Algebraic characterization of quasi-isometric spaces via the Higson compactification

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A B S T R A C T

The purpose of this article is to characterize the quasi-isometry type of a proper metric space via the Banach algebra of Higson functions on it.

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1. Introduction

The Gelfand representation is a contravariant functor from the category whose objects are commutative Banach algebras with unit and whose morphisms are unitary homomorphisms into the category whose objects are compact Hausdorff spaces and whose morphisms are continuous mappings. This functor associates to a unitary Banach algebra its set of unitary characters (they are automatically continuous), topologized by the weak* topology.

That functor (the Gelfand representation) is right adjoint to the functor that to a compact Hausdorff space S assigns the algebra $C(S)$ of continuous complex valued functions on S normed by the supremum norm. If A is a commutative Banach algebra without radical, then A is isomorphic to an algebra of continuous complex valued functions on the space of unitary characters of A.

The Gelfand representation is the base for an algebraic characterization of compactifications of topological spaces. A compactification of a topological space, X, is a pair (X^κ, e) consisting of a compact Hausdorff topological space X^κ and an embedding $e : X \to X^\kappa$ with open dense image. (Thus, only locally compact spaces admit compactifications in this sense.) The complement $X^\kappa \setminus e(X)$ is called the corona (or growth) of the compactification (X^κ, e), and denoted by κX. Usually X

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is identified with its image $e(X)$ and thus regarded as a subspace of X^κ. In this case, the closure $\overline{X} = X^\kappa$ and the boundary $\partial X = \kappa X$.

Let X be a topological space and let (X^κ, e) be a compactification of X. Then $C_b(X^\kappa) = C(X^\kappa)$ is a Banach algebra and the embedding $e : X \to X^\kappa$ induces an algebraic isomorphism of $C_b(X^\kappa)$ into the Banach algebra $C_b(X)$ via composition with the embedding e. The image of $C_b(X^\kappa)$ in $C_b(X)$ consists precisely of all the bounded continuous functions on X that admit a continuous extension to X^κ (via e). It therefore contains the constant functions on X and generates the topology of X in the sense that if E is a compact subset of X and $x \in X \setminus E$, then there is a function in $e^* C_b(X^\kappa)$ that takes on the value 0 at x and is identically 1 on E. Conversely, if A is a Banach subalgebra of $C_b(X)$ that contains the constant functions on X and generates the topology of X, then A is isomorphic to the algebra of (bounded) continuous functions on a compactification of X.

For example, $C_b(X)$, the algebra of bounded continuous functions on X, corresponds to the Stone–Čech compactification of X, and $\mathbb{C} + C_0(X)$, the subalgebra of $C_b(X)$ generated by the constants and the continuous functions that vanish at infinity, corresponds to the one-point compactification of X.

Theorem 1.1 *(Gelfand). Two locally compact Hausdorff spaces, X and Y, are homeomorphic if and only if the Banach algebras $C_b(X)$ and $C_b(Y)$ are algebraically isomorphic.*

In fact, an algebraic isomorphism $C_b(Y) \to C_b(X)$ induces a homeomorphism $X^\beta \to Y^\beta$ that maps X onto Y.

The present paper is motivated by algebraic characterizations of topological structures for which the above theorem is a milestone. Refinements of this milestone that motivated the present paper include the work of Nakai [15] on the algebraic characterization of the holomorphic and quasi-conformal structures of Riemann surfaces.

Let R be a Riemann surface and $M(R)$ the algebra of bounded complex valued functions on R which are absolutely continuous on lines and have finite Dirichlet integral $D(f) = \int_R df \wedge \star \, df$. Endowed with the norm $\|f\| = \sup_{x \in R} |f(x)| + [D(f)]^{1/2}$, $M(R)$ is a commutative Banach algebra. It contains the constant functions and it also contains the compactly supported smooth functions, so it generates the topology of R. Therefore $M(R)$ is the algebra of continuous complex valued functions on a compactification of R, the so called Royden compactification.

Work on the Royden compactification culminated in the following theorem of Nakai [15]:

Theorem 1.2 *(Nakai). Two Riemann surfaces R and R' are quasi-conformally equivalent if and only if the corresponding algebras $M(R)$ and $M(R')$ are algebraically isomorphic.*

Two Riemann surfaces R and R' are conformally equivalent if and only if there is a norm preserving isomorphism between $M(R)$ and $M(R')$.

The Royden algebra can be defined on any locally compact metric space, (X, d), endowed with a Borel measure μ. If f is a complex valued function on X, then its gradient norm is the function $|\nabla f|$ on X given by $|\nabla f|(x) = \limsup_{z \to x} \frac{|f(x) - f(z)|}{d(x,z)}$. A function f on X is a Royden function if it is bounded, continuous, and satisfies $\int_X |\nabla f|^2 \cdot \mu < \infty$. The family of Royden functions on X form a subalgebra of the algebra of bounded continuous functions which contains the constant functions and the compactly supported functions. Its completion with respect to the norm given by $\|f\| = \sup_{x \in X} |f(x)| + (\int_X |\nabla f|^2 \mu)^{1/2}$ is a Banach algebra and it gives rise to a compactification of X, called the Royden compactification of X. If f_n is a Cauchy sequence of Royden functions (with respect to the Royden norm), then f_n converges uniformly to a continuous function on X. Therefore, to each element in the Royden algebra, there corresponds a bounded continuous function on X. This correspondence $M(X) \to C_b(X)$ is a norm decreasing, injective, algebraic homomorphism.

Nakai and others have studied and extended Nakai's theorem on the Royden algebra and Royden compactification of Riemann surfaces to other metric spaces: Riemannian manifolds (Nakai [16], Lelong-Ferrand [6]), and domains in Euclidean spaces (Lewis [14]). A generalization of Nakai's theorem involving Royden p-compactifications was also given in [17]. The following theorem is a representative result of those works.

Theorem 1.3. *Let R, R' be Riemannian manifolds of dimension $\dim R = \dim R' > 2$, endowed with the induced path metric structure and Riemannian measure. Then R and R' are quasi-isometrically homeomorphic if and only if R and R' have homeomorphic Royden compactifications.*

In the present paper we prove an analogous theorem for metric spaces and their coarse quasi-isometries in the sense of Gromov. The algebra of functions that characterizes the coarse quasi-isometry type is the Higson algebra. A Higson function (*cf.* Definition 4.1 below) on a locally compact metric space, (M, d), is a bounded Borel function, f, on M such that, for each $r > 0$, its r-expansion $\nabla_r f(x) = \sup\{|f(x) - f(y)| \mid d(x, y) \leqslant r\}$ is in $\mathcal{B}_0(M)$, the algebra of bounded Borel functions on M that vanish at ∞. The Higson functions on M form a Banach algebra, denoted by $\mathcal{B}_\nu(M)$, and the subalgebra of continuous Higson functions, $C_\nu(M)$, determine the so called Higson compactification M^ν of M. The Higson corona (or growth) is the complement $\nu M = M^\nu \setminus M$. Some topological properties of this Higson compactification were studied in [4,5,13].

Theorem 1.4. *Two proper metric spaces (M, d) and $(M'd')$ are coarsely equivalent if and only if there is an algebraic homomorphism of Higson algebras $\mathcal{B}_\nu(M') \to \mathcal{B}_\nu(M)$ that induces an isomorphism $C(\nu M') \to C(\nu M)$, where νM and $\nu M'$ are the coronas of the respective Higson compactifications of M and M'.*

The "only if" part of Theorem 1.4 has no version with continuous Higson functions, which justifies the use of Borel ones. For instance, **Z** and **R** are coarsely equivalent, but any continuous map $\mathbf{R} \to \mathbf{Z}$ is constant, and therefore no homomorphism $C_\nu(\mathbf{Z}) \to C_\nu(\mathbf{R})$ induces an isomorphism $C(\nu\mathbf{Z}) \to C(\nu\mathbf{R})$.

Other geometric properties of metric spaces have been shown to have a purely algebraic characterization; one example of such properties is illustrated by recent work of Bourdon [3]. To each metric space he associates an algebra of functions based on a Besov norm, and then he proves that two metric spaces are homeomorphic via a quasi-Moebius homeomorphism if and only if those algebras are isomorphic.

It appears of interest to analyze what other geometric structures on a metric space can be inferred from naturally defined Banach algebras of functions on it.

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2. Coarse quasi-isometries

Let (M, d) and (M', d') be arbitrary metric spaces. A mapping $f : M \to M'$ is said to be *Lipschitz* if there is some $C > 0$ such that

$$d'\big(f(x), f(y)\big) \leqslant Cd(x, y)$$

for all $x, y \in M$. Any such constant C is called a *Lipschitz distortion* of f. The map f is said to be *bi-Lipschitz* when there is some $C \geqslant 1$ such that

$$\frac{1}{C}d(x, y) \leqslant d'\big(f(x), f(y)\big) \leqslant Cd(x, y)$$

for all $x, y \in M$. In this case, the constant C will be called a *bi-Lipschitz distortion* of f.

A family of Lipschitz maps is called *equi-Lipschitz* when all the maps in it have some common Lipschitz distortion. A family of bi-Lipschitz maps is said to be *equi-bi-Lipschitz* when all of its maps have some common bi-Lipschitz distortion, which is called an *equi-bi-Lipschitz distortion*.

A *net* in a metric space (M, d) is a subset $A \subset M$ such that $d(x, A) \leqslant K$ for some $K > 0$ and all $x \in M$. On the other hand, a subset A of M is said to be *separated* when there is some $\delta > 0$ such that $d(x, y) > \delta$ for every pair of different points $x, y \in A$. The terms K-net and δ-separated net will be also used.

Lemma 2.1. *Let $K > 0$. There is some K-separated K-net in M. Moreover any K-net of M contains a K-separated $2K$-net of M.*

Proof. Let $\mathcal{S}$ be the family of K-separated subsets of M. By using Zorn's lemma, it follows that there exists a maximal element $A \in \mathcal{S}$. If $d(x, A) > K$ for some $x \in M$, then $A \cup \{x\} \in \mathcal{S}$, contradicting the maximality of A. Hence A is a K-net in M.

Let A be a K-net for M. The above shows that there is a K-separated K-net B for the metric space A. It easily follows that B is a $2K$-net for M.   □

The concept of coarse quasi-isometry was introduced by M. Gromov [8] as follows.[2] A *coarse quasi-isometry* between metric spaces (M, d) and (M', d') is a bi-Lipschitz bijection between some nets $A \subset M$ and $A' \subset M'$; in this case, M and M' are said to have the same *coarse quasi-isometry type* or to be *coarsely quasi-isometric*. A coarse quasi-isometry between M and itself will be called a *coarse quasi-isometric transformation* of M. For some $K > 0$ and $C \geqslant 1$, the pair (K, C) is said to be a *coarsely quasi-isometric distortion* of a coarse quasi-isometry if it is a bi-Lipschitz bijection between K-nets with bi-Lipschitz distortion C. A family of *equi-coarse quasi-isometries* is a collection of coarse quasi-isometries that have a common coarse distortion.

Two coarse quasi-isometries between (M, d) and (M', d'), say $f : A \to A'$ and $g : B \to B'$, are *close* if there are some $r, s > 0$ such that

$$d(x, B) \leqslant r, \qquad d(y, A) \leqslant r,$$
$$d(x, y) \leqslant r \implies d'\big(f(x), g(y)\big) \leqslant s,$$

for all $x \in A$ and $y \in B$. (Such coarse quasi-isometries f and g are said to be (r, s)-*close*.)

It is well known that "being coarsely quasi-isometric" is an equivalence relation on metric spaces. Indeed, this is a consequence of the fact that the "composite" of coarse quasi-isometries makes sense up to closeness [1].

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[2] It is also called *rough isometry* in the context of potential theory [12].

The equivalence classes of the closeness relation on coarse quasi-isometries between metric spaces form a category of isomorphisms. This is indeed the subcategory of isomorphisms of the following larger category. For any set S and a metric space M, with metric d, two maps $f, g : S \to M$ are said to be *close*[3] when there is some $R > 0$ such that $d(f(x), g(x)) \leqslant R$ for all $x \in S$; it may be also said that these maps are *R-close*. If (M', d') is another metric space, a (not necessarily continuous) map $f : M \to M'$ is said to be *large scale Lipschitz* [8] if there are constants $\lambda \geqslant 1$ and $c > 0$ such that

$$d'\big(f(x), f(y)\big) \leqslant \lambda d(x, y) + c$$

for all $x, y \in M$; in this case, the pair (λ, c) will be called a *large scale Lipschitz distortion* of f. The map f is said to be *large scale bi-Lipschitz* if there are constants $\lambda \geqslant 1$ and $c > 0$ such that

$$\frac{1}{\lambda} d(x, y) - c \leqslant d'\big(f(x), f(y)\big) \leqslant \lambda d(x, y) + c$$

for all $x, y \in M$; in this case, the pair (λ, c) will be called a *large scale bi-Lipschitz distortion* of f. The map f will be called a *large scale Lipschitz equivalence* if it is large scale Lipschitz and if there is another large scale Lipschitz map $g : M' \to M$ so that $g \circ f$ and $f \circ g$ are close to the identity maps on M and M', respectively. In this case, if (λ, c) is a large scale Lipschitz distortion of f and g, and $g \circ f$ and $f \circ g$ are R-close to the identity maps for some $R > 0$, then (λ, c, R) will be called a *large scale Lipschitz equivalence distortion* of f. A large scale Lipschitz equivalence is easily seen to be large scale bi-Lipschitz.

It is well known that two metric spaces are coarsely quasi-isometric if and only if they are isomorphic in the category of metric spaces and closeness equivalence classes of large scale Lipschitz maps; this is part of the content of the following two results, where the constants involved are specially analyzed.

Proposition 2.2. *Let $f : A \to A'$ be any coarse quasi-isometry between metric spaces (M, d) and (M', d') with coarse distortion (K, C). Then f is induced by a large scale Lipschitz equivalence $\varphi : M \to M'$ with large scale Lipschitz equivalence distortion $(C, 2CK, K)$.*

Proof. For each $x \in M$ and $x' \in M'$, choose points $h(x) \in A$ and $h'(x) \in A'$ such that $d(x, h(x)) \leqslant K$ and $d'(x', h'(x')) \leqslant K$. Moreover assume that $h(x) = x$ for all $x \in A$, and that $h'(x') = x'$ for all $x' \in A'$. Then f and f^{-1} are respectively induced by the maps $\varphi = f \circ h : M \to M'$ and $\psi = f^{-1} \circ h' : M' \to M$. For all $x, y \in M$,

$$\begin{aligned}
d'\big(\varphi(x), \varphi(y)\big) &\leqslant Cd\big(h(x), h(y)\big) \\
&\leqslant Cd\big(h(x), x\big) + Cd(x, y) + Cd\big(y, h(y)\big) \\
&\leqslant Cd(x, y) + 2CK;
\end{aligned}$$

furthermore

$$d\big(x, \psi \circ \varphi(x)\big) = d\big(x, h(x)\big) \leqslant K.$$

Similarly,

$$d\big(\psi(x'), \psi(y')\big) \leqslant Cd'(x', y') + 2CK,$$
$$d\big(x', \varphi \circ \psi(x')\big) \leqslant K,$$

for all $x', y' \in M'$, and the result follows. □

Proposition 2.3. *Let $\varphi : M \to M'$ be a large scale Lipschitz equivalence with large scale Lipschitz equivalence distortion (λ, c, R). Then, for each $\varepsilon > 0$, the map φ induces a coarse quasi-isometry $f : A \to A'$ between M and M' with coarse distortion*

$$\left(R + 2\lambda R + \lambda c + \lambda \varepsilon + c, \lambda \left(1 + \frac{2R + c}{\varepsilon} \right) \right).$$

Proof. Let $\psi : M' \to M$ be a large scale Lipschitz map with large scale Lipschitz distortion (λ, c) such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are R-close to the identity maps on M and M', respectively.

By Lemma 2.1, there is a $(2R + c + \varepsilon)$-separated $(2R + c + \varepsilon)$-net A of M. Let $A' = \varphi(A)$, and let $f : A \to A'$ denote the restriction of φ. For all $x, y \in M$, the inequality

$$d(x, y) \leqslant d\big(x, \psi \circ \varphi(x)\big) + d\big(\psi \circ \varphi(x), \psi \circ \varphi(y)\big) + d\big(\psi \circ \varphi(y), y\big)$$

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[3] This term is used in [10]. Other terms used to indicate the same property are *coarsely equivalent* [19], *parallel* [8], *bornotopic* [18], and *uniformly close* [2].

implies

$$d(x, y) \leqslant \lambda d'\big(\varphi(x), \varphi(y)\big) + 2R + c. \tag{2.1}$$

In particular, if $\varphi(x) = \varphi(y)$, then $d(x, y) \leqslant 2R + c$. Therefore f is bijective because A is $(2R + c)$-separated.

For any $x' \in M'$, there is some $x \in A$ such that $d(x, \psi(x')) \leqslant 2R + c + \varepsilon$. Then

$$\begin{aligned}
d'\big(x', \varphi(x)\big) &\leqslant d'\big(x', \varphi \circ \psi\big(x'\big)\big) + d'\big(\varphi \circ \psi\big(x'\big), \varphi(x)\big) \\
&\leqslant R + \lambda d\big(\psi\big(x'\big), x\big) + c \\
&\leqslant R + 2\lambda R + \lambda c + \lambda\varepsilon + c.
\end{aligned}$$

So A' is an $(R + 2\lambda R + \lambda c + \lambda\varepsilon + c)$-net of M'.

Because A is $(2R + c + \varepsilon)$-separated, if $x, y \in A$ are distinct, then

$$d'\big(f(x), f(y)\big) \leqslant \lambda d(x, y) + c \leqslant \left(\lambda + \frac{c}{2R + c + \varepsilon}\right)d(x, y).$$

By the same reason and (2.1), it follows that $d'(f(x), f(y)) > \varepsilon/\lambda$. Hence

$$d(x, y) \leqslant \lambda d'\big(f(x), f(y)\big) + 2R + c \leqslant \lambda\left(1 + \frac{2R + c}{\varepsilon}\right)d'\big(f(x), f(y)\big)$$

again by (2.1), which finishes the proof. □

3. Coarse structures

The concept of coarse structure was introduced in Roe [19], and further developed in Higson and Roe [10], as a generalization of the concept of the closeness relation on maps from a set into a metric space. The basic definitions and results pertaining to coarse structures are recalled presently.

Definition 3.1. A *coarse structure* on a set X is a correspondence that assigns to each set S an equivalence relation (called "being *close*") on the set of maps $S \to X$ such that the following compatibility conditions are satisfied:

(i) if $p, q : S \to X$ are close and $h : S' \to S$ is any map, then $p \circ h$ and $q \circ h$ are close;
(ii) if $S = S' \cup S''$ and if $p, q : S \to X$ are maps whose restrictions to both S' and S'' are close, then p and q are close; and
(iii) all constant maps $S \to X$ are close to each other.

A set endowed with a coarse structure is called a *coarse space*.

Definition 3.2. Let X be a coarse space. A subset $E \subset X \times X$ is called *controlled*[4] if the restrictions to E of the two factor projections $X \times X \to X$ are close.

The coarse structure of a coarse space X is determined by its controlled sets: two maps $p, q : S \to X$ are close if and only if the image of $(p, q) : S \to X \times X$ is controlled. Thus a coarse structure can be also defined in terms of its controlled sets [19,10].

A subset $B \subset X$ is called *bounded* if $B \times B$ is controlled, equivalently, if the inclusion mapping $B \hookrightarrow X$ is close to a constant mapping. More generally, a collection $\mathcal{U}$ of subsets of X is said to be *uniformly bounded* if $\bigcup_{U \in \mathcal{U}} U \times U$ is controlled. The coarse space X is called *separable* if it has a countable uniformly bounded cover.

Definition 3.3. A mapping $f : X \to X'$ between coarse spaces is called a *coarse map* if

(i) whenever $p, q : S \to X$ are close maps, the composites $f \circ p, f \circ q : S \to X'$ are close maps; and
(ii) if B is a bounded subset of X', then $f^{-1}(B)$ is bounded in X.

Two coarse spaces, X and X', are *coarsely equivalent* if there are coarse mappings $f : X \to X'$ and $g : X' \to X$ such that $f \circ g$ is close to the identity of X' and $g \circ f$ is close to the identity of X. In this case, f (and g) are called coarse equivalences. The *coarse category* is the category whose objects are coarse spaces and whose morphisms are equivalence classes of coarse mappings, two mappings being equivalent if they are close.

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[4] Controlled sets are called *entourages* in Roe [19].

Definition 3.4. Let X, X' be coarse spaces. A mapping $\varphi : X \to X'$ is *uniformly coarse* if

 (i) for every controlled set $E \subset X \times X$, the image $(\varphi \times \varphi)(E) \subset X' \times X'$ is controlled, and
(ii) for every controlled set $F \subset X' \times X'$, the preimage $(\varphi \times \varphi)^{-1}(F) \subset X \times X$ is controlled.

Proposition 3.5. *Let X and X' be coarse spaces, and let $\varphi : X \to X'$ and $\psi : X' \to X$ be mappings satisfying* (i) *of Definition* 3.3, *and such that $\psi \circ \varphi$ is close to the identity of X and $\varphi \circ \psi$ is close to the identity of X'. Then φ and ψ are uniformly coarse* (*and consequently X and X' are coarsely equivalent*).

Proof. It is plain that (i) of Definition 3.4 is equivalent to (i) of Definition 3.3, and that (ii) of Definition 3.4 implies (ii) of Definition 3.3.

Let p_1 and p_2 denote the projection mappings $X \times X \to X$. If $F \subset X' \times X'$ is controlled, then $(\psi \times \psi)(F) \subset X \times X$ is controlled, and so the mappings $p_1 \circ (\psi \times \psi)$ and $p_2 \circ (\psi \times \psi)$ of F into X are close. Therefore, the mappings $p_1 \circ (\psi \times \psi) \circ (\varphi \times \varphi) = p_1 \circ (\psi \circ \varphi \times \psi \circ \varphi)$ and $p_2 \circ (\psi \times \psi) \circ (\varphi \times \varphi) = p_2 \circ (\psi \circ \varphi \times \psi \circ \varphi)$ from $(\varphi \times \varphi)^{-1}(F)$ into X are also close. Since $\psi \circ \varphi$ is close to the identity on X, the mappings p_1 and p_2 from $(\varphi \times \varphi)^{-1}(F)$ into X are also close, establishing property (ii) of Definition 3.4 for φ. $\square$

Definition 3.6. A coarse structure on a set X is said to be a *proper coarse structure* if

  (i) X is equipped with a locally compact Hausdorff topology;
 (ii) X has a uniformly bounded open cover; and
(iii) every bounded subset of X has compact closure.

A set equipped with a proper coarse structure will be called a *proper coarse space*. Note that bounded subsets of a proper coarse space are those subsets with compact closure.

A metric space, (M, d), has a natural coarse structure, that is defined by declaring two maps $f, g : S \to M$ (where S is any set) to be *close* when $\sup\{d(f(s), g(s)) \mid s \in S\} < \infty$. This closeness relation defines a coarse structure on M, which is called its *metric*[5] coarse structure. The terms *metric closeness* and *metric controlled set* can be used in this case. This coarse structure is proper if and only if the metric space M is proper in the sense that its closed balls are compact. In the case of metric coarse structures, the above abstract coarse notions have their usual meanings for metric spaces.

More generally, following Hurder [11], a *coarse distance* (or *coarse metric*) on a set X is a symmetric map $d : X \times X \to [0, \infty)$ satisfying the triangle inequality; in this case, (X, d) is called a *coarse metric space*. Any coarse distance defines a coarse structure in the same way as a metric does, and will be also called a *metric* coarse structure. In this section and the above one, all notions and properties are given for metric spaces for simplicity, but they have obvious versions for coarse metric spaces.

Definition 3.7. If (M, d) and (M', d') are metric spaces, the two conditions of Definition 3.3 on a map $f : M \to M'$ to be coarse can be written as follows:

 1. (*Uniform expansiveness.*[6]) For each $R > 0$ there is some $S > 0$ such that

$$d(x, z) \leqslant R \quad \Longrightarrow \quad d'\big(f(x), f(z)\big) \leqslant S$$

   for all $x, z \in M$.
 2. (*Metric properness.*[7]) For each bounded subset $B \subset M'$, the inverse image $f^{-1}(B)$ is bounded in M.

The last property admits a uniform version: a map $f : M \to M'$ is said to be *uniformly metrically proper*[8] if for each $R > 0$ there is some $S > 0$ so that

$$d'\big(f(x), f(z)\big) \leqslant R \quad \Longrightarrow \quad d(x, z) \leqslant S$$

for all x, z in M. By using uniform metric properness instead of metric properness, we get what is called the *rough category*. More precisely, a map between metric spaces, $f : M \to M'$, is called a *rough map* if it is uniformly expansive and uniformly metrically proper; if moreover there is a rough map $g : X' \to X$ so that the compositions $g \circ f$ and $f \circ g$ are respectively close to the identity maps on X and X', then f is called a *rough equivalence*; in this case, X and X' are said to be *roughly*

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[5] This term is taken from [10]. It is also called *bounded* coarse structure in [19].
[6] This name comes from Roe [19]. Other terms used to denote the same property are *uniformly bornologous* Roe [18] and *coarsely Lipschitz* Block and Weinberger [2].
[7] This term is used in Roe [19].
[8] This term is used in [19]. Another term used to denote the same property is *effectively proper* [2].

equivalent.[9] Thus rough equivalences are the maps that induce isomorphisms in the rough category. There are interesting differences between the rough category and the coarse category of metric spaces Roe [19], but the following result shows that they have the same isomorphisms.

Proposition 3.8. *Any coarse equivalence between metric spaces is uniformly metrically proper. Moreover the definition of uniform metric properness is satisfied with constants that depend only on the constants involved in the definition of coarse equivalence.*

Proof. Let $f : M \to M'$ and $g : M' \to M$ be coarse maps so that $g \circ f$ and $f \circ g$ are r-close to the identity maps on M and M' for some $r > 0$. Then, because g is uniformly expansive, for any $R > 0$ there is some $S > 0$ such that

$$d'\big(x', z'\big) \leqslant R \quad \Longrightarrow \quad d\big(g\big(x'\big), g\big(z'\big)\big) \leqslant S$$

for all $x', z' \in M'$. It follows that when $x, z \in M$ are such that $d'(f(x), f(z)) \leqslant R$, then

$$d(x, z) \leqslant d\big(x, g \circ f(x)\big) + d\big(g \circ f(x), g \circ f(z)\big) + d\big(g \circ f(z), z\big) \leqslant S + 2r,$$

which establishes that f is uniformly metrically proper. □

It is not possible to define "equi-coarse maps" or "equi-coarse equivalences" between arbitrary coarse spaces, but in the case of metric coarse structures the following related concepts can be defined. A family of maps, $f_i : X_i \to X_i'$, $i \in \Lambda$, is said to be a family of:

- *equi-uniformly expansive maps* if they satisfy the condition of uniform expansiveness involving the same constants;
- *equi-uniformly metrically proper maps* if they satisfy the condition of uniform metric properness involving the same constants;
- *equi-rough maps* if they are equi-uniformly expansive and equi-uniformly metrically proper; and
- *equi-rough equivalences* if they are equi-rough, and there is another collection of equi-rough maps $g_i : X_i' \to X_i$, $i \in \Lambda$, and there is some $r > 0$ so that the composites $g_i \circ f_i$ and $f_i \circ g_i$ are r-close to the identity maps on X_i and X_i', respectively, for all $i \in \Lambda$.

According to Proposition 3.8, a collection of equi-rough equivalences can be also properly called a family of *equi-coarse equivalences*.

Gromov [9, Theorem 1.8.i] characterizes complete path metric spaces (that is, metric spaces where the distance between any two points equals the infimum of the lengths of all paths joining those two points) as those complete metric spaces, (X, d), that satisfy the following property: for all points x, y in X and every $\varepsilon > 0$, there is some point z such that $\max\{d(x, z), d(y, z)\} < \frac{1}{2}d(x, y) + \varepsilon$. This condition can be called "*approximate convexity*:" a subset of $\mathbf{R}^n$ satisfies this property (with respect to the induced metric) if and only if it has convex closure. Gromov [9, Theorem 1.8.i] establishes that a complete, locally compact metric space is approximately convex if and only if it is geodesic: the distance between any two points equals the length of some curve joining those two points.

The following definition is a coarsely quasi-isometric version of the above approximate convexity property.

Definition 3.9. A metric space, (M, d), is said to be *coarsely quasi-convex* if there are constants $a, b, c > 0$ such that, for each $x, y \in M$, there is some finite sequence of points $x = x_0, \ldots, x_n = y$ in M such that $d(x_{k-1}, x_k) \leqslant c$ for all $k \in \{1, \ldots, n\}$, and

$$\sum_{k=1}^{n} d(x_{k-1}, x_k) \leqslant ad(x, y) + b.$$

A family of metric spaces is said to be *equi-coarsely quasi-convex* if all of them satisfy the condition of being coarsely quasi-convex with the same constants a, b, and c.

Remark 3.10. Definition 3.9 can be compared with the concept of monogenic coarse space [20]. In the case of a metric coarse structure, the condition of being monogenic is obtained by removing the constants a, b and the last inequality from Definition 3.9.

A typical example of a coarsely quasi-convex space that is not approximately convex is the set V of vertices of a connected graph G with the metric d_V induced by G. This V satisfies the condition of being coarsely quasi-convex with constants $a = b = c = 1$. This metric on V is the restriction of a metric on G that can be defined as follows. Choose any metric d_e on each edge e of G so that e is isometric to the unit interval. Then the distance between two points $x, y \in G$ is the minimum of the sums of the form

---

[9] The term *uniform closeness* is used in [2] to indicate this equivalence between metric spaces.

$$d_e(x, v) + d_V(v, w) + d_f(w, y),$$

where x, y lie in edges e, f, and v, w are vertices of e, f, respectively. Observe that G is geodesic and V is a $1/2$-net in G. More generally, any net of a geodesic metric space is coarsely quasi-convex. This is a particular case of the following result.

Theorem 3.11. *A metric space, (M, d), is coarsely quasi-convex if and only if there exists a coarse quasi-isometry $f : A \to A'$ between (M, d) and some path metric space (M', d'). In this case, the coarsely quasi-isometric distortion of f depends only on the constants involved in the condition coarse quasi-convexity satisfied by M, and conversely; equivalently, a family of metric spaces is equi-coarsely quasi-convex if and only if they are equi-coarsely quasi-isometric to path metric spaces.*

Proof. Suppose that there is a coarse quasi-isometry $f : A \to A'$, with coarse distortion (K, C), between (M, d) and a path metric space (M', d'). For all $x, y \in M$, there are some $\bar{x}, \bar{y} \in A$ with $d(x, \bar{x}), d(y, \bar{y}) \leqslant K$. Then there is some finite sequence $f(\bar{x}) = x'_0, \dots, x'_n = f(\bar{y})$ in M' such that $d'(x'_{k-1}, x'_k) < 1$ for all $k \in \{1, \dots, n\}$ and

$$\sum_{k=1}^{n} d'(x'_{k-1}, x'_k) = d'(f(\bar{x}), f(\bar{y})).$$

Moreover, we can assume that this is one of the shortest sequences satisfying this condition. If $d'(x'_{k-1}, x'_k) < 1/2$ and $d'(x'_k, x'_{k+1}) < 1/2$ for some k, then the term x'_k could be removed from the sequence, contradicting its minimality. It follows that $d'(x'_{k-1}, x'_k) \geqslant 1/2$ for at least $[n/2]$ indexes k. So

$$(n - 1)/4 \leqslant [n/2]/2 \leqslant \sum_{k=1}^{n} d'(x'_{k-1}, x'_k) = d'(f(\bar{x}), f(\bar{y})),$$

which implies

$$n \leqslant 4d'(f(\bar{x}), f(\bar{y})) + 1. \qquad (3.1)$$

For each $k \in \{0, \dots, n\}$, there is some $\bar{x}'_k \in A'$ with $d'(\bar{x}'_k, x'_k) \leqslant K$, and let $\bar{x}_k = f^{-1}(\bar{x}'_k)$; for simplicity, take $\bar{x}'_0 = x'_0$ and $\bar{x}'_n = x'_n$, and thus $\bar{x}_0 = \bar{x}$ and $\bar{x}_n = \bar{y}$. To simplify the notation, let also $x_0 = x$, $x_n = y$, and $x_k = \bar{x}_k$ for $k \in \{1, \dots, n-1\}$. Then

$$
\begin{aligned}
d(x_{k-1}, x_k) &\leqslant d(x_{k-1}, \bar{x}_{k-1}) + d(\bar{x}_{k-1}, \bar{x}_k) + d(\bar{x}_k, x_k) \\
&\leqslant 2K + Cd'(\bar{x}'_{k-1}, \bar{x}'_k) \\
&\leqslant 2K + C(d'(\bar{x}'_{k-1}, x'_{k-1}) + d'(x'_{k-1}, x'_k) + d'(x'_k, \bar{x}'_k)) \\
&\leqslant 2K + 2CK + C
\end{aligned}
$$

for $k \in \{1, \dots, n\}$, and

$$
\begin{aligned}
\sum_{k=1}^{n} d(x_{k-1}, x_k) &\leqslant d(x, \bar{x}) + \sum_{k=1}^{n} d(\bar{x}_{k-1}, \bar{x}_k) + d(\bar{y}, y) \\
&\leqslant 2K + C \sum_{k=1}^{n} d'(\bar{x}'_{k-1}, \bar{x}'_k) \\
&\leqslant 2K + C \sum_{k=1}^{n} (d'(\bar{x}'_{k-1}, x'_{k-1}) + d'(x'_{k-1}, x'_k) + d'(x'_k, \bar{x}'_k)) \\
&\leqslant 2K + 2CKn + C \sum_{k=1}^{n} d'(x'_{k-1}, x'_k) \\
&\leqslant 2K + 2CK(4d'(f(\bar{x}), f(\bar{y})) + 1) + Cd'(f(\bar{x}), f(\bar{y})) \\
&\leqslant 2K + 2CK + (8CK + C)Cd(\bar{x}, \bar{y}) \\
&\leqslant 2K + 2CK + (8CK + C)C(d(\bar{x}, x) + d(x, y) + d(y, \bar{y})) \\
&\leqslant 2K + 2CK + 2(8CK + C)CK + (8CK + C)Cd(x, y),
\end{aligned}
$$

where (3.1) was used in the fifth inequality. Thus the condition of Definition 3.9 is satisfied with a, b and c depending only on K and C, as desired.

Assume now that (M, d) satisfies the coarsely quasi-convex condition (Definition 3.9) with constants a, b and c. By Lemma 2.1, there is a c-separated c-net A in M. By attaching an edge to any pair of points $x, y \in A$ with $d(x, y) \leqslant 3c$, there results a graph M' whose set of vertices is A. For any $x, y \in A$, there is a finite sequence $x_0 = x, x_1, \ldots, x_n = y$ in M with $d(x_{k-1}, x_k) \leqslant c$ for all $k \in \{1, \ldots, n\}$, and

$$\sum_{k=1}^{n} d(x_{k-1}, x_k) \leqslant ad(x, y) + b.$$

For each k, take some $\bar{x}_k \in A$ with $d(x_k, \bar{x}_k) \leqslant c$; in particular, take $\bar{x}_0 = x$ and $\bar{x}_n = y$. Then there is an edge between each $\bar{x}_{k-1}$ and $\bar{x}_k$ because

$$d(\bar{x}_{k-1}, \bar{x}_k) \leqslant d(\bar{x}_{k-1}, x_{k-1}) + d(x_{k-1}, x_k) + d(x_k, \bar{x}_k) \leqslant 3c.$$

Therefore M' is a connected graph. Let d' denote the path metric on M', defined as above, with each edge having a metric that makes it isometric to the unit interval. Since A is a 1-net in M', it only remains to check that the identity map $(A, d) \to (A, d')$ is bi-Lipschitz with bi-Lipschitz distortion depending only on a, b and c. Fix any pair of different points $x, y \in A$, and take a sequence $x = \bar{x}_0, \ldots, \bar{x}_n = y$ as above; after removing some points of this sequence, if necessary, it may be assumed that $\bar{x}_{k-1} \neq \bar{x}_k$ for all k. Since there is an edge between each $\bar{x}_{k-1}$ and $\bar{x}_k$, it follows that $d'(x, y) \leqslant n$. Since A is c-separated,

$$cn \leqslant \sum_{k=1}^{n} d(\bar{x}_{k-1}, \bar{x}_k) \leqslant ad(x, y) + b \leqslant \frac{ac + b}{c} d(x, y),$$

and so

$$d'(x, y) \leqslant \frac{ac + b}{c^2} d(x, y).$$

On the other hand, if $d'(x, y) = m$, then there is a sequence $x = y_0, \ldots, y_m = y$ in A with the property that each pair y_{k-1}, y_k is joined by an edge; thus $d(y_{k-1}, y_k) \leqslant 3c$ for each k, and so

$$d(x, y) \leqslant \sum_{k=1}^{m} d(y_{k-1}, y_k) \leqslant 3cm = 3cd'(x, y). \qquad \square$$

Remark 3.12. Theorem 3.11 is a coarsely quasi-isometric version of [20, Proposition 2.57], which asserts that the monogenic coarse structures are those that are coarsely equivalent to path metric spaces.

Proposition 3.13. *The following properties hold true*:

 (i) *Any large scale Lipschitz map between metric spaces is uniformly expansive*; *moreover, a family of equi-large scale Lipschitz maps between metric spaces is equi-uniformly expansive.*
 (ii) *Any large scale Lipschitz equivalence is a rough equivalence*; *moreover, a family of equi-large scale Lipschitz equivalences between metric spaces is a family of equi-rough equivalences.*

Proof. Let (M, d) and (M', d') be metric spaces, and let $f : M \to M'$ be a large scale Lipschitz map. If (λ, c) is a large scale Lipschitz distortion of f, then f obviously satisfies the definition of uniform expansiveness with $S = \lambda R + c$ for each $R > 0$. This proves property (i) because S depends only on R, λ and c.

For (ii), suppose that f is a large scale Lipschitz equivalence. Then there is a large scale Lipschitz map $g : M' \to M$, whose large scale Lipschitz distortion can be assumed to be also (λ, c), such that $g \circ f$ and $f \circ g$ are r-close to the identity maps on M and M', for some $r > 0$. Then

$$d(x, y) \leqslant d\big(x, g \circ f(x)\big) + d\big(g \circ f(x), g \circ f(y)\big) + d\big(g \circ f(y), y\big)$$

$$\leqslant \lambda d'\big(f(x), f(y)\big) + 2r.$$

Hence f satisfies the definition of uniform metric properness with $S = \lambda R + 2r$, for each $R > 0$. This proves property (ii) because S depends only on R, λ and r. $\quad \square$

Example 3.14. Let $\mathbf{N}^2 = \{n^2 \mid n \in \mathbf{N}\}$ and $\mathbf{N}^3 = \{n^3 \mid n \in \mathbf{N}\}$ with the restriction of the Euclidean metric on $\mathbf{R}$. Suppose that $\mathbf{N}^2$ and $\mathbf{N}^3$ are large scale Lipschitz equivalent; *i.e.*, there are large scale Lipschitz maps $f : \mathbf{N}^2 \to \mathbf{N}^3$ and $g : \mathbf{N}^3 \to \mathbf{N}^2$ with large scale Lipschitz distortion (λ, c) such that $g \circ f$ and $f \circ g$ are close to identity maps on $\mathbf{N}^2$ and $\mathbf{N}^3$. Let $\sigma, \tau : \mathbf{N} \to \mathbf{N}$ be the maps defined by $f(n^2) = \sigma(n)^3$ and $g(n^3) = \tau(n)^2$. Since $(n+1)^2 - n^2 \to \infty$ and $(n+1)^3 - n^3 \to \infty$ as $n \to \infty$, there is

some $a \in \mathbf{N}$ such that $g \circ f(n^2) = n^2$ and $f \circ g(n^3) = n^3$ for all $n \geqslant a$, and so $\tau \circ \sigma(n) = \sigma \circ \tau(n) = n$ for $n \geqslant a$. Assume for a while that there is some integer $b \geqslant a$ such that $\tau(n) \geqslant n + a + 2$ for all $n \geqslant b$. Thus

$$\tau\big(\{b, b+1, \ldots\}\big) \subset \{b + a + 2, b + a + 3, \ldots\}$$

and therefore

$$\{a, a+1, \ldots, b+a+1\} \subset \tau\big(\{0, 1, \ldots, b-1\}\big)$$

because $\{a, a+1, \ldots\} \subset \tau(\mathbf{N})$. So

$$b + 1 = \#\{a, a+1, \ldots, b+a+1\} \leqslant \#\tau\big(\{0, \ldots, b-1\}\big) \leqslant b,$$

which is a contradiction. Hence there is some sequence $n_k \uparrow \infty$ in $\mathbf{N}$ such that $\tau(n_k) \leqslant n_k + a + 1$ for all k. So

$$\big|\tau(n_k)^2 - n_k^2\big| \leqslant \big(\tau(n_k) + n_k\big)^2 \leqslant (2n_k + a + 1)^2.$$

We can assume that $n_k \geqslant a$ for all k. Then

$$\begin{aligned}
n_k^3 &= f \circ g\big(n_k^3\big) - f \circ g\big(n_1^3\big) + n_1^3 \\
&\leqslant \lambda \big|g\big(n_k^3\big) - g\big(n_1^3\big)\big| + c + n_1^3 \\
&= \lambda \big|\tau(n_k)^2 - \tau(n_1)^2\big| + c + n_1^3 \\
&\leqslant \lambda \big(\big|\tau(n_k)^2 - n_k^2\big| + n_k^2 - n_1^2 + \big|n_1^2 - \tau(n_1)^2\big|\big) + c + n_1^3 \\
&\leqslant \lambda \big((2n_k + a + 1)^2 + n_k^2 - n_1^2 + (2n_1 + a + 1)^2\big) + c + n_1^3 \\
&= 5\lambda n_k^2 + 4\lambda(a+1)n_k + 3\lambda n_1^2 + 4\lambda(a+1)n_1 + 2\lambda(a+1)^2 + c + n_1^3,
\end{aligned}$$

which is a contradiction because $n_k \to \infty$ as $k \to \infty$. Therefore there is no large scale Lipschitz equivalence between $\mathbf{N}^2$ and $\mathbf{N}^3$, and thus these spaces are not coarsely quasi-isometric. But they are coarsely equivalent; indeed, the map $n^3 \mapsto n^2$ is distance decreasing, and the map $n^2 \mapsto n^3$ is coarse: if $0 < |n^2 - m^2| < R$, then $n + m < R$ also, so $|n^3 - m^3| < S$ with $S = R^3$.

Example 3.14 shows that the converse of Proposition 3.13(2) does not hold in general. Nevertheless, the following proposition shows that coarse equivalences coincide with large scale Lipschitz equivalences for metric spaces that are coarsely quasi-convex.

Proposition 3.15. *Any uniformly expansive map of a coarsely quasi-convex metric space to another metric space is large scale Lipschitz; moreover, a family of equi-uniformly expansive maps between metric spaces, whose domains are equi-coarsely quasi-convex, is a family of equi-large scale Lipschitz maps.*

Proof. Let (M, d) and (M', d') be metric spaces, and let $f : M \to M'$ be a uniformly expansive map. Suppose that M satisfies the condition of being coarsely quasi-convex with constants a, b, and c. Fix points $x, y \in M$, and let $x = x_0, \ldots, x_n = y$ be a sequence of smallest length such that $d(x_{k-1}, x_k) \leqslant c$, for $k = 1, \ldots, n$, and

$$\sum_{k=1}^{n} d(x_{k-1}, x_k) \leqslant ad(x, y) + b.$$

If both $d(x_{k-1}, x_k) < c/2$ and $d(x_k, x_{k+1}) < c/2$ for some $k \in \{1, \ldots, n-1\}$, then $d(x_{k-1}, x_{k+1}) < c$, and thus x_k could be removed from the sequence $x_0, x_1, \ldots, x_n$, contradicting that this was a sequence of smallest length. Hence there are at least $(n-1)/2$ indexes $k \in \{1, \ldots, n\}$ such that $d(x_{k-1}, x_k) \geqslant c/2$. So

$$ad(x, y) + b \geqslant \sum_{k=1}^{n} d(x_{k-1}, x_k) \geqslant \frac{(n-1)c}{4},$$

or

$$n \leqslant \frac{4a}{c} d(x, y) + \frac{4b}{c} + 1. \tag{3.2}$$

Since f is uniformly expansive, there is some $S > 0$ such that $d'(f(z), f(z')) \leqslant S$ for all $z, z' \in M$ with $d(z, z') \leqslant c$. So, by (3.2),

$$d'\big(f(x), f(y)\big) \leqslant \sum_{k=1}^{n} d'\big(f(x_{k-1}), f(x_k)\big) \leqslant nS \leqslant \frac{4aS}{c} d(x, y) + \frac{4bS}{c} + S,$$

which establishes that f is large scale Lipschitz with large scale Lipschitz distortion depending only on S, a, b and c.　□

Corollary 3.16. *Any coarse equivalence between coarsely quasi-convex metric spaces is a large scale Lipschitz equivalence; moreover, a family of equi-coarse equivalences between equi-coarsely quasi-convex spaces is a family of equi-large scale Lipschitz equivalences.*

Proof. This is elementary by Proposition 3.15. □

Corollary 3.17. *Two coarsely quasi-convex metric spaces are coarsely quasi-isometric if and only if they are coarsely equivalent; moreover, if M_i and M'_i, $i \in \Lambda$, are families of equi-coarsely quasi-convex metric spaces, then all pairs M_i and M'_i are equi-coarsely quasi-isometric if and only they are equi-coarsely equivalent.*

Proof. This follows from Propositions 2.2, 2.3 and 3.15, and Corollary 3.16. □

4. The Higson compactification

A significant example of coarse structure is induced by any compactification[10] $X^\kappa = \overline{X}$ of a topological space X, with corona $\partial X = \kappa X = X^\kappa \setminus X$. This coarse structure on X is defined by declaring a subset $E \subset X \times X$ to be controlled when

$$\overline{E} \cap \partial(X \times X) \subset \Delta_{\partial X}$$

in $\overline{X} \times \overline{X}$, where

$$\partial(X \times X) = (\partial X \times \overline{X}) \cup (\overline{X} \times \partial X).$$

This is called the *topological*[11] coarse structure associated to the given compactification; it is proper if $\overline{X}$ is metrizable [20,21].

A compactification $\overline{X}$ of a proper coarse space X is said to be a *coarse compactification*, with *coarse corona* $\partial X = \overline{X} \setminus X$, if the identity map from X with its given coarse structure to X endowed with the topological coarse structure arising from $\overline{X}$ is a coarse map. Intuitively, the slices of any controlled subset of $X \times X$ become small when approaching the boundary ∂X; in particular, this holds for the sets of any uniformly bounded family in X.

The structure of coarse compactifications of a proper coarse space X can be described algebraically as follows. Let $\mathcal{B}(X)$ be the Banach algebra of all bounded functions $X \to \mathbf{C}$ with the supremum norm, and let $\mathcal{B}_0(X)$ be the Banach subalgebra of all functions $f \in \mathcal{B}(X)$ that vanish at infinity; *i.e.*, such that, for any $\varepsilon > 0$, there is some compact subset $K \subset X$ so that $|f(x)| < \varepsilon$ for all $x \in X \setminus K$. For any $f \in \mathcal{B}(X)$ and every controlled subset $E \subset X \times X$, let the *E-expansion* of f be the function $\nabla_E f \in \mathcal{B}(X)$ defined by

$$\nabla_E f(x) = \sup\{|f(x) - f(y)| \mid (x, y) \in E\}.$$

Definition 4.1. A function $f \in \mathcal{B}(X)$ is called a *Higson function* if $\nabla_E f \in \mathcal{B}_0(X)$ for all controlled subsets $E \subset X \times X$.

The set $\mathcal{B}_\nu(X)$ of all Higson functions on X is a Banach subalgebra of $\mathcal{B}(X)$ [19,10,20]. If only bounded continuous functions are considered, then the notation $C_b(X)$, $C_0(X)$ and $C_\nu(X)$ will be used instead of $\mathcal{B}(X)$, $\mathcal{B}_0(X)$ and $\mathcal{B}_\nu(X)$, respectively.

The terms $\overline{X}$-*close* maps, $\overline{X}$-*controlled* sets and $\overline{X}$-*coarse* compactification will be used in the case of the topological coarse structure induced by a compactification $\overline{X}$ of a locally compact space X.

The following lemma shows that Higson functions are preserved by coarse maps.

Lemma 4.2. *Let $\overline{X}$ be a compactification of a locally compact space X with boundary ∂X. The following conditions are equivalent for any subset $E \subset X \times X$:*

(i) *E is $\overline{X}$-controlled.*
(ii) *$\nabla_E f \in \mathcal{B}_0(X)$ for every $f \in \mathcal{B}(X)$ having an extension $\bar{f} : \overline{X} \to \mathbf{C}$ that is continuous on the points of ∂X.*
(iii) *$\nabla_E f \in C_0(X)$ for every $f \in C_b(X)$ having a continuous extension to $\overline{X}$.*

Proof. To prove that property (i) implies property (ii), suppose that E is $\overline{X}$-controlled, and assume that some $f \in \mathcal{B}(X)$ has an extension $\bar{f} : \overline{X} \to \mathbf{C}$ that is continuous on the points of ∂X. Since the function $(x, y) \mapsto |\bar{f}(x) - \bar{f}(y)|$ on $\overline{X} \times \overline{X}$ vanishes on $\Delta_{\partial X}$ and is continuous on the points of $\partial X \times \partial X$, there is some open neighborhood Ω of $\Delta_{\partial X}$ in $\overline{X} \times \overline{X}$ such that

---

[10] Only metrizable compactifications are considered in [18], but this kind of coarse structure can be defined for arbitrary compactifications [10,19].

[11] This term is used in [10]. It is also called *continuously controlled* coarse structure in [19].

$|\bar{f}(x) - \bar{f}(y)| < \varepsilon$ for all $(x, y) \in \Omega$. On the other hand, since E is $\overline{X}$-controlled, there is some open neighborhood U of ∂X in $\overline{X}$ such that

$$\overline{E} \cap (U \times \overline{X}) \subset \Omega.$$

If K is the compact set $K = X \setminus U$, then $\nabla_E f(x) < \varepsilon$ for all $x \in X \setminus K = X \cap U$, and so $\nabla_E f \in \mathcal{B}_0(X)$.

Property (iii) is a particular case of property (ii).

To prove that property (iii) implies property (i), assume that $\nabla_E f \in C_0(X)$ for all $f \in C_b(X)$ that admit a continuous extension to $\overline{X}$. If E were not $\overline{X}$-controlled, there would be a pair of different points, $x \in \partial X$ and $y \in \overline{X}$, such that either (x, y) or (y, x) is in $\overline{E}$. Since the family of controlled sets is invariant by transposition [19,10,20], it may be assumed that $(x, y) \in \overline{E}$. Then, for any continuous function $\bar{f} : \overline{X} \to \mathbf{C}$ with $\bar{f}(x) \neq \bar{f}(y)$, the restriction $f = \bar{f}|_X$ would satisfy

$$\liminf_{z \to x} \nabla_E f(z) \geqslant |\bar{f}(x) - \bar{f}(y)| > 0,$$

which would be a contradiction. Therefore E is $\overline{X}$-controlled. $\square$

The following is a direct consequence of Lemma 4.2, which is contained in [20, Proposition 2.39].

Corollary 4.3. *Let $\overline{X}$ be a compactification of a proper coarse space X with boundary ∂X. Then the following conditions are equivalent*:

 (i) *$\overline{X}$ is a coarse compactification of X.*
 (ii) *$\mathcal{B}_\nu(X)$ contains every function in $\mathcal{B}(X)$ that admits an extension to $\overline{X}$ that is continuous on the points of ∂X.*
 (iii) *$C_\nu(X)$ contains every continuous function $X \to \mathbf{C}$ that extends continuously to $\overline{X}$.*

Proposition 4.4. *Let $\overline{X}$ and $\overline{X'}$ be compactifications of locally compact spaces X and X' with boundaries ∂X and $\partial X'$, respectively. Then the following properties hold*:

 (i) *A map $\varphi : X \to X'$ is coarse if it has an extension $\bar{\varphi} : \overline{X} \to \overline{X'}$ that is continuous on the points of ∂X and such that $\bar{\varphi}(\partial X) \subset \partial X'$.*
 (ii) *Let $\varphi, \psi : X \to X'$ be maps with extensions $\bar{\varphi}, \bar{\psi} : \overline{X} \to \overline{X'}$ satisfying the conditions of property (i). Then φ and ψ are $\overline{X'}$-close if and only if $\bar{\varphi} = \bar{\psi}$ on ∂X.*

Proof. Let $\varphi : X \to X'$ be a map satisfying the conditions of property (i). If B is a bounded subset of X', then B has compact closure in X', and thus $\overline{B} \cap \partial X' = \emptyset$. So

$$\bar{\varphi}\big(\overline{\varphi^{-1}(B)} \cap \partial X\big) \subset \overline{\bar{\varphi}(\varphi^{-1}(B))} \cap \partial X' \subset \overline{B} \cap \partial X' = \emptyset$$

because $\bar{\varphi}$ is continuous on the points of ∂X. It follows that $\overline{\varphi^{-1}(B)} \cap \partial X = \emptyset$, and thus $\varphi^{-1}(B)$ has compact closure in X; that is, $\varphi^{-1}(B)$ is bounded in X.

Let E be a controlled subset of $X \times X$, and let $f : X' \to \mathbf{C}$ be a bounded function that admits an extension $\bar{f}$ to $\overline{X'}$ that is continuous on the points of $\partial X'$. Lemma 4.2 implies that $\nabla_{(\varphi \times \varphi)(E)} f = \nabla_E (f \circ \varphi) \in \mathcal{B}_0(X)$ because $\bar{f} \circ \bar{\varphi}$ is an extension of the function $f \circ \varphi$ that is continuous at the points of ∂X. It follows that $(\varphi \times \varphi)(E)$ is a controlled subset of $X' \times X'$ by Lemma 4.2. Therefore φ is a coarse map, which establishes property (i).

Let $\varphi, \psi : X \to X'$ be maps with extensions $\bar{\varphi}, \bar{\psi} : \overline{X} \to \overline{X'}$ satisfying the conditions of property (i). Suppose first that $\bar{\varphi} = \bar{\psi}$ on ∂X, and let $E = \{(\varphi(x), \psi(x)) \mid x \in X\}$. Fix any point $(x', y') \in \overline{E} \cap \Delta_{\partial(X' \times X')}$. Thus, for each neighborhood Ω of (x', y') in $\overline{E}$, there is some point $x_\Omega \in X$ so that $(\varphi(x_\Omega), \psi(x_\Omega)) \in \Omega$; such points x_Ω form a net (x_Ω) in X. Suppose e.g. that $x' \in \partial X'$. Then the net $(\varphi(x_\Omega))$ is unbounded in X', and thus the net (x_Ω) is unbounded in X because φ is a coarse map according to property (i). So there is an accumulation point x of (x_Ω) in ∂X. Since $\bar{\varphi}$ and $\bar{\psi}$ are continuous at x, it follows that $(\bar{\varphi}(x), \bar{\psi}(x))$ is an accumulation point of the net $(\varphi(x_\Omega), \psi(x_\Omega))$, which converges to (x', y'). Hence $(x', y') = (\bar{\varphi}(x), \bar{\psi}(x)) \in \Delta_{\partial X'}$ because $\bar{\varphi} = \bar{\psi}$ on ∂X. This shows that E is $\overline{X'}$-controlled, and thus φ is $\overline{X'}$-close to ψ.

Assume now that φ is $\overline{X'}$-close to ψ; i.e., the set $E = \{(\varphi(x), \psi(x)) \mid x \in X\}$ is $\overline{X}$-controlled. The conditions on $\bar{\varphi}$ and $\bar{\psi}$ imply that

$$
\begin{aligned}
(\bar{\varphi} \times \bar{\psi})(\Delta_{\partial X}) &= (\bar{\varphi} \times \bar{\psi})\big(\overline{\Delta_X} \cap (\partial X \times \partial X)\big) \\
&\subset \overline{(\varphi \times \psi)(\Delta_X)} \cap \big(\partial X' \times \partial X'\big) \\
&= \overline{E} \cap \big(\partial X' \times \partial X'\big) \\
&\subset \Delta_{\partial X'},
\end{aligned}
$$

which establishes that $\bar{\varphi} = \bar{\psi}$ on ∂X, and completes the proof of property (ii). $\square$

Remark 4.5. The above result can be compared with the "if" part of [20, Proposition 2.33]. Continuity on X is not needed in Proposition 4.3, only the continuity on ∂X is used, and properness is replaced by the condition to preserve the boundary of the compactifications. The reciprocal of property (i) holds when the compactifications are first countable[12] [20, Proposition 2.33], and this assumption is necessary for the reciprocal [20, Example 2.34].

According to Corollary 4.3, there is a maximal coarse compactification X^ν, which is the maximal ideal space of $C_\nu(X)$; it is called the *Higson compactification* of X, and its boundary νX is called the *Higson corona*. Since each Higson function on X has a unique extension to X^ν that is continuous on the points of νX, there are canonical isomorphisms

$$C(\nu X) \cong C_\nu(X)/C_0(X) \cong \mathcal{B}_\nu(X)/\mathcal{B}_0(X). \tag{4.1}$$

This isomorphism can be used to define the Higson boundary νX for any coarse space X [20].

For subsets A of X or of $X \times X$, the notation A^ν will be used to indicate the closure of A in X^ν or in $X^\nu \times X^\nu$, respectively. The notation $\nu(X \times X) = (\nu X \times X) \cup (X \times \nu X)$ will be also used.

The following lemma is contained in the proof of [20, Proposition 2.41].

Lemma 4.6. *Let X and X' be proper coarse spaces and let $\varphi : X \to X'$ be a coarse map. Then*:

(i) $f \circ \varphi \in \mathcal{B}_0(X)$ *for all $f \in \mathcal{B}_0(X')$, and*
(ii) $f \circ \varphi \in \mathcal{B}_\nu(X)$ *for all $f \in \mathcal{B}_\nu(X')$.*

Proposition 4.7. *Let X and X' be proper coarse spaces. Any coarse map $\varphi : X \to X'$ has a unique extension $\bar\varphi : X^\nu \to X'^\nu$ that is continuous on the points of νX and such that $\bar\varphi(\nu X) \subset \nu X'$.*

Proof. According to Lemma 4.6, φ induces a homomorphism $\varphi^* : \mathcal{B}_\nu(X') \to \mathcal{B}_\nu(X)$ defined by $\varphi^*(f) = f \circ \varphi$, which maps $\mathcal{B}_0(X')$ to $\mathcal{B}_0(X)$. By (4.1), φ^* induces a homomorphism $C(\nu X') \to C(\nu X)$. Then, by considering maximal ideal spaces, we get a map $\bar\varphi : X^\nu \to X'^\nu$, which extends φ and maps νX into $\nu X'$. The continuity of $\bar\varphi$ on the points of νX is a consequence of the fact that any Higson function has a unique extension to the Higson compactification which is continuous on the Higson corona. $\quad\square$

Remark 4.8. The above result is slightly stronger than [20, Proposition 2.41], which only shows the continuity of the restriction $\bar\varphi : \nu X \to \nu X'$.

Sometimes the Higson compactification can be easily determined, as shown by the following result[13] [20, Proposition 2.48].

Proposition 4.9. *Let X be a proper coarse space with the topological coarse structure induced by a first-countable compactification $\overline{X}$ of X with boundary ∂X. Then $\overline{X}$ and X^ν are equivalent compactifications of X, and thus ∂X is homeomorphic to νX.*

The hypothesis of this proposition, that the coarse structure be induced by a first-countable compactification, is very strong, as the following proposition shows.

Proposition 4.10. *Let (M, d) be a proper metric space, and let M^ν be its Higson compactification. A point p in M^ν is in M if and only if the set $\{p\}$ is a G_δ-set.*

Proof. The "only if" part is elementary. To prove the "if" part, let $p \in M^\nu$ be such that $\{p\}$ is a G_δ set. Then there is a sequence (x_n) in M that converges to p. Suppose that $p \notin M$; *i.e.*, $p \in \nu M$. Passing to a subsequence if necessary, it may be assumed that there is a sequence of positive real numbers $r_n \uparrow \infty$ such that the metric balls $B(x_n, r_n)$ are mutually disjoint. Let $f : M \to \mathbf{R}$ be the function given by

$$f(x) = \begin{cases} (-1)^n \dfrac{r_n - d(x, x_n)}{r_n} & x \text{ in } B(x_n, r_n), \\ 0 & \text{otherwise.} \end{cases}$$

Then f extends to a continuous function $\bar f$ on M^ν, and so $\lim_{n \to \infty} \bar f(x_n) = \bar f(p)$. But the definition of f implies that $\bar f(x_n) = (-1)^n$, so the limit $\lim_{n \to \infty} \bar f(x_n)$ does not exist. $\quad\square$

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[12] Second countability is required in [20, Proposition 2.33], but only first countability is used in the proof.
[13] The statement of [20, Proposition 2.48] requires second countability but, indeed, its proof only uses first countability.

Remark 4.11.

(i) The argument of [20, Example 2.53] can also be used to show that the point p (in proof above) is in M.
(ii) G_δ properties are common in the study of the structure of the Stone–Čech compactification of spaces (*e.g.*, Walker [22]). The property brought to light here also plays a role in Nakai's work on the Royden compactification of Riemann surfaces [15].

Proposition 4.12. *Let* (M, d) *be a non-compact proper metric space. Let* $W \subset M$ *be a subset that contains metric balls of arbitrarily large radius. Then the closure of* W *in* M^ν *is a neighborhood of a point in* νM.

Proof. If $W \subset M$ contains ball of arbitrarily large radius, then, because M is not compact, there is a sequence, (x_n), of points in W without limit point in M, and a sequence of positive real numbers $r_n \uparrow \infty$ such that the metric balls $B(x_n, r_n)$ are mutually disjoint and contained in W. If f is the function constructed in Proposition 4.10, then $g = |f|$ admits a continuous extension, $\bar{g}$, to M^ν that satisfies $\bar{g}(p) = 1$ for any $p \in \nu M$ that is an accumulation point of the sequence (x_n). Therefore $\bar{g}^{-1}(0, 1]$ is an open neighborhood of p contained in the closure of W in M^ν. □

The Higson compactification of a proper coarse space is defined as the maximal ideal space of the algebra of Higson functions on the space. The question arises whether it is possible to construct the Higson compactification directly form the topological structure of the space, or whether the Higson compactification is a Wallman–Frink compactification. A Wallman–Frink compactification can be defined using $\mathcal{H}$-ultrafilters, where $\mathcal{H}$ is the ring of zero sets of Higson functions, topologized in an appropriate manner. The resulting space may not be Hausdorff and has the Higson compactification as a quotient space. Understanding the precise relationship between the two compactifications will lead to an intrinsic characterization of $\mathcal{H}$-set, toward which Proposition 4.12 is a minor contribution.

Even if the statement of Proposition 4.9 was not true when the first-countability axiom is removed, the following result is always true by the maximality of the Higson compactification among all coarse compactifications.

Proposition 4.13. *Proper topological coarse structures are induced by their Higson compactifications.*

The following is a direct consequence of Propositions 4.4, 4.7 and 4.13.

Corollary 4.14. *Let* X *and* X' *be proper topological coarse spaces. Then the following properties hold*:

(i) *A map* $\varphi : X \to X'$ *is coarse if and only if it has an extension* $\varphi^\nu : X^\nu \to X'^\nu$ *that is continuous on the points of* νX *and such that* $\varphi^\nu(\nu X) \subset \nu X'$.
(ii) *Two coarse maps* $\varphi, \psi : X \to X'$ *are close if and only if the extensions* φ^ν *and* ψ^ν, *given by property* (i), *are equal on* νX.

The following result shows that proper metric coarse structures are particular cases of the topological ones (Roe [20, Proposition 2.47]).

Proposition 4.15. *The metric coarse structure of a proper metric space is equal to the topological coarse structure induced by its Higson compactification.*

Proposition 4.15 and Corollary 4.14 have the following consequences.

Theorem 4.16. *Let* X *and* X' *be proper metric spaces. Then a map* $\varphi : X \to X'$ *is a coarse equivalence if and only if it has an extension* $\varphi^\nu : X^\nu \to X'^\nu$ *such that* $\varphi^\nu(\nu X) \subset \nu X'$, φ^ν *is continuous on the points of* νX *and the restriction* $\varphi^\nu : \nu X \to \nu X'$ *is a bijection.*

Proof. The "if" part follows from Corollary 4.14. To prove the "only if" part, assume that $\varphi : X \to X'$ is coarse and admits an extension $\varphi^\nu : X^\nu \to X'^\nu$ that is continuous on the points of νX and takes νX bijectively onto $\nu X'$ (hence φ^ν induces a homeomorphism of νX onto $\nu X'$ because νX is compact and Hausdorff).

The hypotheses imply that φ is uniformly metrically proper. Indeed, if that was not the case, there would be a positive number $R > 0$ and two sequences (x_n) and (z_n) in X such that $d'(\varphi(x_n), \varphi(z_n)) \leqslant R$ but $d(x_n, z_n) \geqslant n$ for all n. Because $\varphi : X \to X'$ is coarse (metric proper and uniformly expansive), it may be assumed, after passing to subsequences if needed, that neither of the sequences (x_n) and (z_n) has accumulation points in X, and that neither $(\varphi(x_n))$ nor $(\varphi(x_n))$ have accumulation points in X'. Because $d(x_n, z_n) \geqslant n$, an application of Proposition 4.12 shows that the set of accumulation points of the sequence (x_n), say P, and of the sequence (z_n), say Q, are disjoint closed subsets of νX. Being continuous on νX, the mapping φ^ν takes P and Q to the set of accumulation points of the corresponding sequences $(\varphi(x_n))$ and $(\varphi(x_n))$, respectively. But, since $d'(\varphi(x_n), \varphi(z_n)) \leqslant R$, it follows that $\varphi^\nu(P) = \varphi^\nu(Q)$. This contradicts that φ^ν induces a homeomorphism of the compact Hausdorff space νX onto $\nu X'$.

It is also true that there is an $N > 0$ such that the image $\varphi(X)$ is N-dense in X'. For if not there would be a sequence (x'_n) in X' such that the union, W, of the metric balls $B(x'_n, n)$ is disjoint from the image $\varphi(X)$. By Corollary 4.12, the closure of W in X'^{ν} is a neighborhood of a point p in $\nu X'$. This clearly contradicts the hypothesis that the mapping φ admits an extension to X' that takes νX onto $\nu X'$ and is continuous at the points of $\nu X'$.

Thus, by the above, there is some number N such that $\varphi(X)$ is N-dense in X', and so a mapping $\psi : X' \to X$ can be defined, by choosing, for each x' in X' a point $\psi(x')$ in X such that $\varphi(\psi(x'))$ is in $B(x', N)$.

The map ψ is uniformly expansive (Definition 3.7 (i)). Let $R > 0$ and let x' and z' in X' be such that $d'(x', z') \leqslant R$. Then, by the definition of ψ, the points $\psi(x')$ and $\psi(z')$ satisfy $d'(\varphi(\psi(x')), \varphi(\psi(z'))) \leqslant d'(x', z') + 2N \leqslant R + 2N$. Because φ is uniformly metrically proper, given $R + 2N > 0$, there is $S = S(R + 2N) > 0$ such that if $d'(\varphi(x), \varphi(z)) \leqslant R + 2N$, then $d(x, z) \leqslant S$. This applies in particular to $x = \psi(x')$ and $z = \psi(z')$.

The map ψ is metrically proper (Definition 3.7 (ii)). Let $B \subset X$ be bounded and suppose that $\psi^{-1}(B) \subset X'$ is not bounded. Then there is a sequence $x'_1, x'_2, \ldots$ in $\psi^{-1}(B)$ such that $d'(x'_{n+1}, x'_1) \geqslant n$ for all n. The points $x_n = \psi(x'_n)$ are all in B, and satisfy $d'(\varphi(x_n), x'_n) \leqslant N$ for all n, by the construction of ψ. Therefore $d'(\varphi(x_1), \varphi(x_n)) \geqslant d(x'_1, x'_n) - 2N \geqslant n - 2N$, so that the sequence $\varphi(x_n)$ is unbounded in X'. Since all the x_n are in the bounded set B, this contradicts the uniform expansiveness of φ.

The composite mapping $\varphi \circ \psi : X' \to X'$ is close to the identity of X' because for any x' in X', the point $\psi(x')$ is such that $d'(\varphi(\psi(x')), x') \leqslant N$.

The composite mapping $\psi \circ \varphi : X \to X$ is close to the identity on X. Indeed, by the definition of ψ, for any x in X, the point $\psi(\varphi(x))$ is such that $d'(\varphi(\psi(\varphi(x))), \varphi(x)) \leqslant N$. Because φ is uniformly metrically proper, there is an $S = S(N)$ such that $d(\psi(\varphi(x)), x) \leqslant S$ for all x in X. □

According to Corollary 3.17, in the case of coarsely quasi-convex metric spaces, the property "coarse equivalence" in this statement can be replaced by the property "coarse quasi-isometry."

Theorem 4.17. *Let (M, d) and (M', d') be proper metric spaces, and suppose that $\varphi : C_\nu(M') \to C_\nu(M)$ is an algebraic isomorphism of their associated Higson algebras. Then M and M' are coarsely equivalent. Furthermore, if M and M' are coarsely quasi-convex, then φ induces a coarse quasi-isometry between M and M'.*

Proof. The algebra $C_\nu(M)$ has trivial radical because it is a Banach subalgebra of $C_b(M)$. Therefore, by Gelfand et al. [7, Theorem 2 of §9], an algebraic isomorphism $\varphi : C_\nu(M') \to C_\nu(M)$ is automatically continuous and induces a homeomorphism of Higson compactifications $F : M^\nu \to M'^\nu$.

That homeomorphism F must send M homeomorphically onto M' because M is first countable but no point in the Higson corona of M is a G_δ-set (Proposition 4.10). Then the induced map $F : M \to M'$ is a coarse equivalence by Theorem 4.16. The last part of the statement now follows by invoking Proposition 3.15, which shows that a coarse mapping between coarsely quasi-convex spaces is a coarse quasi-isometry. □

We now prove a slightly strengthened version of Theorem 1.4 stated in the introduction.

Theorem 4.18. *Two proper coarse metric spaces, (M, d) and (M', d'), are coarsely equivalent if and only if there is an algebraic isomorphism $C(\nu M') \to C(\nu M)$ induced by a homomorphism $\mathcal{B}_\nu(M') \to \mathcal{B}_\nu(M)$. Furthermore, if M and M' are defined by coarsely quasi-convex metrics (or coarse metrics) d and d', then the above condition is equivalent to the existence of a coarse quasi-isometry between (M, d) and (M', d').*

Proof. Let $\varphi : \mathcal{B}_\nu(M') \to \mathcal{B}_\nu(M)$ be an algebraic homomorphism inducing an algebraic isomorphism $C(\nu M') \to C(\nu M)$.

Fix $K > 0$ and apply Lemma 2.1, to obtain K-separated K-nets $A \subset M$ and $A' \subset M'$. The inclusion mapping $A \to M$ induces a norm-decreasing algebraic homomorphism $\mathcal{B}_\nu(M) \to C_\nu(A)$.

There is a Borel partition of M' of the form $\{F_x \mid x \in A'\}$ with $x \in F_x \subset B(x, K)$ for each $x \in A'$. Such a partition can be constructed by induction on n for an enumeration (x_n) of the points of A': take $F_{x_0} = B(x_0, K)$, and

$$F_{x_{n+1}} = \{x_{n+1}\} \cup \big(B(x_{n+1}, K) \setminus (F_{x_0} \cup \cdots \cup F_{x_n})\big)$$

if $F_{x_0}, \ldots, F_{x_n}$ are constructed. Let χ_x denote the characteristic function of F_x for each $x \in A'$. Given a function f on A', $Pf = \sum_{x \in A'} f(x)\chi_x$ is a Higson function on M' by the argument of Roe in [18, Proposition (5.10)]. This defines a homomorphism of algebras $P : C_\nu(A') \to \mathcal{B}_\nu(M')$ because the sets F_x form a partition. Moreover the composition of P with the restriction homomorphism $\mathcal{B}_\nu(M') \to C_\nu(A')$ is the identity on $C_\nu(A')$ because $x \in F_x$ for all $x \in A'$.

It follows from the above that there is a homomorphism of algebras $C_\nu(A') \to C_\nu(A)$ that induces the original isomorphism $C(\nu A') = C(\nu M') \to C(\nu A) = C(\nu M)$. Since $C(\nu A) = C_\nu(A)/C_0(A)$ and $C(\nu A) = C_\nu(A)/C_0(A)$, this homomorphism of algebras induces a continuous mapping $\varphi^\nu : A^\nu \to A'^\nu$ that sends νA into $\nu A'$ homeomorphically, and such that the restriction $\varphi = \varphi^\nu|A$ sends A into A'. It thus follows from Theorem 4.16 that φ induces a coarse equivalence $M \to M'$.

If the metrics d and d' are coarsely quasi-convex, then φ can be improved to a coarse quasi-isometry, because of Corollary 3.17. □

Example 4.19. This result implies that a coarse equivalence between two locally compact metric spaces induces a homeomorphism between their respective coronas. The converse is not true in general. It follows from Example 3.14 that the Higson compactifications of the subspaces $\mathbf{N}^2$ and $\mathbf{N}^3$ of the natural numbers are the same as their Stone–Čech compactifications (which are homeomorphic to the Stone–Čech compactification of the natural numbers).

If the Continuum Hypothesis is accepted, then the Stone–Čech corona of the natural numbers has 2^c automorphisms (Walker [22]). On the other hand, there are at most c maps of $\mathbf{N}$ into $\mathbf{N}$. Therefore, many homeomorphisms of the Higson corona of $\mathbf{N}^2$ into that of $\mathbf{N}^3$ are not induced by a map of $\mathbf{N}^2$ into $\mathbf{N}^3$.

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