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The metric dimension of Cayley digraphs

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Abstract

A vertex x in a digraph D is said to resolve a pair u, v of vertices of D if the distance from u to x does not equal the distance from v to x. A set S of vertices of D is a resolving set for D if every pair of vertices of D is resolved by some vertex of S. The smallest cardinality of a resolving set for D, denoted by $\dim(D)$, is called the metric dimension for D. Sharp upper and lower bounds for the metric dimension of the Cayley digraphs $\mathrm{Cay}(\Delta : \Gamma)$, where Γ is the group $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_m}$ and Δ is the canonical set of generators, are established. The exact value for the metric dimension of $\mathrm{Cay}(\{(0,1),(1,0)\} : \mathbb{Z}_n \oplus \mathbb{Z}_m)$ is found. Moreover, the metric dimension of the Cayley digraph of the dihedral group D_n of order $2n$ with a minimum set of generators is established. The metric dimension of a (di)graph is formulated as an integer programme. The corresponding linear programming formulation naturally gives rise to a fractional version of the metric dimension of a (di)graph. The fractional dual implies an integer dual for the metric dimension of a (di)graph which is referred to as the metric independence of the (di)graph. The metric independence of a (di)graph is the maximum number of pairs of vertices such that no two pairs are resolved by the same vertex. The metric independence of the n-cube and the Cayley digraph $\mathrm{Cay}(\Delta : D_n)$, where Δ is a minimum set of generators for D_n, are established.

Keywords: Metric dimension; Cayley digraphs; Metric independence; Integer programming; Linear programming

1. Introduction

Let G be a connected graph. A vertex x of G is said to *resolve* two vertices u and v of G if the distance $d(u,x)$ from u to x does not equal the distance $d(v,x)$ from v to x. A set S of vertices of G is said to be a *resolving set* for G if, for every two distinct vertices u and v, there is a vertex x of S that resolves u and v. Alternatively, suppose $S=\{x_1,x_2,\ldots,x_k\}$ is a set whose vertices have been assigned the given order. The k-vector $r(v|S)=(d(v,x_1),d(v,x_2),\ldots,d(v,x_k))$ is called the *representation* of v with respect to S. Then S is a resolving set for G if and only if no two vertices of G have the same representation with respect to S. Note that x_i is the only vertex of S for which the ith coordinate of its representation with respect to S is 0. Therefore, when checking if S is a resolving set for G, one need only check that the vertices of $V(G)-S$ have distinct representations with respect to S. The minimum cardinality of a resolving set for G is called the *metric dimension* of G and is denoted by $\dim(G)$. A minimum resolving set is called a *metric basis* for G.

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Harary and Melter [7] and independently Slater in [13,14] introduced this concept. Slater referred to the metric dimension of a graph as its location number and motivated the study of this invariant by its application to the placement of a minimum number of sonar/loran detecting devices in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set. It was noted in [6] that the problem of finding the metric dimension of a graph is NP-hard. Khuller et al. [8] gave a construction that shows that the metric dimension of a graph is NP-hard. Their interest in this invariant was motivated by the navigation of robots in a graph space. A resolving set for a graph corresponds to the presence of distinctively labelled "landmark" nodes in the graph. It is assumed that a robot navigating a graph can sense the distance to each of the landmarks and hence uniquely determine its location in the graph. They also gave approximation algorithms for this invariant and established properties of graphs with metric dimension 2. Motivated by a problem from Pharmaceutical Chemistry, this problem received renewed attention in [1].

The *metric dimension of a connected digraph* D has the expected definition, namely, the smallest cardinality of a set S of vertices with the property that, for every two vertices u, v of D, there is some $x \in S$ such that $d(u,x) \neq d(v,x)$. Since the distance between two vertices in a digraph need not be defined, the metric dimension of a digraph may not be defined. The metric dimension of oriented graphs was first studied by Chartrand et al. in [2] and further in [3]. It was pointed out by these authors that it remains an open problem to determine for which directed graphs the directed distance dimension is defined. In this paper we study the metric dimension of Cayley digraphs for which the metric dimension is defined. These digraphs with their high degree of symmetry are of interest in this context as the metric dimension appears to be related to both local and global symmetry in regular (di)graphs. We establish sharp bounds for this invariant and conclude the paper with an integer programming formulation of this problem as described in [4]. The linear programming relaxation yields a fractional version of the metric dimension whose dual yields a dual for the metric dimension of a graph called the *metric independence* of the graph. This invariant is defined as the maximum number of pairs of vertices in a connected graph G such that no vertex of G simultaneously resolves two distinct pairs in such a set. In [4] a geometric proof was given to show that the metric independence of the n-cube, Q_n, is 2. However, the proof was found to contain a gap. We present here a proof of this fact using the fractional version of the metric dimension and a combinatorial argument. The metric independence of the Cayley digraph for the dihedral group of order $2n$, with a minimum set of generators, is also established.

2. The metric dimension of Cayley digraphs

In this section we focus on determining the dimension of Cayley digraphs. First, recall the definition of the Cayley digraph for a given group with a specified set of generators (see [5]).

Let Γ be a finite group and Δ a set of generators for Γ. The *Cayley digraph of Γ with generating set Δ*, denoted by $\mathrm{Cay}(\Delta : \Gamma)$, is defined as follows:

(1) The vertices of $\mathrm{Cay}(\Delta : \Gamma)$ are precisely the elements of Γ.
(2) For u and v in Γ, there is an arc from u to v if and only if $ug = v$ for some generator $g \in \Delta$.

Note that for a given finite group Γ and a specified set of generators Δ of Γ, every element of the group can be expressed as a product of generators in Γ. Hence, in the graph $G = \mathrm{Cay}(\Delta : \Gamma)$, there exists a path in G from any vertex of G to every other vertex of G. Thus, any Cayley digraph is strongly connected, and the metric dimension of any Cayley digraph is therefore defined.

We now find the metric dimension of some specific Cayley digraphs. Let G be the Cayley digraph for the group $\Gamma = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ with the canonical set of generators $\Delta = \{(1,0), (0,1)\}$. The graph G is shown in Fig. 1(a). The vertices of G are the elements of the group $\mathbb{Z}_2 \oplus \mathbb{Z}_4$, and the arcs between vertices correspond to the generators in Δ. In Fig. 1(a) the dashed arcs correspond to the generator $(1,0)$, and the solid arcs correspond to the generator $(0,1)$. The two shaded vertices in Fig. 1(a) constitute a resolving set for G, and one can verify that no single vertex of G resolves every pair of vertices of G. Thus, the dimension of G is 2.

Now consider the group of symmetries of the regular n-gon, called the *dihedral group of order* $2n$, denoted by D_n. This group consists of n rotations and n reflections. For $n = 4$, this is the group of symmetries of the square, consisting of the four rotations, denoted by R_0, R_{90}, R_{180}, R_{270}, and the four reflections, denoted by A, B, C, D. Let H be the

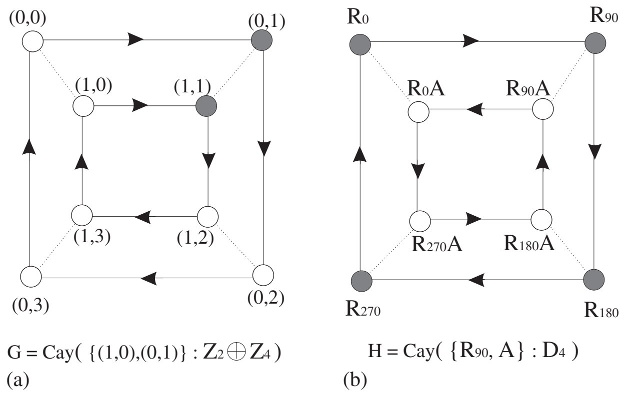

Fig. 1. Cayley graphs for $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ and D_4.

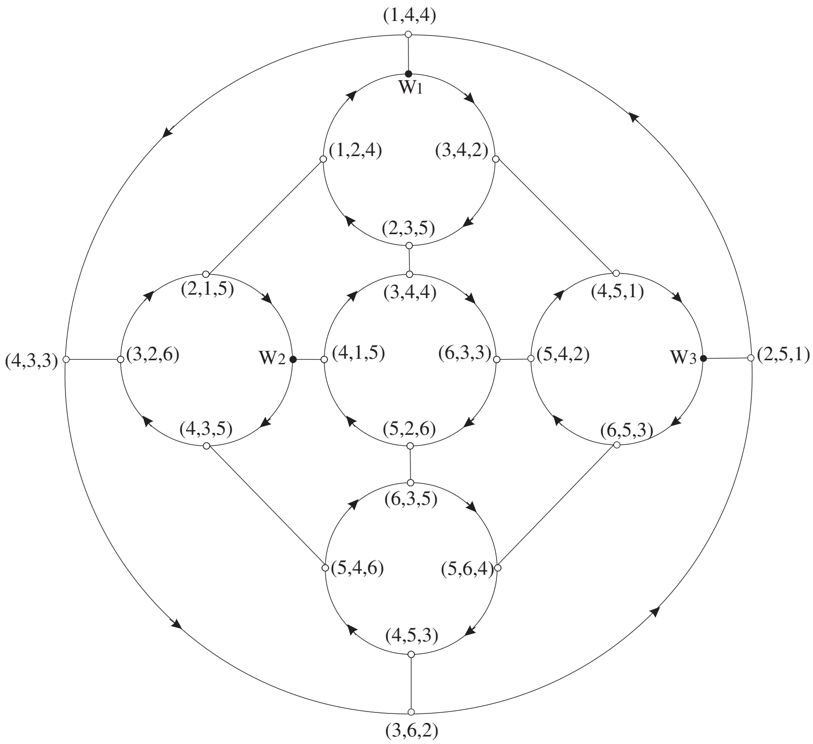

Fig. 2. Cay($\{(1, 2), (1, 2, 3, 4)\} : S_4$).

Cayley digraph for the group D_4 with generating set $\Delta = \{R_{90}, A\}$. The graph H is shown in Fig. 1(b). The structure of this graph is very similar to that of G in Fig. 1(a), except that the two directed 4-cycles in H are oriented in opposite directions, while those of G are oriented in the same direction. Both of these Cayley digraphs are constructed with two generators, and in both cases the minimum order of a generator in Δ is 2. However, while the dimension of G is 2, the dimension of H is 4. The four shaded vertices in Fig. 1(b) constitute a resolving set for H, and it can be verified that no three vertices of H resolve every pair of vertices of H (see Theorem 4).

Fig. 2 shows the Cayley digraph for the non-abelian symmetric group of degree 4, denoted by S_4, with generating set $\Delta = \{(1, 2), (1, 2, 3, 4)\}$. The vertices w_1, w_2, w_3 form a resolving set for this graph as the representations of the vertices of Cay($\{(1, 2), (1, 2, 3, 4)\} : S_4$) with respect to the set $\{w_1, w_2, w_3\}$ are all distinct. These distinct representations are

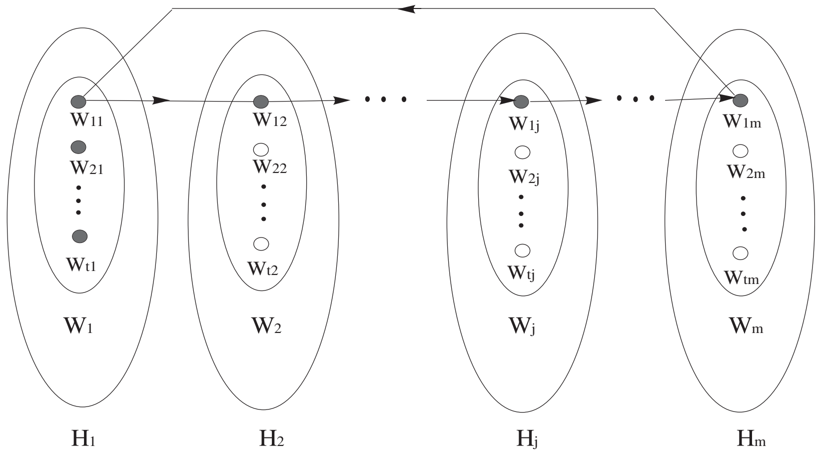

Fig. 3. The graph H'.

displayed in Fig. 2. Furthermore, no two vertices of the graph constitute a resolving set. Thus, the dimension of this graph is 3.

Some familiar graphs are Cayley digraphs. For example, the *n-cube* is the Cayley digraph for the group $\Gamma = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$ (n times), with the canonical set of generators $\Delta = \{(1, 0, 0, \ldots, 0), (0, 1, 0, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)\}$. Later in this section, Corollary 3 shows that $2 \leqslant \dim(Q_n) \leqslant n$. More generally, we will establish bounds on the dimension of the Cayley digraph for the direct product of any number of cyclic groups with the canonical set of generators. To this end, we first look at how the dimension of the Cayley digraph for a group Γ changes when we take the direct product of Γ and the cyclic group of order m, where m is a positive integer. Clearly, the Cayley digraph for the new group $\Gamma \oplus \mathbb{Z}_m$ and its dimension will depend on the set of generators chosen. The following theorem bounds this dimension subject to a specific choice of generators.

Theorem 1. *Let Γ be a group of order n and let $\Delta = \{g_1, g_2, \ldots, g_k\}$ be a generating set for Γ. Let $H = \mathrm{Cay}(\Delta : \Gamma)$. Let $\Delta' = \{(g_1, 0), (g_2, 0), \ldots, (g_k, 0), (e_\Gamma, 1)\}$ be a generating set for the group $\Gamma' = \Gamma \oplus \mathbb{Z}_m$, where $m \geqslant 2$ and e_Γ is the identity element of Γ. Then for $H' = \mathrm{Cay}(\Delta' : \Gamma')$,*

$$\dim(H) \leqslant \dim(H') \leqslant \dim(H) + m - 1.$$

Proof. The graph H' consists of m copies of the graph H. Label these copies $H_1, H_2, \ldots, H_m$. Let $V(H_j) = \{u_{1j}, u_{2j}, \ldots, u_{nj}\}$, for $1 \leqslant j \leqslant m$, where for each i $(1 \leqslant i \leqslant n))$, u_{ij} is in the same position in H_j as u_{ik} is in H_k (for $1 \leqslant j, k \leqslant m$). The arcs between the m copies of H are precisely the arcs on the directed cycles C_i given by

$$C_i : u_{i1}, u_{i2}, \ldots, u_{im}, u_{i1} \quad (\text{for } 1 \leqslant i \leqslant n).$$

That is, H' is constructed by taking m copies of H, $H_1, H_2, \ldots, H_m$, and placing arcs from H_i to H_{i+1} (subscripts modulo m) between corresponding vertices (for $1 \leqslant i \leqslant m$).

To establish the upper bound in the theorem we need to find a resolving set for H' of cardinality $\dim(H) + m - 1$. Let $W = \{w_1, w_2, \ldots, w_t\}$ be a basis for H. Then $\dim(H) = t$, and there exists a corresponding set $W_j = \{w_{1j}, w_{2j}, \ldots, w_{tj}\}$ of vertices of the graph H_j which is a basis for H_j (for $1 \leqslant j \leqslant m$). Let $W' = \{w_{11}, w_{21}, \ldots, w_{t1}, w_{12}, w_{13}, \ldots, w_{1m}\}$. Then $|W'| = t + m - 1 = \dim(H) + m - 1$. We claim that W' is a resolving set for H'.

The graph H' is shown in Fig. 3. Note that, for simplicity, only one of the m-cycles of H' is shown in Fig. 3 and that the shaded vertices in the figure are the vertices of W'.

To demonstrate that W' is a resolving set for H', let u and v be distinct vertices of H'. We show that u and v are resolved by some vertex of W'. We consider two cases.

Case 1: Both u and v are vertices of H_j for some j $(1 \leqslant j \leqslant m)$. Since W_j is a basis for H_j, it follows that $d_{H_j}(u, w_{ij}) \neq d_{H_j}(v, w_{ij})$ for some i $(1 \leqslant i \leqslant t)$. For this i, the structure of H' guarantees that $d_{H'}(u, w_{i1}) = d_{H_j}(u, w_{ij}) + m - j + 1 \neq d_{H_j}(v, w_{ij}) + m - j + 1 = d_{H'}(v, w_{i1})$. Thus, $d_{H'}(u, w_{i1}) \neq d_{H'}(v, w_{i1})$, and so w_{i1} resolves u and v in this case.

Case 2: $u \in V(H_i)$ and $v \in V(H_j)$ for some i and j (where $1 \leqslant i < j \leqslant m$). We consider two subcases.

Subcase 2.1: u and v are in corresponding positions in H_i and H_j, respectively. That is, $u = u_{qi}$ and $v = u_{qj}$ for some q $(1 \leqslant q \leqslant n)$. In this case, w_{11} resolves u and v. To see this note $d_{H_i}(u, w_{1i}) = d_{H_j}(v, w_{1j})$, so if $i = 1$ we have that $d_{H'}(u, w_{11}) = d_{H_i}(u, w_{1i}) < d_{H_j}(v, w_{1j}) + m - j + 1 = d_{H'}(v, w_{11})$, and if $i \neq 1$, since $i < j$, $d_{H'}(u, w_{11}) = d_{H_i}(u, w_{1i}) + m - i + 1 > d_{H_j}(v, w_{1j}) + m - j + 1 = d_{H'}(v, w_{11})$. In either case, $d_{H'}(u, w_{11}) \neq d_{H'}(v, w_{11})$.

Subcase 2.2: u and v are in different positions in H_i and H_j, respectively. That is, $u = u_{si}$ and $v = u_{rj}$ for some s, r where $1 \leqslant s \neq r \leqslant n$. If u and v are resolved by w_{11}, then u and v are resolved by a vertex of W', so we may assume that u and v are not resolved by w_{11}. Then $d_{H'}(u, w_{11}) = d_{H'}(v, w_{11})$. Now if $i = 1$ and $u \in V(H_1)$, then $d_{H'}(u, w_{1j}) = d_{H'}(u, w_{11}) + j - 1 \neq d_{H'}(v, w_{11}) - (m - j + 1) = d_{H'}(v, w_{1j})$, and so $d_{H'}(u, w_{1j}) \neq d_{H'}(v, w_{1j})$. (These distances differ by m.) Thus if $i = 1$, w_{1j} resolves u and v. On the other hand, if $i \neq 1$ and $u \in V(H_i)$ for some $i \in \{2, 3, \ldots, m\}$, then $d_{H'}(u, w_{1i}) = d_{H'}(u, w_{11}) - (m - i + 1) \neq d_{H'}(v, w_{11}) + (i - 1) = d_{H'}(v, w_{1i})$, and so $d_{H'}(u, w_{1i}) \neq d_{H'}(v, w_{1i})$. (Again these distances differ by m.) Thus if $i \neq 1$, w_{1i} resolves u and v. In either case, u and v are resolved by some vertex of W'.

Thus W' is a resolving set for the graph H', as claimed, and so $\dim(H') \leqslant |W'| = t + m - 1 = \dim(H) + m - 1$.

To establish the lower bound in the theorem, let $H_1, H_2, \ldots, H_m$ be the m copies of H in H'. Let W be a basis for H'. Let $W_i = W \cap V(H_i)$ (for $1 \leqslant i \leqslant m$). Let W_i' be the vertices of H_1 that correspond to the vertices of W_i in H_i $(2 \leqslant i \leqslant m)$. Let $U_1 \subseteq V(H_1)$ be the union of W_1 and the sets $W_2', W_3', \ldots, W_m'$. Thus,

$$\begin{aligned} |U_1| &= \left| W_1 \cup \left(\bigcup_2^m W_i' \right) \right| \leqslant |W_1| + |W_2'| + |W_3'| + \cdots + |W_m'| \\ &= |W_1| + |W_2| + |W_3| + \cdots + |W_m| = |W|. \end{aligned}$$

We claim that U_1 is a resolving set for H_1. Let u and v be distinct vertices of H_1. We show that u and v are resolved by some vertex of U_1. Since W is a basis for H', and u and v are vertices of H', there exists a vertex $w \in W$ such that $d_{H'}(u, w) \neq d_{H'}(v, w)$. Recall that $W = W_1 \cup W_2 \cup \cdots \cup W_m$. Thus either $w \in W_1$ or $w \in W_i$ for some $i \in \{2, 3, \ldots, m\}$.

If $w \in W_1$, then $w \in U_1$ since $W_1 \subseteq U_1$, and w resolves u and v in H_1. This follows from the fact that $d_{H_1}(u, w) = d_{H'}(u, w) \neq d_{H'}(v, w) = d_{H_1}(v, w)$.

If $w \in W_i$ for some $i \in \{2, 3, \ldots, m\}$, then let w' be the vertex in $W_i' \subseteq U_1$ corresponding to w. Then w' resolves u and v. This follows from the fact that $d_{H_1}(u, w') = d_{H'}(u, w) - (i - 1) \neq d_{H'}(v, w) - (i - 1) = d_{H_1}(v, w')$.

In either case, u and v are resolved by some vertex of U_1. So U_1 is a resolving set for H_1. This implies that

$$\dim(H) = \dim(H_1) \leqslant |U_1| \leqslant |W| = \dim(H'),$$

from which the lower bound in the theorem follows. □

Recall the Cayley digraph in Fig. 1(a) for the group $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ with the canonical set of generators. This graph has dimension 2. In the following theorem we generalize this result and show that, for positive integers m and n, the dimension of the Cayley digraph for the group $\mathbb{Z}_m \oplus \mathbb{Z}_n$ with the canonical set of generators $\{(1, 0), (0, 1)\}$ is the minimum of m and n.

Theorem 2. *Let m and n be positive integers. Let H' be the Cayley digraph for the group $\mathbb{Z}_n \oplus \mathbb{Z}_m$ with generating set* $\{(1, 0), (0, 1)\}$. *Then* $\dim(H') = \min(m, n)$.

Proof. Suppose that $m \leqslant n$. First we show that $\dim(H') \leqslant \min(m, n) = m$. Let H be the Cayley digraph for the group $\Gamma = \mathbb{Z}_n$ with generating set $\Delta = \{1\}$. Then H is the directed n-cycle, which clearly has dimension 1. Let $\Delta' = \{(1, 0), (e_\Gamma, 1)\} = \{(1, 0), (0, 1)\}$. Then Δ' is a generating set for the group $H' = H \oplus \mathbb{Z}_m = \mathbb{Z}_n \oplus \mathbb{Z}_m$. By Theorem 1, $\dim(H') \leqslant \dim(H) + m - 1 = 1 + m - 1 = m$.

It remains to show that $\dim(H') \geqslant m$. Suppose, to the contrary, that there exists a basis B for H' such that $|B| < m$. As in Theorem 1, H' is constructed from m copies of the directed n-cycle, label them $H_1, H_2, \ldots, H_m$, by placing arcs from H_i to H_{i+1} (subscripts modulo m) between corresponding vertices (for $1 \leqslant i \leqslant m$). Thus, there are n vertex disjoint directed m-cycles in the graph H', as well as m vertex disjoint directed n-cycles. A vertex in the ith n-cycle has first coordinate i (for $0 \leqslant i \leqslant m - 1$), and a vertex in the jth m-cycle has second coordinate j (for $0 \leqslant j \leqslant n - 1$).

Table 1
Table of dimensions of the graph Cay({(1, 0, 0), (0, 1, 0), (0, 0, 1)} : $\mathbb{Z}_m \oplus \mathbb{Z}_n \oplus \mathbb{Z}_k$) for $m, n, k \leqslant 5$

m	n	k	Lower bound	Upper bound	Dimension
2	2	2	2	3	3
2	2	3	2	3	3
2	2	4	2	3	3
2	2	5	2	3	3
2	3	3	3	4	3
2	3	4	3	4	3
2	3	5	3	4	3
2	4	4	4	5	4
2	4	5	4	5	4
2	5	5	5	6	5
3	3	3	3	5	5
3	3	4	3	5	4
3	3	5	3	5	5
3	4	4	4	6	4
3	4	5	4	6	4
3	5	5	5	7	5
4	4	4	4	7	6
4	4	5	4	8	5
4	5	5	5	8	5
5	5	5	5	9	7

Since there are less than m vertices in the basis B, and there are m n-cycles, there must be at least one directed n-cycle which contains no vertex of B. Due to the symmetry of the graph H' we can assume, without loss of generality, that the 0th n-cycle contains no vertex of B. Also, since $|B| < m \leqslant n$, and there are n m-cycles, there is at least one directed m-cycle which contains no vertex of B. Again, by the symmetry of the graph H', we can assume that the 0th m-cycle contains no vertex of B. Now consider the vertices (1, 0) and (0, 1) and any vertex $w \in B$. Since no vertex of B lies on either the 0th m-cycle or the 0th n-cycle, there exists a shortest path from (0, 1) to w, and also one from (1, 0) to w, which contains (1, 1). However, both vertices (0, 1) and (1, 0) are adjacent to (1, 1). Thus, for any vertex w of B, $d((0, 1), w) = d((1, 0), w)$, and so the vertices (0, 1) and (1, 0) are not resolved by any vertex of B, which contradicts the fact that B is a basis for H'. Hence $\dim(H') \geqslant m$. □

Theorem 2 illustrates that the upper bound of Theorem 1 is attained for Cay({(1, 0), (0, 1)} : $\mathbb{Z}_n \oplus \mathbb{Z}_m$). Using the integer programming formulation for (di)graphs, as described in the next section, values for the metric dimension of the Cayley digraphs for the direct product of three cyclic groups with the canonical set of generators are obtained (see Table 1). The upper and lower bounds of Theorem 1 are also included in Table 1.

From these values we conclude that it is possible to have equality for either bound in Theorem 1 and that intermediate values can also be attained. Finding exact values for the metric dimension of Cayley digraphs of abelian groups of the form $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ (for $k \geqslant 3$) with the canonical set of generators $\Delta = \{(1, 0, 0, \ldots, 0), (0, 1, 0, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)\}$ is an open problem. However, the previous two theorems can be used to bound the dimension of these Cayley digraphs. These bounds are given in the following corollary.

Corollary 3. *Let $k, n_1, n_2, \ldots, n_k$ be positive integers where $k \geqslant 2$ and $n_1 \geqslant n_2 \geqslant n_3 \geqslant \cdots \geqslant n_k$. Let $\Gamma = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ and $\Delta = \{(1, 0, 0, \ldots, 0), (0, 1, 0, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)\}$. If $G = \mathrm{Cay}(\Delta : \Gamma)$, then*

$$n_2 \leqslant \dim(G) \leqslant n_2 + \sum_{i=3}^{k} (n_i - 1).$$

Proof. This result follows immediately from repeated applications of Theorem 1 and from Theorem 2. □

The values given in Table 1 support our intuition that there appears to be a correlation between higher degrees of symmetry in a graph and the metric dimension. In particular, if m, n and k are all distinct (so that there is less symmetry) the lower bound of the previous corollary is always achieved.

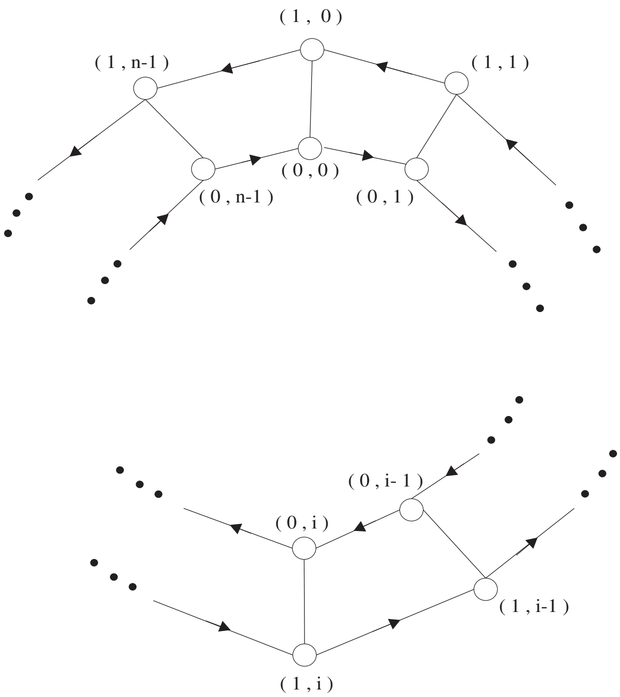

Fig. 4. Cay($\{R_{360/n}, A\} : D_n$).

Recall that the Cayley digraph for the dihedral group D_4 with generating set $\Delta = \{R_{90}, A\}$ has dimension 4. In the next theorem we generalize this result to the dihedral group D_n of order $2n$.

Theorem 4. *Let n be a positive integer, $n \geqslant 3$. Let G be the Cayley digraph for the group D_n with generating set $\{R_{360/n}, A\}$, where A is any reflection in the group D_n. Then* $\dim(G) = n$.

Proof. Label each vertex of G with an ordered pair, where the first coordinate is 0 (or 1) if the vertex is on the "inner" (or "outer") n-cycle, respectively. The outer n-cycle is directed counter-clockwise, and the inner n-cycle is directed clockwise. The second coordinate denotes the position of the vertex on the n-cycle, from 0 to $n - 1$ in the clockwise direction. The resulting Cayley digraph is shown in Fig. 4.

To show that the dimension is at most n, let $W = \{(1, 0), (1, 1), \ldots, (1, n - 1)\}$. All pairs of vertices in $V(G) - W$ (i.e. pairs of vertices on the inner cycle) are resolved since $(0, i)$ is the unique vertex in $V(G) - W$ that is adjacent to $(1, i)$ and is thus the only vertex whose representation has ith coordinate 1. Hence $\dim(G) \leqslant n$.

To establish the lower bound, observe that the only vertices that resolve the pair $\{(0, i - 1), (1, i)\}$ (for $1 \leqslant i \leqslant n$) are the two vertices in the pair (see Fig. 4). Hence, any resolving set contains at least n vertices. Thus, $\dim(G) \geqslant n$. □

3. A fractional version of the metric dimension problem and its dual

Currie and Oellermann in [4] formulated the problem of finding the metric dimension of a graph as an integer programme. This formulation naturally gives rise to a fractional version of the metric dimension of a graph, and its fractional dual implies an integer dual for the metric dimension of a graph. Fractional versions of other graph invariants are discussed in [11].

Let G be a connected graph of order n. Suppose V is the vertex set of G and V_p the collection of all $\binom{n}{2}$ pairs of vertices of G. Let $R(G)$ denote the bipartite graph with partite sets V and V_p such that x in V is joined to a pair $\{u, v\}$ in V_p if and only if x resolves u and v in G. We call $R(G)$ the *resolving graph* of G.

The smallest cardinality of a subset S of V such that the neighborhood $N(S)$ of S in $R(G)$ is V_p is thus the metric dimension of G. Suppose $V = \{v_1, v_2, \ldots, v_n\}$ and $V_p = \{s_1, s_2, \ldots, s_{\binom{n}{2}}\}$. Let $A = (a_{ij})$ be the $\binom{n}{2} \times n$

matrix with

$$a_{ij} = \begin{cases} 1 & \text{if } s_i v_j \in E(R(G)), \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leqslant i \leqslant \binom{n}{2}$ and $1 \leqslant j \leqslant n$.

The integer programming formulation of the metric dimension is given by: minimize

$$f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$$

subject to the constraints

$$Ax \geqslant [1]_{\binom{n}{2}}$$

and

$$x \geqslant [0]_n,$$

where $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}$, $[1]_k$ is the $k \times 1$ matrix all of whose entries are 1, $[0]_n$ is the $n \times 1$ matrix all of whose entries are 0 and $x_i \in \{0, 1\}$ for $1 \leqslant i \leqslant n$.

If we relax the condition that $x_i \in \{0, 1\}$ for every i and require only that $x_i \geqslant 0$ for all i, then we obtain the following linear programming problem: minimize

$$f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$$

subject to the constraints

$$Ax \geqslant [1]_{\binom{n}{2}}$$

and

$$x \geqslant [0]_n.$$

In terms of the resolving graph $R(G)$ of G, solving this linear programming problem amounts to assigning nonnegative weights to the vertices in V so that for each vertex in V_p the sum of the weights in its neighborhood is at least 1 and such that the sum of the weights of the vertices in V is as small as possible. The smallest value for f is called the *fractional dimension* of G and is denoted by $\mathrm{frdim}(G)$.

The dual of this linear programming problem is given by: maximize

$$f(y_1, y_2, \ldots, y_{\binom{n}{2}}) = y_1 + y_2 + \cdots + y_{\binom{n}{2}}$$

subject to the constraints

$$A^{\mathrm{T}} y \leqslant [1]_n$$

and

$$y \geqslant [0]_{\binom{n}{2}},$$

where $y = [y_1, y_2, \ldots, y_{\binom{n}{2}}]^{\mathrm{T}}$.

For the resolving graph $R(G)$ of G this amounts to assigning nonnegative weights to the vertices of V_p so that for each vertex in V the sum of the weights in its neighborhood is at most 1 and subject to this such that the sum of the weights of the vertices in V_p is as large as possible.

The corresponding integer programming problem asks for an assignment of 0's and 1's to the vertices in V_p such that the sum of the weights of the neighbors of every vertex in V is at most 1 and such that the sum of the weights of the vertices in V_p is as large as possible. This integer programming problem, which corresponds to the dual of the fractional form of the metric dimension problem, is equivalent to finding the largest collection of pairs of vertices of G no two of which are resolved by the same vertex. This maximum is called the *metric independence number* of G, denoted by $\mathrm{mi}(G)$. A collection of pairs of vertices of G, no two of which are resolved by the same vertex, is called an *independently resolved collection of pairs*. The *fractional metric independence number* of G is defined in the expected manner and is denoted by $\text{frmi}(G)$. Clearly, $\dim(G) \geqslant \text{frdim}(G)$ and $\text{frmi}(G) \geqslant \text{mi}(G)$. It follows from the Duality Theorem for linear programming that $\text{frdim}(G) = \text{frmi}(G)$. We thus obtain the following string of inequalities:

$$\dim(G) \geqslant \text{frdim}(G) = \text{frmi}(G) \geqslant \text{mi}(G).$$

Note that, for any connected graph G, $\text{mi}(G) \leqslant \text{frdim}(G)$. We now use this fact to show that the metric independence of the n-cube, denoted by Q_n, is 2 for all positive integers $n \geqslant 2$. To this end, the following two lemmas, which can be established in a straightforward manner using induction, are useful.

Lemma 5. *For all positive integers* k,

$$\binom{2k}{k} \leqslant 2^{2k-1}.$$

Lemma 6. *For all positive integers* k,

$$\binom{2k-1}{k-1} \leqslant 2^{2k-2}.$$

We are now ready to prove the following theorem.

Theorem 7. *For all* $n \geqslant 2$,

$$\text{mi}(Q_n) = 2.$$

Proof. To see that $\text{mi}(Q_n) \geqslant 2$, consider the two pairs of vertices that are diametrically opposite to one another on any 4-cycle in Q_n. These two pairs are not resolved by the same vertex and are thus metrically independent. It follows that $\text{mi}(Q_n) \geqslant 2$.

It remains to show that $\text{mi}(Q_n) \leqslant 2$. We recall here two different ways of describing the graph Q_n.

(i) Q_n is the graph whose vertex set consists of all 2^n n-tuples of $0's$ and $1's$, and where two n-tuples are joined by an edge if and only if they differ in exactly one position.
(ii) Q_n can be obtained from two copies of the $(n-1)$-cube, Q_{n-1}, by joining corresponding vertices.

Let Q'_{n-1} and Q''_{n-1} denote two vertex disjoint copies of the $(n-1)$-cube in the graph Q_n. We may assume that all of the vertices of Q'_{n-1} have a 0 in the first position and those in Q''_{n-1} have a 1 in the first position of their n-tuples.

Assign each vertex of Q'_{n-1} a value of $1/2^{n-2}$ and each vertex of Q''_{n-1} a value of 0. Let $R(Q_n)$ be the resolving graph of Q_n defined above. Let V_p be the collection of all pairs of vertices of Q_n. If we can show, with this assignment of fractional values to the vertices of Q_n, that the sum of the values of the neighbors of the vertices in V_p is at least 1, then we have shown that

$$\text{frdim}(Q_n) \leqslant 2^{n-1}\left(\frac{1}{2^{n-2}}\right) + 2^{n-1}(0) = 2.$$

Since $\text{mi}(Q_n) \leqslant \text{frdim}(Q_n)$, the result will follow.

Since Q_n is bipartite, vertices from distinct partite sets are resolved by every vertex of Q_n and hence every vertex of Q'_{n-1}. So, for such a pair, the sum of the values of its neighbors in $R(Q_n)$ is at least $2^{n-1}(1/2^{n-2}) = 2$.

Suppose now that u and v are distinct vertices that belong to the same partite set and suppose that $d(u, v) = d$. Then d is necessarily even. Let $\mathscr{P}$ be the collection of all positions for which the n-tuples of u and v agree. There are $n-d$ such positions. Let $\mathscr{P}'$ be the collection of all positions where the n-tuples for u and v disagree. Then $|\mathscr{P}'| = d$. If a vertex z of Q_n does not resolve u and v, then it is the same distance from u and v. Note that the number of positions in $\mathscr{P}$ where the n-tuple for z differs from the one for u is the same as for v. Suppose that the number of positions in $\mathscr{P}'$ where the n-tuple for z differs from the n-tuple for u is k. Then the number of positions in $\mathscr{P}'$ where the n-tuple for z differs from the n-tuple for v is $d-k$. Since $d(u, z) = d(v, z)$ we have that $k = d-k$. So the n-tuple for z differs from the one for u and the one for v in exactly k positions belonging to $\mathscr{P}'$. We consider two cases.

Case 3: Suppose position 1 belongs to $\mathscr{P}$. Then u and v either both belong to Q'_{n-1} or to Q''_{n-1}. There are $2^{n-2k-1}\binom{2k}{k}$ vertices in Q'_{n-1} that are the same distance from u and v. By Lemma 5, $2^{n-2k-1}\binom{2k}{k}\leqslant 2^{n-2k-1}(2^{2k-1})=2^{n-2}$. So there are at least $2^{n-1}-2^{n-2}=2^{n-2}$ vertices of Q'_{n-1} that resolve u and v. Hence, the sum of the values of the neighbors of $\{u,v\}$ in $R(Q_n)$ is at least $2^{n-2}(1/2^{n-2})=1$.

Case 4: Suppose position 1 belongs to $\mathscr{P}'$. Then one of u and v belongs to Q'_{n-1}, and the other to Q''_{n-1}. Suppose $u\in V(Q'_{n-1})$ and $v\in V(Q''_{n-1})$. Since v necessarily differs in position 1 from all vertices of Q'_{n-1} and as it differs in exactly k positions from z that belong to $\mathscr{P}'$, there are $2^{n-2k}\binom{2k-1}{k-1}$ vertices z in Q'_{n-1} that do not resolve u and v. By Lemma 6, $2^{n-2k}\binom{2k-1}{k-1}\leqslant(2^{n-2k})(2^{2k-2})=2^{n-2}$. So there are at least $2^{n-1}-2^{n-2}=2^{n-2}$ vertices in Q'_{n-1} that resolve the pair $\{u,v\}$. Thus the sum of the values of the neighbors of $\{u,v\}$ in $R(Q_n)$ is at least $2^{n-2}(1/2^{n-2})=1$. Hence $\mathrm{frdim}(Q_n)\leqslant 2$, and the result follows. □

Note that for any connected graph G of order n, any set S of independently resolved pairs of vertices of G must be pairwise disjoint; otherwise, any vertex common to two pairs in the set resolves both pairs, contradicting the fact that the pairs of vertices in S are resolved independently. Thus, $\mathrm{mi}(G)\leqslant\lfloor n/2\rfloor$. This fact and the proof of Theorem 4 lead to the following result.

Theorem 8. *Let n be a positive integer, $n\geqslant 3$. Let G be the Cayley digraph for the dihedral group D_n with generating set $\{R_{360/n}, A\}$, where A is any reflection of D_n. Then* $\mathrm{mi}(G)=n$.

Proof. The proof of Theorem 4 demonstrates that the Cayley digraph G contains n independently resolved pairs of vertices. Thus $\mathrm{mi}(G)\geqslant n$. On the other hand, since G has order $2n$, $\mathrm{mi}(G)\leqslant\lfloor 2n/2\rfloor=n$. □

4. Closing remarks

This paper studies the metric dimension of Cayley digraphs (with minimal generating sets). Bounds for the metric dimension of the Cayley digraphs $\mathrm{Cay}(\Delta:\Gamma)$, where Γ is the group $\mathbb{Z}_{n_1}\oplus\mathbb{Z}_{n_2}\oplus\cdots\oplus\mathbb{Z}_{n_m}$ and Δ is the canonical set of generators, are established and it is shown that these bounds are sharp if $m=2$. The case where $m=3$ has been studied in more depth in [9] but is still partially unresolved. The case where $m\geqslant 4$ remains open. The undirected version was investigated in [10] and was shown to be unrelated to the size of the cyclic groups. More specifically it was shown, for the case $m=2$ and if at least one of the cyclic groups has order at least 3, that the metric dimension is 3 or 4 and depends on the parity of the cyclic groups.

An integer programming formulation of the metric dimension of (di)graphs and its corresponding dual, the metric independence, was studied for Cayley digraphs. It was shown that the metric dimension and the metric independence for the Cayley digraph of the dihedral group D_n, with a minimal set of generators, are both equal to n. On the other hand, it is shown that the metric dimension of the n-cube is 2. The asymptotically exact value for the metric dimension of the n-cube is $2n/\log n$ (see [12]). Thus the ratio of the metric dimension to the metric independence of Cayley (di)graphs maybe arbitrarily large.

The metric dimension of the Cayley digraph for the non-abelian symmetric group S_4, with generating set $\Delta=\{(1,2),(1,2,3,4)\}$, is shown to be 3. However, it remains an open problem to determine the metric dimension and metric independence of these Cayley digraphs for $n>4$.

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