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# LANGFORD SEQUENCES: PERFECT AND HOOKED

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A sequence $\{d, d+1, \ldots, d+m-1\}$ of $m$ consecutive positive integers is said to be perfect if the integers $\{1, 2, \ldots, 2m\}$ can be arranged in disjoint pairs $\{(a_i, b_i)\colon 1 \leq i \leq m\}$ so that $\{b_i - a_i\colon 1 \leq i \leq m\} = \{d, d+1, \ldots, d+m-1\}$. A sequence is hooked if the set $\{1, 2, \ldots, 2m-1, 2m+1\}$ can be arranged in pairs to satisfy the same condition. Well known necessary conditions for perfect sequences are herein shown to be sufficient. Similar necessary and sufficient conditions for hooked sequences are given.

The problem of constructing so-called Langford sequences seems to have originated in the work of Th. Skolem [7] who remarked that Steiner triple systems could be constructed from a sequence of integers $1, \ldots, 2m$ if these integers could be arranged in $m$ disjoint pairs $(a_i, b_i)$, $1 \leq i \leq m$, such that $b_i - a_i = i$. Skolem, also giving credit to Th. Bang, showed in [7] that $m \equiv 0$ or 1 (mod 4) is a necessary and sufficient condition. In [8] he raised the question of whether sequences of integers $1, \ldots, 2m-1$, $2m+1$ could be so arranged for the other two cases, $m \equiv 2$ *or* 3 (mod 4), remarking that this condition is certainly necessary.

One general form of Langford's problem, recast in the notation of Bermond, Brouwer and Germa [2], is to construct a partition of the set $\{1, 2, \ldots, 2m\}$ into $m$ pairs $(a_i, b_i)$ such that the $m$ numbers $b_i - a_i$, $1 \leq i \leq m$, are all of the integers in the set $d, d+1, \ldots, d+m-1$. The original Langford problem, which asked for $b_i - a_i = i+1$ [4], corresponds to $d=2$ in this formulation. The case $d=2$ was solved completely by Priday [5] and Davies [3], who called such sequences perfect. Davies also solved the corresponding problem of partitioning $\{1, 2, \ldots, 2m-1, 2m+1\}$ so that the differences $(b_i - a_i)$ exhaust the set $\{2, 3, \ldots, m+1\}$, calling such sequences hooked. Examples are most easily displayed by interpreting each pair $(a_i, b_i)$ as giving the positions in the sequences of the first and second appearances of the integer $d+i-1$, $1 \leq i \leq m$. Thus 4 2 3 2 4 3 $(d=2, m=3)$ is perfect and 6 4 2 5 2 4 6 3 5 * 3 $(d=2, m=5)$ is hooked. Generalizations in which more than two copies of each integer appear in the sequence have also been considered. Recent work on that problem and other references may be found in [1] and [6], which incidentally contains a correction to one of the theorems in [1].

In [2] among other places, it is shown that necessary conditions for $(d, \ldots, d+m-1)$ to be perfect are

(i) $m \geqslant 2d-1$, and

(ii) $m \equiv 0$ or $1 \pmod 4$, for $d$ odd, $m \equiv 0$ or $3 \pmod 4$ for $d$ even.

These conditions are also shown there to be sufficient for $d=3$ and 4. (Skolem had treated $d=1$ and Davies $d=2$.) Finally, the conditions are shown to be sufficient for $m \equiv 2d-1 \pmod 4$, which covers the cases where $m$ is odd. A number of other special cases are handled, but we will subsume them by giving formulas to establish sufficiency for all even values of $m$ satisfying (i).

**Theorem 1.** *The conditions* (i) *and* (ii) *given above are necessary and sufficient for the existence of perfect sequences.*

**Proof.** All unsettled cases have $m \equiv 0 \pmod 4$. Let $m=4t$. We use tables similar to those in [2].

*Case* 1. Suppose $d \equiv 0$ (mod 4), $d=4s$, $s \geqslant 1$, $t \geqslant 2s$.

| | $a_i$ | $b_i$ | $b_i - a_i$ | $0 \leqslant j \leqslant$ |
|---|---|---|---|---|
| (1) | $2t-3s+1-j$ | $2t+s+2+j$ | $4s+1+2j$ | $t-2s-1$ |
| (2) | $t-2s+1-j$ | $3t+s+1+j$ | $2t+3s+2j$ | $t-2s$ |
| (3) | $6t-s+1-j$ | $6t+3s+1+j$ | $4s+2j$ | $t-2s-1$ |
| (4) | $5t-s+1-j$ | $7t+2s+j$ | $2t+3s-1+2j$ | $t-2s$ |
| (5) | $3t-s+2$ | $5t-s+2$ | $2t$ | — |
| (6) | $2t-j$ | $4t+1+j$ | $2t+1+2j$ | $s-1$ |
| (7) | $4t-s+2+j$ | $6t+s+2+2j$ | $2t+2s+j$ | $s-2$ |
| (8) | $3t+1-j$ | $5t+3+j$ | $2t+2+2j$ | $s-2$ |
| (9) | $3t+s-j$ | $7t+s+1+j$ | $4t+1+2j$ | $s-2$ |
| (10) | $t-s+1-j$ | $5t-s+3+j$ | $4t+2+2j$ | $s-1$ |
| (11) | $2t-3s+2+j$ | $6t-s+3+2j$ | $4t+2s+1+j$ | $2s-2$ |
| (12) | $2t+1+j$ | $6t-s+2+2j$ | $4t-s+1+j$ | $s-1$ |
| (13) | $2t+s+1$ | $6t+3s$ | $4t+2s-1$ | — |

Clearly rows 7, 8 and 9 are to be omitted for $s=1$. Similarly, rows 1 and 3 are omitted when $t=2s$.

*Case* 2. $d \equiv 2 \pmod 4$. Let $d=4s+2$, $s \geqslant 1$, $t \geqslant 2s+1$.

| | $a_i$ | $b_i$ | $b_i - a_i$ | $0 \leqslant j \leqslant$ |
|---|---|---|---|---|
| (1) | $2t-3s-1-j$ | $2t+s+2+j$ | $4s+3+2j$ | $t-2s-2$ |
| (2) | $t-2s-j$ | $3t+s+1+j$ | $2t+3s+1+2j$ | $t-2s-1$ |
| (3) | $6t-s-j$ | $6t+3s+2+j$ | $4s+2+2j$ | $t-2s-2$ |
| (4) | $5t-s-1-j$ | $7t+2s+1+j$ | $2t+3s+2+2j$ | $t-2s-2$ |
| (5) | $5t-s$ | $7t+s+1$ | $2t+2s+1$ | — |
| (6) | $2t-j$ | $4t+1+j$ | $2t+1+2j$ | $s-1$ |
| (7) | $4t-s+1+j$ | $6t+s+3+2j$ | $2t+2s+2+j$ | $s-2$ |
| (8) | $3t+1-j$ | $5t+1+j$ | $2t+2j$ | $s$ |
| (9) | $3t+s-j$ | $7t+s+2+j$ | $4t+2+2j$ | $s-2$ |
| (10) | $t-s-j$ | $5t-s+1+j$ | $4t+1+2j$ | $s-1$ |
| (11) | $2t-3s+j$ | $6t-s+2+2j$ | $4t+2s+2+j$ | $2s-1$ |

*Case* 2. (*contd.*)

| | | | | |
|---|---|---|---|---|
| (12) | $2t+1+j$ | $6t-s+1+2j$ | $4t-s+j$ | $s-1$ |
| (13) | $2t+s+1$ | $6t+3s+1$ | $4t+2s$ | — |
| (14) | $2t-s$ | $6t+s+1$ | $4t+2s+1$ | — |
| (15) | $4t$ | $8t$ | $4t$ | — |

Rows 7 and 9 are omitted for $s=1$. Rows 1, 3 and 4 are omitted when $t=2s+1$. For $s=0$, $d=2$, see Davies [3].

*Case* 3. $d=4s-1$, $s\geq 1$, $t\geq 2s$.

| | $a_i$ | $b_i$ | $a_i-b_i$ | $0\leq j\leq$ |
|---|---|---|---|---|
| (1) | $2t-3s+1-j$ | $2t+s+1+j$ | $4s+2j$ | $t-2s-1$ |
| (2) | $t-2s+2-j$ | $3t+s+j$ | $2t+3s-2+2j$ | $t-2s+1$ |
| (3) | $6t-s-j$ | $6t+3s-1+j$ | $4s-1+2j$ | $t-2s$ |
| (4) | $5t-s+1-j$ | $7t+2s+j$ | $2t+3s-1+2j$ | $t-2s$ |
| (5) | $3t-s+1$ | $5t-s+2$ | $2t+1$ | — |
| (6) | $2t+1-j$ | $4t+1+j$ | $2t+2j$ | $s-1$ |
| (7) | $4t-s+2+j$ | $6t+s+1+2j$ | $2t+2s-1+j$ | $s-2$ |
| (8) | $3t-1-j$ | $5t+2+j$ | $2t+3+2j$ | $s-3$ |
| (9) | $3t+s-1-j$ | $7t+s+j$ | $4t+1+2j$ | $s-1$ |
| (10) | $t-s+1-j$ | $5t-s+3+j$ | $4t+2+2j$ | $s-2$ |
| (11) | $2t-3s+2+j$ | $6t-s+2+2j$ | $4t+2s+j$ | $2s-2$ |
| (12) | $2t+2+j$ | $6t-s+3+2j$ | $4t-s+1+j$ | $s-2$ |
| (13) | $2t-s+1$ | $6t-s+1$ | $4t$ | — |

Omit 5, 7, 8, 10 and 12 when $s=1$; omit 8 when $s=2$; omit 1 for $t=2s$.

*Case* 4. $d=4s+1, s\geq 1$, $t\geq 2s+1$.

| | $a_i$ | $b_i$ | $b_i-a_i$ | $0\leq j\leq$ |
|---|---|---|---|---|
| (1) | $2t-3s-1-j$ | $2t+s+1+j$ | $4s+2+2j$ | $t-2s-2$ |
| (2) | $t-2s-j$ | $3t+s+1+j$ | $2t+3s+1+2j$ | $t-2s-1$ |
| (3) | $6t-s-1-j$ | $6t+3s+j$ | $4s+1+2j$ | $t-2s-1$ |
| (4) | $5t-s-j$ | $7t+2s+j$ | $2t+3s+2j$ | $t-2s-1$ |
| (5) | $t-s$ | $3t+s$ | $2t+2s$ | — |
| (6) | $2t+1-j$ | $4t+1+j$ | $2t+2j$ | $s-1$ |
| (7) | $4t-s+1+j$ | $6t+s+2+2j$ | $2t+2s+1+j$ | $s-2$ |
| (8) | $3t-1-j$ | $5t+j$ | $2t+1+2j$ | $s-1$ |
| (9) | $3t+s-1-j$ | $7t+s+j$ | $4t+1+2j$ | $s-1$ |
| (10) | $t-s-1-j$ | $5t-s+1+j$ | $4t+2+2j$ | $s-2$ |
| (11) | $2t-3s+j$ | $6t-s+1+2j$ | $4t+2s+1+j$ | $2s-1$ |
| (12) | $2t+2+j$ | $6t-s+2+2j$ | $4t-s+j$ | $s-2$ |
| (13) | $2t-s+1$ | $6t-s$ | $4t-1$ | — |
| (14) | $2t-s$ | $6t+s$ | $4t+2s$ | — |
| (15) | $4t$ | $8t$ | $4t$ | — |

Omit 7, 10 and 12 when $s=1$. Omit 1 for $t=2s+1$. The case $s=0$, $d=1$, is covered by Skolem [7].

Following Davies we define a hooked sequence as a sequence $\{d, d+1, \ldots, d+m-1\}$ for which there is a partition of the set $\{1, 2, \ldots, 2m-1, 2m+1\}$ into $m$ pairs $(a_i, b_i)$ such that the $m$ numbers $b_i-a_i$, $1\leq i\leq m$ are all of the integers

$d, d+1, \ldots, d+m-1$. Simple examples are 4 8 5 7 4 3 6 5 3 8 7 * 6 and 6 4 7 5 8 4 6 3 5 7 3 * 8, using $d=3$, $m=6$. As before, $a_i$ and $b_i$ are interpreted as the two positions in the sequence where $b_i - a_i$ appears.

**Theorem 2.** *Necessary and sufficient conditions for the sequence* $\{d, d+1, \ldots, d+m-1\}$ *to be hooked are*

(iii) $m(m+1-2d)+2 \geqslant 0$, *and*

(iv) $m \equiv 2$ *or* 3 (mod 4) *for d odd*, $m \equiv 2$ *or* 1 (mod 4) *for d even.*

**Proof.** Necessity flows from the observation that $D = \frac{1}{4}[m(m+1-2d)+2]$ must be a non-negative integer. This is a consequence of the following computations:

$$\sum_{i=1}^{m} a_i + \sum_{i=1}^{m} b_i = \left(\sum_{j=1}^{2m-1} j\right) + 2m + 1,$$

$$\sum_{i=1}^{m} (b_i - a_i) = \sum_{j=d}^{d+m-1} j = \tfrac{1}{2}m(m+2d-1).$$

Solving for $\sum_{i=1}^{m} a_i$ yields

$$\sum_{i=1}^{m} a_i = \tfrac{1}{2}[2m^2 + m + 1 - \tfrac{1}{2}m(m+2d-1)] = \tfrac{1}{4}m(3m-2d+3) + \tfrac{1}{2}.$$

The minimum possible value of $\sum a_i$ would be obtained if $\{a_i\} = \{1, \ldots, m\}$, or if $\sum_{i=1}^{m} a_i = \frac{1}{2}m(m+1)$. The difference $D$ between the calculated value of $\sum a_i$ and this minimum is necessarily a non-negative integer:

$$D = \tfrac{1}{4}m(3m-2d+3) + \tfrac{1}{2} - \tfrac{1}{2}m(m+1) = \tfrac{1}{4}(m-2d+1) + \tfrac{1}{2}.$$

We remark that the same proof gives the necessary conditions for perfect sequences, except that the final $\frac{1}{2}$ is not present in the expression for $D$.

Sufficiency is demonstrated by providing tables, as before.

*Case* 1. Suppose $m \equiv 2d+1$ (mod 4), with $d \geqslant 3$, so that $m = 2d+1+4r$. Also, let $q=0$ if $d \equiv 2$ or 3 (mod 4) and $q=1$ if $d \equiv 0$ or 1 (mod 4). This covers all cases where $m$ is odd, $m \geqslant 7$. Suppose first that $d = 2e$ is even:

| | $a_i$ | $b_i$ | $b_i - a_i$ | $0 \leqslant j \leqslant$ |
|---|---|---|---|---|
| (1) | $2r+1-j$ | $2r+d+1+j$ | $d+2j$ | $r$ |
| (2) | $r-j$ | $3r+d+3+j$ | $2r+d+3+2j$ | $r-1$ |
| (3) | $m+2r+1-j$ | $m+2r+d+2+j$ | $d+1+2j$ | $r-1$ |
| (4) | $m+r+1-j$ | $m+3r+d+3+j$ | $2r+d+2+2j$ | $r$ |
| (5) | $3r+d+2$ | $m+3r+d+2$ | $m$ | — |
| (6) | $4r+d+3$ | $2m+1$ | $m+d-1$ | — |
| (7) | $2r+d-j$ | $m+2r+2+j$ | $m-d+2+2j$ | $e-2$ |
| (8) | $m-j$ | $2m-d+3+j$ | $m-d+3+2j$ | $e-1$ |
| (9) | $m-e-1-j$ | $2m-e+3+j$ | $m+4+2j$ | $e-4$ |
| (10) | $m-e$ | $m+2r+e+1$ | $2r+d+1$ | — |
| (11) | $2r+e$ | $m+2r+e+2$ | $m+2$ | — |

*Case* 1. (*contd.*)

| | $a_i$ | $b_i$ | $b_i-a_i$ | $0\le j\le$ |
|---|---|---|---|---|
| (12) | $2r+e+1-2j$ | $m+2r+e+4+2j$ | $m+3+4j$ | $\lfloor\frac{1}{2}(e+1)\rfloor-2$ |
| (13) | $2r+e-2-2j$ | $m+2r+e+3+2j$ | $m+5+4j$ | $\lfloor\frac{1}{2}e\rfloor-2$ |
| (14) | $2r+2+q$ | $m+2r+d+q$ | $m+d-2$ | — |

(Here $\lfloor x\rfloor$ is the 'floor' or greatest integer function of $x$.)

For $d$ odd $(d=2e+1)$, replace lines 7 through 13 by:

| | | | | |
|---|---|---|---|---|
| (7′) | $2r+d-j$ | $m+2r+2+j$ | $m-d+2+2j$ | $e-1$ |
| (8′) | $m-j$ | $2m-d+3+j$ | $m-d+3+2j$ | $e-2$ |
| (9′) | $m-e-j$ | $2m-e+1+j$ | $m+1+2j$ | $e-2$ |
| (10′) | $m-e+1$ | $m+2r+e+3$ | $2r+d+1$ | — |
| (11′) | $2r+e$ | $m+2r+e+2$ | $m+2$ | — |
| (12′) | $2r+e+1-2j$ | $m+2r+e+5+2j$ | $m+4+4j$ | $\lfloor\frac{1}{2}(e+1)\rfloor-2$ |
| (13′) | $2r+e-2-2j$ | $m+2r+e+4+2j$ | $m+6+4j$ | $\lfloor\frac{1}{2}e\rfloor-2$ |

Omit 2 and 3 when $r=0$. Omit 13′ when $d=7$. Omit 9 and 13 when $d=6$. Omit 12′ and 13′ when $d=5$. For $d=4$, besides dropping 9, 12 and 13, replace 8, 10 and 11 by

| | | | | |
|---|---|---|---|---|
| (8*) | $m$ | $2m-1$ | $m-1$ | — |
| (10*) | $m-1$ | $m+2r+4$ | $2r+5$ | — |
| (11*) | $2r+2$ | $m+2r+3$ | $m+1$ | — |

Omit $8', 9', 11', 12'$ and $13'$ for $d=3$. For $d=2, r\ge 1$, use the perfect 3-to-$(m+1)$ sequence with $2*2$ attached. For $d=1, r\ge 2$, use the perfect 3-to-$m$ sequence with $112*2$ attached. These may be found in [2, Theorem 1, p. 36], using $m'=m-1$ or $m'=m-2$ for $m$. For $r=0, d=2$, we have $m=5$, which may be solved by $2\,5\,2\,4\,6\,3\,5\,4\,3*6$. Solve $r=0$, $d=1$, $m=3$ by $1\,1\,2\,3\,2*3$. Finally, solve $r=1$, $d=1$, $m=7$ by $2\,6\,2\,1\,1\,7\,4\,6\,3\,5\,4\,3\,7*5$. These special cases are needed because the formulas in [2] do not provide perfect sequences for such small values of $m'$.

The remaining cases all concern $m\equiv 2 \pmod 4$. Henceforth $m=4t+2$.

*Case* 2. Let $d=4s$, with $t-2s=r\ge 0$.

| | $a_i$ | $b_i$ | $b_i-a_i$ | $0\le j\le$ |
|---|---|---|---|---|
| (1) | $2r+s+1-j$ | $2r+5s+1+j$ | $4s+2j$ | $r$ |
| (2) | $r+1-j$ | $3r+7s+2+j$ | $2r+7s+1+2j$ | $r$ |
| (3) | $m+2r+3s-j$ | $m+2r+7s+1+j$ | $4s+1+2j$ | $r-1$ |
| (4) | $m+r+s+1-j$ | $m+3r+8s+1+j$ | $2r+7s+2j$ | $r$ |
| (5) | $m+s$ | $2m+1$ | $m-s+1$ | — |
| (6) | $2t-1-j$ | $m+1+j$ | $2t+4+2j$ | $s-2$ |
| (7) | $m-s+2+j$ | $m+2t+s+3+2j$ | $2t+2s+1+j$ | $s-2$ |
| (8) | $m-s+1$ | $m+2t-s+3$ | $2t+2$ | — |
| (9) | $3t-j$ | $5t+3+j$ | $2t+3+2j$ | $s-2$ |
| (10) | $3t+s-j$ | $m+3t+s+1+j$ | $m+1+2j$ | $s-1$ |
| (11) | $3t+s+1$ | $5t+s+2$ | $2t+1$ | — |
| (12) | $t-s-j$ | $m+t-s+2+j$ | $m+2+2j$ | $s-2$ |
| (13) | $2t+1-j$ | $m+2t-s+1+j$ | $m-s+2j$ | 1 |

*Case* 2. (*contd.*)

| | $a_i$ | $b_i$ | $b_i-a_i$ | $0\le j\le$ |
|---|---|---|---|---|
| (14) | $2t+2+j$ | $m+2t-s+5+2j$ | $m-s+3+j$ | $s-3$ |
| (15) | $2t+s$ | $m+2t+3s$ | $m+2s$ | — |
| (16) | $2t-3s+3+j$ | $m+2t-s+4+2j$ | $m+2s+1+j$ | $2s-3$ |
| (17) | $2t-3s+2$ | $m+2t+s+1$ | $m+4s-1$ | — |

Omit 3 when $r=0$. Omit 14 for $s=2$. For $s=1$, omit 6, 7, 9, 12, 14, 15 and 17, and replace 13 and 16 by

| | | | | |
|---|---|---|---|---|
| (13*) | $2t+1$ | $m+2t$ | $m-1$ | — |
| (16*) | $2t-1+j$ | $m+2t+1+2j$ | $m+2+j$ | 1 |

*Case* 3. Let $d=4s+1$, with $t-2s=r\ge 0$. Suppose $s=2e$ is even:

| | $a_i$ | $b_i$ | $b_i-a_i$ | $0\le j\le$ |
|---|---|---|---|---|
| (1) | $2r+s+2-j$ | $2r+5s+3+j$ | $4s+1+2j$ | $r-1$ |
| (2) | $r+2-j$ | $3r+7s+2+j$ | $2r+7s+2j$ | $r$ |
| (3) | $m+2r+3s+1-j$ | $m+2r+7s+3+j$ | $4s+2+2j$ | $r-1$ |
| (4) | $m+r+s+1-j$ | $m+3r+8s+2+j$ | $2r+7s+1+2j$ | $r-1$ |
| (5) | $m+s+1$ | $2m+1$ | $m-s$ | — |
| (6) | $2t+2-j$ | $m+2+j$ | $2t+2+2j$ | $s-2$ |
| (7) | $4t-s+3+j$ | $m+2t+s+1+2j$ | $2t+2s+j$ | $s-1$ |
| (8) | 1 | $m+1$ | $m$ | — |
| (9) | $3t+2-j$ | $m+t+3+j$ | $2t+3+2j$ | $s-2$ |
| (10) | $3t+s+1-j$ | $m+3t+s+3+j$ | $m+2+2j$ | $s-2$ |
| (11) | $3t-s+3$ | $m+t-s+2$ | $2t+1$ | — |
| (12) | $t-s+2-j$ | $m+t-s+3+j$ | $m+1+2j$ | $s-1$ |
| (13) | $2t+4$ | $m+2t-s+3$ | $m-s-1$ | — |
| (14) | $2t+5+j$ | $m+2t-s+6+2j$ | $m-s+1+j$ | $s-3$ |
| (15) | $2t+3$ | $m+2t+3s+1$ | $m+3s-2$ | — |
| (16) | $2t-2s+2+j$ | $m+2t+s+2+2j$ | $m+3s+j$ | $s$ |
| (17) | $2t-s+3$ | $m+2t-s+2$ | $m-1$ | — |
| (18) | $2t-3s+3$ | $m+2t-s+4$ | $m+2s+1$ | — |
| (19) | $2t-3s+4+2j$ | $m+2t-s+7+4j$ | $m+2s+3+2j$ | $e-2$ |
| (20) | $2t-3s+5+2j$ | $m+2t-s+5+4j$ | $m+2s+2j$ | $e-2$ |

If $s=2e+1$ is odd, replace lines 18, 19 and 20 by:

| | | | | |
|---|---|---|---|---|
| (18′) | $2t-3s+4$ | $m+2t-s+4$ | $m+2s$ | — |
| (19′) | $2t-3s+3+2j$ | $m+2t-s+5+4j$ | $m+2s+2+2j$ | $e-1$ |
| (20′) | $2t-3s+6+2j$ | $m+2t-s+7+4j$ | $m+2s+1+2j$ | $e-2$ |

Omit 1, 3 and 4 if $r=0$. Omit 20′ if $s=3$. Omit 14, 19 and 20 if $s=2$. For $s=1$ omit 6, 9, 10, 14, 17, 18′, 19′, and 20′, and replace 13 and 15 by

| | | | | |
|---|---|---|---|---|
| (13*) | $2t+3$ | $m+2t+1$ | $m-2$ | — |
| (15*) | $2t+2$ | $m+2t+4$ | $m+2$ | — |

For $s=0$ attach 11 to the hooked 2-to-$m$ sequence (Case 1 above).

*Case* 4. Let $d=4s+2$, with $t-2s-1=r\geqslant 0$. (The case $t=2s$ is only possible if $t=0$, which is the sequence $2\,3\,2*3$.)

| | $a_i$ | $b_i$ | $b_i-a_i$ | $0\leqslant j\leqslant$ |
|---|---|---|---|---|
| (1) | $2r+s+2-j$ | $2r+5s+4+j$ | $4s+2+2j$ | $r$ |
| (2) | $r+2-j$ | $3r+7s+6+j$ | $2r-7s+4+2j$ | $r$ |
| (3) | $m+2r+3s+2-j$ | $m+2r+7s+5+j$ | $4s+3+2j$ | $r-1$ |
| (4) | $m+r+s+2-j$ | $m+3r+8s+5+j$ | $2r+7s+3+2j$ | $r$ |
| (5) | $m+s+1$ | $2m+1$ | $m-s$ | — |
| (6) | $2t+1-j$ | $m+2+j$ | $2t+3+2j$ | $s-2$ |
| (7) | $4t-s+3+j$ | $m+2t+s+2+2j$ | $2t+2s+1+j$ | $s-1$ |
| (8) | $1$ | $m+1$ | $m$ | — |
| (9) | $3t+1-j$ | $m+t+1+j$ | $2t+2+2j$ | $s-1$ |
| (10) | $3t+s+1-j$ | $m+3t+s+2+j$ | $m+1+2j$ | $s-1$ |
| (11) | $3t+s+2$ | $m+t+s+1$ | $2t+1$ | — |
| (12) | $t-s-j$ | $m+t-s+2+j$ | $m+2+2j$ | $s-2$ |
| (13) | $2t+2$ | $m+2t-s+1$ | $4t-s+1$ | — |
| (14) | $2t+3+j$ | $m+2t-s+4+2j$ | $m-s+1+j$ | $s-2$ |
| (15) | $2t-s+2$ | $m+2t+3s+2$ | $m+4s$ | — |
| (16) | $2t-3s+1$ | $m+2t-s+2$ | $m+2s+1$ | — |
| (17) | $2t-3s+2+2j$ | $m+2t-s+5+4j$ | $m+2s+3+2j$ | $s-1$ |
| (18) | $2t-3s+3+2j$ | $m+2t-s+3+4j$ | $m+2s+2j$ | $s-1$ |

Omit 3 if $r=0$. Omit 6, 12 and 14 if $s=1$. For $s=0$, which is $d=2$, attach $2*2$ to the corresponding perfect sequence starting with 3, given in [2, p. 36].

*Case* 5. Let $d=4s+3$, with $t-2s-1=r\geqslant 0$.

| | $a_i$ | $b_i$ | $b_i-a_i$ | $0\leqslant j\leqslant$ |
|---|---|---|---|---|
| (1) | $2r+s+2-j$ | $2r+5s+5+j$ | $4s+3+2j$ | $r-1$ |
| (2) | $r+1-j$ | $3r+7s+5+j$ | $2r+7s+4+2j$ | $r$ |
| (3) | $m+2r+3s+2-j$ | $m+2r+7s+6+j$ | $4s+4+2j$ | $r-1$ |
| (4) | $m+r+s+1-j$ | $m+3r+8s+6+j$ | $2r+7s+5+2j$ | $r-1$ |
| (5) | $m+s+1$ | $2m+1$ | $m-s$ | — |
| (6) | $2t-j$ | $m+1+j$ | $2t+3+2j$ | $s-1$ |
| (7) | $m-s+1+j$ | $m+2t+s+3+2j$ | $2t+2s+2+j$ | $s-1$ |
| (8) | $m-s$ | $m+2t-s+2$ | $2t+2$ | — |
| (9) | $3t+1-j$ | $m+t+3+j$ | $2t+4+2j$ | $s-2$ |
| (10) | $3t+s+1-j$ | $m+3t+s+3+j$ | $m+2+2j$ | $s-1$ |
| (11) | $3t-s+2$ | $m+t-s+1$ | $2t+1$ | — |
| (12) | $t-s+1-j$ | $m+t-s+2+j$ | $m+1+2j$ | $s$ |
| (13) | $2t+2$ | $m+2t-s+1$ | $m-s-1$ | — |
| (14) | $2t+3+j$ | $m+2t-s+4+2j$ | $m-s+1+j$ | $s-1$ |
| (15) | $2t+1$ | $m+2t+3s+3$ | $m+3s+2$ | — |
| (16) | $2t-3s+1+j$ | $m+2t-s+3+2j$ | $m+2s+2+j$ | $s-1$ |
| (17) | $2t-2s+1+j$ | $m+2t+s+4+2j$ | $m+3s+3+j$ | $s-1$ |

Omit 1, 3 and 4 if $r=0$. Omit 9 if $s=1$. For $r\geqslant 1$, $s=0$, hook $3**3$ onto the corresponding hooked 4 to $m+2$ sequence (Case 1). Examples for $r=0=s$ have already been given. This completes the proof.

The number $D$ that appears in this proof, and its variation for perfect sequences, has a very concrete interpretation. Let positions 1 through $m$ be the first half of the sequence, and the others the second half of the sequence. For some of the integers in $A=\{d, d+1, \ldots, d+m-1\}$, both appearances are in the same half. That is, either $b_i \leqslant m$ or $a_i > m$. Using $(a_j, b_j)$ as the two locations where $j \in A$ appears, let $Q_1=\{j \in A: b_j \leqslant m\}$ and $Q_2=\{j \in A: a_j > m\}$. Then $D=\sum_{j \in Q_2} a_j - \sum_{j \in Q_1} b_j$. That is, $D$ is the sum of the differences between first location of those $j$ that appear twice in the second half and second location of those $j$ that appear twice in the first half.

It is apparent that the argument for necessity given above could be repeated for constructions required to yield one (or more) gaps in other locations. Each such argument would produce an expression for $D$ leading to conditions like those in Theorems 1 and 2. It seems likely that such conditions would also generally be sufficient, with only a few exceptions for small values of $d$ and $m$. In fact Priday [5] calls certain sequences looped, and we may modify his definition to fit our notation by saying that $A=\{d, d+1, \ldots, d+m-1\}$ is looped if there is an arrangement of $\{1, 2, \ldots, 2m-1, 2m+2\}$ and an arrangement of $\{1, 2, \ldots, 2m-2, 2m, 2m+1\}$, each satisfying $\{b_i - a_i\} = A$. Both conditions lead to the same value of $D=\frac{1}{4}m(m-2d+1)+1$. For this $D$ to be a non-negative integer is equivalent to (i) and (ii) above, except when this $D=0$. One may conjecture therefore that a sequence is perfect iff it is looped, with the following exceptions:

(1) $\{3,4,5,6\}$ is looped (5 3 6 4 3 5 * 4 6 and 4 5 3 6 4 3 5 * * 6) but is not perfect.

(2) $\{1\}$ is perfect but not looped.

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