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Properties of h-convex functions related to the Hermite–Hadamard–Fejér inequalities

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ABSTRACT

In this paper we prove the Hermite–Hadamard–Fejér inequalities for an h-convex function and we point out the results for some special classes of functions. Also, some generalization of the Hermite–Hadamard inequalities and some properties of functions H and F which are naturally joined to the h-convex function are given. Finally, applications on p-logarithmic mean and mean of the order p are obtained.

1. Introduction

In the paper [1] a large class of non-negative functions, the so-called h-convex functions is considered. This class contains several well-known classes of functions such as non-negative convex functions, s-convex in the second sense, Godunova–Levin functions and P-functions. Let us repeat the definition of an h-convex function.

Definition 1. Let I, J be intervals in $\mathbb{R}$, $(0, 1) \subseteq J$, and let $h : J \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. A non-negative function $f : I \to \mathbb{R}$ is called h-convex if for all $x, y \in I, \alpha \in (0, 1)$ we have

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y). \tag{1}$$

If the inequality in (1) is reversed, then f is said to be h-concave.

In the above-mentioned paper [1], a structure of that class is described, some examples are given and the Jensen-type inequality is obtained. This paper is devoted to the Hermite–Hadamard inequalities for h-convex functions. But before that, let us say a few words about assumptions about functions f and h. The referee remarked that not all convex functions belong to the class of h-convex ones and suggested that this inconvenience can be avoided omitting the assumption that f is non-negative. So, in the further text we assume that h and f are real functions without assumption of non-negativity.

The most well-known inequalities related to the integral mean of a convex function f are the Hermite–Hadamard inequalities or its weighted versions, the so-called Hermite–Hadamard–Fejér inequalities.

Theorem 2 (*The Hermite–Hadamard–Fejér Inequalities*). *If $f : [a, b] \to \mathbb{R}$ is convex, and $w : [a, b] \to \mathbb{R}$, $w \geq 0$, integrable and symmetric about $\frac{a+b}{2}$, then*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)\mathrm{d}x \leq \int_a^b f(x)w(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)\mathrm{d}x. \tag{2}$$

If f is a concave function, then the reversed inequalities in (2) hold.

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If $w \equiv 1$, then we are talking about the Hermite–Hadamard inequalities. More about those inequalities can be found in a number of papers and monographies (for example, see [2,3]). Here we research which properties connected with the integral mean of the function f still remain if the class of convex functions is expanded to the class of h-convex functions. In the second section we prove both the Hermite–Hadamard–Fejér inequalities for an h-convex function and we point out results for some special classes of functions. Also we proved that the left-hand side inequality is stronger than the right-hand side inequality. At the end of the second section we give some generalization of the Hermite–Hadamard inequalities. In the third section we give some properties of functions H and F which are naturally joined to the function f. Finally, in the last section we give some applications on p-logarithmic mean and mean of the order p.

Throughout this paper we assume that intervals I, J satisfy assumptions from Definition 1. Also, we assume that all considered integrals exist.

2. The Hermite–Hadamard–Fejér inequalities for an *h*-convex function

Theorem 3 (*The Second Hermite–Hadamard–Fejér Inequality for an h-convex Function*). *Let $f : [a,b] \to \mathbb{R}$ be h-convex, $w : [a,b] \to \mathbb{R}$, $w \geq 0$, symmetric with respect to $\frac{a+b}{2}$. Then*

$$\frac{1}{b-a}\int_a^b f(t)w(t)\mathrm{d}t \leq [f(a)+f(b)]\int_0^1 h(t)w\left(ta+(1-t)b\right)\mathrm{d}t. \tag{3}$$

If f is an h-concave function, then the inequality in (3) is reversed.

Proof. For any $x \in (a,b)$ exists $\alpha \in (0,1)$ such that $x = \alpha a + \bar{\alpha} b$, $\bar{\alpha} = 1-\alpha$.

From the definition of an h-convex function we have

$$f(\alpha a+\bar{\alpha}b)w(\alpha a+\bar{\alpha}b) \leq \left(h(\alpha)f(a)+h(\bar{\alpha})f(b)\right)w(\alpha a+\bar{\alpha}b) \tag{4}$$

$$f(\bar{\alpha}a+\alpha b)w(\bar{\alpha}a+\alpha b) \leq \left(h(\bar{\alpha})f(a)+h(\alpha)f(b)\right)w(\overline{\alpha}a+\alpha b). \tag{5}$$

After adding (4) and (5), and integrating we obtain

$$\begin{aligned}
&\int_0^1 f(\alpha a+\bar{\alpha}b)w(\alpha a+\bar{\alpha}b)\mathrm{d}\alpha + \int_0^1 f(\bar{\alpha}a+\alpha b)w(\bar{\alpha}a+\alpha b)\mathrm{d}\alpha \\
&\quad \leq \int_0^1 \left[h(\alpha)f(a)w(\alpha a+\bar{\alpha}b)+h(\overline{\alpha})f(b)w(\alpha a+\bar{\alpha}b)+h(\bar{\alpha})f(a)w(\bar{\alpha}a+\alpha b)+h(\alpha)f(b)w(\bar{\alpha}a+\alpha b)\right]\mathrm{d}\alpha \\
&\quad = \int_0^1 \{f(a)\left[h(\alpha)w(\alpha a+\bar{\alpha}b)+h(\bar{\alpha})w(\bar{\alpha}a+\alpha b)\right]+f(b)\left[h(\overline{\alpha})w(\alpha a+\bar{\alpha}b)+h(\alpha)w(\bar{\alpha}a+\alpha b)\right]\}\mathrm{d}\alpha \\
&\quad = 2f(a)\int_0^1 h(t)w(ta+(1-t)b)\mathrm{d}t + 2f(b)\int_0^1 h(t)w((1-t)a+tb)\mathrm{d}t \\
&\quad = 2[f(a)+f(b)]\int_0^1 h(t)w(ta+(1-t)b)\mathrm{d}t,
\end{aligned}$$

where we use the symmetricity of the weight w.

After suitable substitutions we obtain that both the integrals in the first line are equal to $\frac{1}{b-a}\int_a^b f(t)w(t)\mathrm{d}t$, and the theorem has been established. □

Remark 4. (a) If $h(t) = t$ in Theorem 3 i.e. if f is a convex function we have the right-hand side of the classical inequality (2).

(b) For $h(t) = t^s$, $s \in (0,1)$, i.e. if f is an s-convex function in the second sense, then we have a result of Theorem 2.1 from [4]

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \leq \frac{f(a)+f(b)}{s+1}.$$

Theorem 5 (*The First Hermite–Hadamard–Fejér Inequality for an h-Convex Function*). *Let h be defined on $[0, \max\{1, b-a\}]$ and $f : [a,b] \to \mathbb{R}$ be h-convex, $w : [a,b] \to \mathbb{R}$, $w \geq 0$, symmetric with respect to $\frac{a+b}{2}$ and $\int_a^b w(t)\mathrm{d}t > 0$. Then*

$$f\left(\frac{a+b}{2}\right) \leq C\int_a^b f(t)w(t)\mathrm{d}t, \tag{6}$$

where $C = \frac{2h\left(\frac{1}{2}\right)}{\int_a^b w(t)\mathrm{d}t}$.

Furthermore, if $\int_a^{\frac{a+b}{2}}\int_{\frac{a+b}{2}}^b h(y-x)w(y)w(x)\mathrm{d}y\mathrm{d}x \neq 0$, $h(x)\neq 0$ *for* $x \neq 0$ *and*

(i) *h is multiplicative or*
(ii) *h is supermultiplicative and f is non-negative*

and if f is an h-convex function, then inequality (6) holds for

$$C = \min\left\{\frac{2h\left(\frac{1}{2}\right)}{\int_a^b w(t)\mathrm{d}t}, \frac{\int_0^{\frac{b-a}{2}} h(x)w\left(x+\frac{a+b}{2}\right)\mathrm{d}x}{\int_a^{\frac{a+b}{2}}\int_{\frac{a+b}{2}}^b h(y-x)w(y)w(x)\mathrm{d}y\mathrm{d}x}\right\}. \tag{7}$$

Proof. Let f be an h-convex function. If $\alpha = \frac{1}{2}$, $x = ta+(1-t)b$ and $y=(1-t)a+tb$, from the definition of an h-convex function we have the following

$$f\left(\frac{a+b}{2}\right) \le h\left(\frac{1}{2}\right)(f(ta+(1-t)b)+f((1-t)a+tb)).$$

Now we multiply it with $w(ta+(1-t)b) = w((1-t)a+tb)$ and integrate by t over $[0, 1]$ to obtain inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{2h\left(\frac{1}{2}\right)}{\int_a^b w(t)\mathrm{d}t}\int_a^b f(t)w(t)\mathrm{d}t, \tag{8}$$

which holds in general case.

Let h be supermultiplicative $h(x) \neq 0$ for $x \neq 0$. Then $h(x) > 0$ for $x > 0$. For $x, y \in [a, b]$ such that $a \le x < \frac{a+b}{2} < y \le b$ we have

$$\frac{a+b}{2} = \left(\frac{y-\frac{a+b}{2}}{y-x}\right)x + \left(\frac{\frac{a+b}{2}-x}{y-x}\right)y.$$

Denote $\alpha = \frac{y-\frac{a+b}{2}}{y-x} > 0$. Then $\bar{\alpha} = 1-\alpha = \frac{\frac{a+b}{2}-x}{y-x}$ and $\frac{a+b}{2} = \alpha x + \bar{\alpha} y$, and $f(\frac{a+b}{2}) = f(\alpha x+\bar{\alpha}y) \le h(\alpha)f(x)+h(\bar{\alpha})f(y)$.

Since h is supermultiplicative, we have

$$h(\alpha) = h\left(\frac{y-\frac{a+b}{2}}{y-x}\right) \le \frac{h\left(y-\frac{a+b}{2}\right)}{h(y-x)} \quad \text{and} \quad h(\bar{\alpha}) \le \frac{h\left(\frac{a+b}{2}-x\right)}{h(y-x)}.$$

So, when $f > 0$ we have

$$f\left(\frac{a+b}{2}\right) \le \frac{h\left(y-\frac{a+b}{2}\right)}{h(y-x)}f(x) + \frac{h\left(\frac{a+b}{2}-x\right)}{h(y-x)}f(y),$$

$$h(y-x)f\left(\frac{a+b}{2}\right) \le h\left(y-\frac{a+b}{2}\right)f(x) + h\left(\frac{a+b}{2}-x\right)f(y). \tag{9}$$

This inequality also holds if h is multiplicative, regardless the positivity of f.

Multiplying (9) with $w(x)$ and integrating over interval $\left[a, \frac{a+b}{2}\right]$ with respect to dx, and after that multiplying with $w(y)$ and integrating over interval $\left[\frac{a+b}{2}, b\right]$ with respect to dy we get

$$f\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^b\left(\int_a^{\frac{a+b}{2}} h(y-x)w(x)\mathrm{d}x\right)w(y)\mathrm{d}y \le \int_{\frac{a+b}{2}}^b h\left(y-\frac{a+b}{2}\right)w(y)\mathrm{d}y\int_a^{\frac{a+b}{2}} f(x)w(x)\mathrm{d}x$$
$$+ \int_{\frac{a+b}{2}}^b f(y)w(y)\mathrm{d}y\int_a^{\frac{a+b}{2}} h\left(\frac{a+b}{2}-x\right)w(x)\mathrm{d}x.$$

After substitution $y-\frac{a+b}{2} = t$ in the first integral on the right-hand side and substitution $\frac{a+b}{2}-x = t$ in the integral in the second term of sum, we get

$$\begin{aligned}f\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^b\int_a^{\frac{a+b}{2}} h(y-x)w(x)w(y)\mathrm{d}x\mathrm{d}y &\le \int_0^{\frac{b-a}{2}} h(t)w\left(t+\frac{a+b}{2}\right)\mathrm{d}t\int_a^{\frac{a+b}{2}} f(x)w(x)\mathrm{d}x \\ &\quad + \int_0^{\frac{b-a}{2}} h(t)w\left(\frac{a+b}{2}-t\right)\mathrm{d}t\int_{\frac{a+b}{2}}^b f(y)w(y)\mathrm{d}y \\ &= \int_0^{\frac{b-a}{2}} h(t)w\left(t+\frac{a+b}{2}\right)\mathrm{d}t\cdot\int_a^b f(x)w(x)\mathrm{d}x,\end{aligned}$$

where in the first equality we use that the function w is symmetric on the interval $[a, b]$, i.e. $w\left(\frac{a+b}{2} - t\right) = w\left(\frac{a+b}{2} + t\right)$ for $t \in \left[0, \frac{b-a}{2}\right]$. □

Remark 6. Under the conditions of Theorem 5:

(a) If f is an h-concave function, then the inequality in (6) is reversed.

(b) If h submultiplicative, $\int_a^{\frac{a+b}{2}} \int_{\frac{a+b}{2}}^b h(y-x)w(y)w(x)\mathrm{d}y\mathrm{d}x \neq 0$, $h \geq 0$, and if f is an h-concave function then the inequality in (6) is reversed, with constant C as in (7) with change min → max.

Remark 7. (a) If f is convex, i.e. $h(t) = t$, then inequality (6) becomes the left-hand side of inequality (2).
(b) Let $w = 1$ and let f be an s-convex function in the second sense, i.e. f be an h-convex function with multiplicative $h(t) = t^s$, $s \in (0, 1)$. Then the constant C from Theorem 5 has a form

$$C = \min\left\{\frac{2^{1-s}}{b-a}, \frac{\int_0^{\frac{b-a}{2}} t^s \mathrm{d}t}{\int_a^{\frac{a+b}{2}} \int_{\frac{a+b}{2}}^b (y-x)^s \mathrm{d}y\mathrm{d}x}\right\} = \min\left\{\frac{2^{1-s}}{b-a}, \frac{s+2}{(b-a)(2^{s+1}-1)}\right\}.$$

In [5] Jagers shows that

$$2^{s-1} < \frac{2^{s+1}-1}{s+2}, \quad \text{for } s \in (0, 1).$$

So, the first Hermite–Hadamard inequality for s-convex function in the second sense states:

$$\frac{2^{s+1}-1}{s+2} \cdot f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t. \tag{10}$$

The inequality (10) can be found in [5] and it is an improvement of the Dragomir–Fitzpatrick result from [4] where they used constant 2^{s-1} instead of $\frac{2^{s+1}-1}{s+2}$.

In the following text we will deal with the non-weighted Hermite–Hadamard inequalities for h-convex function in the form

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \leq (f(a)+f(b))\int_0^1 h(t)\mathrm{d}t, \tag{11}$$

where $h\left(\frac{1}{2}\right) > 0$.

Let us define:

$$L : [a, b] \to \mathbb{R}, L(y) = (f(a)+f(y))(y-a)\int_0^1 h(t)\mathrm{d}t - \int_a^y f(t)\mathrm{d}t$$

$$P : [a, b] \to \mathbb{R}, P(y) = \int_a^y f(t)\mathrm{d}t - f\left(\frac{a+y}{2}\right)\frac{y-a}{2h\left(\frac{1}{2}\right)}.$$

Theorem 8. *If the function f is h-convex, $f \geq 0$, $h\left(\frac{1}{2}\right) > 0$ and $\frac{1}{4h\left(\frac{1}{2}\right)} \geq \int_0^1 h(t)\mathrm{d}t$, then*

$$L(y) \geq P(y) \geq 0 \quad \textit{for all } y \in [a, b]. \tag{12}$$

Proof. The second non-weighted Hermite–Hadamard inequality (11) on intervals $\left[a, \frac{a+y}{2}\right]$ and $\left[\frac{a+y}{2}, y\right]$ gives us:

$$\int_a^{\frac{a+y}{2}} f(t)\mathrm{d}t \leq \frac{f(a)+f\left(\frac{a+y}{2}\right)}{2}(y-a)\int_0^1 h(t)\mathrm{d}t \tag{13}$$

$$\int_{\frac{a+y}{2}}^y f(t)\mathrm{d}t \leq \frac{f\left(\frac{a+y}{2}\right)+f(y)}{2}(y-a)\int_0^1 h(t)\mathrm{d}t. \tag{14}$$

Adding (13) and (14) we obtain:

$$\int_a^y f(t)\mathrm{d}t \leq (y-a)\int_0^1 h(t)\mathrm{d}t\left[\frac{f(a)+f(y)}{2} + f\left(\frac{a+y}{2}\right)\right]$$

which is (after multiplying by 2) equivalent to

$$\int_a^y f(t)\mathrm{d}t-(y-a)\int_0^1 h(t)\mathrm{d}t\cdot(f(a)+f(y))\leq 2(y-a)\int_0^1 h(t)\mathrm{d}t\cdot f\left(\frac{a+y}{2}\right)-\int_a^y f(t)\mathrm{d}t.$$

Now,

$$\begin{aligned}P(y)&=\int_a^y f(t)\mathrm{d}t-f\left(\frac{a+y}{2}\right)\frac{y-a}{2h\left(\frac{1}{2}\right)}\\&\leq\int_a^y f(t)\mathrm{d}t-f\left(\frac{a+y}{2}\right)2\int_0^1 h(t)\mathrm{d}t\cdot(y-a)\\&\leq(y-a)(f(a)+f(y))\int_0^1 h(t)\mathrm{d}t-\int_a^y f(t)\mathrm{d}t=L(y).\end{aligned}$$

The second inequality in (12) is a simple consequence of the first non-weighted Hermite–Hadamard inequality (11). □

Remark 9. If $y=b$ and under the same conditions of Theorem 8 we get that the first inequality in (11) is stronger than the second inequality in the non-weighted Hermite–Hadamard inequalities, i.e. we have the following inequality

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\leq(f(a)+f(b))\int_0^1 h(t)\mathrm{d}t-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t. \tag{15}$$

Similar result for convex functions is given in [2, p. 140].

Next theorem gives us some results on errors in trapezoidal and mid-point formulae. Let us define

$$T_n(f;a,b)=\frac{\delta}{2}\left(\sum_{k=0}^{n-1}f(a+k\delta)+\sum_{j=1}^{n}f(a+j\delta)\right)$$

and

$$M_n(f;a,b)=\delta\sum_{k=0}^{n-1}f\left(a+\left(k+\frac{1}{2}\right)\delta\right),$$

where $\delta=\frac{b-a}{n}$.

The Hermite–Hadamard inequalities (11) can be written as

$$\frac{1}{2h\left(\frac{1}{2}\right)}M_1(f;a,b)\leq\int_a^b f(t)\mathrm{d}t\leq 2T_1(f;a,b)\int_0^1 h(t)\mathrm{d}t. \tag{16}$$

Next theorem extends (15) and (16).

Theorem 10. *Let $f:[a,b]\to\mathbb{R}^+$ be an h-convex function, $h\left(\frac{1}{2}\right)>0$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}M_n(f;a,b)\leq\int_a^b f(t)\mathrm{d}t\leq 2T_n(f;a,b)\cdot\int_0^1 h(t)\mathrm{d}t \tag{17}$$

and if $\frac{1}{4h\left(\frac{1}{2}\right)}\geq\int_0^1 h(t)\mathrm{d}t$, then

$$0\leq\int_a^b f(t)\mathrm{d}t-\frac{1}{2h\left(\frac{1}{2}\right)}M_n(f;a,b)\leq 2T_n(f;a,b)\int_0^1 h(t)\mathrm{d}t-\int_a^b f(t)\mathrm{d}t. \tag{18}$$

Proof. Applying (11) to the segment $[a+k\delta,a+(k+1)\delta]$, $\delta=\frac{b-a}{n}$, we obtain:

$$\begin{aligned}\frac{\delta}{2h\left(\frac{1}{2}\right)}\cdot f\left(\frac{a+(k+1)\delta+a+k\delta}{2}\right)&\leq\int_{a+k\delta}^{a+(k+1)\delta}f(t)\mathrm{d}t\\&\leq 2\delta\cdot\frac{f(a+k\delta)+f(a+(k+1)\delta)}{2}\cdot\int_0^1 h(t)\mathrm{d}t,\end{aligned}$$

and summing up for $k = 0$ to $k = n - 1$

$$\frac{\delta}{2h\left(\frac{1}{2}\right)} \cdot \sum_{k=0}^{n-1} f\left(a + \left(k + \frac{1}{2}\right)\delta\right) \leq \int_a^b f(t)\mathrm{d}t$$

$$\leq 2\delta \cdot \int_0^1 h(t)\mathrm{d}t \cdot \sum_{k=0}^{n-1} \frac{f(a + k\delta) + f(a + (k+1)\delta)}{2}.$$

Since $\delta \sum_{k=0}^{n-1} f\left(a + (k + \frac{1}{2})\delta\right) = M_n(f; a, b)$ and $\delta \sum_{k=0}^{n-1} \frac{f(a+k\delta)+f(a+(k+1)\delta)}{2} = T_n(f; a, b)$, this proves (17). The proof of (18) is based on the identity

$$T_{2n}(f; a, b) = \frac{M_n(f; a, b) + T_n(f; a, b)}{2}.$$

From (17) we have

$$\int_a^b f(t)\mathrm{d}t \leq 2T_{2n}(f; a, b) \int_0^1 h(t)\mathrm{d}t = (M_n(f; a, b) + T_n(f; a, b)) \int_0^1 h(t)\mathrm{d}t,$$

$$\int_a^b f(t)\mathrm{d}t - 2M_n(f; a, b) \int_0^1 h(t)\mathrm{d}t \leq 2T_n(f; a, b) \int_0^1 h(t)\mathrm{d}t - \int_a^b f(t)\mathrm{d}t.$$

Since $\frac{1}{4h\left(\frac{1}{2}\right)} \geq \int_0^1 h(t)\mathrm{d}t$, we get the second inequality from (18). □

Remark 11. The result for a convex function f is due to Allasia, Giordano and Pečarić, [6].

3. Mappings *H* and *F*

Let us define two functions on the interval $[0, 1]$

$$H(t) = \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) \mathrm{d}x$$

and

$$F(t) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx + (1-t)y)\mathrm{d}x\mathrm{d}y.$$

Obviously $H(0) = f\left(\frac{a+b}{2}\right)$, $H(1) = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$. Some properties of these two mappings for convex functions and s-convex functions are given in [4,7] respectively. Here we investigate which of these properties can be generalized for h-convex functions.

Theorem 12. *Let f be h-convex on the interval $[a, b]$, $h : J \to \mathbb{R}$, $[0, 1] \subseteq J$. Then the function H is h-convex on $[0, 1]$ and for $t \in [0, 1]$*

$$H(0) \leq C_1(t)H(t) \tag{19}$$

where

$$C_1(t) = \begin{cases} 2h\left(\frac{1}{2}\right), & \textit{in general case} \\ \min\left\{2h\left(\frac{1}{2}\right), \frac{2\int_0^1 h\left(\frac{b-a}{2}tx\right)\mathrm{d}x}{\int_0^1\int_0^1 h\left(\frac{b-a}{2}t(y-x+1)\right)\mathrm{d}y\mathrm{d}x}\right\}, & h \textit{ satisfies } \text{(i) } \textit{or } \text{(ii) } \textit{of Theorem 5}. \end{cases}$$

Proof. The h-convexity of the function H is a consequence of the h-convexity of the function f. Namely, we have

$$\begin{aligned} H(\alpha t + \beta u) &= \frac{1}{b-a}\int_a^b f\left((\alpha t + \beta u)x + (1 - \alpha t - \beta u)\frac{a+b}{2}\right)\mathrm{d}x \\ &= \frac{1}{b-a}\int_a^b f\left(\alpha\left(tx + (1-t)\frac{a+b}{2}\right) + \beta\left(ux + (1-u)\frac{a+b}{2}\right)\right)\mathrm{d}x \\ &\leq \frac{1}{b-a}\int_a^b \left[h(\alpha)f\left(tx + (1-t)\frac{a+b}{2}\right) + h(\beta)f\left(ux + (1-u)\frac{a+b}{2}\right)\right]\mathrm{d}x \\ &= h(\alpha)H(t) + h(\beta)H(u). \end{aligned}$$

After changing the variable $u = tx + (1-t)\frac{a+b}{2}$ we have

$$\begin{aligned} C_1(t)H(t) &= \frac{C_1(t)}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)\mathrm{d}x \\ &= \frac{C_1(t)}{b-a}\int_{u_L}^{u_U} f(u)\frac{b-a}{u_U-u_L}\mathrm{d}u = \frac{C_1(t)}{u_U-u_L}\int_{u_L}^{u_U} f(u)\mathrm{d}u \\ &\geq f\left(\frac{u_U+u_L}{2}\right) = f\left(\frac{a+b}{2}\right) \end{aligned}$$

where $\frac{C_1(t)}{u_U-u_L}$ is a constant defined in Theorem 5 but with respect to the interval $[u_L, u_U]$ where $u_L = ta + (1-t)\frac{a+b}{2}$ and $u_U = tb + (1-t)\frac{a+b}{2}$. □

Remark 13. If f is a convex function, then we get $H(t) \geq H(0)$. It is known result for a convex function. If f is an s-convex function in the second sense, then $C_1(t) = \frac{s+2}{2^{s+1}-1}$ and we have the following refinement of the result from [4]:

$$H(t) \geq \frac{2^{s+1}-1}{s+2}H(0) \geq 2^{s-1}H(0).$$

Theorem 14. *Let f be an h-convex function on the interval $[a, b]$, $h : J \to \mathbb{R}$, $[0, 1] \subseteq J$. Then the function F is symmetric with respect to $\frac{1}{2}$ and h-convex on $[0, 1]$. Also, the following inequalities hold*

$$2h\left(\frac{1}{2}\right)F(t) \geq F\left(\frac{1}{2}\right), \qquad C_1(t)F(t) \geq H(1-t),$$

where C_1 is defined as in the previous theorem.

Proof. Let us prove the first inequality. From

$$\frac{x+y}{2} = \frac{1}{2}(tx + (1-t)y) + \frac{1}{2}((1-t)x + ty), \quad x, y \in [a, b],\ t \in [0, 1],$$

we obtain

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2}\right)f(tx + (1-t)y) + h\left(\frac{1}{2}\right)f((1-t)x + ty).$$

Integrating over $x \in [a, b]$ and over $y \in [a, b]$ and using the fact that

$$\int_a^b\int_a^b f(tx + (1-t)y)\mathrm{d}x\mathrm{d}y = \int_a^b\int_a^b f((1-t)x + ty)\mathrm{d}x\mathrm{d}y$$

we get

$$\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)\mathrm{d}x\mathrm{d}y \leq 2h\left(\frac{1}{2}\right)\int_a^b\int_a^b f\,(tx + (1-t)y)\,\mathrm{d}x\mathrm{d}y = 2h\left(\frac{1}{2}\right)F(t)(b-a)^2$$

which established the proof.

To get the second inequality we define a function

$$H_y(t) = \frac{1}{b-a}\int_a^b f(tx + (1-t)y)\mathrm{d}x$$

for fixed y. Using the substitution $u = tx + (1-t)y$, we obtain

$$H_y(t) = \frac{1}{u_U-u_L}\int_{u_L}^{u_U} f(u)\mathrm{d}u.$$

Using the result from Theorem 5 for h-convex function f we get

$$C_1(t)H_y(t) \geq f\left(\frac{u_U+u_L}{2}\right) = f\left(t\cdot\frac{a+b}{2} + (1-t)y\right).$$

Integrating over $y \in [a, b]$ and dividing by $(b-a)$ we get that

$$C_1(t)F(t) \geq H(1-t). \quad \square$$

Remark 15. If $C_1(t) > 0$, then we have $F(t) \geq \frac{1}{C_1(t)} H(1-t)$ and for variable $1-t$ we have similar: $F(1-t) \geq \frac{1}{C_1(1-t)} H(t)$. But F is symmetric, i.e. $F(t) = F(1-t)$, so we have $F(t) \geq \max\left\{\frac{1}{C_1(t)} H(1-t), \frac{1}{C_1(1-t)} H(t)\right\}$.

If f is a convex function, then we get a known result $F(t) \geq \max\{H(1-t), H(t)\}$.

If h is a multiplicative function, then $C_1(t) = C_1(1-t)$ and

$$F(t) \geq \frac{1}{C_1(t)} \cdot \max\left\{H(1-t), H(t)\right\}.$$

Especially, if $h(t) = t^s$, then we get a refinement of the Dragomir–Fitzpatrick result from [4]:

$$F(t) \geq \frac{2^{s+1}-1}{s+2} \cdot \max\{H(1-t), H(t)\} \geq 2^{s-1} \cdot \max\{H(1-t), H(t)\}.$$

4. Applications

It is interesting to consider a situation when the function f is concave and h-convex simultaneously, or vice versa, when f is convex and h-concave. If f is a concave and h-convex function with $\int_0^1 h(t)\mathrm{d}t > 0$, then the classical Hermite–Hadamard inequalities, Theorems 3 and 5 give us

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \leq f\left(\frac{a+b}{2}\right) \leq C\int_a^b f(t)\mathrm{d}t \tag{20}$$

and

$$\frac{1}{(b-a)\int_0^1 h(t)\mathrm{d}t}\int_a^b f(t)\mathrm{d}t \leq f(a)+f(b) \leq \frac{2}{b-a}\int_a^b f(t)\mathrm{d}t. \tag{21}$$

If f is a convex and h-concave function simultaneously, then reversed signs in inequalities (20) and (21) hold.

Putting for f and h special functions we obtain new results for inequalities between p-logarithmic mean and mean of the order p. Let us recall the definition of these means. If $p \in \mathbb{R} \setminus \{0, -1\}$, p-logarithmic mean L_p of two different numbers $a, b \in \mathbb{R}$ is defined as

$$L_p = \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}$$

and the mean of the order p is defined as $M_p = \left(\frac{a^p+b^p}{2}\right)^{1/p}$. If $a = b$, then $L_p = M_p = a$.

In [1] the following result for functions f and h_k defined as $h_k(x) = x^k, f(x) = x^p, x > 0, k, p \in \mathbb{R}$ is given:

- the function f is h_k-convex if
  1. $p \in (-\infty, 0] \cup [1, \infty)$ and $k \leq 1$;
  2. $p \in (0, 1)$ and $k \leq p$;
- the function f is h_k-concave if
  1. $p \in (0, 1)$ and $k \geq 1$;
  2. $p > 1$ and $k \geq p$.

So, for $p \in (0, 1)$ and $0 \leq k \leq p$ we have the following inequalities:

$$\left(\frac{k+2}{2^{k+1}-1}\right)^{1/p} L_p \geq M_1 \geq L_p,$$

$$L_p \geq M_p \geq \left(\frac{k+1}{2}\right)^{1/p} L_p.$$

If $p > 1$ and $k \geq p$, then reversed signs in the previous inequalities hold.

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