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Some properties on quadratic infinite programs of integral type

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Abstract

In this work, we investigate the properties of a class of quadratic infinite programs where the objective is a quadratic functional of integral type and the feasible region is a subset of the infinite dimensional space $L^p([0, 1])$. We first derive a dual problem of the primal problem to demonstrate that there is no duality gap between them. Then we prove that the objective function depends continuously on the design function. Two existence theorems for this kind of optimization problem are presented. These theoretical results may prove useful in the design of efficient algorithms for this class of infinite programming problem.

Keywords: Infinite programs; Existence of solutions; Quadratic programs; Duality gap

1. Introduction

Recently, the importance of quadratic semi-infinite programming (QSIP) has been well recognized. In [3,5–9] and [11] etc., the authors developed a series of theoretical results, proposed many efficient algorithms and found a lot of applications in the field of engineering for QSIP. For comprehensive survey on the early contribution in this connection, one can refer to [13]. As a natural extension of QSIP, we consider the following infinite dimensional quadratic programming problem:

$$\begin{aligned}(QIP):\quad &\min_h \frac{1}{2}\int_0^1\int_0^1 f(s,t)h(s)h(t)\,\mathrm{d}s\,\mathrm{d}t + \int_0^1 c(s)h(s)\,\mathrm{d}s\\ &\text{s.t.}\quad \int_0^1 \phi(s,y)h(s)\,\mathrm{d}s \le g(y),\quad \forall y\in Y=[0,T],\\ &h\in L^p([0,1]),\quad h\ge 0, a.e.,\end{aligned} \tag{1}$$

where $h \in L^p([0, 1])$ $(1 < p < \infty)$ is the design (or control) function, and $f : [0, 1] \times [0, 1] \to R$, $c : [0, 1] \to R$, $\phi : [0, 1] \times [0, T] \to R$ and $g : [0, T] \to R$ are given continuously differentiable functions.

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In fact, the above infinite programs of integral type belong to the functional optimization problems, which can model a variety of tasks arising in emerging fields of interest in Operations Research (see [2,10,12,16]). One can find interesting examples and more detailed knowledge in [1,4,17] and the references therein.

We say that f is symmetric and positive definite with respect to the cone

$$\{h \in L^p([0,1]) : h \geq 0\}$$

if and only if f has the following properties:

(1) $f(s,t) = f(t,s)$, for any $(s,t) \in [0,1] \times [0,1]$.
(2) For all $h(\in L^p([0,1])) > 0$,

$$\int_0^1 \int_0^1 f(s,t)h(s)h(t)\,\mathrm{d}s\,\mathrm{d}t > 0. \tag{2}$$

Throughout this work, we assume that f is symmetric and positive definite.

Obviously, if $h \in L^p([0,1])$ is restricted to being a piecewise constant function, then the problem (1) turns out to be an ordinary quadratic semi-infinite programming except for the additional bound constraints $h \geq 0$.

Because both the dimension of the design variables and the number of the constraints are infinite, the above problem QIP (1) is a complicated nonconvex optimization problem even if the objective is positive definite. Hence, the question of the existence of the solutions to (1) is usually not easy to answer. To our knowledge, there is no specific consideration for this kind of mathematical programming problem in the literature.

In this work, we first intend to derive a dual problem of (1) such that there is no duality gap between them. Then we will prove that the objective function of integral type in (1) has some continuity in the space $L^p([0,1])$ in the next section. Two existence theorems for this kind of optimization problem will be presented in Section 3. Final remarks are given in the last section.

We introduce some notation as follows. Denote by $C(Y)$ the Banach space of all continuous real functions on Y equipped with the supremum norm, and by $M(Y)$ the space of all signed finite regular Borel measures on Y. It is known that $M(Y)$ is the dual space of $C(Y)$. Denote by $M^+(Y)$ the cone consisting of all nonnegative elements of $M(Y)$.

2. Duality and continuity

Before we derive the duality theorem, we first state the Slater constraint qualification, which we assume problem (1) to be subject to.

We say that the Slater constraint qualification holds in the problem (1) if there exist some $\bar{h} \in L^p([0,1])$ and $\bar{h} \geq 0$ such that

$$\int_0^1 \phi(s,y)\bar{h}(s)\,\mathrm{d}s < g(y), \quad \text{for all } y \in Y = [0,T]. \tag{3}$$

Associated with the problem (1), let us define the Lagrangian $\mathcal{L} : L^p([0,1]) \times L^q([0,1]) \times M^+(Y) \to R$:

$$\begin{aligned}\mathcal{L}(h,\lambda,\mu) = {} & \frac{1}{2}\int_0^1 \int_0^1 f(s,t)h(s)h(t)\,\mathrm{d}s\,\mathrm{d}t + \int_0^1 c(s)h(s)\,\mathrm{d}s - \int_0^1 \lambda(s)h(s)\,\mathrm{d}s \\ & + \int_Y \left[\int_0^1 (\phi(s,y)h(s)\,\mathrm{d}s - g(y))\right]\mathrm{d}\mu(y)\end{aligned} \tag{4}$$

where q is a constant scalar satisfying

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{5}$$

and $\lambda : [0,1] \to R$ and $\mu \in M^+(Y)$ are the so called Lagrangian multipliers corresponding to the ordinary and infinite dimensional constraints in (1), respectively.

If (4) is equivalently rewritten into the following form:

$$\mathcal{L}(h,\lambda,\mu) = -\frac{1}{2}\int_0^1\int_0^1 f(s,t)h(s)h(t)\,\mathrm{d}s\,\mathrm{d}t - \int_Y g(y)\,\mathrm{d}\mu(y) + \int_0^1 h(s)\left[\int_0^1 f(s,t)h(t)\,\mathrm{d}t + c(s) - \lambda(s) + \int_Y \phi(s,y)\,\mathrm{d}\mu(y)\right]\mathrm{d}s,$$

then the variation of $\mathcal{L}$ with respect to h is

$$\nabla_h\mathcal{L}(h,\lambda,\mu) = \int_0^1 f(s,t)h(t)\,\mathrm{d}t + c(s) - \lambda(s) + \int_Y \phi(s,y)\,\mathrm{d}\mu(y), \quad s \in [0,1]. \tag{6}$$

Therefore, on the basis of the idea of Dorn's dual (see [14,15]), we get the dual problem of (1) as follows:

$$\begin{aligned}(DQIP):\ & \min_{h,\lambda,\mu} \frac{1}{2}\int_0^1\int_0^1 f(s,t)h(s)h(t)\,\mathrm{d}s\,\mathrm{d}t + \int_Y g(y)\,\mathrm{d}\mu(y)\\ & \text{s.t.}\quad \int_0^1 f(s,t)h(t)\,\mathrm{d}t + c(s) - \lambda(s) + \int_Y \phi(s,y)\,\mathrm{d}\mu(y) = 0, \quad \forall s \in [0,1],\\ & \lambda(s) \ge 0, \quad \lambda \in L^q([0,1]),\\ & \mu \in M^+(Y).\end{aligned} \tag{7}$$

It is easy to see that (7) is equivalent to the following problem:

$$\begin{aligned}(DQIP'):\ & \min_{h,\mu} \frac{1}{2}\int_0^1\int_0^1 f(s,t)h(s)h(t)\,\mathrm{d}s\,\mathrm{d}t + \int_Y g(y)\,\mathrm{d}\mu(y)\\ & \text{s.t.}\quad \int_0^1 f(s,t)h(t)\,\mathrm{d}t + c(s) + \int_Y \phi(s,y)\,\mathrm{d}\mu(y) \ge 0, \quad \forall s \in [0,1],\\ & \mu \in M^+(Y).\end{aligned} \tag{8}$$

Like with the approach in [15], we can prove the following result.

Theorem 1 (*Duality*). *Assume that the Slater constraint qualification holds in (1). If $h^* \in L^p([0,1])$ is the minimizer of (1), then there exist Lagrangian multipliers $\lambda^* \in L^q([0,1])$ and $\mu^* \in M^+(Y)$ such that the tuple (h^*,λ^*,μ^*) solves the problem (7), and there is no duality gap between (1) and (7).*

Obviously, (h^*,μ^*) solves the problem (8), and there is no duality gap between (1) and (8).

Remark. The above duality theorem is helpful in designing an algorithm for finding the numerical solution of the primal problem (1).

The following theorem shows that the objective function of the problem (1) has continuity of a certain meaning on the space $L^p([0,1])$.

Theorem 2 (*Continuity*). *Let $\{h^k\}$ be one function sequence in the space $L^p([0,1])$. If this sequence is weak (or weak*) compact, i.e. there exists one subsequence $\{h^{k_l}\}$ of $\{h^k\}$ such that h^{k_l} weakly converges to some function h^*, then*

$$\left|\int_0^1\int_0^1 f(s,t)h^{k_l}(s)h^{k_l}(t)\,\mathrm{d}s\,\mathrm{d}t - \int_0^1\int_0^1 f(s,t)h^*(s)h^*(t)\,\mathrm{d}s\,\mathrm{d}t\right| \to 0$$

as $k_l \to +\infty$.

Proof. From the weak convergence of the sequence $\{h^k\}$ in $L^p([0,1])$, it follows that for each $\beta(t) \in L^q([0,1])$ where q satisfies (5), we have

$$\int_0^1 (h^{k_l}(t) - h^*(t))\beta(t)\,\mathrm{d}t \to 0$$

as $k_l \to +\infty$.

Because $f : [0, 1] \times [0, 1] \to R$ is continuously differentiable, for each $t \in [0, 1]$, $f(\cdot, t) : [0, 1] \to R$ and $\beta^{k_l} : [0, 1] \to R$ are also continuously differentiable over the interval [0, 1], where β^{k_l} is defined as

$$\beta^{k_l}(t) = \int_0^1 f(s, t)h^{k_l}(s)\,\mathrm{d}s.$$

Thus, $\beta^{k_l} \in L^q([0, 1])$ and $f(\cdot, t) \in L^q([0, 1])$ for each $t \in [0, 1]$. From the assumption on the sequence $\{h^k\}$, it follows that for any $t \in [0, 1]$,

$$\left|\int_0^1 f(s, t)(h^{k_l}(s) - h^*(s))\,\mathrm{d}s\right| \to 0 \tag{9}$$

and

$$\left|\int_0^1 (h^{k_l}(t) - h^*(t))\beta^{k_l}(t)\,\mathrm{d}t\right| \to 0 \tag{10}$$

as $k_l \to +\infty$.

Since

$$\begin{aligned}
&\left|\int_0^1\int_0^1 f(s, t)h^{k_l}(s)h^{k_l}(t)\,\mathrm{d}s\,\mathrm{d}t - \int_0^1\int_0^1 f(s, t)h^*(s)h^*(t)\,\mathrm{d}s\,\mathrm{d}t\right| \\
&\quad= \left|\int_0^1 h^{k_l}(t)\left[\int_0^1 f(s, t)h^{k_l}(s)\,\mathrm{d}s\right]\mathrm{d}t - \int_0^1 h^*(t)\left[\int_0^1 f(s, t)h^*(s)\,\mathrm{d}s\right]\mathrm{d}t\right| \\
&\quad= \left|\int_0^1\left[h^{k_l}(t)\int_0^1 f(s, t)h^{k_l}(s)\,\mathrm{d}s - h^*(t)\int_0^1 f(s, t)h^*(s)\,\mathrm{d}s\right]\mathrm{d}t\right| \\
&\quad= \left|\int_0^1\left[(h^{k_l}(t) - h^*(t))\int_0^1 f(s, t)h^{k_l}(s)\,\mathrm{d}s + h^*(t)\int_0^1 f(s, t)(h^{k_l}(s) - h^*(s))\,\mathrm{d}s\right]\mathrm{d}t\right| \\
&\quad\le \left|\int_0^1 (h^{k_l}(t) - h^*(t))\int_0^1 f(s, t)h^{k_l}(s)\,\mathrm{d}s\,\mathrm{d}t\right| + \int_0^1 |h^*(t)|\left|\int_0^1 f(s, t)(h^{k_l}(s) - h^*(s))\,\mathrm{d}s\right|\mathrm{d}t \\
&\quad= \left|\int_0^1 (h^{k_l}(t) - h^*(t))\beta^{k_l}(t)\,\mathrm{d}t\right| + \int_0^1 |h^*(t)|\left|\int_0^1 f(s, t)(h^{k_l}(s) - h^*(s))\,\mathrm{d}s\right|\mathrm{d}t.
\end{aligned}$$

As $k_l \to +\infty$, we can deduce that the last summation converges to zero from (9) and (10).

The proof is complete. □

Remark. The result in Theorem 2 has been an assumed condition for dealing with a class of quadratic infinite programs on measure spaces in [15]. However, here we have proved that the result in Theorem 2 is an intrinsic property of problem (1).

3. Existence

On the basis of Theorem 2, we can derive the conditions for guaranteeing the existence of an optimal solution of the problem (1). For this, we first define one univariate function $\Phi_\mu : [0, 1] \to R$ as follows:

$$\Phi_\mu(s) = \int_Y \phi(s, y)\,\mathrm{d}\mu(y), \tag{11}$$

where μ is some given measure in the cone $M^+(Y)$.

Because $\phi(\cdot, y)$ is continuous in the interval [0, 1], it is easy to prove that $\Phi_\mu(\cdot)$ is also continuous, i.e. $\Phi_\mu \in C([0, 1])$ for every $\mu \in M(Y)$. Hence, Φ_μ is bounded for every $\mu \in M^+(Y)$.

In the following, we present two existence theorems.

Theorem 3 (*Existence*). *Suppose that there exists a constant $M > 0$ such that $\|h\|_{L_p} \leq M$ for all $h \in \mathcal{F}_1$. Then, $\mathcal{F}_1$ is weakly compact. Furthermore, there exists an $h^* \in L^p([0, 1])$ which solves the problem (1).*

Proof. We only need to prove that $\mathcal{F}_1$ is weakly compact.

By applying the Banach–Alaoglu theorem, we know $B_M = \{h : h \in L^p[0, 1] \text{ and } \|h\|_{L_p} \leq M\}$ is weakly compact. From the assumption, we know $\mathcal{F}_1 \subseteq B_M$.

It is obvious that $\mathcal{F}_1$ is weakly closed and hence it is weakly compact. From Theorem 2, it follows that there exists an $h^* \in L^p([0, 1])$ which solves the problem (1).

The desired conclusion holds. □

Theorem 4 (*Existence*). *Suppose that there exists an L_p-integrable function g such that $h \leq g$ for all $h \in \mathcal{F}_1$. Then, $\mathcal{F}_1$ is weakly compact. Hence there exists $h^* \in L^p([0, 1])$ which solves the problem (1).*

Proof. For each $h \in \mathcal{F}_1$, we have $h \leq g$. Thus $\|h\|_{L_p} \leq \|g\|_{L_p}$. The result follows from Theorem 3. □

4. Final remarks

In this work, a dual problem for a class of infinite programming problem has been derived, and there is no duality gap between the dual and the primal problem.

Secondly, the continuity of the objective function of the integral type has been proved in the L^p space.

Lastly, we gave two sufficient conditions which can guarantee that the optimal solutions exist for the infinite programs. However, the computability of those conditions needs further investigation in practice.

The above theoretical results provide the possibility of designing some efficient algorithms for the proposed optimization problem.

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