Logarithmic integrals and system dynamics: an analogue of Bode’s sensitivity integral for continuous-time, time-varying systems

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Abstract

A new time-domain interpretation of Bode’s integral is presented. This allows for a generalization to the class of time-varying systems which possess an exponential dichotomy. It is shown that the sensitivity function is constrained, on average, by the spectral values in the dichotomy spectrum of the antistable component of the open-loop dynamics. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

It should not be surprising that the open loop dynamics of a system impose limits on the achievable performance in a closed-loop system. Deriving the characterizations of these constraints has been a topic of enduring interest in mathematical control theory. One of the earliest is due to Bode [1], who showed that for a single-input single-output, stable, open loop system \( L(s) \), the sensitivity function \( S(s) = 1/(1 + L(s)) \) must satisfy

\[
\int_0^\infty \log |S(i\omega)| d\omega = 0. \tag{1.1}
\]
This integral has very important practical implications. For example, in the tracking problem depicted in Fig. 1, one seeks to keep the magnitude of the tracking error $E(i\omega)$ below $\varepsilon < 1$ for reference signals with frequency content $\omega \in [0, \Omega]$. Thus, the sensitivity function must satisfy

$$\log |S(i\omega)| < \log \varepsilon < 0 \quad \forall \omega \in [0, \Omega].$$

If one imposes a bandwidth requirement on the closed-loop system, this integral constraint must be satisfied in $\omega \in (\Omega, \Omega_{BW})$. Since the integral of $\log |S(i\omega)|$ is zero, it must follow that $\log |S(i\omega)| > 0$ for some $\omega \notin (\Omega, \Omega_{BW})$. In fact, a simple calculation shows that

$$0 \leq (-\log \varepsilon)\Omega \leq (\Omega_{BW} - \Omega) \sup_{\omega \in (\Omega, \Omega_{BW})} \log |S(i\omega)| \leq (\Omega_{BW} - \Omega) \log \|S\|_{\infty}$$

so that

$$\|S\|_{\infty} \geq \exp\left(\frac{\Omega}{\Omega_{BW} - \Omega}(-\log \varepsilon)\right).$$

Thus, a smaller $\varepsilon$ or a larger $\Omega$ will increase the corresponding values of $|S(i\omega)|$ in $\omega \notin [0, \Omega]$.

Bode’s integral has now been extended in numerous ways. Probably the most important generalization is that of Freudenberg and Looze [2], who dispensed with the stability assumption on $L(s)$. In particular, they showed that if $L(s)$ has unstable poles $\{p_i\}_{i=1}^{n_u}$, then the integral is now equal to

$$\int_{0}^{\infty} \log |\det S(i\omega)| \, d\omega = \pi \sum_{i=1}^{n_u} \text{Re} \, p_i > 0. \quad (1.2)$$

The appearance of the unstable poles in the right-hand side of (1.2) serves only to worsen the bandwidth and magnitude tradeoffs mentioned with regards to (1.1). Freudenberg and Looze also provided similar constraints on the integral of the complementary sensitivity function $T(s) = I - S(s)$. Discrete-time and sampled-data versions of these integrals are found in [3–6]. While the result in (1.2) applies to multivariable systems, there has also been considerable effort in deriving sensitivity integrals on the individual singular values of $S(i\omega)$, rather than the weighted integral one obtains from the determinant. Contributions in this vein include [7–11].

These extensions all have one thing in common: they rely on Poisson-type integrals of the system’s sensitivity transfer function. Extending these results to other classes of systems that do not allow for straightforward transfer functions, for example time-varying or nonlinear systems, presents several difficulties. The first is that
the integral in the left-hand side of (1.1) may not be computable. A second related difficulty is that the right-hand side of (1.1), that requires the poles of the system, also needs further interpretation. For discrete-time, time-varying systems, an extension of Bode’s integral was made by the author in [12]; for nonlinear systems an extension is found in [13].

The goal of this paper is to present a generalization of Bode’s integral relation to continuous-time, time-varying systems. This work can be seen as a counterpart of the discrete-time results presented in [12].

What makes possible these extensions is the connection that exists between the logarithmic integral found in Bode’s relationship and several cost functions used in control theory. One of these is the minimum entropy control problem considered by Mustafa and Glover [14]. A time-varying analogue of this cost function was considered by Iglesias in [15]. Glover and Doyle [16] showed that in the linear time-invariant case, the minimum entropy cost function is equivalent to that considered in stochastic risk-sensitive control problems. While this relationship does not hold for very general linear-time varying systems, it does hold for both discrete and continuous-time systems that admit a state-space realization [17]. These connections serve to provide a time-domain interpretation to (1.1). Moreover, they allow extensions to time-varying systems.

The rest of the paper is organized as follows. Section 2 provides some necessary preliminary material. This material includes a consideration of the spectrum of linear-time varying systems. It is worth pointing out that in the right-hand side of (1.2), it is not so much the value of the poles that matter, but the value of the real part of the poles. This small difference is significant, as it has been known since Lyapunov’s work that there exist natural generalizations to these, namely, the Lyapunov exponents or characteristic numbers of the system. Definitions and properties of the Lyapunov exponents, exponential dichotomies and the dichotomy spectrum are provided.

To consider the sensitivity operator, it will be necessary to perform an inner/outer factorization of the sensitivity input/output operator. In Section 3 we show that for the particular class of systems considered in this paper, these factorizations always exist. We then provide a state-space realization of both factors.

In Section 4 we present several time-domain characterizations of Bode’s integral. This is done by relating the integral to two cost functions of robust control. The time-domain characterization allows for a generalization to time-varying systems. In Section 5 we show how this generalization can be expressed in terms of the unstable dynamics of the open-loop system. Finally, some concluding remarks are provided in Section 6.

2. Preliminaries

We will consider linear time-varying systems $\Sigma_G$ admitting a finite-dimensional, state-space representation
\[
\Sigma_G := \begin{bmatrix}
\dot{x}(t) = A(t)x(t) + B(t)w(t) \\
y(t) = C(t)x(t) + D(t)w(t)
\end{bmatrix} =
\begin{bmatrix}
A(t) & B(t) \\
C(t) & D(t)
\end{bmatrix}.
\]

We will assume that the matrix functions \( A(t) : \mathbb{R}^+ \mapsto \mathbb{R}^{n \times n} \), \( B(t) : \mathbb{R}^+ \mapsto \mathbb{R}^{n \times m} \), \( C(t) : \mathbb{R}^+ \mapsto \mathbb{R}^{p \times n} \), and \( D(t) : \mathbb{R}^+ \mapsto \mathbb{R}^{p \times m} \) are all continuous and bounded. With this system we associate an operator \( G \) mapping the input \( w \) to output \( y \). This operator has an integral representation
\[
y(t) = \int_{-\infty}^{t} g(t, \tau) w(\tau) \, d\tau,
\]
where the kernel \( g(t, \tau) \) equals
\[
g(t, \tau) = D\delta(t - \tau) + C(t)\Phi_A(t, \tau)B(\tau) \quad \text{if } t \geq \tau
\]
and zero otherwise. The matrix function \( \Phi(t, \tau) \) is the transition matrix which equals
\[
\Phi_A(t, \tau) = X(t)X^{-1}(\tau),
\]
where \( X(t) \) is the fundamental solution to the matrix differential equation
\[
\dot{X}(t) = A(t)X(t), \quad X(0) = X_0,
\]
and \( X_0 \) is invertible. The following result is standard, see e.g. [18], and will be needed in the sequel.

**Lemma 1** (Liouville’s formula). The transition matrix for \( A(t) \) satisfies
\[
\log \det \Phi_A(t, \tau) = \int_{\tau}^{t} \text{trace}[A(\sigma)] \, d\sigma
\]
for every \( t \) and \( \tau \).

### 2.1. Lyapunov exponents and exponential dichotomies

In the Freudenberg and Looze generalization of Bode’s integral (1.2), the real part of the unstable poles are needed. For time-varying systems, the Lyapunov exponents or characteristic numbers serve the same rôle as the real parts of the eigenvalues of the time-invariant matrix \( A(t) \equiv A \). We now present, following [19], some basic results on Lyapunov exponents.

#### 2.1.1. Lyapunov exponents

Consider the \( n \)-dimensional homogeneous system
\[
\dot{x}(t) = A(t)x(t).
\]
Suppose that \( X(t) \) is the fundamental solution with an orthogonal initial condition \( X_0 \), and let \( \{p_i\}_{i=1}^{n} \) be an orthonormal basis for \( \mathbb{R}^n \). Then the characteristic numbers
\[ \lambda_i(p_i) = \limsup_{t \to \infty} \frac{1}{t} \log \| X(t) p_i \| \]  

(2.3)

are well defined.

Suppose that the orthonormal basis \( \{ p_i \}_{i=1}^n \) is chosen so as to minimize \( \sum_{i=1}^n \lambda_i(p_i) \); the basis is then said to be normal and the corresponding \( \lambda_i \) are called the Lyapunov exponents.

For now we will write \( \lambda_i \) to be the Lyapunov exponents associated with a normal basis. It is well known that, in this case

\[ \sum_{i=1}^n \lambda_i \geq \limsup_{t \to \infty} \frac{1}{t} \int_0^t \text{trace}(A(s)) \, ds. \]  

(2.4)

In Bode’s result, it is important to differentiate between the stable poles of \( L(s) \)—which do not contribute to the right-hand side of (1.2)—and the unstable poles, which do. In our context, uniform exponential stability is required.

**Definition 2.** The matrix function \( A(t) \) is uniformly exponentially stable (UES) if there exist positive constants \( \gamma, \delta \) such that

\[ \| \Phi(t, \tau) \| \leq \gamma e^{-\delta(t-\tau)} \]

for all \( t \) and \( \tau \) such that \( t \geq \tau \).

**Definition 3.** The matrix function \( A(t) \) is uniformly exponentially antistable (UEA) if there exist positive constants \( \gamma, \delta \) such that

\[ \| \Phi(t, \tau) \| \leq \gamma e^{\delta(t-\tau)} \]

for all \( t \) and \( \tau \) such that \( \tau \geq t \).

It is straightforward to check that \( A(t) \) is antistable if and only if \( -A'(t) \) is stable.

For the rest of the paper, ‘stability’ will refer to uniform exponential stability. Similarly, an ‘antistable’ system is the one that is uniformly exponentially antistable.

2.1.2. Exponential dichotomy

A time-invariant system whose \( A \) matrix has no eigenvalues on the imaginary axis is similar to a block-diagonal matrix consisting of stable and antistable blocks. These systems are said to possess an exponential dichotomy. A similar notion is found for time-varying systems [20,21]. In particular, the linear system (2.2) is said to possess an exponential dichotomy if there exists a projection \( P \), and real constants \( \gamma > 0, \lambda > 0 \) such that

\[ \| X(t)PX^{-1}(\tau) \| \leq \gamma \exp(-\lambda(t-\tau)) \quad \text{for } t \geq \tau, \]

\[ \| X(t)(I-P)X^{-1}(\tau) \| \leq \gamma \exp(-\lambda(t-\tau)) \quad \text{for } \tau \geq t. \]
Note that, if \( \text{rank}(P) = n_s \), exponential dichotomy implies that \( n_s \) fundamental solutions are UES, whereas \( n_u = n - n_s \) are UEA.

The existence of an exponential dichotomy allows us to define a stability preserving state-space transformation (a Lyapunov transformation) that separates the stable and antistable parts of \( A(t) \).

**Lemma 4** [22, Chapter 5]. If the function \( A(t) \) in realization (2.1) admits an exponential dichotomy, then there exists a bounded matrix function \( T(t) \) with bounded inverses such that

\[
\begin{pmatrix}
(T(t) + T(t)A(t))T^{-1}(t) & T(t)B(t) \\
C(t)T^{-1}(t) & D(t)
\end{pmatrix} =:
\begin{pmatrix}
A_s(t) & 0 & B_s(t) \\
0 & A_u(t) & B_u(t) \\
C_s(t) & C_u(t) & D(t)
\end{pmatrix},
\]

where \( A_s(t) \) is UES and \( A_u(t) \) is UEA.

Whenever two matrices \( A_1(t) \) and \( A_2(t) \) are related by a Lyapunov transformation, we say that they are *kinematically similar* and this will be denoted \( A_1 \simeq A_2 \).

2.1.3. Dichotomy spectrum

Exponential dichotomies permit us to present another form of spectral representation for linear time-varying systems related to Lyapunov exponents. The dichotomy or Sacker-Sell spectrum \( S_{\text{dich}} \) of the system (2.2) is the set of real values \( \gamma \) for which the translated systems

\[
\dot{x}(t) = (A(t) - \lambda I)x(t)
\]

fail to have an exponential dichotomy [23].

In general, the spectrum is a collection of compact noninterlapping intervals:

\[
S_{\text{dich}} = \bigcup_{i=1}^{m} [\lambda_i, \bar{\lambda}_i],
\]

where \( m \leq n \) and \( \lambda_1 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \lambda_m \).

**Remark.** The dichotomy spectrum arises naturally from the resolvent set \( \rho(L) \) of the differential operator \( L \):

\[
(Lx)(t) = \dot{x}(t) - A(t)x(t).
\]

It can be shown that \( S_{\text{dich}} = \mathbb{R} - \rho(L) \) [23].

Suppose that \( \lambda_0, \lambda_1, \ldots, \lambda_m \) are chosen in the resolvent set \( \rho(L) \) so that

\[
\lambda_0 < \lambda_1 \leq \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_{m-1} < \lambda_{m-1} < \lambda_m \leq \lambda_m < \lambda_m.
\]

(2.7)

It is straightforward to check that for \( \lambda_0 \), all trajectories of (2.5) are unbounded. Similarly, for \( \lambda_m \), all trajectories of (2.1) are bounded.

Now, the matrix \( A(t) - \lambda_1 I \) admits an exponential dichotomy and thus, from Lemma (4), is kinematically similar to a block-diagonal matrix. Equivalently,
\[ A(t) \simeq \begin{bmatrix} A_1(t) & 0 \\ 0 & \bar{A}_1(t) \end{bmatrix}, \]

where \( \dim A_1(t) = n_1 \). Repeating this process with \( \lambda_2 \) leads to a resulting

\[ \bar{A}_1(t) \simeq \begin{bmatrix} A_2(t) & 0 \\ 0 & \bar{A}_2(t) \end{bmatrix}, \]

and \( n_2 = \dim(A_2) \). Continuing this procedure will lead to a sequence of matrices \( A_k(t) \) of dimension \( n_k \), for \( k = 1, \ldots, m \), so that \( A(t) \simeq \text{diag}(A_1(t), \ldots, A_m(t)) \) and \( \dim A_k(t) = n_k \), where \( n_1 + \cdots + n_m = n \). It should be stated that the resulting matrices \( A_k(t) \) do not depend on the particular choice of \( \lambda_k \), provided that (2.7) holds.

Since \( A(t) - \lambda_0 I \) is unstable, there exists an \( \epsilon > 0 \) and a \( K \geq 0 \), both depending on \( \lambda_0 \) such that, for \( t \geq s \):

\[
\Phi_{A - \lambda_0 I}(t, s) \Phi_{A - \lambda_0 I}(t, s) \geq K^2 \exp(-2\epsilon(s - t))I
\]

However,

\[
\Phi_{A - \lambda_0 I}(t, s) = \Phi_A(t, s)e^{-\lambda_0(t-s)}
\]

so that

\[
\Phi'_A(t, s)\Phi_A(t, s) \geq K^2 \exp(2(\lambda_0 + \epsilon)(t - s))I
\]

and thus,

\[
\log \det \Phi_A(t, s) \geq n \log K + n(\lambda_0 + \epsilon)(t - s).
\]

Divide by \( t - s \) and let this ratio go to infinity. Moreover, by Lemma 1,

\[
\lim_{t-s \to \infty} \frac{1}{t-s} \int_s^t \text{trace}[A(\sigma)]d\sigma \geq n(\lambda_0 + \epsilon).
\]

We can do this for any \( \lambda_0 < \lambda_1 \). Taking the limit as \( \lambda_0 \to \lambda_1 \), and setting \( t = T \) and \( s = -T \), we have

\[
\lim_{t \to \infty} \frac{1}{2T} \int_{-T}^T \text{trace}[A(\sigma)]d\sigma \geq n\lambda_1.
\]

For \( \lambda_m \), we have that \( A(t) - \lambda_m I \) is stable so that

\[
\Phi'_{A - \lambda_m I}(t, s)\Phi_{A - \lambda_m I}(t, s) \leq K^2 \exp(-2\epsilon(t - s))
\]

for some \( K \geq 0 \) and \( \epsilon > 0 \). Proceeding as above, we have that

\[
\lim_{t \to \infty} \frac{1}{2T} \int_{-T}^T \text{trace}[A(\sigma)]d\sigma \leq n\lambda_m.
\]

This procedure can be repeated for different \( \lambda_k \in (\bar{\lambda}_k, \tilde{\lambda}_{k+1}) \), \( k = 1, \ldots, m - 1 \), to yield the following.

**Lemma 5.** Suppose that the matrix function \( A(t) \) has dichotomy spectrum \( \mathcal{S}_{\text{dich}} = \sum_{k=1}^m [\bar{\lambda}_k, \tilde{\lambda}_k] \) satisfying (2.7), and suppose that the corresponding \( A_k(t) \) have dimensions \( n_k \), \( k = 1, \ldots, m \). Then
\[ \sum_{k=1}^{m} n_k \lambda_k \leq \lim_{t \to \infty} \frac{1}{2T} \int_{-T}^{T} \text{trace}[A(\sigma)] \, d\sigma \leq \sum_{k=1}^{m} n_k \bar{\lambda}_k. \] (2.8)

2.1.4. Regular systems

The special case where each of these intervals is a point (not necessarily unique), the spectrum is known as a point spectrum. In this case, each \( \lambda_i \) in the point spectrum equals a Lyapunov exponent, and the system is said to be regular. Moreover, in this case the lim sups in (2.3) and (2.4) can be replaced by limits [19].

Clearly, time-invariant systems are regular and the elements of the point spectrum are the real part of the eigenvalues of \( A \). Similarly, if the matrix function \( A(t) \) is periodic, then the Floquet theory (see [18] for a description) states that there exists a change of variable so that the resultant equation is time-invariant. The resultant point spectrum coincides with the Floquet spectrum.

In general, however, systems will not have point spectrum. An example of a \( 2 \times 2 \) real matrix with almost periodic coefficients is given in [24]. Unfortunately, regularity is hard to verify for any particular system, though all time-invariant and periodic systems are regular. In these two cases, the spectral values are the magnitude of the eigenvalues and Floquet characteristic exponents of the system. For systems involving a flow with an invariant probability measure, Oseledeč’s multiplicative ergodic theory states that regularity occurs with probability 1; see [25].

2.2. Assumptions on plant

We consider the system of Fig. 1, in which the open-loop system \( \Sigma_L \) has a uniformly stabilizable and detectable state-space representation

\[ \Sigma_L = \begin{bmatrix} A(t) & B(t) \\ C(t) & 0 \end{bmatrix}. \] (2.9)

It should be noted that this system will include both the controller and plant subsystems. Using (2.9) we have a corresponding state-space representation for the sensitivity system:

\[ \Sigma_S = \begin{bmatrix} A(t) - B(t)C(t) & B(t) \\ -C(t) & I \end{bmatrix}. \] (2.10)

We will need the following assumptions. First of all, we want to ensure that the sensitivity operator \( S \) associated with \( \Sigma_S \) is bounded. It is known that, if a realization is stabilizable and detectable, then the input–output operator is bounded if and only

---

1 Recall that a realization is uniformly stabilizable if there exists a bounded matrix function \( F(t) \) such that \( A(t) + B(t)F(t) \) is UES. Similarly, it is uniformly detectable if there exists a bounded matrix function \( L(t) \) such that \( A(t) + L(t)C(t) \) is UES.
if the ‘$A(t)$’ matrix is UES [26]. Since realization (2.9) is stabilizable and detectable, it is straightforward to check that so is that of (2.10) for $\Sigma_S$. Thus, we can assume that:

**Assumption 1.** $A(t) - B(t)C(t)$ is UES. Equivalently, $S$ is bounded.

Note that this assumption would be a necessary requirement of any internally stabilizing controller.

We will also need to differentiate between the stable and antistable dynamics of $\Sigma_L$. To this purpose, we assume:

**Assumption 2.** The open-loop dynamics $A(t)$ admits an exponential dichotomy of rank $n_s$. Moreover, the antistable component has dichotomy spectrum

$$A_u = [\bar{\lambda}_1, \lambda_1] \cup [\bar{\lambda}_2, \lambda_2] \cup \cdots \cup [\bar{\lambda}_m, \lambda_m]$$

with dimensions $n_1, \ldots, n_m$ and $\sum_{k=1}^m n_k = n_u$, where $n_s + n_u = n$.

In the time-invariant case, Bode’s result requires that the open-loop system has relative degree 2 or higher. In terms of state-space matrices, this amounts to requiring that $CB = 0$. Thus, we assume:

**Assumption 3.** The open-loop system has relative degree of at least 2; that is $C(t)B(t) = 0$ for all $t$.

3. **Inner/outer factorizations**

In the sequel, we will need to compute an inner/outer factorization of the sensitivity operator $S$. That is, we seek two systems with associated input/output operators $S_i$ and $S_o$ such that $S = S_iS_o$, where $S_i, S_o$ and $S_o^{-1}$ are all bounded and $\|S_o w\|_2 = \|w\|_2$ for any $w \in L^2$.

For the sensitivity operator arising from (2.10), we can derive state-space representations of its inner and outer factors in terms of the solution of a related Riccati differential equation. In the following result, we say that $(A, B)$ is *uniformly completely controllable* if there exist positive constants $\sigma, \alpha, \beta$ such that

$$\alpha I \leq W(t, t + \sigma) \leq \beta I$$

for all $t$, where

$$W(t, t + \sigma) := \int_t^{t+\sigma} \Phi_A(t, \tau)B(\tau)B'(\tau)\Phi_A(t, \tau) \, d\tau$$

(3.1)

is the controllability Gramian of $(A, B)$. Moreover, the Riccati differential equation

$$-\dot{X}(t) = A'(t)X(t) + X(t)A(t) - X(t)B(t)B'(t)X(t)$$

(3.2)
has a stabilizing solution if \( X(t) = X'(t) \geq 0 \) satisfies (3.2), \( X(t) \) is bounded, and \( A(t) - B(t)B'(t)X(t) \) is UES.

**Lemma 6.** Suppose that \( A(t) \) admits an exponential dichotomy and that the pair \((A, B)\) is stabilizable. Then (3.2) has a stabilizing solution \( X(t) \) and:

(i) If \( A(t) \) is UES, then \( X(t) \equiv 0 \) for all \( t \).

(ii) If \( A(t) \) is UEA, then \((\exists \epsilon > 0)\) such that \( \epsilon I \leq X(t) \) for all \( t \).

(iii) If

\[
A(t) = \begin{bmatrix} A_s(t) & 0 \\ 0 & A_u(t) \end{bmatrix}, \quad B(t) = \begin{bmatrix} B_s(t) \\ B_u(t) \end{bmatrix}
\]

admits an exponential dichotomy as in Lemma 4, then

\[
X(t) = \begin{bmatrix} 0 & 0 \\ 0 & X_u(t) \end{bmatrix}
\]

and \((\exists \epsilon > 0)\) such that \( \epsilon I \leq X_u(t) \) for all \( t \).

**Proof.** Note that the solution to the general equation (3.2) and that of item (iii) are related by \( X(t) \mapsto T'(t)X(t)T(t) \), where \( T(t) \) is the Lyapunov transformation of Lemma 4. Thus, to prove the existence of a general solution it is enough to prove (i)—which is trivial—and (ii). Note that existence of the solution follows from the general result of [27] that the Riccati differential equation has a solution iff the corresponding Hamiltonian has an exponential dichotomy. In this specific case, the Hamiltonian is block-diagonal and hence has an exponential dichotomy iff \( A(t) \) does. However, as we need to show boundedness (above and below) of the solution we will first show the existence and this will lead to the required bounds.

To prove (ii) we use the result of [28] that states that \((A, B)\) is uniformly stabilizable iff the pair \((A_u, B_u)\) is uniformly completely controllable. Note that in (ii), \( A_u \equiv A \) and \( B_u \equiv B \).

To prove item (ii), our approach follows that of [29]; see also [30]. Consider the finite-horizon matrix equation

\[
\dot{Q}_T(t) = A(t)Q_T(t) + Q_T(t)A'(t) - B(t)B'(t), \quad Q_T(T) = 0.
\]

The solution is \( Q_T(t) = W(t, T) \), where \( W(t, T) \) is given by (3.1). Note that

\[
\alpha I \leq Q_T(t) \leq \beta I \quad \text{for } t < T - \sigma.
\]

For two terminal times \( T_2 > T_1 \) it is straightforward to check that \( Q_{T_2}(t) \geq S_{T_1}(t) \) for all \( t < T_1 - \sigma \). Thus, \( \{Q_T(t)\} \), indexed by \( T \), is a nondecreasing sequence of continuous functions that is bounded below. Thus, a (unique) bounded function \( Q(t) \) defined as \( \lim_{T \to \infty} Q_T(t) = Q(t) \) exists. As in [29], it follows that \( Q(t) \) satisfies

\[
\dot{Q}(t) = A(t)Q(t) + Q(t)A'(t) - B(t)B'(t)
\]

and \( \alpha I \leq S(t) \leq \beta I \). Thus \( Q(t) \) is invertible. It is straightforward to check that \( X(t) = Q^{-1}(t) \) is the solution to (3.2) and that it is bounded above and below.
It remains to show that $A(t) - B(t)B'(t)X(t)$ is UES. This follows from the fact that due to the bounds on $Q(t)$, it is a Lyapunov transformation and thus $A(t) - B(t)B'(t)X(t)$ is kinematically similar to $-A'(t)$:

$$\dot{Q}(t) = (A(t) - B(t)B'(t)X(t))Q(t) - Q(t)(-A'(t))$$

and $-A'(t)$ is UES since $A(t)$ is UEA. □

**Corollary 7.** Under the assumptions of Lemma 6, the sensitivity operator has an inner/outer factorization $S = S_iS_o$, where the two factors have state-space representations:

$$S_i = \begin{bmatrix} A(t) - B(t)B'(t)X(t) & B(t) \\ -B'(t)X(t) & I \end{bmatrix}$$

(3.3)

and

$$S_o = \begin{bmatrix} A(t) - B(t)C(t) & B(t) \\ B'(t)X(t) - C(t) & I \end{bmatrix}.$$  

(3.4)

**Proof.** The proof is a straightforward extension of the time-invariant result which can be obtained by the general formulae for inner/outer factorizations that are found in [31, p. 369]. □

### 4. Bode’s integral in the time-domain

The usual interpretation of Bode’s integral is that it represents a (geometric) mean of the gain of the transfer function $S(s)$ over the different sinusoidal inputs $e^{j\omega t}$. This interpretation does not generalize easily to the time-varying case. Instead, we look at an alternative interpretation.

#### 4.1. Connections with entropy integral

We will first outline the connection between the entropy integral and Bode’s integral for discrete-time systems. For a stable function $G(z)$ with norm bound $\|G\|_{\infty} < 1$, the entropy is defined as

$$E(G, \gamma) := -\frac{\gamma^2}{2\pi} \int_{-\pi}^{\pi} \ln \det(I - \gamma^{-2}G'(e^{j\omega})G(e^{j\omega})) \, d\omega.$$  

(4.1)

Suppose that the sensitivity function has norm bound $\|S\|_{\infty} < \alpha$. This allows one to factor

$$I - \alpha^{-2}S'(1/\bar{z})S(z) = G'(1/\bar{z})G(z).$$  

(4.1)
Moreover,
\[ \ln |\det S(e^{i\omega})| = \frac{1}{2} \ln |\det (I - G'(e^{i\omega}) G(e^{i\omega}))| + m \ln \alpha, \]
where \( m \) is the matrix dimension of \( S \). The sensitivity integral can be written as
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |\det S(e^{i\omega})| \, d\omega = m \ln \alpha - \frac{1}{2} E(G, 1). \tag{4.2} \]
This shows that Bode’s integral can be computed by means of an entropy computation.

In [16], it was shown that the entropy cost function is equivalent to the infinite horizon risk-sensitive cost function. We can use this connection to provide the time-domain characterization of Bode’s integral that we seek.

Suppose that the input \( r_k \) is white, Gaussian stochastic process with unit covariance, and that \( e = Sr \). Moreover, let \( \Pi \) be a positive definite matrix. A risk-sensitive cost function can be defined as
\[ L(G, \gamma, N) := \frac{2\gamma^2}{N} \ln \mathbb{E} \exp \left( \frac{1}{2\gamma^2} \left( x_N' \Pi x_N + \sum_{k=0}^{N-1} (r_k'r_k - e_k'e_k) \right) \right), \]
where \( \mathbb{E} \) denotes an expectation and \( x_N \) terminal state.

As \( N \uparrow \infty \), the term \( x_N' \Pi x_N \) can be disregarded since \( G \) is stable and thus \( x_N \downarrow 0 \). Moreover, the summation is then nothing but the \( \ell^2 \) norm squared of \( r \) and \( e \). In this case, we can show, as in [16], that for stable linear time-invariant discrete-time system \( S(z) \),
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |\det S(e^{i\omega})| \, d\omega = -\lim_{N \uparrow \infty} \frac{1}{2N} \ln \mathbb{E} \exp \left( \frac{1}{2} ( \|r\|_2^2 - \|e\|_2^2 ) \right) \tag{4.3} \]
This expression serves as a time-domain characterization of Bode’s integral.

As shown in [12], this characterization of Bode’s integral shows how the frequency domain trade-offs translate into the time-domain. Suppose that we divide the time line into two regions \( 0, \ldots, M \) and \( M + 1, \ldots, N \) then the argument of the exponent:
\[ \exp \left( \frac{1}{2} ( \|r\|_2^2 - \|e\|_2^2 ) \right) \]
\[ = \exp \left( \frac{1}{2} \sum_{k=0}^{M} (r_k'r_k - e_k'e_k) \right) \exp \left( \frac{1}{2} \sum_{k=M+1}^{N} (r_k'r_k - e_k'e_k) \right). \]
Now, if Bode’s integral is zero, this means that
\[ \mathbb{E} \exp \left( \frac{1}{2} \sum_{k=0}^{M} (r_k'r_k - e_k'e_k) \right) \exp \left( \frac{1}{2} \sum_{k=M+1}^{N} (r_k'r_k - e_k'e_k) \right) = 1. \]
If the \( e_k \) is small as \( k \uparrow \infty \), then it will necessarily be larger in \( k = 0, \ldots, M \).

In the time-varying case, a relationship between the entropy and risk-sensitive cost functions analogous to that obtained in [16] exists for time-varying systems [32]. This connection was used in [12] to extend the time-domain characterization of Bode’s integral to time-varying systems.
Suppose that the input signal $r_{[N]}$ is a windowed white noise sequence defined over the interval $k = 0, \pm 1, \ldots, \pm N$ and let $e_{[N]}$ be the corresponding output of the sensitivity operator. Unlike the time-invariant case, where every signal gives rise to the same cost, in time-varying systems this set of signals will give rise to a sequence of costs. In the limit, we can consider an average cost function

$$\limsup_{n \to \infty} -\frac{1}{2N+1} \ln \delta \exp \left( \frac{1}{2} \left( \|r_{[N]}\|_2^2 - \|e_{[N]}\|_2^2 \right) \right)$$

(4.4)

and treat this as the time-varying analogue to Bode’s integral.

Let $S$ be a linear time-varying discrete-time system matrix associated with the sensitivity operator. Then, using a standard computation, we can show that (4.4) equals

$$\lim_{N \to \infty} \frac{1}{2N+1} \ln \det(S_N^* S_N),$$

where $S_N : \mathbb{R}^{(2N+1)m} \mapsto \ell_m^2$ is the finite-rank operator

$$(S_N)_{i,j} = S_{i,j}, \quad j = 0, \pm 1, \ldots, \pm N, \quad i = 0, \pm 1, \ldots$$

Let $S_o$ be the outer factor in an inner/outer factorization of $S$. Then, as shown in [12]:

$$\lim_{N \to \infty} \frac{1}{2N+1} \ln \det(S_N^* S_N) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{k=-N}^{N} \ln \det(S_o)_{k,k}.$$  (4.5)

4.2. Continuous-time systems

Even though the connection between the minimum entropy control problem and the related stochastic risk-sensitive control problem was shown for linear time-invariant systems in [33], it is not possible to carry out the steps of the previous section without some changes. The difficulty is that, because the continuous-time integral is unbounded, the right-hand side of the analogue for (4.2) would be unbounded.

In the time-varying case, a generalization of the minimum entropy control problem was presented in [32]. In the discrete-time case, we took the logarithm of the diagonal term of the outer factor of $S$: $\ln(S_o)_{k,k}$; and averaged over all time $k$. This is the direct feedthrough term of the outer factor of the system.

This interpretation cannot be carried out without modification. As shown in (3.4) the direct feedthrough term is the identity. To show what modification needs to be made, we first consider an outer LTI system with impulse response $s_o(t)$. Define the unit-norm input:

$$\delta_\alpha(t) = \begin{cases} 1/\sqrt{\alpha} & \text{for } t \in [0, \alpha], \\ 0 & \text{elsewhere,} \end{cases}$$

and let $p_i, i = 1, \ldots, m$, be the $i$th standard basis vector for $\mathbb{R}^m$. Now, consider the signal
$e_{i,\alpha}(t) := (S_0 * \delta_\alpha p_i)(t)$

$$= \left[ \delta_\alpha(t) I + \int_0^\alpha s_0(t - \sigma)\delta_\alpha(\sigma)\,d\sigma \right] p_i$$

$$\approx \left[ \delta_\alpha(t) I + \sqrt{\alpha} s_0(t) \right] p_i,$$

where the approximation holds as $\alpha \downarrow 0$ by continuity of the kernel $s_0(t)$.

As our analogue for the integral (1.1) we take the logarithm of this gain:

$$\mathcal{B}_0 := \lim_{\alpha \downarrow 0} \frac{1}{2\alpha} \sum_{i=1}^m \log \int_0^\alpha \|e_{i,\alpha}(t)\|^2\,dt.$$  

Now, as $\alpha \downarrow 0$,

$$\|e_{i,\alpha}(t)\|^2 \approx \delta_\alpha^2(t) + 2\sqrt{\alpha} \delta_\alpha(t) p'_i s_0(t) p_i + \alpha \|s_0(t) p_i\|^2.$$

Hence,

$$\int_0^\alpha \|e_{i,\alpha}(t)\|^2\,dt \approx 1 + 2\alpha p'_i s_0(0) p_i + \alpha^2 \|s_0(0) p_i\|^2$$

and thus

$$\mathcal{B}_0 = \lim_{\alpha \downarrow 0} \frac{1}{2\alpha} \sum_{i=1}^m \log (1 + 2\alpha p'_i s_0(0) p_i + O(\alpha^2)) = \text{trace} s_0(0).$$

The time-varying case requires one modification. Obviously, when the input is applied it will have a bearing on the cost function. For this reason we vary the input

$$\delta_{\alpha,t}(\tau) = \begin{cases} 1/\sqrt{\alpha} & \text{for } \tau \in [t, t + \alpha), \\ 0 & \text{elsewhere}, \end{cases}$$

with corresponding output $e_{t,i,\alpha}$. This leads to a cost function

$$\mathcal{B}_t := \frac{1}{2\alpha} \sum_{i=1}^m \int_0^\alpha \log \|e_{t,i,\alpha}(\tau)\|^2\,d\tau.$$  

Finally, we average over all possible ‘initial’ times $t$, so that the analogue for the integral is

$$\mathcal{B} := \frac{1}{T} \int_0^T \mathcal{B}_t\,dt.$$  

(4.6)

Proceeding as above, we have that

$$\mathcal{B}_t := \lim_{\alpha \downarrow 0} \frac{1}{4\alpha} \sum_{i=1}^m \log \left( 1 + \alpha p'_i \hat{S}_o(t, t) p_i + \alpha p'_i \hat{S}'_o(t, t) p_i + \alpha^2 p'_i \hat{S}_o(t, t) \hat{S}_o(t, t) p_i \right)$$

$$= \lim_{\alpha \downarrow 0} \frac{1}{4\alpha} \log \det \left( I + \alpha \hat{S}_o(t, t) + \alpha \hat{S}'_o(t, t) + \alpha^2 \hat{S}_o(t, t) \right)$$

$$= \frac{1}{2} \text{trace} [\hat{S}_o(t, t)]$$
and thus
\[
\mathcal{B} = \lim_{T \to \infty} \frac{1}{4T} \int_{-T}^{T} \text{trace}\left[\hat{S}_o(t, t)\right] \, dt.
\]

4.2.1. Generalization of Szegö’s limit theorem

We have argued that one way in which our generalization of the logarithmic integral \( \mathcal{B} \) arises is as an extension of the entropy cost function of [32]. We now present a second view, following [34].

Let \( M_N \) be the \( N \times N \) block Toeplitz matrix arising from the continuous \( m \times m \) matrix function
\[
F(z) = \sum_{k=-\infty}^{\infty} F_k z^k.
\]
Provided that \( \det F(z) \neq 0 \) on \( |z| = 1 \), Szegö’s limit theorem says that [35, p. 621]
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log F(e^{i\omega}) \, d\omega = \lim_{N \to \infty} \frac{1}{N} \log \det M_N.
\]
The discrete-time characterization (4.5) is directly analogous to this limit, by setting \( F(z) = S'(1/\bar{z})S(z) \).

For continuous-time systems, Dym and Ta’assan provided an abstract version of Szegö’s result that matches that considered here [34]. In particular, let \( M \) be a bounded integral operator with \( m \times m \) matrix valued kernel \( M(t,s) \) acting on \( L^2(0, \infty) \) and let \( P_T \) be the projection:
\[
(P_T x)(s) = \begin{cases} x(s) & \text{if } 0 \leq s \leq T, \\ 0 & \text{otherwise}, \end{cases}
\]
for \( 0 \leq T < \infty \). Then \( \det(I - P_T M P_T) \) serves as a natural continuous analogue of Szegö’s limit.

Note that, since the sensitivity operator is of the form \( I + \hat{S} \), it follows that
\[
(I + \hat{S})^*(I + \hat{S}) = I + M = (I + \hat{S}_o)^*(I + \hat{S}_o). \tag{4.7}
\]
For our interest, the main result of [34] is the following.

**Theorem 8** [34]. Let \( M \) be a bounded integral operator with continuous kernel \( M(t,s) \) and if \( P_T M P_T \) is trace class and \( I + P_T M P_T \) is invertible for all \( 0 \leq t \leq T \), then
\[
\lim_{T \to \infty} \frac{1}{T} \log \det(I + P_T M P_T) = \lim_{T \to \infty} \frac{1}{T} \text{trace}[P_T \hat{S}_o P_T + P_T \hat{S}_o P_T], \tag{4.8}
\]
where \( I + M = (I + \hat{S}_o)^*(I + \hat{S}_o) \).

To use this result on the operator \( I + M \) in (4.7) we need to show that \( M \) is trace class.
Lemma 9. Let $I + G$ be the integral operator associated with the state-space representation (2.1) with $D(t) = 0$ and where $B(t)$ and $C(t)$ are bounded, continuous functions of $t$, and $A(t)$ and $\dot{B}(t)$ are also bounded. Then $P^T GP^T$ is a trace-class operator.

Proof. To show that the operator is trace-class, we use a result of Stinespring, see [36, p. 119], which states that the operator is trace-class if the kernel $G(t, s) = C(t) \Phi_A(t, s)B(s)$ is Lip$\alpha$ in the variable $s$; i.e.

$$\|G(t, s_2) - G(t, s_1)\| \leq \beta |s_2 - s_1|^\alpha$$

for some $\alpha \in (1/2, 1)$ and the constant $\beta$ is independent of $t \in [0, T]$.

Now, assume without loss of generality that $s_2 = s_1 + h$ and $h > 0$. Then

$$\|G(t, s_2) - G(t, s_1)\|$$

$$= \|C(t) \Phi_A(t, s_1 + h) (B(s_2) - \Phi_A(s_1 + h, s_1)B(s_1))\|$$

$$\leq \|C(t)\| \|\Phi_A(t, s_1 + h)\| \|B(s_1 + h) - \Phi_A(s_1 + h, s_1)B(s_1)\|.$$

From Taylor’s theorem, there exist $\theta \in (0, h)$ such that $B(s_1 + h) = B(s_1) + h \dot{B}(s_1 + \theta)$. Thus

$$\|G(t, s_2) - G(t, s_1)\|$$

$$\leq \|C(t)\| \|\Phi_A(t, s_1 + h)\| \|B(s_1) - \Phi_A(s_1 + h, s_1)B(s_1) + h \dot{B}B(s_1 + \theta)\|$$

$$\leq \|C(t)\| \|\Phi_A(t, s_1 + h)\| \|B(s_1)\| \|I - \Phi_A(s_1 + h, s_1)\| + h \|\dot{B}(s_1 + \theta)\|.$$

Moreover, because of the boundedness of $A(t)$, if $\|A(t)\| \leq \gamma$, we can write

$$\Phi_A(s + \delta, s) = \exp(\tilde{A}_\delta(s)\delta) + R(\delta, s), \quad \delta > 0,$$

where

$$\tilde{A}_\delta(s) := \frac{1}{\delta} \int_s^{s+\delta} A(\tau) \, d\tau,$$

where $R(\delta, s)$ satisfies $\|R(\delta, s)\| \leq \gamma^2 \delta^2 e^{\gamma \delta}$; see [18, p. 73]. Thus

$$\|I - \Phi_A(s_1 + h, s_1)\| = \|I - \exp(\tilde{A}_\delta(s_1)h) - R(\delta, s_1)\|$$

$$\leq \exp(\|\tilde{A}_\delta(s_1)h\|) - 1 + \|R(\delta, s_1)\|$$

$$\leq \exp(\|A\| h) - 1 + \gamma^2 h^2 e^{\gamma h}$$

$$\leq \beta h,$$

where
Thus, the operator is trace-class. □

Corollary 10. Let $M$ be the operator defined in (4.7). Assume that $\dot{B}(t)$ and $\dot{C}(t)$ are bounded. Then

$$\lim_{T \to \infty} \frac{1}{T} \log \det (I + P_T M P_T^T) = \lim_{T \to \infty} \frac{2}{T} \int_{-T}^{T} \text{trace}[\hat{S}_o(t, t)] \, dt.$$ 

Proof. The system associated with the input–output operator $M$ in (4.7) has a state-space representation given by

$$\Sigma_M = \begin{bmatrix} \tilde{A}(t) & \tilde{B}(t) \\ \tilde{C}(t) & 0 \end{bmatrix}.$$ 

Boundedness of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\dot{\tilde{C}}$ follows from the boundedness of the corresponding constituent matrices. Similarly, continuity of $\tilde{B}$ and $\tilde{C}$ follows from the continuity of $B$ and $C$. Thus, from the previous lemma, $M$ is trace class. That $I + P_T M P_T^T$ is invertible follows from the state-space description so that we can apply Theorem 8. Furthermore, we can evaluate the right-hand side of (4.8), as in the proof of Theorem 3.1 in [34] (see also [36, Chapter III.10]) as

$$\text{trace}[P_T \hat{S}_o P_T + P_T \hat{S}_o P_T] = 2 \int_{0}^{T} \text{trace}[\hat{S}_o(t, t)] \, dt.$$ 

□

In the following section we show how the analogue of the integral $\mathcal{B}$ is also tied to the antistable dynamics of the open loop system as in the time-invariant case.

5. Main results

Having provided a time-domain generalization of the integral found in Bode’s integral, we now show the connection with the spectrum of the matrix $A(t)$.

Theorem 11. Suppose that the system $\Sigma_L$ satisfies Assumptions 1–3. Then
\[ 0 < \sum_{i=1}^{n_u} n_i \lambda_i \leq \lim_{T \to \infty} \frac{1}{4T} \int_{-T}^{T} \text{trace}[\hat{S}_o(t,t)] \, dt \leq \sum_{i=1}^{n_u} n_i \bar{\lambda}_i, \]  

(5.1)

where \( \lambda_i \) and \( \bar{\lambda}_i \) are the spectral values of the antistable component of \( A(t) \).

**Proof.** Recall from (3.4) that the kernel of the outer factor of the sensitivity operator equals

\[ \hat{S}_o(t, \tau) = \left[ B'(t)X(t) - C(t) \right] \Phi_{A-BC}(t, \tau) B(\tau), \quad t \geq \tau, \]

and thus

\[ \hat{S}_o(t, t) = \left[ B'(t)X(t) - C(t) \right] B(t) \]

\[ = B'(t)X(t)B(t) \]

\[ = B'(t)X(t)B(t), \]  

(5.2)

\[ \text{trace} \left[ B'_u(t)X_u(t)B_u(t) \right] = \text{trace} \left[ X_u(t)B_u(t)B'_u(t) \right]. \]

Moreover, \( \hat{S}_o(t, \tau) = \left[ B'(t)X(t) - C(t) \right] \Phi_{A-BC}(t, \tau) B(\tau) \), \( t \geq \tau \).

\[ \hat{S}_o(t, t) = \left[ B'(t)X(t) - C(t) \right] B(t) \]

\[ = B'(t)X(t)B(t) \]

= \( B'(t)X(t)B(t) \),  

(5.3)

Now, noting that the Riccati differential equation for \( X_u \) can be rewritten as

\[ \dot{X}_u(t) = \left( -A_u(t) + B_u(t)B'_u(t)X_u(t) \right)' X_u(t) + X_u(t)(-A_u(t)). \]

The solution of this equation is, for any \( t \) and \( \tau \),

\[ X_u(t) = \Phi_{-A_u-B_uB'_uX_u}(t, \tau) X_u(\tau) \Phi_{A_u}(\tau, t). \]

In particular, let \( t = T \) and \( \tau = -T \). Now, taking the logarithm of the determinant of both sides of this equation leads to

\[ \log \det X_u(T) = \log \det X_u(-T) \]

\[ + \log \det \Phi_{-A_u-B_uB'_uX_u}(T, -T) + \log \det \Phi_{A_u}(-T, T). \]

Furthermore,

\[ \lim_{T \to \infty} \frac{1}{2T} \left( \log \det X_u(T) - \log \det X_u(-T) \right) = 0 \]

since \( X_u(t) \) is bounded above and below. Thus

\[ \lim_{T \to \infty} \frac{1}{2T} \left( \log \det \Phi_{-A_u-B_uB'_uX_u}(T, -T) + \log \det \Phi_{A_u}(-T, T) \right) = 0. \]
Applying Lemma 1 to both transition matrices yields

\[
\log \det \Phi_{-\left[A_u - B_u B_u' X_u\right]}(T, -T) = \int_{-T}^T \operatorname{trace} \left[ B_u(t) B_u'(t) X_u(t) - A_u(t) \right] \, dt
\]

\[
= \int_{-T}^T \operatorname{trace} \left[ B_u(t) B_u'(t) X_u(t) \right] \, dt - \int_{-T}^T \operatorname{trace} [A_u(t)] \, dt
\]

and

\[
\log \det \Phi_{A_u}(0, T) = -\int_{-T}^T \operatorname{trace}[A_u(t)] \, dt.
\]

Thus

\[
\mathcal{B} = \lim_{T \to \infty} \frac{1}{4T} \int_{-T}^T \operatorname{trace} \left[ B_u(t) B_u'(t) X_u(t) \right] \, dt
\]

\[
= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T \operatorname{trace} [A_u(t)] \, dt.
\]

Finally, by Lemma 5 we obtain (5.1).

**Corollary 12.** If, in addition to the assumptions of Theorem 11, the antistable component of the open loop dynamics \( A_u \) is regular, then

\[
\lim_{T \to \infty} \frac{1}{4T} \int_{-T}^T \operatorname{trace} \left[ \hat{S}_o(t, t) \right] \, dt = \sum_{k=1}^m \lambda_i,
\]

where \( \lambda_i \) are the Lyapunov exponents of the antistable component \( A_u \).

**Proof.** This follows directly from the definition of regularity and Theorem 11.

**Remark.** In [37], Gohberg et al. show that the Szegö limits can be expressed in terms of spectral properties of the realizations of the symbol. The results presented in this paper serve as counterparts of those in [37] for the Szegö-type limit theorem considered by Dym and Ta’assan in [34].

6. Conclusions

By means of the connection between Bode’s integral and the entropy cost function, we have provided a time-domain characterization of Bode’s sensitivity integral. The traditional frequency domain interpretation is that, if the sensitivity of a closed-loop system is decreased over a particular frequency range—typically the low frequencies—the designer ‘pays’ for this in increased sensitivity outside this frequency range. This interpretation is also valid for the characterization presented here provided one deals with time horizons rather than frequency ranges.
Having reinterpreted Bode’s sensitivity integral in the time domain, it is possible to consider extensions to systems not admitting frequency domain representations, including linear time-varying systems. In the generalization presented here, the poles of the open loop system are replaced by the lower and upper spectral values in the dichotomy spectrum of the time-varying differential equation.

Bode’s integral is but one of several known mathematical characterizations of the analytic constraints found in feedback control systems. The corresponding counterparts for time-varying systems are unknown but will be the subject of future research.

References