional since (7) holds. Hence by a dimension argument the chain

$$J^j \subseteq S_R \subseteq S_{R+1} \subseteq \cdots$$

becomes stationary, hence (6) holds.

It remains to show that (7) $\Rightarrow$ (1) which follows from an argument similar to that given above. Let

$$H_1 \subseteq H_2 \subseteq \cdots \subseteq H_r \subseteq \cdots$$

be a chain of subideals of $A \otimes B$. By 4.1 there exist ideals I_r of A and integers π_r such that

$$I_r^{\pi_r} \otimes B \subseteq H_r \subseteq I_r \otimes B.$$

As above, we see that

$$J^j \otimes B \subseteq H_i \subseteq J \otimes B$$

for $i \geqslant R$. J/J^j is finite-dimensional by (7), and B is finite-dimensional, so $J \otimes B/J^j \otimes B$ has finite dimension and the chain $\{H_i\}$ must become stationary.

Similarly we can prove a "minimum condition" version of 5.2:

THEOREM 5.3. *The properties* (1)–(7) *of* 5.2 *are equivalent if* max *is replaced throughout by* min.

Remark. For Lie algebras it is known ([14]) that min-si and min-$\lhd^3$ are equivalent. For characteristic zero min-si and min-$\lhd^2$ are equivalent. In fact, the proof in [14] shows that in characteristic 0 if every ideal of a Lie algebra satisfies min-$\lhd$ then the algebra satisfies min-si. The corresponding questions for maximal conditions are open [13, p. 95, Question 8]. 5.2 shows that the algebras $A \otimes B$ do not settle them in the negative.

Another result of this type which we shall find useful is

THEOREM 5.4. *If A, B are as in 4.1 and B is finite-dimensional; and if further every nonzero ideal of A is not nilpotent and is of finite codimension in A; then every nonzero subideal of $A \otimes B$ is of finite codimension.*

Proof. Let $0 \neq S$ si $A \otimes B$. Then $J^j \otimes B \subseteq S \subseteq J \otimes B$. $J^j \neq 0$ since A has no nilpotent nonzero ideals. Then $J^j \lhd A$ so is of finite codimension, hence S is of finite codimension. Note that the conclusion implies that $A \otimes B$ satisfies max-si; but max-si is strictly weaker since there exist infinite-dimensional simple algebras.

6. SOME ASSOCIATIVE THEORY

In this section we shall collect some results about associative algebras. These are presumably well known, but it is convenient to develop them in a form suitable for later applications.

The following result should be compared with the well-known theorem of Hopkins [3] that for a ring-with-1 min-$\lhd$ implies max-$\lhd$.

PROPOSITION 6.1. *For associative algebras (not necessarily having an identity)* min-si $\Rightarrow$ max-si.

Proof. Let A be associative with min-si. The radical N of A is nilpotent and satisfies min-$\lhd$, so is finite-dimensional. A/N is a finite direct sum of simple algebras so satisfies max-si and min-si. Consequently, A satisfies max-si (as in [13, p. 90]).

max-si plus min-si is equivalent to the existence of a *composition series.* We shall prove that the polynomial algebra $\mathfrak{k}[t]$ satisfies max-si, so that max-si does not imply min-$\lhd$. We shall show that min-$\lhd$ plus max-$\lhd$ does not imply either min-$\lhd^2$ or max-$\lhd^2$; but first we need a method of constructing associative algebras.

Let X, Y be associative $\mathfrak{k}$-algebras, and suppose that X is a Y-bimodule (i.e., a left Y-module and a right Y-module such that the actions commute). Suppose further that the action is compatible with the algebra structure of X in the sense that

$$(y_1x_1)(x_2y_2) = y_1(x_1x_2)y_2$$

if $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. The Cartesian product $X \times Y$ is an associative algebra under the multiplication

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 + x_1y_2 + y_1x_2, y_1y_2).$$

Identifying X with $\{(x, 0): x \in X\}$ and Y with $\{(0, y): y \in Y\}$ $X \times Y$ is a split extension of X by Y; i.e. $X \lhd X \times Y$, $X \cap Y = 0$, and $X + Y = X \times Y$. Further, all split extensions of X by Y arise in this way (cf. the analogous case of Lie algebras; [4, p. 17]).

Let $\mathfrak{k}$ be a field, and $\mathfrak{K}$ an extension field of infinite degree. Let $\mathfrak{K}^0$ denote the algebra formed from $\mathfrak{K}$ by making all products zero. Then $\mathfrak{K}^0$ has a natural compatible $\mathfrak{K}$-bimodule structure and is irreducible as $\mathfrak{K}$-module. The split extension $\mathfrak{K}^0 \times \mathfrak{K}$ satisfies max-$\lhd$ and min-$\lhd$, but its nilradical $\mathfrak{K}^0$ is an infinite-dimensional zero algebra, so does not satisfy max-$\lhd$ or min-$\lhd$. Hence $\mathfrak{K}^0 \times \mathfrak{K}$ does not satisfy max-si or min-si.

THEOREM 6.2. *The polynomial algebra* $\mathfrak{k}[t]$ *in one indeterminate* t *satisfies* max-si. *Indeed, every ideal is of finite codimension.*

Proof. Let $R = \mathfrak{k}[t]$, $I \lhd R$. R is a principal ideal domain so $I = (f)$ for $f \in R$. By the division algorithm $R/(f)$ has dimension equal to the degree of f, which is finite.

Polynomial algebras in more than 1 indeterminate do not satisfy max-si. The ideal (xy) of $\mathfrak{k}[x, y]$ contains a strictly increasing chain of ideals

$$I_n = (xy^2, \ldots, xy^n).$$

Another ring of particular interest in applications is the power series ring $\mathfrak{k}[[t]]$. It is a *complete discrete valuation ring*, so that its only ideals are the powers of its unique maximal ideal (t). Clearly $\mathfrak{k}[[t]]/(t^n)$ is finite-dimensional.

7. EXAMPLES

If B is a finite-dimensional central simple Lie (Jordan) algebra and $A = \mathfrak{k}[t]$ then since A is commutative $A \otimes B$ is also Lie (Jordan). By 5.2 and 6.2 $A \otimes B$ satisfies max-si, and every subideal is of finite codimension. By 5.1, $A \otimes B$ is residually finite.

For a sharper example we replace $\mathfrak{k}[t]$ by its ideal (t). Then $A \otimes B$ is residually nilpotent, infinite-dimensional, but has all subideals of finite codimension. This example should be compared with the results of [16, p. 329].

If we let $A = \mathfrak{k}[[t]]$ then by 2.2 $A \otimes B$ has a unique descending chain of ideals. By 4.1 every subideal contains a member of this chain. If instead we replace $\mathfrak{k}[[t]]$ by its ideal (t) then $A \otimes B$ becomes residually nilpotent. In the Lie case we have shown that *there exists an infinite-dimensional Lie algebra L having a unique descending chain of ideals, all of finite codimension, and such that every subideal contains a member of the chain.*

We can use these methods to construct Lie algebras satisfying min-si, but by 6.1 these will have a composition series. Lie algebras satisfying min-si but without a composition series can be found in [14, 17]. The methods of [17] provide examples of Jordan algebras satisfying min-si. We can also use these methods for min-$\triangleleft$ or max-$\triangleleft$. Lie algebras satisfying max-$\triangleleft$ are known (Hartley [2, p. 269], Moody [8, p. 226]). We have not attempted to decide whether or not Moody's algebras satisfy max-si.

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