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# THE ELEMENTARY THEORY OF ABELIAN GROUPS

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## Introduction

In this paper we make a comprehensive survey of the first-order properties of abelian groups. The principal method is the investigation of saturated abelian groups. As a result of our determination of the structure of saturated groups we are able to give new model-theoretic proofs of the results of W. Szmielew [12]; moreover we obtain new results on the existence of saturated models of complete theories of abelian groups; and we also generalize our results to modules over Dedekind domains.

One of the principal results of Szmielew is the determination of group-theoretic invariants which characterize abelian groups up to elementary equivalence (The decidability of the theory of abelian groups follows relatively easily from this result). Now elementarily equivalent saturated groups of the same cardinality are isomorphic; so our method is to look for invariants which characterize saturated abelian groups up to isomorphism. We prove that any $\kappa$-saturated group $A (\kappa \geq \omega_1)$ is built up in a specified way from the groups $Z(p^\infty)$, $Z_p$, $Z(p^n)$ and $Q$ and that the number of copies of these groups which occur are determined by the elementarily definable dimensions $\dim(p^{n-1}A[p])$, $\dim(p^{n-1}A/p^nA)$, and $\dim(p^{n-1}A[p]/p^nA[p])$ and by the exponent of $A$ (for explanations of the notation and more details, see § 1). These dimensions, which arise

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naturally in considering the structure of saturated groups, are, respectively, the Szmielew invariants $\rho^{(1)}[p,n](A)$, $\rho^{(2)}[p,n](A)$, and $\rho^{(3)}[p,n](A)$. Thus we are led to Szmielew's theorem. In fact, we work with slightly different invariants viz.

$$\mathrm{D}(p;A) = \lim_{n\to\infty} \dim(p^n A[p]) ,$$

which determines the number of copies of the divisible group $Z(p^\infty)$ in a *saturated* group $A$;

$$\mathrm{Tf}(p;A) = \lim_{n\to\infty} \dim(p^n A/p^{n+1} A) ,$$

which determines the number of copies of the torsion-free group $Z_p$ in $A$;

$$\text{and } \mathrm{U}(p,n-1;A) = \dim(p^{n-1} A[p]/p^n A[p]) ,$$

the Ulm invariants, which determine the number of copies of $Z(p^n)$ in $A$.

We begin in section 0 by reviewing the theory of saturated structures and take advantage of the opportunity to distinguish carefully between two notions of homogeneity which have sometimes been confused in the literature. Section 1 deals with the structure of $\kappa$-saturated groups; in section 2 we prove the criterion for elementary equivalence of groups and the decidability result and also give a criterion for elementary embedding of groups which is implicit in [12]. In section 2 we also prove that any group which has the structure described in section 1 is in fact $\kappa$-saturated. Using this we are able, in section 3, to determine precisely the cardinals in which a given complete theory of abelian groups has saturated models. The table in section 3 summarizes this information and gives a complete analysis of the categoricity and $\omega_1$-stability of complete theories of abelian groups. In section 4 we give a model-theoretic proof of the existence of an elimination of quantifiers for a conservative extension of the theory of abelian groups. Finally in section 5 we generalize many of the preceding results to theories of modules over Dedekind domains.

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## §0. Preliminaries

In this section we will review some of the basic facts about saturated and homogeneous-universal models. For even more basic model-theoretic notions such as "elementary substructure", "formula", "1-type", etc., we refer the reader to Shoenfield [11], Bell and Slomson [1], and Chang and Keisler [3]. All results not otherwise credited can be found in the papers of Morley and Vaught [9] and Vaught [13].

We will use latin capitals $A, B, C, \ldots$ to denote sets and German capitals, $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ to denote structures on these respective sets. A structure $\mathfrak{A}$ is said to be *for* a language L or to be an *L-structure* if $\mathfrak{A}$ has an interpretation for each non-logical constant of L (and no other relations, functions, and individuals). In later sections, when we deal with abelian groups we will follow custom and use latin capitals for both sets and structures. We think of a language as the set of (first-order) formulas in its vocabulary; thus, a language always has infinite cardinality. $\kappa$ and $\lambda$ will be used exclusively to denote cardinals, and Card$(A)$ will denote the cardinality of $A$. $\mathcal{P}_\kappa(A)$ stands for $\{S \subset A \mid \text{Card}(S) < \kappa\}$.

Given a structure $\mathfrak{A}$ we say that $\mathfrak{A}$ is *elementarily $\kappa$-universal* or (since no confusion can arise for our purposes) just *$\kappa$-universal* if every structure $\mathfrak{B}$ which is elementarily equivalent to $\mathfrak{A}$ and of cardinality less than $\kappa$ can be elementarily embedded in $\mathfrak{A}$. $\mathfrak{A}$ is said to be (elementarily) *$\kappa$-homogeneous* if every isomorphism between elementary substructures of $\mathfrak{A}$ of cardinality less than $\kappa$ can be extended to an automorphism of $\mathfrak{A}$.

Given subsets $S$ and $T$ of structures $\mathfrak{A}$ and $\mathfrak{B}$ respectively for a common language L we say that a function $f: S \to T$ is a *local elementary isomorphism* if for every formula $\varphi(v_1, \ldots, v_n)$ of the language L and every $s_1, \ldots, s_n \in S$,

$$\mathfrak{A} \models \varphi[s_1, \ldots, s_n] \quad \text{iff} \quad \mathfrak{B} \models \varphi[f(s_1), \ldots, f(s_n)] \ .$$

Clearly such a local elementary isomorphism admits a unique extension to a local elementary isomorphism between the substructures generated by $S$ and $T$, i.e., the closures of $S$ and $T$ under the functions (if any) named in L.

A structure $\mathfrak{A}$ is said to be *pointwise $\kappa$-homogeneous* if for every pair

$S, T \in \mathcal{P}_\kappa(A)$, element $a \in A$, and local elementary isomorphism $f: S \to T$, there exist $a' \in A$ and a local elementary isomorphism $g: S \cup \{a\} \to T \cup \{a'\}$ which extends $f$.

A *κ-expansion* of a language L is a language which is obtained from L by the addition of $< \kappa$ new individual constants. A κ-expansion of an L-structure is defined analogously. We say that a structure $\mathfrak{A}$ is *κ-saturated* if for every κ-expansion $\mathfrak{A}'$ of $\mathfrak{A}$, $\mathfrak{A}'$ realizes every type of $\mathrm{Th}(\mathfrak{A}')$. An elementary theory $T$ is said to be *κ-stable* if for every κ-expansion $\mathfrak{A}'$ of a model $\mathfrak{A}$ of $T$ there are $< \kappa$ 1-types in $\mathrm{Th}(\mathfrak{A}')$.

A structure $\mathfrak{A}$ is said to be *universal* if it is $\mathrm{Card}(A)^+$-universal, *homogeneous* if it is $\mathrm{Card}(A)$-homogeneous, *pointwise homogeneous* if it is pointwise $\mathrm{Card}(A)$-homogeneous, and *saturated* if it is $\mathrm{Card}(A)$-saturated.

These notions are related through the following well-known theorems. We assume throughout that $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ are L-structures and that $\kappa \geq \lambda = \mathrm{Card}(\mathrm{L}) \leq \mathrm{Card}(A)$.

**0.1. Theorem.** *$\mathfrak{A}$ is κ-saturated iff $\mathfrak{A}$ is pointwise κ-homogeneous and $\kappa^+$-universal.*

**0.2. Theorem.** *$\mathfrak{A}$ is saturated iff $\mathfrak{A}$ is pointwise homogeneous and universal.*

**0.3. Corollary.** *Any two elementarily equivalent saturated structures of the same cardinality are isomorphic*

**0.4. Theorem.** *If $\mathfrak{A}$ is pointwise κ-homogeneous and $a \in A$, then every local elementary isomorphism between subsets $S$ and $T$ of $A$ such that* $\mathrm{Card}(S) = \mathrm{Card}(T) < \kappa$ *can be extended to an isomorphism between elementary substructures $\mathfrak{B}$ and $\mathfrak{C}$ of $\mathfrak{A}$ such that $a \in B$.*

**0.5. Theorem.** *If* $\mathrm{Card}(A) > \lambda$, *then $\mathfrak{A}$ is saturated iff $\mathfrak{A}$ is homogeneous and universal.*

It should be clear that every structure $\mathfrak{A}$ for a countable language such that $\mathrm{Card}(A) = \aleph_0$ is homogeneous, but not every such structure is pointwise homogeneous. We now give a counter-example, due to Morley, to the possibility that these two notions coincide for uncountable structures in countable languages.

Let $A = [0, 1] \cup \{-1\}$. For each positive integer $n$, let $U_n$ be the subset of $A$ consisting of those real numbers in $[0, 1]$ which have a 1 in the $n$-th place under some binary expansion (the binary rationals $\neq 0$ have two binary expansions). Let $\equiv$ be an equivalence relation on $A$ such that $0 \not\equiv -1$ and for all $a \in A$, $a \equiv 0$ or $a \equiv -1$, and $\{a | a \equiv 0\}$ and $\{a | a \equiv -1\}$ are both dense in $[0, 1]$. We claim that $\mathfrak{A} = \langle A, \equiv, U_1, U_2, \ldots\rangle$ is homogeneous, but not pointwise 2-homogeneous.

Indeed, let $f: \mathfrak{B} \to \mathfrak{C}$ be an isomorphism between elementary substructures of $\mathfrak{A}$. Then $f$ must clearly leave every element of $B \cap (0, 1]$ fixed. Also, if $0 \in B$, then there exists $a \in B \cap (0, 1]$ such that $0 \equiv a$. Then $f(0) \equiv f(a) = a$, and $f(0) = 0$. Similarly if $-1 \in B$, then $f(-1) = -1$. In any case, $\mathfrak{B} = \mathfrak{C}$ and $f$ is the restriction of the identity automorphism of $\mathfrak{A}$. On the other hand, 0 and $-1$ satisfy the same 1-type. This is perhaps best seen by taking a saturated elementary extension $\mathfrak{A}^*$ of $\mathfrak{A}$ and constructing an automorphism of $\mathfrak{A}^*$ which takes 0 to $-1$.

**0.6. Theorem.** *Let $\mathfrak{A}$ be $\kappa^+$-universal, and let $\Phi$ be a set of formulas such that for all $\varphi \in \Phi$ the free variables of $\varphi$ are in $\{v_\xi | \xi < \kappa\}$. Then if $\Phi$ is consistent with* Th($\mathfrak{A}$), *$\Phi$ is realized by a $\kappa$-sequence in $\mathfrak{A}$.*

**Proof.** Let $\Sigma$ be the result of replacing the free variables in $\Phi$ by new individual constants $\{c_\xi : \xi < \kappa\}$. Then Th($\mathfrak{A}$) $\cup\ \Sigma$ has a model of cardinality $\kappa$. By $\kappa^+$-universality, this model can be elementarily embedded in $\mathfrak{A}$, and the theorem is proved.

The previous theorem is particularly useful in conjunction with the next one.

**0.7. Theorem.** *If $\mathfrak{A}$ is $\kappa$-saturated, then every $\kappa$-expansion of $\mathfrak{A}$ is $\kappa$-saturated.*

**0.8 Theorem.** *If $\mathfrak{A}$ is $\kappa$-saturated, then every reduct of $\mathfrak{A}$ is $\kappa$-saturated.*

**0.9. Theorem.** *If $\mathfrak{A}$ is $\kappa$-saturated and $\mathfrak{B}$ is a substructure of $\mathfrak{A}$ such that $B = \{a \in A \mid \mathfrak{A} \models \varphi[a]\}$ for some formula $\varphi$, then $\mathfrak{B}$ is $\kappa$-saturated.*

**0.10. Theorem.** *Every structure has a $\kappa$-saturated elementary extension.*

Corollary 0.3 above shows that if there exists a saturated model of a complete theory $T$ with cardinality $\kappa \geq \text{Card}(T)$, then it is unique up to isomorphism. Existence is provided by the following theorem.

Let $\kappa^{\underset{\smile}{\lambda}} = \sum_{\alpha<\lambda} \kappa^{\alpha}$, and call a cardinal $\kappa$ a *saturation cardinal* if $\kappa^{\underset{\smile}{\kappa}} = \kappa$.

**0.11. Theorem.** *Every theory T having infinite models has a saturated model in every saturation cardinal* $\kappa > \text{Card}(T)$.

Every strongly inaccessible cardinal and every cardinal such that $\kappa = \lambda^+ = 2^{\lambda}$ is a saturation cardinal; moreover, it is consistent to assume that $2^{\aleph_0}$, for example, is very large and yet is a saturation cardinal.

The reader who needs a justification for an appeal to the existence of arbitrarily large saturated models for a theory $T$ can follow one of four routes. First, he can assume that there are arbitrarily large inaccessibles. Secondly, he can assume something like the generalized continuum hypothesis. Thirdly, he can assume no more than the usual axioms for set theory and employ an argument involving the constructible universe. For example, suppose that $\mathfrak{A} \subset \mathfrak{B}$, and that one can show that $\mathfrak{A} \prec \mathfrak{B}$ if a saturated elementary extension of $\mathfrak{B}$ exists in some cardinal $\kappa > \text{Card}(B)$. Let $T'$ be $\text{Th}(\mathfrak{B}')$ where $\mathfrak{B}'$ is an expansion of $\mathfrak{B}$ in which every element of $B$ has a name. Let $S$ be the transitive closure of $\{\mathfrak{B}', T'\}$ and let $\text{L}(S)$ be the Gödel universe obtained by constructing sets in the usual way but starting with $S$ instead of the empty set. In $\text{L}(S)$ it is true that

$$\forall\kappa(\kappa \geq \text{Card}(S) \rightarrow 2^{\kappa} = \kappa^{+}).$$

Reasoning in $\text{L}(S)$ we conclude painlessly that $\mathfrak{A} \prec \mathfrak{B}$ holds in $\text{L}(S)$ and therefore absolutely, i.e., in V. A last approach which the reader can attempt to follow is to modify the original proof and use $\kappa$-saturated structures which are not saturated: in other words, a rich supply of local elementary automorphisms must replace the automorphism in the original argument.

**0.12. Theorem.** *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* L-*structures, and let* $\kappa$ *be a saturation cardinal* $\geq \text{Card}(A) + \text{Card}(B) + \text{Card}(\text{L})^{+}$. *Then the saturated elementary extensions* $\mathfrak{A}'$ *of* $\mathfrak{A}$ *and* $\mathfrak{B}'$ *of* $\mathfrak{B}$ *of cardinality* $\kappa$ *are such that* $\mathfrak{A}' \times \mathfrak{B}'$ *is a saturated elementary extension of* $\mathfrak{A} \times \mathfrak{B}$.

**Proof.** We may assume that $A$, $B$, and $A \times B$ are pairwise disjoint. Let

$$\mathfrak{C} = \langle A \times B \cup A \cup B, V_{A\times B}, V_A, V_B, \pi_1, \pi_2, \ldots \rangle,$$

where $V_{A\times B}$, $V_A$, and $V_B$ are unary relations holding for the members of $A \times B$, $A$ and $B$ respectively, $\pi_1$ and $\pi_2$ are the projections $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$, and the remaining relations are those of $\mathfrak{A} \times \mathfrak{B}$, $\mathfrak{A}$, and $\mathfrak{B}$. Now a saturated elementary extension $\mathfrak{C}'$ of $\mathfrak{C}$ exists having cardinality $\kappa$ and the obvious relativized reducts are readily seen to be elementary extensions of $\mathfrak{A} \times \mathfrak{B}$, $\mathfrak{A}$, and $\mathfrak{B}$ which are saturated by Theorems 0.8 and 0.9 above.

**0.13. Corollary** (Feferman-Vaught [4]). *Given* L-*structures* $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$, *if* $\mathfrak{A}_1 \equiv \mathfrak{A}_2$ *and* $\mathfrak{B}_1 \equiv \mathfrak{B}_2$ *then* $\mathfrak{A}_1 \times \mathfrak{B}_1 \equiv \mathfrak{A}_2 \times \mathfrak{B}_2$, *and if* $\mathfrak{A}_1 \prec \mathfrak{A}_2$ *and* $\mathfrak{B}_1 \prec \mathfrak{B}_2$ *then* $\mathfrak{A}_1 \times \mathfrak{B}_1 \prec \mathfrak{A}_2 \times \mathfrak{B}_2$.

The method used in the proof of Theorem 0.12 can be employed to show that the theory of the limit of any finite diagram (in the category-theoretic sense) is determined by the theories of the structures in the diagram.

Our last application will be needed in a later section.

**0.14. Theorem** (Macintyre [7]). *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* L-*structures such that* Th($\mathfrak{A}$) *and* Th($\mathfrak{B}$) *are* $\kappa$-*stable for* $\kappa >$ Card(L), *then* Th($\mathfrak{A} \times \mathfrak{B}$) *is* $\kappa$-*stable.*

**Proof.** We may assume that Card($A$) = Card($B$) = a saturation cardinal $\geq \kappa$ and that $\mathfrak{A}$ and $\mathfrak{B}$ are saturated. Since $\mathfrak{A} \times \mathfrak{B}$ is saturated (by 0.12) it suffices to show that, for every subset $S$ of $A \times B$ with Card($S$) $< \kappa$, there are fewer than $\kappa$ 1-types realized in the $\kappa$-expansion of $\mathfrak{A} \times \mathfrak{B}$ in which the members of $S$ have names. We may assume that $S$ is of the form $S_1 \times S_2$ with $S_1 \subset A$ and $S_2 \subset B$. Define $(a, b) \sim (a', b')$ if $a$ satisfies the same type as $a'$ in the $S_1$-expansion of $\mathfrak{A}$ and $b$ satisfies the same type as $b'$ in the $S_2$-expansion of $\mathfrak{B}$. The number of equivalence classes is clearly less than $\kappa$, while on the other hand, given $(a, b) \sim (a', b')$, there exist automorphisms $f_1$ and $f_2$ of $\mathfrak{A}$ and $\mathfrak{B}$ which fix the members of $S_1$ and $S_2$ and move $a$ to $a'$ and $b$ to $b'$, respectively. $f_1 \times f_2$ thus fixes the members of $S = S_1 \times S_2$ and moves $(a, b)$ to $(a', b')$, showing that $(a, b)$ and $(a', b')$ realize the same type in the $S$-expansion of $\mathfrak{A} \times \mathfrak{B}$, and completing the proof.

The following two results will be useful in a later section.

**0.15. Theorem** (Vaught [13]). *An $\omega_1$-stable theory in a countable language has a countable saturated model.*

If $f_1 : \mathfrak{A}_1 \to \mathfrak{B}_1$ and $f_2 : \mathfrak{A}_2 \to \mathfrak{B}_2$ are elementary monomorphisms, we say that $f_1$ is equivalent to $f_2$ if there exist isomorphisms $g: \mathfrak{A}_1 \to \mathfrak{A}_2$ and $h: \mathfrak{B}_1 \to \mathfrak{B}_2$ such that

$$\begin{array}{ccc} \mathfrak{A}_1 & \xrightarrow{f_1} & \mathfrak{B}_1 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \mathfrak{A}_2 & \xrightarrow{f_2} & \mathfrak{B}_2 \end{array}$$

commutes.

**0.16. Theorem.** *Let $T$ be a theory in a countable language. Suppose that there exists a countable set $S$ of triples $\langle \mathfrak{A}, f, \mathfrak{B} \rangle$ such that every elementary monomorphism $g : \mathfrak{A}' \to \mathfrak{B}'$ between countable models of $T$ is equivalent to some $f : \mathfrak{A} \to \mathfrak{B}$ with $\langle \mathfrak{A}, f, \mathfrak{B} \rangle \in S$. Then $T$ is $\omega_1$-stable.*

**Proof.** If $T$ were not $\omega_1$-stable there would exist a countable model $\mathfrak{A}$ of $T$ with an $\omega_1$-expansion $\mathfrak{A}'$ such that uncountably many 1-types are in $\mathrm{Th}(\mathfrak{A}')$. Putting together the facts that (i) every 1-type can be realized in some countable elementary extension of $\mathfrak{A}$, (ii) only countably many types can be realized in a given countable elementary extension, and (iii) equivalent elementary extensions realize the same types, we get the desired contradiction.

## § 1. The structure of saturated abelian groups

Our goal is to characterize all the complete theories of abelian groups. This problem is equivalent (given the existence of saturated models of every complete theory) to characterizing the isomorphism classes of saturated groups (of a fixed cardinality). In order to solve this problem, we proceed, through a series of lemmas, to give a complete structure theory, and to isolate a complete set of elementarily definable invariants, for saturated abelian groups.

Most of the algebraic tools we need can be found in the very readable book by Kaplansky [6]. We will review many of the algebraic definitions and theorems we use, but we refer the reader to [6] for further details. (References in parentheses are to [6] unless otherwise specified). We begin by recalling some fundamental definitions. First, "group" will always mean abelian group. If $A$ is a group and $n \in \mathbf{Z}$, $nA = \{na : a \in A\}$. If $p$ is a prime, $A[p] = \{a \in A : pa = 0\}$. (We will write $nA[p]$ instead of $(nA)[p]$). If $a \in A$, $n \in \mathbf{Z}$, we say *$n$ divides $a$* (we write $n|a$) if there exists $b \in A$ (not necessarily unique) such that $nb = a$. $A$ is called *divisible* if for every $a \in A$ and every $n \neq 0$, $n$ divides $a$ (§5). A subgroup $B$ of $A$ is called a divisible subgroup of $A$ if $B$ is a divisible group in its own right. If $B$ is a divisible subgroup of $A$, $B$ is a direct summand of $A$ (Theorem 2). $A$ is called *reduced* if $A$ has no non-zero divisible subgroups.

Throughout this section we will be working towards the goal of determining the structure of a $\kappa$-saturated group $S$, where $\kappa$ is an uncountable cardinal. However, some of the results along the way will be true under weaker hypotheses, which we will point out for the purposes of later sections. In fact our first structural results use only the fact that $S$ is $\omega_1$-equationally compact i.e. any countable system of equations (in any number of unknowns) with constants from $S$ which is finitely solvable in $S$ is solvable in $S$. It is known that a group is $\omega_1$-equationally compact if and only if it is a direct summand of every group in which it is contained as a pure subgroup ([5], Theorem 38.1 and Exercise 5, p. 162). Such groups are called *pure-injective* (or *algebraically compact*) and their structure is known ([5], Prop. 40.1). However in order to keep our discussion as self-contained as possible we will give a simple direct analysis of the structure of an $\omega_1$-equationally compact group $S$.

Let $S_d$ be the maximal divisible subgroup of $S$ i.e. the union of all the divisible subgroups of $S$. We may write $S = R \oplus S_d$ where $R$ is reduced (Theorem 3). $S_d$ is uniquely determined; $R$ is not uniquely determined in general but it is unique up to isomorphism because it is isomorphic to $S/S_d$.

Let $S_r = S/S_d$; we will frequently without further remark identify $S_r$ with a (not uniquely determined) subgroup of $S$ so that $S = S_r \oplus S_d$. On occasion it will be necessary to identify $S_r$ with a more explicitly defined subgroup of $S$.

Let $D$ = the set of all elements of $S$ which are divisible by every integer $\neq 0$. Clearly $S_d \subseteq D$, but the opposite inclusion is false in general. However it is true in an $\omega_1$-equationally compact group:

**1.1. Lemma.** *Let $S$ be an $\omega_1$-equationally compact group. Then $S_d$, the maximal divisible subgroup of $S$, equals $D$, the set of all elements of $S$ divisible by every integer $\neq 0$.*

**Proof.** $D$ is clearly a subgroup of $S$. To prove $D \subseteq S_d$ it suffices to prove $D$ is a divisible subgroup of $S$. Given $a \in D$, $n \in \mathbf{Z} - \{0\}$, consider the set of equations $\mathcal{E} = \{my_m = x : m \in \mathbf{Z} - \{0\}\} \cup \{nx = a\}$. $\mathcal{E}$ is finitely solvable in $S$; indeed, it suffices to prove that for any $m \in \mathbf{Z} - \{0\}$, $my_m = x$ and $nx = a$ are simultaneously solvable in $S$, which is clear since $mn$ divides $a \in D$. Therefore $\mathcal{E}$ is solvable in $S$ i.e. there exists $b \in S$ such that $b \in D$ and $nb = a$. This completes the proof of Lemma 1.1.

Consider the reduced group $S_r = S/S_d$ (which we identify with a subgroup of $S$). For each prime $p$, define a "semi-norm" $|\ |_p$ on $S_r$ by: $|a|_p = 0$ if $p^n$ divides $a$ for every $n > 0$; otherwise, $|a|_p = p^{-n}$ if $n$ is the largest integer $\geq 0$ such that $p^n$ divides $a$. Define, for every $a \in S_r$, $|a| = \sum_p |a|_p 2^{-p}$; this defines a norm on $S_r$, i.e.,

(i) $|a| = 0 \Longleftrightarrow a = 0$,

(ii) $|a + b| \leq |a| + |b|$,

(iii) $|a| = |-a|$;

(i) follows from Lemma 1.1 because if $a \in S_r$ and $|a| = 0$ then $p^n$ divides $a$ for every prime $p$ and every $n \geq 0$; hence $a \in S_r \cap S_d = \{0\}$.

It is easy to see that a sequence $(a_n)_{n<\omega}$ in $S_r$ is Cauchy with respect to $|\ |$ if and only if it is Cauchy with respect to every semi-norm $|\ |_p$; and

$a$ is the limit of $(a_n)_{n<\omega}$ iff it is the limit with respect to every $|\ |_p$. In the following lemma, we are assuming $S$ is $\omega_1$-equationally compact and $S_r$ is defined as above:

**1.2. Lemma.** *$S_r$ is complete in the topology induced by $|\ |$.*

**Proof.** Let $(a_n)_{n<\omega}$ be a Cauchy sequence in $S_r$. Then for each $p$ and each $r > 0$ there exists $N_{p,r}$ such that $n, m \geq N_{p,r}$ implies $|a_n - a_m|_p < p^{-r}$ It suffices to show that there is an $x$ in $S$ such that for each $p$ and $r$, $|x - a_{N_{p,r}}|_p < p^{-r}$; if this is the case, then for any $n \geq N_{p,r}$, $|x - a_n|_p \leq \max(|x - a_{N_{p,r}}|_p, |a_{N_{p,r}} - a_n|_p) < p^{-r}$ i.e. $x$ is a limit of $(a_n)_{n<\omega}$ with respect to $|\ |_p$. Thus we consider the set of equations $\mathcal{E} = \{p^r y_{p,r} = x - a_{N_{p,r}}\}_{p,r}$. $\mathcal{E}$ is finitely solvable in $S$: indeed, given primes $p_1, \ldots, p_n$ and integers $r_1, \ldots, r_n > 0$, let $m = \max\{N_{p_i,r_i} : i = 1, \ldots, n\}$; then $a_m$ satisfies $p_i^{r_i} b_i = a_m - a_{N_{p_i,r_i}}$ $(i = 1, \ldots, n)$ for some $b_i \in S$. Therefore since $S$ is $\omega_1$-equationally compact $\mathcal{E}$ is solvable in $S$ and $S_r$ is complete. This completes the proof of Lemma 1.2.

Now consider the subset $\bar{S}_p$ of all elements of $S_r$ divisible by every integer relatively prime to $p$ i.e. $\bar{S}_p = \{a \in R : |a|_q = 0$ for every prime $q \neq p\}$. (The reason for the notation $\bar{S}_p$ will become clear later.)

By an argument like that in Lemma 1.1 we can prove that, since $S$ is $\omega_1$-equationally compact, if $a \in \bar{S}_p$ and if $q \neq p$, there exists $b \in \bar{S}_p$ such that $qb = a$ i.e. $\bar{S}_p$ can be regarded as a module over $\mathrm{Z}_{p} = \{\frac{n}{m} : (m,p) = 1\}$ (the valuation ring of the $p$-adic valuation on **Q**). The topology induced on $\bar{S}_p$ by $|\ |_p$ is called the *p-adic topology* on $\bar{S}_p$; the submodules $p^n \bar{S}_p$ form a fundamental system of neighborhoods of 0; $|\ |_p$ is a norm on $\bar{S}_p$ i.e. for any $a \in \bar{S}_p$, $|a|_p = 0 \iff a = 0$. Since $\bar{S}_p$ is closed in $S_r$, $\bar{S}_p$ is complete in the $p$-adic topology.

**1.3. Lemma.** *Let $S$ be $\omega_1$-equationally compact. For any $p$ and any $a \in S_r$, there is a unique $a_p \in \bar{S}_p$ such that $|a - a_p|_p = 0$.*

**Proof.** Uniqueness is easy since if $a_p, b_p \in \bar{S}_p$ such that $|a - a_p|_p = 0 = |a - b_p|_p$, then $|a_p - b_p|_p = 0$; therefore since $|a_p - b_p|_q = 0$ for $p \neq q$ (by definition of $\bar{S}_p$), we have $a_p = b_p$ by (i). To prove existence it suffices to prove $\{p^n y_n = a - x : n < \omega\} \cup \{m z_m = x : (m,p) = 1\}$ is finitely solvable in $S$. But for any $n < \omega$ and any $m$ relatively prime to $p$, there exist $s, t$ such that $sm + tp^n = 1$. Taking $x = msa$ we see that $m$ divides $x$ and $a - x = p^n(ta)$.

**1.4. Lemma.** *If $S$ is $\omega_1$-equationally compact, $S_r$ is isomorphic to the direct product* $\Pi_p \bar{S}_p$.

**Proof.** Define $f: S_r \to \Pi_p \bar{S}_p$ by : $f(a) = (a_p)$, where $a_p$ is the unique element of $\bar{S}_p$ such that $|a - a_p|_p = 0$. Clearly $f$ is a homomorphism. $f$ is one-one since if $(a_p) = f(a) = f(b)$ then for each $p$, $0 \leq |a-b|_p \leq |a-a_p|_p + |a_p - b|_p = 0$, i.e. $|a-b| = 0$ or $a = b$. To prove $f$ is onto, consider an element $(a_p) \in \Pi_p \bar{S}_p$. For any $n$ let $s_n = \sum_{i=1}^n a_{p_i}$, where $p_1, ..., p_n$ are the first $n$ primes. The sequence $(s_n)_{n<\omega}$ is Cauchy because for any $p$, if $n$, $m$ are such that $p_n, p_m \geq p$, then $|s_n - s_m|_p = 0$. Let $a$ be the limit of $(s_n)_{n<\omega}$. Then $f(a) = (a_p)$, because for any $r > 0$ there is $n_r$ such that $p_{n_r} > p$ and $|a - s_{n_r}|_p < 1/r$; hence $|a - a_p|_p = |a - s_{n_r}|_r < 1/r$ for every $r$. This completes the proof of Lemma 1.4.

To sum up: we have proved that if $S$ is $\omega_1$-equationally compact, then $S$ is isomorphic to the direct sum of a divisible group and a direct product, over the primes $p$, of modules complete over $Z_p$ : $S = \Pi_p \bar{S}_p \oplus S_d$. It may be proved that, conversely, any group with this structure is $\omega_1$-equationally compact i.e. pure-injective ([5], Theorem 38.1). We will not use this result, but we will use the term pure-injective to designate a group with the structure described above.

Now we turn our attention to $\bar{S}_p$, for a fixed prime $p$. As we have observed, $\bar{S}_p$ has no elements of infinite height, and it is complete in its $p$-adic topology. There is a complete structure theory for such modules: $\bar{S}_p$ is the completion of a direct sum of cyclic $Z_p$-modules $\oplus_n Z(p^n)^{(\alpha_n)} \oplus Z_p^{(\beta)}$ (where $Z(p^n)$ is the cyclic group of order $p^n$) and the cardinal numbers $\alpha_n$, $\beta$ are uniquely determined by $\bar{S}_p$ and form a complete set of invariants for $\bar{S}_p$ (Theorem 22). Since we will have to analyze this structure in more detail, we will sketch briefly the basic ideas which are involved in this structure theory. (The structure theory may be developed for complete modules over any discrete valuation ring $\Lambda$. For our purpose we may take $\Lambda = Z_p$ for a fixed prime $p$).

We begin with the notion of a pure submodule (§7). A submodule $B$ of a $Z_p$-module $A$ is *pure* in $A$ if for any $n$, $B \cap p^n A = p^n B$. It may be proved that any $Z_p$-module which is not divisible contains a non-zero pure cyclic submodule (Lemma 20). $A$ subset $\{x_i\}_{i \in I}$ of elements of $A$ is a *pure independent* subset if it is independent (i.e $\sum_i n_i x_i = 0$ implies $n_i x_i = 0$ for all $i$) and the submodule generated by $\{x_i\}_i$ is pure in $A$.

A submodule $B$ of a $Z_p$-module $A$ is called *basic* if (1) $B$ is a direct sum of cyclic $Z_p$-modules; (2) $B$ is pure in $A$; and (3) $A/B$ is divisible (§ 16). Any $Z_p$-module contains a basic submodule; in fact any maximal pure independent subset of $A$ generates a basic submodule (Lemma 21(a)).

Now if $A/B$ is divisible, then $B$ is dense in $A$ (because for any $a \in A$ and any $n$, there exists $b \in B$ and $x \in A$ such that $a - p^n x = b$ i.e. $p^n$ divides $a - b$). Hence if $A$ is a complete $Z_p$-module without elements of infinite height, $A$ is the completion of any basic submodule of $A$. Moreover it can be proved that if $A$ is the completion of a direct sum of cyclic modules $B$, then $B$ is a basic submodule of $A$ (proof of Theorem 22).

To see that any two basic submodules are isomorphic, it suffices to show that the $\alpha_n$'s and $\beta$ are invariants of $A$. We do this by expressing these cardinals in terms of dimensions of certain vector spaces (over the field $Z/pZ$) associated with $A$.

One can readily check that $\alpha_n = \dim(p^{n-1}B[p]/p^n B[p]) = \dim(p^{n-1}A[p]/p^n A[p]) =$ the $n-1$-st Ulm invariant of $A = f(p, n-1; A)$ (proof of Lemma 21 (b)).

Let $T =$ the torsion submodule of $A$. To see that $B$ is uniquely determined by $A$, one observes that $\beta = \dim(B/(T \cap B) + pB) = \dim(Z_p{}^{(\beta)}/pZ_p{}^{(\beta)})$ and one proves that $B/(T \cap B) + pB$ is isomorphic to $A/T + pA$ (proof of Lemma 21 (b)). Thus $\beta = \dim(A/T + pA)$.

We now return to our consideration of the pure-injective group $S = \Pi_p \bar{S}_p \oplus S_d$. Let

$$S_p = \oplus_n Z(p^n)^{(\alpha_{p,n})} \oplus Z_p{}^{(\beta_p)}$$

be a basic submodule of $\bar{S}_p$, so that $\bar{S}_p$ is the completion of $S_p$. $S_p$ is not uniquely determined but we have just seen that the cardinals $\alpha_{p,n}$, $\beta_p$ are completely determined by $\bar{S}_p$. In the case that $S$ is $\kappa$-saturated we are going to derive additional information about the structure of $S$. We maintain the above notation throughout the rest of this section. As far as the cardinals $\alpha_{p,n}$ are concerned we have the following lemma:

**1.5. Lemma.** *Let $S = \Pi_p \bar{S}_p \oplus S_d$ be a pure-injective group, and let $\alpha_{p,n}$ be defined as above.*

(a) $\alpha_{p,n} = \dim(p^{n-1}S_p[p]/p^nS_p[p]) = \dim(p^{n-1}\bar{S}_p[p]/p^n\bar{S}_p[p]) = \dim(p^{n-1}S[p]/p^nS[p])$.

(b) *If $S$ is $\kappa$-saturated and $\alpha_{p,n}$ is infinite, then $\alpha_{p,n} \geq \kappa$.*

(a) The first two equalities were already observed above. As for the third equality, we have to prove that $f(p, n; S) = f(p, n; \bar{S}_p)$ for every $n < \omega$. First observe that if $S = S_r \oplus S_d$ where $S_r$ is reduced and $S_d$ divisible, then $f(p, n; S) = f(p, n; S_r)$ because $p^kS_d = S_d$ for all $k$. Now $S_r = \Pi_q\bar{S}_q$ and $f(p, n; S_r) = f(p, n; \Pi_q\bar{S}_q) = \sum_q f(p, n; \bar{S}_q) = f(p, n; \bar{S}_p)$ because $p^k\bar{S}_q = \bar{S}_q$ for all $q \neq p$ and all $k$.

(b) Let $\{x_\nu : \nu < \kappa\}$ be a set of $\kappa$ free variables and let $\mathcal{F}$ be the set of all formulas

$$\exists y(p^{n-1}y = x_\nu) \wedge px_\nu = 0$$

for each $\nu < \kappa$ plus all formulas of the form

$$\forall y\,[p^n y \neq \sum_{i=1}^{k} m_i x_{\nu_i}]$$

where $k \geq 1$, $\nu_1 < \nu_2 < \ldots < \nu_k$ and $(m_1, \ldots, m_k) \in \{0, 1, \ldots, p-1\}^k - \{(0, 0, \ldots, 0)\}$.

$\mathcal{F}$ is clearly finitely satisfiable because $\dim(p^{n-1}S[p]/p^nS[p])$ is infinite. Therefore since $S$ is $\kappa$-saturated, $\mathcal{F}$ is satisfiable in $S$, which means $\dim(p^{n-1}S[p]/p^nS[p] \geq \kappa$.

We now turn our attention to $\beta_p$ = the number of copies of $Z_p$ in the direct decomposition of $S_p$. We have observed above that $\beta_p = \dim(S_p/(T \cap S_p) + pS_p) = \dim(\bar{S}_p/(T \cap \bar{S}_p) + p\bar{S}_p)$ where $T$ = the torsion subgroup of $S$. The canonical projection $\pi_p : S_r = \Pi_q\bar{S}_q \to \bar{S}_p$ induces an isomorphism: $S_r/(T \cap S_r) + pS_r \to \bar{S}_p/(T \cap \bar{S}_p) + p\bar{S}_p$, since $p\bar{S}_q = \bar{S}_q$ for all $q \neq p$. Furthermore the projection $\pi_r : S \to S_r$ induces an isomorphism: $S/T + pS \to S_r/(T \cap S_r) + pS_r$, because $pS_d = S_d$. Therefore $\beta_p = \dim(S/T + pS)$.

For any $k \geq 1$, multiplication by $p$ defines an isomorphism: $p^{k-1}S/p^{k-1}T + p^kS \to p^kS/p^kT + p^{k+1}S$, so that $\beta_p = \dim(p^kS/p^kT + p^{k+1}S)$. In general we would not expect this dimension to be elementarily definable, but if $p^k T = p^{k+1} T$ for some $k$, then $\dim(p^k S/p^k T + p^{k+1} S) = \dim(p^k S/p^{k+1} S)$; and there is a first-order sentence (or set of sentences) which expresses the fact that this dimension is a given finite cardinal (or is infinite). We will prove that in the opposite case – i.e. if for all $k$, $p^k T \neq p^{k+1} T$ – if $S$ is $\kappa$-saturated then $\beta_p \geq \kappa$. First we investigate what it means to have $p^k T = p^{k+1} T$. (Note that if $p^k T = p^{k+1} T$ then $p^n T = p^k T$ for all $n \geq k$).

**1.6. Lemma.** *Let $A$ be a group and let $T$ = the torsion subgroup of $A$. For any integer $k \geq 0$:*

(i) $p^k T = p^{k+1} T$ *if and only if for every* $n \geq k$, $\dim(p^n T[p]/p^{n+1} T[p]) = 0$.

(ii) $\dim(p^k A/p^{k+1} A) = \dim(p^{k+1} A/p^{k+2} A) + \dim(p^k A[p]/p^{k+1} A[p])$.

(iii) *If* $p^k T = p^{k+1} T$, *then* $\dim(p^k A/p^{k+1} A) = \dim(p^n A/p^{n+1} A)$ *for all* $n \geq k$.

(iv) *If for all integers* $k \geq 0$, $p^k T \neq p^{k+1} T$, *then* $\dim(p^n A/p^{n+1} A)$ *is infinite for all integers* $n \geq 0$.

**Proof.** (i) The implication from left to right is obvious. To prove the converse, first write $T = \oplus_q T_q$ where $T_q$ is the $q$-primary part of $T$ (Theorem 1). If $p^k T \neq p^{k+1} T$, then also $p^k T_p \neq p^{k+1} T_p$ (because $pT_q = T_q$ for $q \neq p$). Choose $x \in p^k T_p - p^{k+1} T_p$ such that the order of $x$ is minimal, say $p^n$. We assert $p^{n-1} x \in p^{k+n-1} T[p] - p^{k+n} T[p]$; if not, then $p^{n-1} x = p^{k+n} a$ for some $a \in T$ and $p^{n-1}(x - p^{k+1} a) = 0$, a contradiction of the choice of $x$.

(ii) (cf. [12], Theorem 1.7) The following sequence is exact: $0 \to p^k A[p]/p^{k+1} A[p] \to p^k A/p^{k+1} A \xrightarrow{f} p^{k+1} A/p^{k+2} A \to 0$ where $f$ is multiplication by $p$. The results (ii) follows immediately.

(iii) follows from (ii) and the fact that $p^n A[p]/p^{n+1} A[p]$ is isomorphic to $p^n T[p]/p^{n+1} T[p]$ for all $n$.

(iv) We can prove by induction, using (ii), that

$$\dim(p^n A/p^{n+1} A) \geq \sum_{j=n}^{m} \dim(p^j T[p]/p^{j+1} T[p])$$

for any integer $m \geq n$. Then (iv) follows immediately from (i). This completes the proof of Lemma 1.6.

For any group $A$, since $\dim(p^k A/p^{k+1}A)$ is a monotonically decreasing function of $k$, we can define

$$\mathrm{Tf}(p;A) = \begin{cases} \text{eventual value of } \dim(p^k A/p^{k+1}A) \text{ if that value is finite} \\ \infty \quad \text{otherwise} \end{cases}$$

**1.7. Lemma.** *Let $S$ be a pure-injective group and let $T$ be the torsion subgroup of $S$. Let $\beta_p$ be defined as in the remarks preceding Lemma* 1.5.

(a) *For any integer $k \geq 0$, $\beta_p = \dim(p^k S/p^k T + p^{k+1}S)$.*

(b) *If $p^k T = p^{k+1}T$ for some $k \geq 0$, then $\beta_p$ = the eventual value of* $\dim(p^n S/p^{n+1}S)$.

(c) *If* $\mathrm{Tf}(p;S)$ *is finite, then* $\beta_p = \mathrm{Tf}(p;S)$.

(d) *If $S$ is $\kappa$-saturated ($\kappa \geq \omega$) and* $\mathrm{Tf}(p;S) = \infty$, *then* $\beta_p \geq \kappa$.

**Proof.** (a) was proved in the remarks preceding Lemma 1.6.

(b) If $p^k T = p^{k+1}T$, then by (a), $\beta_p = \dim(p^n S/p^{n+1}S)$ for any $n \geq k$.

(c) If $\mathrm{Tf}(p;S)$ is finite then because of Lemma 1.6(iv) there is a $k$ such that $p^k T = p^{k+1}T$; the result follows from (b).

(d) Let $\{x_\nu : \nu < \kappa\}$ be a set of $\kappa$ free variables and let $\mathcal{F}$ be the set of all formulas of the form

$$\forall y\, [n((\sum_{i=1}^{t} m_i x_{\nu_i}) - py) \neq 0]$$

where $n > 0$, $t > 0$, $\nu_1 < \ldots < \nu_t$, $(m_1, \ldots, m_t) \in \{0, 1, \ldots, p-1\}^t - \{(0, \ldots, 0)\}$. We prove that $\mathcal{F}$ is finitely satisfiable in $S$; to prove this it suffices to prove that for a fixed $n$ and $t$ the set of formulas

$$\forall y\, [n(\sum_{i=1}^{t} m_i x_i - py) \neq 0]$$

is satisfiable in $S$ (where $(m_1, \ldots, m_t)$ ranges over all non-trivial $t$-tuples, as above). Suppose $n = p^k d$, where $(d, p) = 1$; choose $a_1, \ldots, a_t \in S$ such that $p^k a_1, \ldots, p^k a_n$ represent independent elements of $p^k S/p^{k+1}S$ (this is possible by hypothesis). Because

$$p^{k+1} \nmid \sum_{i=1}^{t} n\, m_i a_i,$$

we can conclude that $\mathcal{F}$ is finitely satisfiable in $S$. Therefore since $S$ is $\kappa$-saturated, $\mathcal{F}$ is satisfiable in $S$ which means that $\beta_p = \dim(S/T+pS) \geq \kappa$.

*Remark.* Note that (d) is false without the assumption that $S$ is $\kappa$-saturated. In fact if $S$ is the completion of $\oplus_n Z(p^n)$ in the $p$-adic topology, then $S$ is pure-injective, and $\mathrm{Tf}(p;s) = \infty$ (by Lemma 1.6 (iv)) but $\beta_p = 0$.

Next we look at $S_d$, the maximal divisible subgroup of $S$. $S_d$ is isomorphic to a direct sum of copies of the rational numbers $\mathbf{Q}$ and of $\mathbf{Z}(p^\infty)$ for various primes $p$:

$$S_d = \oplus_p \mathbf{Z}(p^\infty)^{(\gamma_p)} \oplus \mathbf{Q}^{(\delta)}$$

where $\mathbf{Z}(p^\infty)$ = the group of all $p^n$-th roots of unity, $n = 0, 1, 2, \ldots$ (Theorem 4). We consider first the number of copies, $\gamma_p$, of $\mathbf{Z}(p^\infty)$ in the direct decomposition of $S_d$. Clearly $\gamma_p = \dim S_d[p]$. (Recall that "dim" means dimension over $\mathbf{Z}/p\mathbf{Z}$.) Let $T$ = the torsion subgroup of $S$ and write $T = \oplus_q T_q$, where $T_q$ = the $q$-primary part of $T$ (Theorem 1). Hence for any $k \geq 0$, $p^k T = p^{k+1} T \iff p^k T_p = p^{k+1} T_p \iff p^k T_p$ is divisible. Thus if $p^k T = p^{k+1} T$, then $\gamma_p = \dim(p^k T[p]) = \dim(p^k S[p])$. If for all $k \geq 0$, $p^k T \neq p^{k+1} T$ and if $S$ is $\kappa$-saturated, we will prove that $\gamma_p \geq \kappa$. First we prove a preliminary lemma about the meaning of $p^k T = p^{k+1} T$.

**1.8. Lemma.** *Let $A$ be a group and $T$ the torsion subgroup of $A$. For any integer $k > 0$:*

(i) *If $p^k T = p^{k+1} T$, then* $\dim(p^k A[p]) = \dim(p^n A[p])$ *for all $n \geq k$.*

(ii) $\dim(p^k A[p]) = \dim(p^{k+1} A[p]) + \dim(p^k A[p]/p^{k+1} A[p])$.

(iii) *If for all integers $k \geq 0$, $p^k T \neq p^{k+1} T$, then* $\dim(p^n A[p])$ *is infinite for all integers $n \geq 0$.*

**Proof.** (i) is easy because $p^n A[p] = p^n T[p]$ for all $n \geq 0$, and if $p^k T = p^{k+1} T$, then $p^n T = p^k T$ for all $n \geq k$. (ii) (cf. [12], Theorem 1.7) follows immediately from the exactness of the sequence

$$0 \to p^{k+1} A[p] \to p^k A[p] \to p^k A[p]/p^{k+1} A[p] \to 0$$

(iii) By induction, using (ii), we prove

$$\dim(p^n A[p]) \geq \sum_{j=n}^{m} \dim(p^j A[p]/p^{j+1} A[p])$$

for any $m \geq n$. Then the result follows easily from Lemma 1.6 (i). This completes the proof of Lemma 1.8.

Since $\dim(p^k A[p])$ is a monotonically decreasing function of $k$ it makes sense to define

$$D(p; A) = \begin{cases} \text{eventual value of } \dim(p^k A[p]) \text{ if that value is finite} \\ \infty \qquad \text{otherwise} \end{cases}$$

**1.9. Lemma.** *Let $S = \Pi_q \bar{S}_q \oplus S_d$ be a pure-injective group and $T$ the torsion subgroup of $S$. Let $\gamma_p$ be defined as in the remarks preceding* Lemma 1.8.

(a) *For any integer $k \geq 0$, $\gamma_p$* $= \dim(p^k S_d[p])$

(b) *If $p^k T = p^{k+1} T$ for some $k \geq 0$, then $\gamma_p$ = the eventual value of* $\dim(p^n S[p])$.

(c) *If $D(p; S)$ is finite, then $\gamma_p$* $= D(p; S)$.

(d) *If $S$ is $\kappa$-saturated ($\kappa \geq \omega$) and $D(p; S) = \infty$, then $\gamma_p \geq \kappa$.*

**Proof.** (a) was proved in the remakrs preceding Lemma 1.8.
(b) If $p^k T = p^{k+1} T$, then $p^n T \subseteq S_d$ for all $n \geq k$ and hence $p^n S[p] = p^n T[p] = p^n S_d[p]$.
(c) If $D(p; S)$ is finite, then because of Lemma 1.8 (iii) $p^k T = p^{k+1} T$ for some integer $k \geq 0$; the result follows from (b).
(d) We have to prove that $\dim(S_d[p]) \geq \kappa$.
Because $S$ is $\kappa$-saturated it suffices to prove that the set of formulas

$$\mathcal{F} = \{p x_\nu = 0 : \nu < \kappa\}$$

$$\cup \{\textstyle\sum_{i=1}^{t} m_i x_{\nu_i} \neq 0 : t > 0; \nu_1 < \ldots < \nu_t < \kappa; (m_1, \ldots, m_t)$$

$$\in \{0, 1, \ldots, p-1\}^t - \{(0, \ldots, 0)\}\}$$

$$\cup \{\exists y (p^r y = x_\nu) : \nu < \kappa; \ 1 \leq r < \omega\}$$

is finitely satisfiable in $S$. (Because of Lemma 1.1, the last set of formulas insures that the $x_\nu$ are in $S_d$). But this follows from the assumption that $\dim(p^k S[p])$ is infinite for all integers $k$ (cf. the proof of Lemma 1.7(d)).

*Remark*. Lemma 1.9 (d) is false without the assumption that $S$ is $\kappa$-saturated. (Use the same example as in the remark following Lemma 1.7).

As the last stage in an analysis of the structure of $S$, we consider $\delta$ = the number of copies of $\mathbf{Q}$. A group $A$ is said to be of *bounded order* if there is an integer $n$ such that $nA = 0$. Define

$$\operatorname{Exp}(A) = \begin{cases} 0 \text{ if } A \text{ is of bounded order} \\ \infty \text{ otherwise} \end{cases}$$

**1.10. Lemma.** *Let $S$ be pure-injective and $\delta$ be as above.*

(a) *If* $\operatorname{Exp}(S) = 0$, *then* $\delta = 0$.

(b) *If $S$ is $\kappa$-saturated* ($\kappa \geq \omega$) *and* $\operatorname{Exp}(S) = \infty$, *then* $\delta \geq \kappa$.

In the proof of Lemma 1.10 (b) we will use the following *Well-Known-Fact:* Let $V$ be the vector space of dim $n$ over an infinite field $F$ and let $H_1, \ldots, H_r$ be hyperplanes in $V$ (i.e. $H_i$ is the set of zeroes of a non-zero linear polynomial $f_i \in F[X_1, \ldots, X_n]$). Then $\bigcup_{i=1}^r H_i \neq V$.

This well-known fact is implied by the even-better-known fact that the only polynomial over $F$ which vanishes at every $n$-tuple of elements of $F$ is the zero polynomial.

**Proof of Lemma 1.10.** (a) is obvious.

(b) We have to prove that there is a set of independent torsion-free elements in $S_d$ of cardinality $\kappa$. Since $S$ is $\kappa$-saturated it suffices to prove that

$$\mathcal{F} = \{\exists y(ry = x_\nu) : 1 \leq r < \omega; \nu < \kappa\} \cup$$

$$\{\sum_{i=1}^{t} m_i x_{\nu_i} \neq 0 : t < \omega; \nu_1 < \ldots < \nu_t < \kappa; (m_1, \ldots, m_t) \neq (0, \ldots, 0)\}$$

is finitely satisfiable in $S$. To prove this, it suffices to prove that for a fixed $r$ and $n$ and any $(m_1^{(j)}, ..., m_t^{(j)}) \neq (0, ..., 0)$, $j = 1, ..., k$,

$$\{\sum_{i=1}^{t} m_i^{(j)} x_i \neq 0 : j = 1, ..., k\} \cup \{\exists y(ry = x_i) : i = 1, ..., t\}$$

is satisfiable in $S$.

By the well-known fact, we may choose $r_1, ..., r_t \in \mathbf{Z}$ so that $\sum_{i=1}^{t} m_i^{(j)} r_i \neq 0$ for $j = 1, ..., k$. Let $s = \prod_{j=1}^{k} \sum_{i=1}^{t} m_i^{(j)} rr_i$ and choose $a \in S$ such that $sa \neq 0$. Then if $x_i = rr_i a$, we have

$$\sum_{i=1}^{t} m_i^{(j)} x_i = \left(\sum_{i=1}^{t} m_i^{(j)} rr_i\right) a \neq 0$$

for each $j = 1, ..., k$; since $r$ divides $x_i$, for $i = 1, ..., t$, the proof of Lemma 1.10 is complete.

We are now going to summarize the information we have gained about the structure of a $\kappa$-saturated group $S$. First, we define, for any group $A$

$$\mathrm{U}(p, k; A) = \begin{cases} \dim(p^k A[p]/p^{k+1} A[p]) & \text{if finite} \\ \infty & \text{otherwise} \end{cases}$$

$\mathrm{Tf}(p; A)$ and $\mathrm{D}(p; A)$ have been defined before Lemmas 1.7 and 1.9 respectively.

**1.11. Theorem.** *Let $\kappa$ be an uncountable cardinal and let $S$ be a $\kappa$-saturated group. Then $S$ is a pure-injective group i.e. it is isomorphic to a product $\Pi_p \bar{S}_p \oplus S_d$, where $S_d$ is divisible and $\bar{S}_p$ is the completion in the p-adic topology of a direct sum of cyclic groups,*

$$S_p = \oplus_n \mathrm{Z}(p^n)^{(\alpha_{p,n})} \oplus \mathrm{Z}_p^{(\beta_p)} .$$

*Furthermore*

$$\begin{cases} \alpha_{p,n} = \mathrm{U}(p, n-1; S) & \textit{if } \mathrm{U}(p, n-1; S) \textit{ is finite} \\ \alpha_{p,n} \geq \kappa & \textit{otherwise} \end{cases}$$

$$\begin{cases} \beta_p = \mathrm{Tf}(p;S) \text{ if } \mathrm{Tf}(p;S) \text{ is finite} \\ \beta_p \geq \kappa \quad \text{otherwise} \end{cases}$$

*and*

$$S_d \cong \oplus_p \mathbf{Z}(p^\infty)^{(\gamma_p)} \oplus \mathbf{Q}^{(\delta)}$$

*where*

$$\begin{cases} \gamma_p = \mathrm{D}(p;S) \text{ if } \mathrm{D}(p;S) \text{ is finite} \\ \gamma_p \geq \kappa \quad \text{otherwise} \end{cases}$$

$$\begin{cases} \delta = 0 \quad \text{if} \quad \mathrm{Exp}(S) = 0 \\ \delta \geq \kappa \quad \text{otherwise} \end{cases}$$

The proof is contained in the sequence of lemmas preceding the theorem. The results about $\alpha_{p,n}$, $\beta_p$, $\gamma_p$ and $\delta$ are contained, respectively, in Lemmas 1.5, 1.7, 1.9 and 1.10.

**1.12. Corollary.** *Let $S_1$ and $S_2$ be saturated groups of cardinality $\kappa \geq \omega$. $S_1$ is isomorphic to $S_2$ if and only if* $\mathrm{Exp}(S_1) = \mathrm{Exp}(S_2)$ *and for every $p$, $n$,* $\mathrm{U}(p,n;S_1) = \mathrm{U}(p,n;S_2)$, $\mathrm{Tf}(p;S_1) = \mathrm{Tf}(p;S_2)$ *and* $\mathrm{D}(p;S_1) = \mathrm{D}(p;S_2)$.

## § 2. Elementary embeddings and decidability

We now take up the task of giving a convenient criterion for deciding when one abelian group is elementarily embedded in another. As a by-product of this work, we will be able to deduce quickly the main results of Szmielew's paper. The finer details of her paper seem to require a little more delicate work, which we pursue in section 4.

Given a group $A$ we refer to the associated invariants $\mathrm{U}(p, n-1; A)$, $\mathrm{Tf}(p; A)$, $\mathrm{D}(p; A)$ and $\mathrm{Exp}(A)$ defined in Section 1 as the *elementary invariants* of $A$. This terminology will now be justified.

**2.1. Theorem.** *If $A$ is elementarily equivalent to $B$, then the elementary invariants of $A$ and $B$ are the same.*

**Proof.** $\mathrm{U}(p, n-1; A) \geq k$ iff there exist $a_1, ..., a_k \in A$ such that $p^{n-1}a_1, ..., p^{n-1}a_k$ are each of order $p$ and independent modulo $p^n$. In other words, the following sentence holds in $A$ if and only if $\mathrm{U}(p, n-1) \geq k$ :

$$\exists x_1, ..., x_k \left[\bigwedge_{i=1}^{k} (p^{n-1} | x_i \wedge px_i = 0) \wedge \right.$$

$$\left. \wedge \bigwedge_{(m_1,...,m_k)\in S} \neg\left(p^n | \sum_{i=1}^{k} m_i x_i\right)\right] \tag{2.1.1}$$

where $S$ is the set of all non-trivial $k$-tuples of natural numbers such that $0 \leq m_i < p$.

If $A$ is elementarily equivalent to $B$, then clearly for each $k$, $\mathrm{U}(p, n-1; A) \geq k$ iff $\mathrm{U}(p, n-1; B) \geq k$. Thus, $\mathrm{U}(p, n-1; A) = \mathrm{U}(p, n-1; B)$.

(ii) Similarly, $\dim(p^{n-1}A/p^nA) \geq k$ iff the following sentence holds in $A$:

$$\exists x_1, ..., x_k \left[\bigwedge_{i=1}^{k} p^{n-1} | x_i \wedge \bigwedge_{(m_1,...,m_k)\in S} \neg\left(p^n | \sum_{i=1}^{k} m_i x_i\right)\right] . \tag{2.1.2}$$

Thus, for each $n$ $\dim(p^{n-1}A/p^nA) = \dim(p^{n-1}B/p^nB)$, in the finite-$\infty$ sense, and the eventual values $\mathrm{Tf}(p;A)$ and $\mathrm{Tf}(p;A)$ are equal.

(iii) Also, $\dim(p^{n-1}A[p]) \geq k$ iff the following sentence holds in $A$:

$$\exists x_1, \ldots, x_k \left[ \bigwedge_{i=1}^{k} (p^{n-1} | x_i \wedge p x_i = 0) \wedge \right.$$

$$\left. \wedge \bigwedge_{(m_1, \ldots, m_k) \in S} \sum_{i=1}^{k} m_i x_i \neq 0 \right] \qquad (2.1.3)$$

Thus, for each $n$ $\dim(p^{n-1}A[p]) = \dim(p^{n-1}B[p])$, in the finite-$\infty$ sense, and the eventual values $D(p; A)$ and $D(p; B)$ are equal.

(iv) Finally, $A$ has exponent dividing $n$ iff the following sentence holds:

$$\forall x (nx = 0) ;$$

and therefore $\operatorname{Exp}(A) = \operatorname{Exp}(B)$.

The proof of Theorem 2.1 is thus complete.

The converse of Theorem 2.1 is an immediate consequence of Corollary 1.12, given that any group is elementarily equivalent to a saturated group. One may eliminate the latter assumption in the manner indicated in section 0. However we will prove the converse of Theorem 2.1 below (as Theorem 2.6) without making any assumption about the existence of saturated groups. Later, in section 3, we will prove that saturated models of any complete theory of groups exist in all cardinals $\kappa$ such that $\kappa^{\aleph_0} = \kappa$.

Let us note that if $A$ is an elementary substructure of $B$, then $A$ is elementarily equivalent to $B$ and $A$ is a pure subgroup of $B$. The next two lemmas are needed to show the converse of this statement, which we derive as Corollary 2.5 below.

**2.2. Lemma.** *Let $A$ and $B$ be pure-injective groups with $A$ a pure subgroup of $B$. Then given a decomposition of $A$ of the form $A = \Pi_p \bar{A}_p \oplus A_d$, there exists a divisible group $C_d$ and for each prime $p$ a direct sum of cyclic $\mathbf{Z}_p$-modules $C_p$ such that $B = \Pi_p (\overline{A_p \oplus C_p}) \oplus (A_d \oplus C_d)$.*

**Proof.** Clearly $A_d$ is a divisible subgroup of $B_d = \{b \in B : \forall n \neq 0, n | b\}$ and therefore there exists a $C_d$ such that $B_d = A_d \oplus C_d$.

Let $A_r = \Pi_p \bar{A}_p$. By purity $A_r \cap B_d = \{0\}$, and letting $B_r$ be a maximal subgroup of $B$ such that $A_r \subset B_r$ and $B_r \cap B_d = \{0\}$, we have $B = B_r \oplus B_d$.

Letting $\bar{B}_p = \{b \in B_r : \forall m, (m, p) = 1 \Rightarrow m|b\}$, we know from the results of section 1 that we can identify $B_r$ with $\Pi_p \bar{B}_p$. Given $A_p$, a basic submodule of $\bar{A}_p$, we can find a basic submodule $B_p$ of $\bar{B}_p$, of which $A_p$ is a direct summand, by Lemma 21 (c) of [6]. We let $C_p$ be the complementary summand of $A_p$ in $B_p$, and the proof of Lemma 2.2 is complete.

Given a group $A$ we say that $A$ *appears to be κ-saturated* (for $\kappa \geq \aleph_0$) if $A$ satisfies the conclusion of Theorem 1.11, i.e., $A$ is pure-injective and if we write $A = \Pi_p \bar{A}_p \oplus A_d$ with $A_p = \oplus_{n=1}^{\infty} Z(p^n)^{(\alpha_{p,n})} \oplus Z_p^{(\beta_p)}$ and $A_d = \oplus_p Z(p^\infty)^{(\gamma_p)} \oplus Q^{(\delta)}$, then for each $p, n$:

$$\begin{cases} \alpha_{p,n} = U(p, n-1; A) & \text{if } U(p, n-1; A) \text{ is finite} \\ \alpha_{p,n} \geq \kappa & \text{otherwise} \end{cases}$$

$$\begin{cases} \beta_{p,n} = Tf(p; A) & \text{if } Tf(p; A) < \infty \\ \beta_{p,n} \geq \kappa & \text{otherwise} \end{cases}$$

$$\begin{cases} \gamma_p = D(p; A) & \text{if } D(p; A) < \infty \\ \gamma_p \geq \kappa & \text{otherwise} \end{cases}$$

$$\begin{cases} \delta = 0 & \text{if } Exp(A) = 0 \\ \delta \geq \kappa & \text{if } Exp(A) = \infty \end{cases}$$

**2.3. Lemma.** *Let $A$ be a pure subgroup of $B$. Suppose that $A$ and $B$ have the same elementary invariants, and both appear to be κ-saturated for $\kappa > \aleph_0$. Let $S \subset A$ and $T \subset B$ be subsets having cardinality $< \kappa$. Then there exists an automorphism $\varphi$ of $B$ such that* (i) *for all $a \in S$, $\varphi(a) = a$, and* (ii) $\varphi(T) \subset A$.

**Proof.** By Lemma 2.2, we can write $A = \Pi_p \bar{A}_p \oplus A_d$ and $B = \Pi_p (\overline{A_p \oplus C_p}) \oplus (A_d \oplus C_d)$. On cardinality grounds, there exist divisible groups $E_d$, $E'_d$, $F_d$ and $F'_d$, and for each prime $p$, direct sums of cyclic $Z_p$-modules $E_p$, $E'_p$, $F_p$ and $F'_p$ such that

(i) $E_d \oplus E'_d = A_d$, and $F_d \oplus F'_d = C_d$; for each $p$,
$E_p \oplus E'_p = A_p$, and $F_p \oplus F'_p = C_p$;
(ii) $|E_d| < \kappa$ and $|F_d| < \kappa$; for each $p$, $|E_p| < \kappa$ and $|F_p| < \kappa$.
(iii) $S \subset \Pi_p \overline{E}_p \oplus E_d$ and $T \subset \Pi_p (\overline{E_p \oplus F_p}) \oplus E_d \oplus F_d$.

[Note that in (ii) we can claim that $|E_p| \leq |S| \cdot \aleph_0 < \kappa$ since every element of $\overline{A}_p$ is a limit of a *countable* sequence of elements of $A_p$.] Thus we have

$$B = \Pi_p (\overline{\underbrace{E_p \oplus E'_p}_{A_p} \oplus \underbrace{F_p \oplus F'_p}_{C_p}}) \oplus \underbrace{E_d \oplus E'_d}_{A_d} \oplus \underbrace{F_d \oplus F'_d}_{C_d}.$$

We now claim further that there exists groups $E''_d$, $E'''_d$ and $E''_p$, $E'''_p$ such that $E'''_d \cong F_d$ (say, by $\chi_d$), $E'_d = E''_d \oplus E'''_d$, $E'''_p \cong F_p$ (say, by $\chi_p$), and $E'_p = E''_p \oplus E'''_p$. We illustrate the reasoning for a prime $p$. Let $F_p = \bigoplus_n Z(p^n)^{(\alpha'_{p,n})} \oplus Z_p^{(\beta'_p)}$. Then $\alpha_{p,n}(B) \geq \alpha_{p,n}(A) + \alpha'_{p,n}$ and $\beta_p(B) \geq \beta_p(A) + \beta'_p$. Thus, if $\alpha'_{p,n}$ is greater than 0, then $\alpha_{p,n}(A) \geq \kappa$; for otherwise $\alpha_{p,n}(A) = \mathrm{U}(p, n-1; A) = \mathrm{U}(p, n-1; B) = \alpha_{p,n}(B) > \alpha_{p,n}(A)$, a contradiction. Similarly, if $\beta'_p$ is greater than 0, then $\beta_p(A) \geq \kappa$. Thus, since $|E_p| < \kappa$ there is ample room in $E'_p$ to split off a copy of $F_p$.

Our desired automorphism can now be obtained by putting together automorphisms $\varphi_d$ of $B_d$ and $\varphi_p$ of $\overline{B}_p$, for each prime $p$ as follows:

$\varphi_d$ is the sum of the identity maps on $E_d$, $E''_d$ and $F'_d$ and the maps $\chi_d$ and $\chi_d^{-1}$ on $E'''_d$ and $F_d$ respectively. $\varphi_p$ is defined by taking the sum of the identity maps on $E_p$, $E''_p$ and $F'_p$ and the maps $\chi_p$ and $\chi_p^{-1}$ on $E'''_p$ and $F_p$ respectively, and extending by continuity to the completion $\overline{B}_p$ of $B_p$. Schematically

$$B = \Pi_p (\overline{E_p \oplus E''_p \oplus \underset{\chi_p^{-1}}{\overset{\chi_p}{E'''_p \oplus F_p}} \oplus F'_p}) \oplus E_d \oplus E''_d \oplus \underset{\chi_d^{-1}}{\overset{\chi_d}{E'''_d \oplus F_d}} \oplus F'_d$$

That this automorphism has the required properties is immediate.

**2.4. Theorem.** *Let A be a pure subgroup of B, and let A and B have the same elementary invariants. Then A is an elementary substructure of B.*

**Proof.** By blowing up the situation $A \subset B$ (for example, by an ultrapower) we get the following commutative diagram:

$$\begin{array}{ccc} A' & \subset & B' \\ \curlyvee & & \curlyvee \\ A & \subset & B \end{array},$$

where $A'$ and $B'$ are $\omega_1$-saturated, and all inclusions are pure. It thus suffices to establish the result under the hypothesis that $A$ and $B$ are $\omega_1$-saturated.

By Tarski's Lemma it suffices to show that given $a_1, \ldots, a_n \in A$ and $b \in B$, there exists an automorphism $\varphi$ of $B$ such that $\varphi(a_1) = a_1, \ldots, \varphi(a_n) = a_n$ and $\varphi(b) \in A$. But this is immediate from Lemma 2.3., using the fact that, by Theorem 1.11, an $\omega_1$-saturated group appears to be $\omega_1$-saturated.

**2.5. Corollary.** *If A is a pure subgroup of B which is elementarily equivalent to B, then A is an elementary substructure of B.*

**2.6. Theorem** (Szmielew [12]). *If A and B have the same elementary invariants then A is elementarily equivalent to B.*

**Proof.** By taking $\omega_1$-saturated elementary extensions of $A$ and $B$, we see that we need only prove the result for $\omega_1$-saturated groups $A$ and $B$. Let $A'$ be obtained from $A$ by choosing a decomposition for $A$ according to Theorem 1.11 and throwing away all but $\aleph_1$ copies of any summand of $A_p$ or $A_d$ which occurs at least $\aleph_1$ times. Clearly $A'$ is a pure subgroup of $A$ with the same elementary invariants; thus, by Theorem 2.4, $A' \prec A$. Let $B'$ be obtained from $B$ in the same way. Clearly $B' \prec B$ and moreover $A' \cong B'$. Finally $A$ is elementarily equivalent to $B$.

**2.7. Theorem.** *Let $\kappa$ be uncountable. Every group which appears to be $\kappa$-saturated is $\kappa$-saturated.*

**Proof.** Let $A$ appear to be $\kappa$-saturated. Let $B$ be an elementary extension of $A$ which is $\kappa$-saturated. Given a subset $S$ of $A$ with cardinality $< \kappa$ we must show that any type of Th($A$, $S$) is realized in $A$. Since $B$ is a $\kappa$-saturated elementary extension of $A$, any such type is realized by an element $b$ of $B$. By Lemma 2.3, there is an automorphism of $B$ which leaves $S$ fixed and moves $b$ into $A$. This shows that the type is realized in $A$, and the proof is complete.

The reader who is familiar with infinitary languages should recognize that Lemma 2.3 actually establishes the following: if $A$ is a pure subgroup of $B$, $A$ and $B$ have the same elementary invariants, and $A$ and $B$ both appear to be $\kappa$-saturated, then $A \prec_{\infty,\kappa} B$. 2.4 and 2.7 are thus immediate consequences of 2.3.

We are now in a position to demonstrate the decidability of the theory of abelian groups. Let us define a *core sentence* to be a sentence of one of the forms 2.1.1, 2.1.2, 2.1.3 (see Theorem 2.1) or of the form $\forall x(nx = 0)$, or a negation of one of the above.

**2.8. Theorem** (Szmielew [12]). *Let T be the theory of abelian groups. Then every complete extension of T can be axiomatized by adding a set of core sentences to T. Also, each sentence which is consistent with T is a consequence of T together with a consistent finite conjunction of core sentences*

**Proof.** This is a trivial consequence of Theorem 2.6 and the compactness theorem.

Let us now define a *Szmielew-group A* to be one which can be written in the following form:

$$(*) \qquad A = \oplus_{p,n} Z(p^n)^{(\alpha_{p,n})} \oplus \oplus_p Z_p{}^{(\beta_p)} \oplus \oplus_p Z(p^\infty)^{(\gamma_p)} \oplus Q^{(\delta)} ,$$

where each $\alpha_{p,n}$, $\beta_p$ and $\gamma_p$ is finite or countably infinite and $\delta$ is 0 or 1.

**2.9. Theorem** (Szmielew [12]). *Every group is elementarily equivalent to a Szmielew group.*

**Proof.** One can routinely check that the dimensions of the vector spaces which we associate with groups add over direct sums. Similarly one can check that the dimensions of the building blocks used in (*) above are just what one would expect (see [12], p. 219). Now given a group $B$ we define:

$$\alpha_{p,n} = \begin{cases} \mathrm{U}(p, n-1; B) & \text{if finite} \\ \aleph_0 & \text{otherwise} \end{cases}$$

$$\beta_p = \begin{cases} \mathrm{Tf}(p; B) & \text{if finite} \\ \aleph_0 & \text{otherwise} \end{cases}$$

$$\gamma_p = \begin{cases} \mathrm{D}(p; B) & \text{if finite} \\ \aleph_0 & \text{otherwise} \end{cases}$$

$$\delta = \begin{cases} 0 & \text{if } \mathrm{Exp}(B) = 0 \\ 1 & \text{if } \mathrm{Exp}(B) = \infty \end{cases}$$

Using these exponents we form the group $A$ in (*) above. $A$ thus has the same invariants as $B$, and by Corollary 2.6, $A$ is elementarily equivalent to $B$.

A *Szmielew-group of finite rank* is a Szmielew-group in which every exponent is finite and all but finitely many are zero.

**2.10. Theorem** (Szmielew [12]). *If $\sigma$ is a consistent sentence in the theory of abelian groups, then there exists a Szmielew group of finite rank which is a model of $\sigma$.*

**Proof.** By Theorem 2.8 there exists a consistent finite set of core sentences $C$ which implies $\sigma$. Let $A$ as in (*) above be a Szmielew-group in which $C$ holds. We will define a "truncation" of $A$ of finite rank in which the sentences of $C$ hold. If $\sigma'$ is a sentence of $C$ of the form 2.1.2 (resp. 2.1.3) – i.e. if $\sigma'$ says "$\dim(p^{n-1}A/p^nA) \geq k$" (resp. "$\dim(p^{n-1}A[p]) \geq k$") – then since $\sigma'$ holds in $A$

$$\sum_{j \geq n} \alpha_{pj} + \beta_p \geq k.$$

$$(\text{resp. } \sum_{j \geq n} \alpha_{pj} + \gamma_p \geq k)$$

(where $\alpha_{p,j}, \beta_p, \gamma_p, \delta$ are as in (*)). Hence $\exists M$ such that

$$\sum_{j=n}^{M} \alpha_{p,j} + \beta_p \geq k$$

$$(\text{resp. } \sum_{j=n}^{M} \alpha_{p,j} + \gamma_p \geq k).$$

Choose an $M$ which works for each $\sigma'$ of the form 2.1.2 or 2.1.3 in $C$; then let $N$ be the maximum of $M$ and of the $n$'s occurring in sentences of $C$ which are of the form 2.1.1, 2.1.2 or 2.1.3; and let $K$ be the maximum of the $k$'s occurring in such sentences (where $n$ and $k$ are as in 2.1.1, 2.1.2 or 2.1.3). Let $F$ be the finite set of primes which pertain to sentences of $C$. Then let

$$A' = \bigoplus_{p \in F} \left( \bigoplus_{n=1}^{N} \mathrm{Z}(p^n)^{(\alpha'_{p,n})} \right) \oplus \bigoplus_{p \in F} \mathrm{Z}_p^{(\beta'_p)} \oplus \bigoplus_{p \in F} \mathrm{Z}(p^\infty)^{(\gamma'_p)} \oplus \mathrm{Q}^{(\delta)}$$

where $\alpha'_{p,n} = \min(\alpha_{p,n}, K)$, $\beta'_p = \min(\beta_p, K)$, and $\gamma'_p = \min(\gamma_p, K)$. A routine verification shows that $C$ holds in $A'$, and the theorem is proved.

**2.11. Theorem** (Szmielew [12]). *The elementary theory of abelian groups is a decidable theory.*

**Proof.** It suffices to show that the set of consistent sentences is r.e. For this we effectively enumerate sequences $\langle S, \delta \rangle$ where $\delta$ is 0 or 1 and $S$ is a set of the form:

$$\{\langle p, \bar{\alpha}_p, \beta_p, \gamma_p \rangle : p \in F\};$$

where $F$ is a finite set of primes, each $\beta_p$ and $\gamma_p$ is a natural number, and each $\overline{\alpha}_p$ is a finite sequence $\langle \alpha_{p,1}, ..., \alpha_{p,n} \rangle$ of natural numbers. For each sequence $\langle S, \delta \rangle$ we can (uniformly) effectively enumerate the finite conjunctions of core sentences (and hence by Theorem 2.8 all sentences) which hold in the Szmielew-group of finite rank with invariants given by $\langle S, \delta \rangle$. Finally, by Theorem 2.10 and a diagonalization, we see that the set of consistent sentences is r.e., and Szmielew's theorem is proved.

For the sake of completeness we give one more result which will be improved considerably in section 4 below.

**2.12. Theorem** (Szmielew [12]). *Let $\sigma$ be a sentence in the language of group theory. Then $\sigma$ is equivalent in the theory of abelian groups to a sentence which is a disjunction of conjunctions of core sentences.*

**Proof.** Consider the set $S$ of all finite conjunctions of core sentences which imply $\sigma$. Suppose that $\sigma$ is not equivalent to any finite disjunction of members of $S$. Then $\sigma \cup \{\neg \varphi : \varphi \in S\}$ is consistent with the theory of groups. If we now let $A$ be a group which is a model of this set of sentences we see, as in Theorem 2.8, that some conjunction $\varphi$ of core sentences true in $A$ implies $\sigma$. Thus $\varphi \in S$, contradicting the fact that $A$ is a model of $\{\neg \varphi : \varphi \in S\}$.

## §3. The existence of saturated abelian groups

Let $A$ be an infinite group (we exclude the finite groups since their theories are trivial). In this section we are going to determine the cardinals in which the theory of $A$, Th($A$), has a saturated model. If $\kappa$ is an infinite cardinal, let $S = S_{A,\kappa}$ be the $\kappa$-saturated model of Th($A$) defined by

$$S_{A,\kappa} = \prod_p \bar{S}_p \oplus S_d$$

where

$$S_p = \oplus_n \mathbf{Z}(p^n)^{(\alpha_{p,n})} \oplus \mathbf{Z}_p^{(\beta_p)}$$

$$S_d = \oplus_p \mathbf{Z}(p^\infty)^{(\gamma_p)} \oplus \mathbf{Q}^{(\delta)}$$

where

$$\alpha_{p,n} = \min\{\mathrm{U}(p, n-1; A), \kappa\}$$

$$\beta_p = \min\{\mathrm{Tf}(p; A), \kappa\}$$

$$\gamma_p = \min\{\mathrm{D}(p; A), \kappa\}$$

($\infty$ is defined to be $> \kappa$) and

$$\delta = \begin{cases} 0 \text{ if } A \text{ is of bounded order} \\ \kappa \text{ otherwise.} \end{cases}$$

By Theorem 2.7, $S_{A,\kappa}$ is $\kappa$-saturated and by Corollary 2.6, it is a model of Th($A$). It is clear that Th($A$) has a saturated model of cardinality $\kappa$ if and only if the cardinality of $S_{A,\kappa}$ is $\kappa$. For example, the condition $\kappa^{\aleph_0} = \kappa$ is easily seen to be sufficient for Th($A$) to have a saturated model of cardinality $\kappa$. Whether or not it is necessary depends on properties of the group $A$. We sum up the results in the following table in which we also determine for each group $A$ whether or not Th($A$) is $\aleph_0$-categorical, $\aleph_1$-categorical and $\omega_1$-stable. The results on $\aleph_1$-categoricity and $\omega_1$-stability are due to A. Macintyre [7]. An explanation follows the table. (By the "reduced part of $A$" we mean $A/A_d$, where $A_d$ is the maximal divisible subgroup of $A$).

Table

| $A$ is an INFINITE group satisfying: | Th($A$) is $\aleph_0$-categorical | Th($A$) is $\aleph_1$-categorical | Th($A$) is $\omega_1$-stable | Th($A$) has a saturated model of Card $\kappa$ iff |
|---|---|---|---|---|
| (I) $A$ is of bounded order | yes | – | yes | $\kappa \geq \aleph_0$ |
| (a) U($p$, $n$; $A$) = ∞ for only one pair ($p$, $n$) | " | yes | " | " |
| (b) not (a) | " | no | " | " |
| (II) Not (I) i.e. $A$ is not of bounded order | no | – | – | – |
| (a) reduced part of $A$ is of bounded order | " | – | yes | $\kappa \geq \aleph_0$ |
| (i) ∀$p$ (the number of elements of order $p$ is finite) | " | yes | " | " |
| (ii) not (i) | " | no | " | " |
| (b) not (a) | " | " | no | – |
| (i) ∀$p$, Tf($p$; $A$) ≠ ∞ *and* ∃ only finitely many $p$ such that ∃$n$ for which U($p$, $n$; $A$) = ∞ | " | " | " | $\kappa \geq 2^{\aleph_0}$ |
| (ii) not (i) | " | " | " | $\kappa^{\aleph_0} = \kappa$ |

*Explanation of the table*:

First, observe that for any infinite group $A$, all the models of Th($A$) belong to the same row of the table as $A$. The conditions on $A$ stated in the first column are not elementary statements in general, but if a given group $A$ satisfies one of these conditions it satisfies an elementary statement, or set of statements, which imply the condition. For example, if the reduced part of $A$ is of bounded order, then there is an $n$ such that $nA$ is divisible; for a fixed $n$, the latter is an elementary statement and so every model $G$ of Th($A$) is such that $nG$ is divisible i.e. the reduced part of $G$ is of bounded order.

(I) If $A$ is of bounded order, then so is every model of Th($A$), and every model of Th($A$) is a direct sum of finite cyclic groups (Theorem 6 of [6]). Say $A = \oplus_{p,n} Z(p^n)^{(\alpha_{p,n})}$; there are only a finite number of pairs $(p, n)$ such that $\alpha_{p,n} \neq 0$. For each $(p, n)$ there is a sentence (or set of sentences) of Th($A$) which says that $U(p, n-1; A)$ is a given finite cardinal (or is infinite). If $B$ is a countable model of Th($A$) and if $U(p, n-1; A) = U(p, n-1; B) = \infty$, then necessarily $\dim p^{n-1}B[p]/p^nB[p] = \aleph_0$ i.e. the number of copies of $Z(p^n)$ in the direct decomposition of $B$ is $\aleph_0$. Hence the countable model of Th($A$) is uniquely determined up to isomorphism.

In order to prove that Th($A$) is $\omega_1$-stable it suffices to prove that if $B = \oplus_{p,n} Z(p^n)^{(\beta_{p,n})}$ is the countable model of Th($A$) there are only countably many inequivalent elementary embeddings of $B$ into itself (Theorem 0.16). If $f: B \to B$ and $g: B \to B$ are elementary embeddings they are pure embeddings and since $B$ is of bounded order, $f(B)$ and $g(B)$ are direct summands of $B$ (Theorem 7 of [6]): $B = f(B) \oplus C$, $B = g(B) \oplus C'$. Clearly $f$ and $g$ are equivalent if and only if $C$ and $C'$ are isomorphic. But $C \cong \oplus_{p,n} Z(p^n)^{(\gamma_{p,n})}$, where $\gamma_{p,n} \leq \beta_{p,n}$; since $\beta_{p,n} = 0$ for all but a finite number of pairs $(p, n)$ there are only a countable number of isomorphism classes of groups to which $C$ can belong.

We shall prove below that if the reduced part of $A$ is of bounded order, then Th($A$) has a saturated model in every cardinal $\kappa \geq \aleph_0$.

(I) (a) Since $A$ is infinite but of bounded order, there is at least one pair $(p, n)$ such that $U(p, n-1; A) = \infty$. If there is a unique $(p_0, n_0)$ such that $U(p_0, n_0 - 1; A) = \infty$, then the unique (up to isomorphism) model of Th($A$) of cardinality $\kappa \geq \aleph_1$ is

$$\bigoplus_{(p,n) \neq (p_0, n_0)} Z(p^n)^{(\alpha_{p,n})} \oplus Z(p_0^{\,n_0})^{(\kappa)}.$$

(I) (b) On the other hand, if there exist $(p_0, n_0) \neq (p_1, n_1)$ such that $U(p_0, n_0 - 1; A) = \infty = U(p_1, n_1 - 1; A)$, then the following are two non-isomorphic models of Th($A$) of cardinality $\kappa \geq \aleph_1$ (Let $I = \{(p_0, n_0), (p_1, n_1)\}$ and $\alpha'_{p,n} = \min\{\alpha_{p,n}, \aleph_0\}$):

$$\bigoplus_{(p,n) \notin I} Z(p^n)^{(\alpha'_{p,n})} \oplus Z(p_0^{\,n_0})^{(\aleph_0)} \oplus Z(p_1^{\,n_1})^{(\kappa)}$$

and

$$\bigoplus_{(p,n) \notin I} Z(p^n)^{(\alpha'_{p,n})} \oplus Z(p_0^{\,n_0})^{(\kappa)} \oplus Z(p_1^{\,n_1})^{(\aleph_0)}$$

(II) If $A$ is not of bounded order, let $\alpha_{p,n} = \min\{U(p, n; A), \aleph_0\}$, $\beta_p = \min\{Tf(p; A), \aleph_0\}$, $\gamma_p = \min\{D(p; A), \aleph_0\}$ (where we define $\infty > \aleph_0$).
Then the following are two non-isomorphic countable models of Th($A$) (see §2):

$$B = \oplus_{p,n} Z(p^n)^{(\alpha_{p,n})} \oplus \oplus_p Z_p^{(\beta_p)} \oplus \oplus_p Z(p^\infty)^{(\gamma_p)} \oplus \mathbf{Q}$$

and

$$B' = B \oplus \mathbf{Q}$$

($B$ and $B'$ are both models of Th($A$) because they are not of bounded order and have the same invariants as $A$).

(II) (a) Write $A = A_r \oplus A_d$ where $A_r$ is reduced and $A_d$ is divisible. Condition (II) (a) says $A_r$ is of bounded order. We have proved above that Th($A_r$) is $\omega_1$-stable. In order to prove Th($A$) is $\omega_1$-stable, it suffices, by Theorem 0.14, to prove that Th($A_d$) is $\omega_1$-stable. Let $D_0, D_1, D_2$ be countable models of Th($A_d$) with $D_0 \subseteq D_1$, $D_0 \subseteq D_2$. We will prove that if $a_i \in D_i$ ($i = 1, 2$) then $a_1$ and $a_2$ have the same type over $D_0$ if and only if $\{n \in \mathbf{Z} : na_1 \in D_0\} = \{n \in \mathbf{Z} : na_2 \in D_0\}$ ($= d\mathbf{Z}$, say) and $da_1 = da_2$. If this is the case then clearly there are only a countable number of types in Th$(D_0, x)_{x \in D_0}$. Now the conditions are obviously necessary; on the other hand, it $a_1$ and $a_2$ satisfy the conditions, then there is an isomorphism $f : D_0 + \mathbf{Z}a_1 \to D_0 + \mathbf{Z}a_2$ (where $D_0 + \mathbf{Z}a_i$ is the subgroup of $D_i$ generated by $D_0$ and $a_i$) such that $f(d + na_1) = d + na_2$ ($d \in D_0$). $f$ extends to an isomorphism

$$f' : E(D_0 + \mathbf{Z}a_1) \to E(D_0 + \mathbf{Z}a_2)$$

of injective envelopes of $D_0 + \mathbf{Z}a_1$ and $D_0 + \mathbf{Z}a_2$ contained in $D_1$ and $D_2$, respectively. $E(D_0 + \mathbf{Z}a_i)$ is an elementary substructure of $D_i$ ($i = 1, 2$) (by Theorem 2.4) and $f'(a_1) = a_2$ and $f'(d) = d$ for $d \in D_0$, so $a_1$ and $a_2$ have the same type over $D_0$. Therefore Th($A$) is $\omega_1$-stable.

Since Th($A$) is $\omega_1$-stable it has a countable saturated model (Theorem 0.15). For $\kappa \geq \aleph_1$, we prove that $S_{A,\kappa}$ (see the remarks before the table) is of cardinality $\leq \kappa$. (Since $A$ is infinite, $S_{A,\kappa}$ has cardinality $\geq \kappa$). Indeed if we write $S_{A,\kappa} = \Pi_p \bar{S}_p \oplus S_d$ then certainly $S_d$ has cardinality $\leq \kappa$. For all but a finite number of $p$, $\bar{S}_p = \{0\}$ and for all $p$, $S_p$ is of bounded order and has cardinality $\leq \kappa$. Since $S_p$ is of bounded order, $S_p$ is complete in its $p$-adic topology. (Any Cauchy sequence is eventually constant because $p^n S_p = 0$ for sufficiently large $n$.) Therefore $\bar{S}_p = S_p$ for all $p$, and we conclude that $\mathrm{Card}(S_{A,\kappa}) = \kappa$.

(II) (a) (i) If for every $p$ there are only a finite number of elements of order $p$, then $\mathrm{U}(p, n; A) < \infty$ and $\mathrm{D}(p; A) < \infty$ for all $p$, $n$. Therefore the unique model of $\mathrm{Th}(A)$ of cardinality $\kappa \geq \aleph_1$ is:

$$\oplus_{p,n} \mathrm{Z}(p^n)^{(\alpha_{p,n})} \oplus \oplus_p \mathrm{Z}(p^\infty)^{(\gamma_p)} \oplus \mathrm{Q}^{(\kappa)}$$

where $\alpha_{p,n} = \mathrm{U}(p, n-1; A)$, $\gamma_p = \mathrm{D}(p; A)$.

(II) (a) (ii) On the other hand if there is a $p_0$ such that $\mathrm{D}(p_0; A) = \infty$ (respectively: such that there exists $n_0$ such that $\mathrm{U}(p_0, n_0 - 1; A) = \infty$), then if $\alpha_{p,n} = \min\{\mathrm{U}(p, n-1; A), \aleph_0\}$, $\gamma_p = \min\{\mathrm{D}(p; A), \aleph_0\}$, the following are non-isomorphic models of $\mathrm{Th}(A)$ of cardinality $\kappa \geq \aleph_1$:

$$\oplus_{(p,n)} \mathrm{Z}(p^n)^{(\alpha_{p,n})} \oplus \oplus_p \mathrm{Z}(p^\infty)^{(\gamma_p)} \oplus \mathrm{Q}^{(\kappa)}$$

and

$$\oplus_{(p,n)} \mathrm{Z}(p^n)^{(\alpha_{p,n})} \oplus \bigoplus_{p \neq p_0} \mathrm{Z}(p^\infty)^{(\gamma_p)} \oplus \mathrm{Z}(p_0{}^\infty)^{(\kappa)}$$

(respectively: $\displaystyle \bigoplus_{(p,n) \neq (p_0, n_0)} \mathrm{Z}(p^n)^{(\alpha_{p,n})} \oplus \mathrm{Z}(p_0{}^{n_0})^{(\kappa)} \oplus \oplus_p \mathrm{Z}(p^\infty)^{(\gamma_p)} \oplus \mathrm{Q}$)

(II) (b) If the reduced part of $A$ is not of bounded order we are going to prove that $\mathrm{Th}(A)$ does not have a countable saturated model and therefore certainly $\mathrm{Th}(A)$ is neither $\omega_1$-stable nor $\omega_1$-categorical. There are two cases; if the reduced part of $A$ is not of bounded order either: (1) there are infinitely many $p$ such that there is an $n$ for which $\mathrm{U}(p, n-1; A) \neq 0$; or (2) there is a $p$ such that $\mathrm{Tf}(p; A) \neq 0$. (To see that these are the only cases look at an $\omega_1$-saturated model $S = \Pi_p \bar{S}_p \oplus S_d$ of $\mathrm{Th}(A)$ and note that $\Pi_p \bar{S}_p$ is of unbounded order if and only if there are infinitely many $p$ such that $S_p \neq \{0\}$ or there is a $p$ such that $S_p$ is of unbounded order; in the latter case it follows from Lemma 1.6 (iv) that we must have $\mathrm{Tf}(p; A) \neq 0$). In Case (1), if $\Delta = \{p : \exists n(\mathrm{U}(p, n-1; A) \neq 0)\}$, $\Delta$ is infinite and for any subset $\Delta'$ of $\Delta$

$$\Sigma_{\Delta'}(x) = \{\exists y(py = x) : p \in \Delta'\} \cup \{\neg \exists y(py = x) : p \in \Delta - \Delta'\}$$

is consistent (if $S = \Pi_p \bar{S}_p \oplus S_d$ is an $\omega_1$-saturated model of Th($A$), the element $f = f_{\Delta'} \in \Pi_p S_p \subseteq S$ defined by: $f(p) = 0$ for $p \in \Delta'$, $f(p)$ = an element of $S_p$ of height 0 for $p \in \Delta - \Delta'$, satisfies $\Sigma_{\Delta'}(x)$). Hence there are uncountably many 1-types in Th($A$). If we are in Case (2), let $B$ be a model of Th($A$) such that $\mathbf{Z}_p \subsetneq B$. Let $b \in B$ be a generator of a copy of $Z_p$ in $B$. We are going to prove that Th($B$, $b$) has uncountably many 1-types, from which it follows that there is no countable saturated model. In fact we define a consistent set of formulas $\Sigma_\sigma(x)$ for any Cauchy sequence $\sigma = (a_n)$ in $\mathbf{Z}_p$ such that if $\sigma$ and $\sigma'$ have different limits in the completion $\bar{\mathbf{Z}}_p$ of $\mathbf{Z}_p$, then $\Sigma_\sigma(x) \cup \Sigma_{\sigma'}(x)$ is inconsistent. Let $\sigma = (a_n)$ be a Cauchy sequence in $\mathbf{Z}_p$; for any $k > 0$, there exists $n_k$ such that $m, n \geq n_k$ implies $p^k$ divides $a_n - a_m$. If $a_{n_k} = r_k/s_k$ where $r_k, s_k \in \mathbf{Z}$, $(s_k, p) = 1$, let

$$\Sigma_\sigma(x) = \{\exists y \exists z[(p^k y = x - z) \wedge (s_k z = r_k b)] : 1 \leq k < \omega\} .$$

Then $a$ realizes $\Sigma_\sigma(x)$ in a model $G$ of Th($A$) extending $B$, if and only if $a$ is a limit of the Cauchy sequence $(a_n \cdot b)_{n<\omega}$ in $\mathbf{Z}_p b \subsetneq B$. Clearly, if $\sigma$ and $\sigma'$ have different limits, $\Sigma_\sigma(x) \cup \Sigma_{\sigma'}(x)$ is inconsistent. Since $\bar{\mathbf{Z}}_p$ is uncountable, there are uncountable many 1-types in Th($B$, $b$).

(II) (b) (i) Let $A$ be a group satisfying (II) (b) (i). Consider $S_{A,\kappa} = \Pi_p \bar{S}_p \oplus S_d$, as in the introduction to this section, for $\kappa \geq \aleph_1$. For each $p$, $S_p = \oplus_n \mathbf{Z}(p^n)^{(\alpha_{p,n})} + \mathbf{Z}_p^{(\beta_p)}$ where $\beta_p$ is finite. The completion $\bar{S}_p$ of $S_p$ equals the direct sum of the completion of $\oplus_n \mathbf{Z}(p^n)^{(\alpha_{p,n})}$ and the completion of $\mathbf{Z}_p^{(\beta_p)}$. Since Tf($p$; $A$) $< \infty$, $\oplus_n \mathbf{Z}(p^n)^{(\alpha_{p,n})}$ is of bounded order and is therefore complete (cf. (II) (a)). The completion of $\mathbf{Z}_p^{(\beta_p)}$ has cardinality $\leq 2^{\aleph_0}$ since $\beta_p$ is finite. Since there are only finitely many $p$ for which there is an $n$ such that U($p$, $n$; $A$) $= \infty$, Card($\bar{S}_p$) $\leq 2^{\aleph_0}$ for all but a finite number of $p$, and for the remaining $p$'s we have Card($\bar{S}_p$) $\leq \kappa + 2^{\aleph_0}$. On the other hand, since $\Pi_p \bar{S}_p$ does not have bounded order, either there are infinitely many $p$ such that $\bar{S}_p \neq \{0\}$ or $\exists p$ such that Card($\bar{S}_p$) $= 2^{\aleph_0}$. Therefore Card($\Pi_p \bar{S}_p$) $\geq 2^{\aleph_0}$, and if $\kappa \geq 2^{\aleph_0}$, Card $\Pi_p \bar{S}_p = \kappa$.

(II) (b) (ii) If $A$ satisfies the hypotheses of (II) (b) (ii) and $\kappa \geq \aleph_1$ we shall prove that $S_{A,\kappa}$ has cardinality $\kappa^{\aleph_0}$. Consider $S_{A,\kappa} = \Pi_p \bar{S}_p \oplus S_d$; if there are infinitely many $p$ for which there exists $n$ such that U($p$, $n$; $A$) $= \infty$, then since $\mathbf{Z}(p^{n+1})^{(\kappa)} \subseteq S_p$ for any pair ($p$, $n$) such that U($p$, $n$; $A$) $= \infty$, we see that there are infinitely many $p$ such that $\operatorname{Card} S_p = \kappa$. Therefore $\operatorname{Card}(\Pi_p \bar{S}_p) = \kappa^{\aleph_0}$. In the other case, i.e. if there is a $p$ such that $\mathrm{Tf}(p; A) = \infty$, then $Z_p{}^{(\kappa)} \subseteq \bar{S}_p$ (as a pure submodule), and it will suffice to prove that the completion of $Z_p{}^{(\kappa)}$ has cardinality $\kappa^{\aleph_0}$. Let $a_\nu$ be a generator of the $\nu$-th copy of $Z_p$ in $Z_p{}^{(\kappa)} (\nu < \kappa)$. For any $\sigma \in \kappa^{\aleph_0}$ define $s_\sigma = (s_{\sigma,n})$ where $s_{\sigma,n} = \sum_{k=1}^{n} p^k a_{\sigma(k)}$. Clearly $s_\sigma$ is a Cauchy sequence in $Z_p{}^{(\kappa)}$, and we claim that if $\tau \in \kappa^{\aleph_0}$ such that $\sigma \neq \tau$, then $s_\sigma$ and $s_\tau$ have different limits. Indeed if $m$ is minimal such that $\sigma(m) \neq \tau(m)$ then for any $n \geq m$,

$$s_{\sigma,n} - s_{\tau,n} = p^m (a_{\sigma(m)} - a_{\tau(m)}) + \sum_{k=m+1}^{n} p^k (a_{\sigma(k)} - a_{\tau(k)}) \text{ so}$$

so that $s_{\sigma,n} - s_{\tau,n}$ is not divisible by $p^{m+1}$, and the same applies to the two limits.

## § 4. Elimination of quantifiers

In this section we will prove that every formula in the language of abelian groups is equivalent (relative to the theory of abelian groups) to a formula which has a particularly simple form, viz., a boolean combination of core sentences (see § 2) and formulas of the form "$p^k \mid \sum_{i=1}^{n} r_i x_i$". This result is due to Szmielew ([12], Theorem 4.22) and it is the basis for her proof that the theory of abelian groups is decidable. Our approach will be model-theoretic, exploiting heavily the pure-injectivity of $\omega_1$-saturated groups. In order to keep the discussion as self-contained as possible, we will first prove a series of lemmas of an algebraic sort which have analogues for modules over arbitrary rings. The above result will then be cast in the form of an elimination-of-quantifiers theorem for a natural extension of the theory of abelian groups.

Our first lemma expresses the fact that $\omega_1$-saturated groups satisfy the usual defining property of pure-injective groups.

**4.1. Lemma.** *If $A$ is $\omega_1$-saturated and $B$ is a pure extension of $A$, then $A$ is a direct summand of $B$.*

**Proof.** Choosing an $\omega_1$-saturated elementary (and hence pure) extension $B'$ of $B$, we find (as in the proof of Lemma 2.2) that given a decomposition $A = \Pi_p \bar{A}_p \oplus A_d$, there exists a decomposition of $B'$ of the form:

$$B' = \Pi_p (\bar{A}_p \oplus \bar{C}_p) \oplus (A_d \oplus C_d) .$$

Letting $C = \Pi_p C_p \oplus C_d$, we have $B' = A \oplus C$, and clearly $B = A \oplus (C \cap B)$, proving the lemma.

A *positive primitive* (*p.p*) *formula* $\varphi$ is a formula of the form

$$\exists y_1, ..., y_m \left[ \bigwedge_{k=1}^{t} \varphi_k(x_1, ..., x_n, y_1, ..., y_m) \right],$$

where each $\varphi_k$ is of the form $\sum_{i=1}^{n} r_i x_i + \sum_{j=1}^{m} s_j y_j = 0$.

Note that every such formula is preserved under homomorphisms (into) and that a conjunction or existential quantification of positive primitive formulas is logically equivalent to a positive primitive formula.

**4.2. Corollary.** *If $A$ is a pure subgroup of $B$, $\varphi(x_1, \ldots, x_n)$ is a positive primitive formula, and $a_1, \ldots, a_n \in A$; then $\varphi(a_1, \ldots, a_n)$ is true in $A$ if and only if it is true in $B$.*

**Proof.** From the fact that positive existential formulas are preserved under homomorphisms, it should be clear that the conclusion holds if $A$ is a direct summand of $B$. Blowing up the situation to an $\omega_1$-saturated extension, we have

$$\begin{array}{ccc} A' & \subset & B' \\ \curlyvee & & \curlyvee \\ A & \subset & B \end{array},$$

where $A'$ and $B'$ are $\omega_1$-saturated and all inclusions are pure. By Lemma 4.1, $A'$ is a direct summand of $B$ and the corollary is proved after a short diagram chase.

For a more standard proof of this corollary, see Fuchs ([15], Theorem 28.5).

If $A$, $A'$, $B$ and $B'$ are groups with $A' \subset A$ and $B' \subset B$ and $f: A' \to B'$ is a homomorphism, then we say that $f$ is a *strong homomorphism* (relative to $A$ and $B$) if for every $a \in A'$ and every $n \in \mathbf{Z}$, $n|a$ in $A$ only if $n|f(a)$ in $B$. $f$ is called a *strong isomorphism* if $f$ is an isomorphism and both $f$ and $f^{-1}$ are strong homomorphisms.

**4.3. Lemma.** *Given homomorphisms $f_i: B \to A_i$, $i = 1, 2$, there exists a group $C$ and homomorphisms $g_i: A_i \to C$ such that*

$$\begin{array}{ccc} A_1 & \xrightarrow{g_1} & C \\ {\scriptstyle f_1}\uparrow & & \uparrow{\scriptstyle g_2} \\ B & \xrightarrow[f_2]{} & A_2 \end{array}$$

*commutes and*

(i) *if* $f_1$ *is an embedding,* $g_2$ *is also;*
(ii) *if, furthermore,* $f_2$ *is a strong homomorphism relative to* $A_1$ *and* $A_2$, $g_2$ *is a pure embedding;*
(iii) *if, furthermore,* $f_2$ *is a strong isomorphism,* $g_1$ *is a pure embedding.*

**Proof.** We take $C$ to be the push-out $A_1 \oplus A_2/\{(f_1(b), -f_2(b)) : b \in B\}$ and $g_i$ to be the inclusion map of $A_i$ in $A_1 \oplus A_2$ followed by the canonical projection on $C$. That $g_2$ is one-one if $f_1$ is one-one is well-known, and, in any case, is easily checked. Now suppose that $f_1$ is an embedding and that $f_2$ is a strong homomorphism relative to $A_1$ and $A_2$ (where we regard $B$ as a subgroup of $A_1$ via $f_1$). We want to prove $g_2$ is a pure embedding. Suppose $n | g_2(a_2)$; then there exists $b \in B$ and $(x, y) \in A_1 \oplus A_2$ such that $n(x, y) = (0, a_2) + (f_1(b), -f_2(b))$. Thus $nx = f_1(b)$, and since $f_2$ is strong, $n$ divides $f_2(b)$. Hence, since $ny = a_2 - f_2(b)$, we conclude that $n$ divides $a_2$. This finishes the proof of (ii), and (iii) follows from (i) and (ii).

Given a group $A$ and names for the members of $A$, we let $D^*(A)$, *the divisibility diagram of* $A$, be the set of all sentences of the form $a + b = c$, $a + b \neq c$, and $n \nmid a$ which are true in $A$.

Let $T$ be the theory of abelian groups. It should be clear that the reducts (to the language of $T$) of models of $T \cup D^*(A)$ are just the pure extensions of $A$.

Given names for the members of a set $S$, a *positive primitive* (*p.p.*) *system* $\Sigma(x)$ *over* $S$ is a set of positive primitive formulas involving the names for the members of $S$ and a single free variable $x$. Given a subset $S$ of a group $A$ and names for the members of $A$, we say that such a system $\Sigma(x)$ over $S$ is *consistent* (relative to $A$) if $T \cup D^*(A) \cup \Sigma(x)$ is consistent, i.e., if $\Sigma(x)$ is realized in a pure extension of $A$.

Note that by the compactness theorem such a p.p. system $\Sigma(x)$ over $S$ is consistent relative to $A$ if and only if the existential closure of the conjunction of every finite subset of $\Sigma(x)$ is true in $A$; and moreover, if consistent, $\Sigma(x)$ can be extended to a maximal consistent p.p. system over $S$ relative to $A$.

**4.4. Lemma.** *If $A \subset B$ are groups such that every maximal consistent p.p. system over $A$ relative to $B$ is realized in $A$, then $A$ is a direct summand of $B$.*

**Proof.** Consider the set of strong homomorphisms $f$ from subgroups of $B$ into $A$ which extend the identity map on $A$. Clearly a maximal member $f$ of this set exists. We need only show that the domain of $f$ is $B$.

If not, let $D$ be the domain of $f$ and choose $b \in B - D$. Let $\Sigma(x)$ be the set of positive primitive formulas true of $b$ with parameters in $D$. Let $\Sigma'(x)$ be the set of formulas obtained from those in $\Sigma(x)$ by replacing each occurrence of a name for a member of $D$ by a name for its image under $f$. Let $\mathcal{F}(x)$ be a finite subset of $\Sigma'(x)$. Since $\mathcal{F}'(x)$ arises from a finite subset $\mathcal{F}(x)$ of $\Sigma$ and $\exists x \bigwedge \mathcal{F}(x)$ is true in $B$ and since $f$ is strong, $\exists x \bigwedge \mathcal{F}'(x)$ is true in $B$. Thus $\Sigma'(x)$ is consistent relative to $B$.

Let $\Sigma''(x)$ be a maximal consistent p.p. system over $A$ relative to $B$ which extends $\Sigma'(x)$. By assumption there exists $a \in A$ which realizes $\Sigma''(x)$ and thus $\Sigma'(x)$. We can now define $g : D + \mathbf{Z}b \to A$ according to $g(d + rb) = f(d) + ra$. We need only check that $g$ is well-defined and strong, since it will then be clear that $g$ is a homomorphism which contradicts the maximality of $f$.

Suppose that $d_1 + r_1 b = d_2 + r_2 b$. Then $d_1 - d_2 + (r_1 - r_2)b = 0$, so that "$d_1 - d_2 + (r_1 - r_2)x = 0$" belongs to $\Sigma(x)$. Thus "$f(d_1) - f(d_2) + (r_1 - r_2)x = 0$" belongs to $\Sigma'(x)$, i.e., $f(d_1) + r_1 a = f(d_2) + r_2 a$ and $g$ is well-defined.

Finally suppose that $n | d + rb$, i.e., "$\exists y(ny - d - rx = 0)$" belongs to $\Sigma(x)$. This implies that "$\exists y(ny - f(d) - rx = 0)$" belongs to $\Sigma'(x)$, i.e., $n | f(d) + ra$.

**4.5. Lemma.** *Let $S$ be a subgroup of a group $A$ and let $\Sigma(x)$ be a consistent p.p. system over $S$ relative to $A$. There exists a countable subsystem $\Sigma'(x)$ of $\Sigma(x)$ such that if an element $b$ realizes $\Sigma'(x)$ in a pure extension $B$ of $A$, then $b$ realizes $\Sigma(x)$ in $B$.*

**Proof.** Let us call two formulas in $\Sigma(x)$ equivalent if they differ only in the choice of parameters from $S$. There are clearly only countably many equivalence classes, and we let $\Sigma'(x)$ be obtained by choosing a representative for each equivalence class from $\Sigma(x)$.

Let $b$ realize $\Sigma'(x)$ in a pure extension $B$ of $A$. Consider a formula

$\varphi(x)$ from $\Sigma(x)$ and the equivalent formula $\chi(x)$ from $\Sigma'(x)$. Since $\Sigma(x)$ is consistent, there exists $a \in A$ such that $\varphi(a)$ and $\chi(a)$ are true in $A$. Now let us assume that $\varphi(x)$ is of the form:

$$\exists y_1, ..., y_n \left( \bigwedge_{j=1}^{k} \left( \sum_{i=1}^{n} r_j^i y_i + rx = c_j \right) \right),$$

while $\chi(x)$ is of the form:

$$\exists y_1, ..., y_n \left( \bigwedge_{j=1}^{k} \left( \sum_{i=1}^{n} r_j^i y_i + rx = d_j \right) \right),$$

where the $c_j$'s and $d_j$'s name elements of $S$.

Putting everything together we can write.

(i) $$\exists y_1, ..., y_n \left( \bigwedge_{j} \left( \sum_{i} r_j^i y_i + rb = d_j \right) \right),$$

(ii) $$\exists y_1, ..., y_n \left( \bigwedge_{j} \left( \sum_{i} r_j^i y_i + ra = c_j \right) \right),$$

and

(iii) $$\exists y_1, ..., y_n \left( \bigwedge_{j} \left( \sum_{i} r_j^i y_i + ra = d_j \right) \right).$$

(i) and (iii) imply:

(iv) $$\exists y_1, ..., y_n \left( \bigwedge_{j} \left( \sum_{i} r_j^i y_i + r(b-a) = 0 \right) \right);$$

and (ii) and (iv) imply:

(v) $$\exists y_1, ..., y_n \left( \bigwedge_{j} \left( \sum_{i} r_j^i y_i + rb = c_j \right) \right),$$

i.e., $\varphi(b)$ is true.

**4.6. Lemma.** *If $A$ and $B$ are $\omega_1$-saturated, $A' \subset A$, $B' \subset B$, and $f: A' \to B'$ is a strong isomorphism, then there exist direct summands $A''$ and $B''$ of $A$ and $B$ respectively and an isomorphism $g: A'' \to B''$ which extends $f$. $A''$ and $B''$ may be chosen to have cardinality* $\leq \mathrm{Card}(A')^{\aleph_0}$.

**Proof.** If $A'$ is not a direct summand of $A$, then by Lemma 4.4, there exists a p.p. system $\Sigma(x)$ over $A'$ which is maximal consistent relative to $A$ and which is not realized in $A'$. By Lemma 4.1 it is clear that there must exist a member $a$ of $A$ which realizes $\Sigma(x)$.

If $\Sigma'(x)$ is obtained from $\Sigma(x)$ by replacing each occurrence of a name for a member of $A'$ by a name for its image under $f$, we see by a repetition of the argument in the proof of Lemma 4.4 that $\Sigma'(x)$ is consistent relative to $B$. Thus by Lemma 4.1, there exists $b \in B$ which realizes $\Sigma'(x)$.

We claim that $\Sigma'(x)$ is the set of *all* positive primitive formulas over $B'$ true of $b$. If not, the amalgamation property expressed in Lemma 4.3 would give us a contradiction to the maximality of $\Sigma(x)$.

If we now set $f_1(a' + ra) = f(a') + rb$, we see by a repetition of the remainder of the argument in the proof of Lemma 4.4 that $f_1$ is a strong isomorphism property extending $f$.

We can continue this process of extending $f$ (by transfinite induction, taking unions at limit ordinals) until a direct summand $A''$ of $A$ is obtained. The cardinality of $A''$ must be $\leq \mathrm{Card}(A)^{\aleph_0}$, since every maximal consistent system is equivalent by Lemma 4.5 to a countable subsystem and the cofinality of $\mathrm{Card}(A')^{\aleph_0}$ is $> \aleph_0$.

We claim finally that the range of the map $g$ which we have obtained is a direct summand $B''$ of $B$. Otherwise there would be a maximal consistent p.p. system over $B''$ of $B$ which would give rise by Lemma 4.3 again to a corresponding system over $A''$, contradicting the fact that $A''$ is a direct summand of an $\omega_1$-saturated group.

**4.7. Lemma.** *If $A_1 = B_1 \oplus C_1$ is an uncounatble saturated group with* $\mathrm{Card}(B_1) < \mathrm{Card}(A_1)$, *and if $A_1 \cong A_2$ and $B_1 \cong B_2$ with $A_2 = B_2 \oplus C_2$, then $C_1 \cong C_2$.*

**Proof.** We will show that $C_1$ and $C_2$ are saturated and elementarily equivalent. Let $d$ stand for the dimension of one of the vector spaces which we associate with abelian groups. Since these dimensions add over direct sums, we have $d_{B_1} + d_{C_1} = d_{A_1} = d_{A_2} = d_{B_2} + d_{C_2}$. If $d_{A_1}$ is finite, we can cancel and obtain $d_{C_1} = d_{C_2} < \aleph_0$. If $d_{A_1}$ is infinite, it must be the cardinality of $A_1$ while $d_{B_1} = d_{B_2}$ is $<$ Card($A_1$), since $A_1$ is saturated and Card($B_1$) $<$ Card($A_1$). Now we can conclude $d_{C_1} = d_{C_2}$ = Card($C_1$) = Card($C_2$).

Clearly $C_1$ and $C_2$ are saturated and elementarily equivalent, hence isomorphic.

**4.8. Theorem.** *If $A$ is elementarily equivalent to $B$, $a_1, ..., a_n \in A$, $f$ is a strong isomorphism from the subgroup generated by $\{a_1, ..., a_n\}$ into $B$, and $\varphi(x_1, ..., x_n)$ is an arbitrary formula in the language of abelian groups, then $\varphi(a_1, ..., a_n)$ is true in $A$ if and only if $\varphi(f(a_1), ..., f(a_n))$ is true in $B$.*

**Proof.** By first taking elementary extensions, we may assume that $A$ and $B$ are isomorphic saturated groups of cardinality $> 2^{\aleph_0}$. By Lemma 4.6 $f$ can be extended to an isomorphism between direct summands of $A$ and $B$ of cardinality at most $2^{\aleph_0}$. By Lemma 4.7 $g$ can be further extended to an isomorphism between $A$ and $B$. The conclusion is immediate.

We now introduce an expansion of the language of group theory with appropriate new axioms. For each core sentence $\sigma$ (see § 2), we introduce a propositional constant $p_\sigma$ together with the axiom $p_\sigma \longleftrightarrow \sigma$. For each formula of the form $p^k | x$ (with $p$ a prime), we introduce a unary relation symbol $D_{p,k}(x)$ together with the axiom $\forall x (D_{p,k}(x) \longleftrightarrow p^k | x)$. Let us call this the *extended language of abelian groups*.

Since every abelian group has a (unique) expansion to a model of the extended theory, the new theory $T'$ is an inessential extension of $T$, in the sense that no new theorems in the language of $T$ are derivable. We will call $T'$ the *extended theory of abelian groups*.

Let $A$ be a group and let $a_1, ..., a_n \in A$. Let us call the set of all (open) formulas of the extended language of abelian groups which are true of $\langle a_1, ..., a_n \rangle$ in the unique expansion of $A$ *the extended theory (respectively, extended open theory)* of $\langle A, a_1, ..., a_n \rangle$.

**4.9. Theorem.** *Given a group $A$ and elements $a_1, ..., a_n$ of $A$, the extended theory of $\langle A, a_1, ..., a_n \rangle$ is equivalent to the union of the extended theory of abelian groups and the extended open theory of $\langle A, a_1, ..., a_n \rangle$.*

**Proof.** We need only show that if $A$ and $A'$ are groups with $a_1, \ldots, a_n \in A$ and $a'_1, \ldots, a'_n \in A'$ such that the extended open theory of $\langle A, a_1, \ldots, a_n \rangle$ is the same as that of $\langle A', a'_1, \ldots, a'_n \rangle$, then the (full) extended theories are the same. But using open formulas we can say that $A \equiv B$ and that the corresponding members $\sum_{i=1}^n r_i a_i$ and $\sum_{i=1}^n r_i a'_i$ of the subgroups generated by $\{a_1, \ldots, a_n\}$ and $\{a'_1, \ldots, a'_n\}$ have the same divisibility properties. Theorem 4.8 then yields the desired conclusion.

We can now state and prove our elimination-of-quantifiers theorem.

**4.10. Theorem.** *Every formula in the extended language of abelian groups is equivalent relative to the extended theory of abelian groups to an open formula.*

**Proof.** Let $\varphi(x_1, \ldots, x_n)$ be an arbitrary formula of the extended language. Let $\Phi$ be the set of all open formulas $\chi(x_1, \ldots, x_n)$ such that $T' \vdash \forall x_1, \ldots, x_n(\chi(x_1, \ldots, x_n) \to \varphi(x_1, \ldots, x_n))$. If $\varphi$ is not equivalent relative to $T'$ to one of the members of $\Phi$, then since $\Phi$ is closed under disjunctions, $\varphi(c_1, \ldots, c_n) \cup \{\neg\chi(c_1, \ldots, c_n) : \chi \in \Phi\} \cup T'$ is consistent, where the $c_i$'s are new individual constants. Let $\langle A, a_1, \ldots, a_n \rangle$ be a model for this theory. By Theorem 4.9 if $\Psi$ is the set of all open statements in the extended language of abelian groups which are true of $\langle a_1, \ldots, a_n \rangle$ in $A$, then $\Psi \vdash \varphi(c_1, \ldots, c_n)$. By the compactness theorem, a finite conjunction $\chi(c_1, \ldots, c_n)$ of members of $\Psi$ is such that $T' \vdash \chi(c_1, \ldots, c_n) \to \varphi(c_1, \ldots, c_n)$ and hence $T' \vdash \forall x_1, \ldots, x_n(\chi(x_1, \ldots, x_n) \to \varphi(x_1, \ldots, x_n))$. Thus $\chi \in \Phi$, contradicting the fact that $\langle A, a_1, \ldots, a_n \rangle$ is a model of $\{\neg\chi(c_1, \ldots, c_n) : \chi \in \Phi\}$.

**4.11. Corollary** (Szmielew [12], Theorem 4.22). *Every formula in the language of abelian groups is equivalent relative to the theory of abelian groups to a formula which is a disjunction of conjunctions of core sentences and formulas of the form "$p^k \mid \sum_{i=1}^n r_i x_i$" and "$p^k \mid \sum_{i=1}^n r_i x_i$", where $p$ is a prime and $k$ is a positive integer.*

**Proof.** By Theorem 4.10 every formula is equivalent relative to $T'$ to a formula of the desired type. Since $T'$ is an inessential expansion of $T$, the equivalence is a theorem of $T$.

## §5. Modules over a Dedekind domain

In this section we discuss how to generalize our results to modules over a Dedekind domain. For the reader who is only interested in such a generalization to principal ideal domains, we remark that the only new aspect of the problem is expressed by Lemma 5.1 below; Theorem 5.2 is then a trivial generalization of Theorem 1.11.

We refer the reader to the standard sources [2] and [17] for the fundamental definitions and properties of Dedekind domains. In short, a commutative integral domain $\Lambda$ is a Dedekind domain if and only if one of the following equivalent conditions holds:

(i) every ideal is a product of prime ideals ([17], p. 270),

(ii) every ideal is invertible ([17], p. 275),

(iii) $\Lambda$ is a Noetherian Prüfer ring ([2], pp. 133–134),

(iv) every divisible module is injective ([2], p. 134),

(v) $\Lambda$ is Noetherian, every prime ideal is maximal, and $\Lambda$ is integrally closed in its quotient field ([17], p 275).

Let $\Lambda$ be a fixed Dedekind domain. The language L in which we formulate the elementary theory of modules over $\Lambda$ has a binary function symbol + together with a unary function symbol $f_\lambda$ corresponding to each $\lambda \in \Lambda$. The following axioms are adequate to define the theory $T$ of modules over $\Lambda$:

$$\forall x, y, z((x + y) + z = x + (y + z))$$

$$\forall x, y(x + y = y + x)$$

$$\forall x, y(f_\lambda(x + y) = f_\lambda(x) + f_\lambda(y)),\ \lambda \in \Lambda$$

$$\forall x(f_1(x) = x)$$

$$\forall x, y(f_0(x) = f_0(y))$$

$$\forall x(f_{\lambda_1+\lambda_2}(x) = f_{\lambda_1}(x) + f_{\lambda_2}(x)),\ \lambda_1, \lambda_2 \in \Lambda$$

$$\forall x(f_{\lambda_1\lambda_2}(x) = f_{\lambda_1}(f_{\lambda_2}(x))),\ \lambda_1, \lambda_2 \in \Lambda .$$

We would now like to describe the complete extensions of $T$. Our method is the same as before, viz., we look at saturated modules, but now we consider $\kappa$-saturated modules for $\kappa > \lambda = \mathrm{Card}(\Lambda) + \aleph_0$. If $M$ is

$\kappa$-saturated, then, of course, it is $\kappa$-equationally-compact, and it can be shown that the $\lambda^+$-equationally compact modules are precisely the algebraically compact or pure-injective modules ([5], p. 178). It is known that any pure-injective module $M$ over $\Lambda$ is the direct sum of a divisible (i.e., injective) module $M_d$ and a product $M_r = \Pi_P \overline{M}_P$ of modules $\overline{M}_P$ over the local rings $\Lambda_P$, where $P$ ranges over the prime (i.e., maximal) ideals of $\Lambda$ and each $\overline{M}_P$ is Hausdorff and complete in the $P$-adic topology ([15], Corollary 8). One may check that $\overline{M}_P = \{x \in M_r : \lambda \text{ divides } x$ for each $\lambda \in \Lambda - P\}$ and that $M_d = \{x \in M : \lambda \text{ divides } x \text{ for all } \lambda \in \Lambda - \{0\}\}$.

Now $\Lambda_P$ is a discrete valuation ring ([17], Vol. II, pp. 38–39), and so the structure theorem of Kaplansky for complete modules applies ([6], Theorem 22), i.e., $\overline{M}_P$ is the completion of a direct sum

$$M_P = \oplus_n \Lambda_P/P^n \Lambda_P{}^{(\alpha_{P,n})} \oplus \Lambda_P{}^{(\beta_P)}$$

of cyclic modules over $\Lambda_P$. Also, $M_d$, the maximal divisible submodule of $M$ is a direct sum

$$\oplus_P E(\Lambda/P)^{(\gamma_P)} \oplus F^{(\delta)}$$

where $E(\Lambda/P)$ denotes the injective envelope of $\Lambda/P = \Lambda_P/P\Lambda_P$ and $F$ the quotient field of $\Lambda$ ([8], 2.5 and 3.1). As before we define

$$\mathrm{U}(P, n; M) = \begin{cases} \dim(P^n M[P]/P^{n+1}M[P]) & \text{if finite} \\ \infty & \text{otherwise} \end{cases}$$

$$\mathrm{Tf}(P; M) = \begin{cases} \text{ev. value } \dim(P^n M/P^{n+1}M) & \text{if finite} \\ \infty & \text{otherwise} \end{cases}$$

$$\mathrm{D}(P; M) = \begin{cases} \text{ev. value } \dim(P^n M[P]) & \text{if finite} \\ \infty & \text{otherwise} \end{cases}$$

where $N[P] = \{x \in N : Px = \{0\}\}$ and "dim" means dimension over $\Lambda/P$.

Note that $\Lambda_P$ is a principal ideal domain and that if $p \in \Lambda_P$ is a generator of the maximal ideal of $\Lambda_P$, then $p\Lambda_P = P\Lambda_P$ and $\dim(P^n M[P]/P^{n+1}M[P])$ $\dim(p^n \overline{M}_P[p]/p^{n+1}\overline{M}_P[p]) = \dim(p^n M_P[p]/p^{n+1}M_P[p])$, since for any $Q \neq P$, $P^{n+1}\overline{M}_Q = \overline{M}_Q$. Similarly for the other dimensions under consideration. Moreover, one can check that Lemmas 1.5–1.10 continue to hold, with appropriate changes in notation. For example (see the paragraph before Lemma 1.6), to check that $P^{k-1}M/P^{k-1}T + P^kM$ has the same dimension as $P^kM/P^kT + P^{k+1}M$ (where $T$ is the torsion submodule of $M$), one should first note that each side is isomorphic to the corresponding quotient with $P$ replaced by $p$, $M$ replaced by $M_P$ and $T$ replaced by $T \cap M_P$, and then use the natural map induced by multiplication by $p$.

However, this is not sufficient for the purposes of characterizing elementary equivalence of modules over $\Lambda$. The additional complication is due to the fact that if $\Lambda/P = \Lambda_P/P\Lambda_P$ is infinite, then there is *no* elementary formula $\varphi(x, y)$ which says that $x$ and $y$ are independent modulo $P$. (We would need an infinite conjunction of the formulas $\lambda_1 x + \lambda_2 y \neq 0$ where $(\lambda_1, \lambda_2)$ ranges over all pairs such that $\lambda_1 \notin P$ or $\lambda_2 \notin P$). Thus the invariants $\mathrm{U}(P, n; M)$, $\mathrm{Tf}(P; M)$, $\mathrm{D}(P; M)$ are not necessarily elementarily definable. However we can say elementarily that

$$\dim(P^n M[P]/P^{n+1}M[P]) \neq 0,$$

$$\dim(P^n M/P^{n+1}M) \neq 0,$$

$$\dim(P^n M[P] \neq 0.$$

It turns out that for saturated modules this is enough to determine their structure up to isomorphism. Indeed, we have the following results.

**5.1. Lemma.** *Let $\Lambda$ be a Dedekind domain and $P$ a prime ideal such that $\Lambda/P$ is infinite. Let $M$ be an $\omega$-saturated $\Lambda$-module. Then* $\mathrm{U}(P, n; M)$, $\mathrm{Tf}(P; M)$ *and* $\mathrm{D}(P; M)$ *are either* 0 *or* $\infty$.

**Proof.** We prove that $\mathrm{D}(P; M) = 0$ or $\infty$; the other proofs are similar. It suffices to prove for any $k$ that $\dim(P^k M[P]) \neq 0$ implies $\dim(P^k M[P]) \geq \aleph_0$.

In the following, let us write $I^n$ for the $n$-th power of an ideal $I$ and

$I^{(n)}$ for the set of $n$-tuples from $I$. Now let $\alpha_1, ..., \alpha_r$ be a basis of $P^k$. It suffices to prove that for any $n$ the set of formulas

$$\mathcal{F}_n = \{\exists y_1, ..., y_r \left(\sum_{j=1}^{r} \alpha_j y_j = x_i\right) : i = 1, ..., n\}$$

$$\cup \{\lambda x_i = 0 : \lambda \in P, i = 1, ..., n\}$$

$$\cup \{\sum_{i=1}^{n} \lambda_i x_i \neq 0 : (\lambda_1, ..., \lambda_n) \in \Lambda^{(n)} - P^{(n)}\}$$

in the free variables $x_1, ..., x_n$ is finitely satisfiable, hence satisfiable, in $M$. But given a finite number of $n$-tuples $(\lambda_1^{(1)}, ..., \lambda_n^{(1)}), ..., (\lambda_1^{(t)}, ..., \lambda_n^{(t)})$ in $\Lambda^{(n)} - P^{(n)}$, it follows as in the proof of Lemma 1.10 that there exist $\rho_1, ..., \rho_n \in \Lambda$ such that $\sum_{i=1}^{n} \lambda_i^{(j)} \rho_i \notin P$ for $j = 1, ..., t$. Let $a$ be a non-zero element of $P^k M[P]$ and let $x_i = \rho_i a$. Then $\sum_{i=1}^{n} \lambda_i^{(j)} x_i = (\sum_{i=1}^{n} \lambda_i^{(j)} \rho_i) a \neq 0$ for $j = 1, ..., t$, so $\mathcal{F}_n$ is finitely satisfiable.

If $P$ is a prime ideal of $\Lambda$ such that $\Lambda/P$ is infinite, define

$$U^*(P, n; M) = \begin{cases} 0 \text{ if } U(P, n; M) = 0 \\ \infty \quad \text{otherwise} \end{cases}$$

$$Tf^*(P; M) = \begin{cases} 0 \text{ if } Tf(P; M) = 0 \\ \infty \quad \text{otherwise} \end{cases}$$

$$D^*(P; M) = \begin{cases} 0 \text{ if } D(P; M) = 0 \\ \infty \quad \text{otherwise}. \end{cases}$$

If $\Lambda/P$ is finite, let $U^*(P, n; M) = U(P, n; M)$, $Tf^*(P; M) = Tf(P; M)$, $D^*(P; M) = D(P; M)$. We say $M$ is *of bounded order* if there exists $0 \neq \lambda \in \Lambda$ such that $\lambda M = 0$.

Then we have:

**5.2. Theorem.** *Let $\Lambda$ be a Dedekind domain and $M$ a $\kappa$-saturated $\Lambda$-module, where $\kappa > \mathrm{Card}(\Lambda) + \aleph_0$. Then*

$$M \cong \Pi_P \bar{M}_P \oplus M_d$$

*where $\bar{M}_P$ is the completion in the P-adic topology of*

$$M_P = \oplus_n (\Lambda/P^n)^{(\alpha_{P,n})} \oplus \Lambda_P^{(\beta_P)}$$

*and*

$$M_d = \oplus_P E(\Lambda/P)^{(\gamma_P)} \oplus F^{(\delta)}$$

*where $F$ = quotient field of $\Lambda$, and*

$$\alpha_{P,n} \begin{cases} = \mathrm{U}^*(P, n-1; M), & \text{if finite} \\ \geq \kappa, & \text{otherwise} \end{cases}$$

$$\beta_P \begin{cases} = \mathrm{Tf}^*(P; M), & \text{if finite} \\ \geq \kappa, & \text{otherwise} \end{cases}$$

$$\gamma_P \begin{cases} = \mathrm{D}^*(P; M), & \text{if finite} \\ \geq \kappa, & \text{otherwise} \end{cases}$$

*and*

$$\delta \begin{cases} = 0, & \text{if } M \text{ is of bounded order} \\ \geq \kappa, & \text{otherwise} \end{cases}$$

**5.3. Corollary.** *Let $\Lambda$ be a Dedekind domain. If $M_1$ and $M_2$ are $\Lambda$-modules, then $M_1$ is elementarily equivalent to $M_2$ if and only if for every prime $P$ and every $n \geq 0$*

$$\mathrm{U}^*(P, n; M_1) = \mathrm{U}^*(P, n; M_2)$$

$$\mathrm{Tf}^*(P; M_1) = \mathrm{Tf}^*(P; M_2)$$

$$\mathrm{D}^*(P; M_1) = \mathrm{D}^*(P; M_2)$$

*and $M_1$ and $M_2$ are both of bounded order or both are not of bounded order.*

(The corollary follows from the theorem given the existence of saturated elementary extensions of $M_1$ and $M_2$. This assumption can be avoided in a number of ways as has been indicated previously.)

We can use Corollary 5.3 to prove decidability results as we did in §2. Of course, we will need to assume that $\Lambda$ is "effectively given" in some sense. We define: $\Lambda$ is *computable* if $\Lambda$ is finite or there is a bijection between $\Lambda$ and the natural numbers under which the operations on $\Lambda$ correspond to recursive functions of natural numbers (cf. [10]).

**5.4. Theorem.** *Let $\Lambda$ be a computable Dedekind domain such that:* (i) *there is a recursive enumeration of the finite sets of generators of prime ideals of $\Lambda$; and* (ii) *for every prime ideal $P$, we can decide whether the residue class field $\Lambda/P$ is finite or not, and if it is finite, we can effectively choose a set of representatives of $\Lambda/P$. Then the theory of $\Lambda$-modules is decidable.*

**Proof.** The proof of Theorem 5.4 proceeds like the proof of Theorem 2.11. That is to say, we first prove, using Corollary 5.3, that any consistent sentence of the theory of $\Lambda$-modules is true in a module of the form

$$M = \oplus_P \{\oplus_n \Lambda_P/P^n\Lambda_P{}^{(\alpha_{P,n})} \oplus \Lambda_P{}^{(\beta_P)} \oplus E(\Lambda/P)^{(\gamma_P)}\} \oplus F^{(\delta)}$$

where only a finite number of the coefficients $\alpha_{P,n}$, $\beta_P$, $\gamma_P$ and $\delta$ are nonzero and all are finite. Using hypothesis (i) above we can effectively enumerate all such $M$, and for each $M$, because $\Lambda$ is computable and satisfies (ii), we can effectively enumerate all the sentences true in $M$ (i.e. all the sentences which are consequences of the complete axiomatization of Th($M$) which Corollary 5.3 yields).

*Examples.* Let us look at some examples of rings $\Lambda$ such that the hypotheses of Theorem 5.4 are satisfied.

(1)(1) If $K$ is a finite field, then $\Lambda = K[X]$ is a computable P.I.D. such that (i) the prime elements (= irreducible polynomials of $\Lambda$) are effectively enumerable (because there are only a finite number of polynomials of each degree) and (ii) if $f(X)$ is irreducible of degree $m$ the polynomials of degree $< m$ represent all the non-zero elements of $\Lambda/\langle f(x)\rangle$.

(2) If $K$ is a computable infinite field with a splitting algorithm ([10],

Definition 9) – for example, $K = Q$ ([14], §25) or $K$ = any countable algebraically closed field ([10], Theorem 7) – then $\Lambda = K[X]$ is a computable P.I.D. such that (i) is satisfied; and (ii) the residue class fields are all infinite.

(3) Let $\Lambda$ be the ring of integers in a quadratic number field $\mathbf{Q}(\sqrt{m})$ ($m$ square-free). Thus ([13], Theorem 6-1-1) $\Lambda = \mathbf{Z}\cdot 1 + \mathbf{Z}\sqrt{m}$ if $m \equiv 1 \pmod 4$; and $\Lambda = \mathbf{Z}\cdot 1 + \mathbf{Z}(\frac{1}{2} + \frac{1}{2}\sqrt{m})$ if $m = 2$ or $3 \pmod 4$; clearly $\Lambda$ is computable. Moreover, because of ([16], Theorem 6-2-1), we can effectively enumerate the prime ideals $P$ of $\Lambda$ in such a way that we can effectively determine $P \cap \mathbf{Z} = p\mathbf{Z}$. Then the residue class field $\Lambda/P = (\mathbf{Z}/p\mathbf{Z})[X]/(X^2 - m)$ (except possibly for $p = 2$, which case can be handled separately, see ([16], p. 235)) and clearly we can choose a set of representatives.

(4) More generally, if $K = Q(\theta)$ is an algebraic number field of degree $n/\mathbf{Q}$ such that the ring of integers $\Lambda$ of $K$ is $\mathbf{Z}\cdot 1 + \mathbf{Z}\cdot\theta + \ldots + \mathbf{Z}\cdot\theta^{n-1}$ then a theorem of Kummer ([16], Theorem 4-9-1) enables us to effectively enumerate the primes of $\Lambda$ and to determine the residue class fields.

Let L and $T$ be as in the third paragraph of this section. It is easy to see that the results (and proofs) of section 4 generalize readily to Dedekind domains. For this purpose, let us define the core sentences of L to be those which express the following assertions:

$$\dim(P^{k-1}m[P]/P^kM[P]) \leq n,$$

$$\dim(P^{k-1}M/P^kM) \leq n,$$

$$\dim(P^{k-1}M[P]) \leq n\ ,$$

$$\forall x(\lambda x = 0)\ ,$$

for $\lambda \in \Lambda$, $P$ a prime ideal, $k$ a positive integer, and if $\Lambda/P$ is infinite $n = 0$, otherwise $n$ a non-negative integer.

For each prime ideal $P$, let $\varphi_{P,k}(x)$ express the relation $x \in P^kM$.

Let L′ be the language which has a propositional constant $p_\sigma$ corresponding to each core sentence $\sigma$ and a unary relation symbol $D_{P,k}(x)$

corresponding to each formula $\varphi_{P,k}$. Let $T'$ be the extension of $T$ in the language L′ with the following additional axioms:

$$p_\sigma \leftrightarrow \sigma$$

$$\forall x\,(\mathrm{D}_{P,k}(x) \leftrightarrow \varphi_{P,k}(x))$$

**5.5. Theorem.** *Let* L′ *and* $T'$ *be as above. Then* $T'$ *is an inessential extension of* $T$ *such that every formula of* L′ *is* $T'$*-equivalent to an open formula.*

**Proof.** As in section 4.

Let $N$ be an infinite $\Lambda$-module. We consider the problem of constructing a saturated model of Th($N$) of a given cardinality. Define

$$\mathcal{S}(N) = \{\kappa : \kappa > \mathrm{Card}(\Lambda) \text{ and there exists a saturated model of } \mathrm{Th}(N) \text{ of cardinality } \kappa\}.$$

Let

$$\mathcal{N}_1 = \{P : \mathrm{Tf}^*(P;N) = \infty \;\text{ or }\; \exists n \text{ s.t. } \mathrm{U}^*(P,n;N) = \infty\},$$

$$\mathcal{N}_2 = \{P : P \notin \mathcal{N}_1,\ \mathrm{Tf}^*(P;N) \neq 0 \;\text{ or }\; \exists n \text{ s.t. } \mathrm{U}^*(P,n;N) \neq 0\},$$

and let $\eta_i = \mathrm{Card}(\mathcal{N}_i)$.

**5.6. Theorem.** (i) *If the reduced part of* $N$ *is of bounded order, then* $\mathcal{S}(N) = \{\kappa : \kappa > \mathrm{Card}(\Lambda)\}$

(ii) *If the reduced part of* $N$ *is not of bounded order and if* $\exists P$ *such that* $\mathrm{Tf}^*(P;N) = \infty$ *or there exist infinitely many* $P$ *such that* $\exists n$ *such that* $\mathrm{U}^*(P,n;N) = \infty$, *then* $\mathcal{S}(N) = \{\kappa : \kappa \geq 2^{\eta_2}$ *and* $\kappa^{\eta_1 + \aleph_0} = \kappa\}$

(iii) *If* $N$ *does not satisfy* (i) *or* (ii), *then* $\mathcal{S}(N) = \{\kappa : \kappa \geq 2^{\eta_2 + \aleph_0}\}$ .

**Proof.** Since the idea of the proof is the same as that for abelian groups in §3 we will content ourselves with a sketch, indicating mainly the differences with the case of **Z**-modules. We consider

$$S_{N\kappa} = \Pi_P \bar{S}_P \oplus \oplus_P\, E(\Lambda)P)^{(\gamma_P)} \oplus F^{(\delta)}$$

where

$$S_P = \oplus_n \Lambda/P^{n^{(\alpha_{P,n})}} \oplus \Lambda_P{}^{(\beta_P)}$$

where

$$\alpha_{P,n} = \min\{U^*(P, n-1; N), \kappa\}$$

$$\beta_P = \min\{Tf^*(P; N), \kappa\}$$

$$\gamma_P = \min\{D^*(P; N), \kappa\}$$

$$\delta = \begin{cases} 0 & \text{if } N \text{ has bounder order} \\ \kappa & \text{otherwise.} \end{cases}$$

As in §3 we have that Th($N$) has a saturated model of cardinality $\kappa \Longleftrightarrow \mathrm{Card}(S_{N,\kappa}) = \kappa$.

(i) If the reduced part of $N$ is of bounded order then there are only a finite number of $P$ such that $\bar{S}_P \neq 0$, and for each such $P$,

$$S_P = \bar{S}_P = \bigoplus_{n \leq N_P} \Lambda/P^{n^{(\alpha_{P,n})}}.$$

Clearly $|S_{N,\kappa}| \leq \kappa$, and, since $N$ is infinite, $|S_{N,\kappa}| = \kappa$.

(ii) Let $\mathcal{I} = \{P : \Lambda/P \text{ is infinite}\}$. Notice that $\mathcal{I} \cap \mathcal{N}_2 = \emptyset$, so that if $P \in \mathcal{N}_2$, then Card $\bar{S}_P \leq 2^{\aleph_0}$. [To see this, note that if $\Lambda/P$ is finite, the completion $\bar{\Lambda}_P$ of $\Lambda_P$ has cardinality $2^{\aleph_0}$ because every element of $\bar{\Lambda}_P$ is uniquely represented in the form

$$\sum_{i \geq N} s_i p^i$$

where $s_i \in S$ = a complete set of representatives of $\Lambda/P$, and $P\Lambda_P = p\Lambda_P$ ([16], Theorem 1-9-1, p. 35). Also note that the torsion part of $S_P$ is countable and of finite length]. If $Tf^*(P; N) = \infty$ then since the completion of $\Lambda_P{}^{(\kappa)}$ has cardinality $\kappa^{\aleph_0}$ (cf. §3), $|\bar{S}_P| = \kappa^{\aleph_0}$. By considering all the possibilities for Card$\{P \in \mathcal{N}_1 : Tf^*(P; N) = \infty\}$ and Card$\{P \in \mathcal{N}_1 : Tf^*(P; N) \neq \infty\}$, one can prove that $\mathrm{Card}(\Pi_{P \in \mathcal{N}_1} \bar{S}_P) = \kappa^{\eta_1 + \aleph_0} \geq 2^{\aleph_0}$.

Also it is not hard to see that if $\eta_2$ is infinite, $\mathrm{Card}(\Pi_{P\in\mathcal{N}_2}\bar{S}_P) = 2^{\eta_2}$ (because each $\bar{S}_P$ for $P\in\mathcal{N}_2$ satisfies $0 < \mathrm{Card}\,\bar{S}_p \leq 2^{\aleph_0}$). Therefore $\mathrm{Card}(\Pi_P\bar{S}_P) = \kappa^{\eta_1+\aleph_0} + 2^{\eta_2}$.

(iii) We have already noted that $\mathrm{Card}(\bar{S}_P) \leq 2^{\aleph_0}$ for $P$ in $\mathcal{N}_2$. Also, since we are assuming that $\Pi_P\bar{S}_P$ is not of bounded order, either there exists $P$ such that $\mathrm{Tf}^*(P;N) \neq 0$ or there exist infinitely many $P$ such that $\bar{S}_P \neq 0$. Using these observations and considering all the possibilities for $\mathrm{Card}\,\{P\in\mathcal{N}_2 : \mathrm{Tf}(P;N) \neq 0\}$ and $\mathrm{Card}\,\{P\in\mathcal{N}_2 : \mathrm{Tf}(P;N) = 0\}$ one can prove that $\mathrm{Card}(\Pi_{P\in\mathcal{N}_2}\bar{S}_P) = 2^{\eta_2+\aleph_0}$. Because we are in case (iii) $\mathcal{N}_1$ is a finite set of $P$ such that $\mathrm{Tf}^*(P;N) = 0$ and $\exists n$ such that $\mathrm{U}^*(P,n;N) = \infty$. Therefore $\mathrm{Card}(\Pi_{P\in\mathcal{N}_1}\bar{S}_P) = \kappa$. So $\mathrm{Card}(S_{N,\kappa}) = \kappa + 2^{\eta_2+\aleph_0}$.

**Added in proof.** An earlier attempt to give a new proof of some of Szmielew's results was made by Kargapolov [6a]. This work made use of Robinson's test for model-completeness, but it was later observed that the proof contained an error (see the review by Mennicke [8a]). A more recent paper by Kozlov and Kokorin [6b] makes use of Robinson's test to give a proof of a generalization of Szmielew's criterion for elementary equivalence. We have not checked the details of the proof, but it appears to avoid Kargapolov's error.

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