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On the Equivalence between Matrix Riccati Equations and Fredholm Resolvents*

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ABSTRACT

It is shown that the solution to every matrix Riccati equation can be generated by the resolvent of a certain Fredholm integral operator and, conversely, this resolvent can be determined from the corresponding Riccati solution. This result leads to a computational scheme, based on initial-value methods, for solving a large class of Fredholm integral equations. A connection between this theory and the factorization of integral operators is also described.

1. INTRODUCTION

In an important paper on Filtering Theory, Kalman and Bucy [*1*] showed that there was a basic equivalence between a certain matrix Riccati equation and the Wiener–Hopf integral equation for optimum filtering. Other examples of this relationship are found in the theory of stochastic processes [*2*] and radiative transfer [*3*].

In this paper we prove, without recourse to any underlying physical model, that the solution to *every* matrix Riccati equation can be generated by the resolvent of a certain Fredholm integral operator and, in turn, this resolvent can be determined from the solution of the corresponding Riccati equation. Besides completing the picture of equivalence, this result leads to a computational scheme, based on initial-value methods, for solving a large class of Fredholm integral equations.

The ideas presented here are rooted in the theory of "embedding" of integral operators as developed by Bellman, Kalaba, and Krein (cf. [*4*]–[7]).

In Section 2, the main theorem is stated, the proof being given in Section 5. Section 3 contains some examples to show the applicability of the method. A connection between this theory and the factorization of integral operators is described in Section 4.

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2. Main Theorem

By a matrix Riccati equation we mean the initial-value problem:

$$\text{(I)}\qquad \begin{cases} \dfrac{d}{dx} R = A(x) + B(x)\, R + RC(x) + RD(x)\, R, \\ R(a) = F, \end{cases}$$

where A, B, C, D are $n \times m$, $n \times n$, $m \times m$, $m \times n$ matrices, respectively, whose elements are complex-valued continuous functions of a real variable x on the interval $[a, \infty)$ and where F is an arbitrary $n \times m$ matrix of complex numbers. Since the function $A + BR + RC + RDR$ is locally Lipschitz, the solution to (I) is unique wherever it exists. Let m_{I} denote the maximal interval of existence, i.e.,

$$m_{\mathrm{I}} = [a, b); \qquad b = \sup\{s : \text{(I) has a solution on } [a, s]\}.$$

We now construct the associated integral operator. The matrices A, B, C, D, and F will be the same as in (I).

A continuous $n \times m$ matrix kernel $k(t, s)$, $t, s \geqslant a$, will be called *fundamental* for the matrices $A(t)$, $B(t)$, $C(t)$, and F, if k is continuously differentiable on each of the sets $t > s$, $t < s$ and $t = s$, and satisfies the system of equations

$$\begin{aligned} &\text{(a)}\ \frac{\partial}{\partial t} k(t, s) = B(t)\, k(t, s), \qquad t \geqslant s, \\ &\text{(b)}\ \frac{\partial}{\partial s} k(t, s) = k(t, s)\, C(s), \qquad t \leqslant s, \\ &\text{(c)}\ \frac{d}{dt} k(t, t) = A(t) + B(t)\, k(t, t) + k(t, t)\, C(t), \qquad t \geqslant a, \\ &\text{(d)}\ k(a, a) = F. \end{aligned} \tag{2.1}$$

The above equations specify the function $k(t, s)$ uniquely. The precise form of this function is not important. However, the construction is given below for completeness:

Let $\Phi(t, \xi)$ and $\Psi(\xi, s)$ be the $n \times n$ and $m \times m$ matrix solutions to the equations

$$\frac{\partial}{\partial t} \Phi(t, \xi) = B(t)\, \Phi(t, \xi), \qquad \Phi(\xi, \xi) = I_{n \times n},$$

$$\frac{\partial}{\partial s} \Psi(\xi, s) = \Psi(\xi, s)\, C(s), \qquad \Psi(\xi, \xi) = I_{m \times m},$$

where I is the identity matrix. Then the function

$$k(t, s) = \Phi(t, a) F \Psi(a, s) + \int_a^{\min(t, s)} \Phi(t, \xi)\, A(\xi)\, \Psi(\xi, s)\, d\xi, \quad t, s \geqslant a,$$

has the desired properties.

Now suppose $D(t)$ is factored in the form

$$D(t) = G(t)\, H(t),$$

where G and H are continuous matrix functions on $[a, \infty)$ of orders $m \times p, p \times n$. (One such factorization is always possible, i.e., $p = n$, $G = D$, $H = I$.) Define the operator T_x on the space $C_p[a, x]$ of p-vectors, whose components are complex-valued continuous functions on $[a, x]$, by

$$(T_x f)(t) = H(t) \int_a^x k(t, s)\, G(s) f(s)\, ds, \; t \in [a, x].$$

T_x is an integral operator on $C_p[a, x]$ with kernel

$$T(t, s) = H(t)\, k(t, s)\, G(s). \tag{2.2}$$

(The kernel in the Wiener–Hopf equation for optimum filtering is of this form [1].)

Let m_{II} denote the maximal interval of existence of the inverse of $(I - T_x)$; more precisely

$$\begin{aligned} m_{\mathrm{II}} &= [a, c), \\ c &= \sup\{y : (I - T_x) \text{ has an inverse in the algebra} \\ &\qquad \text{of bounded linear operators on } C_p[a, x], \text{ for each } x \in [a, y]\}. \end{aligned}$$

Finally, for $x \in m_{\mathrm{II}}$ and $a \leqslant t, s \leqslant x$, let $K(t, s, x)$ denote the Fredholm resolvent of T_x, that is, the kernel of the operator $(I - T_x)^{-1} - I$.

Our main result is the following.

THEOREM 1. (a) *Let $R(x)$ satisfy the Riccati equation* (I) *on m_{I}:*

$$\text{(I)} \qquad \begin{cases} \dfrac{d}{dx} R = A(x) + B(x)\, R + RC(x) + RD(x)\, R, \\ R(a) = F. \end{cases}$$

For each fixed $t, s \in m_{\mathrm{I}}$, define the functions $U(t, x)$ and $V(s, x)$ as the (unique) solutions to the initial-value problems:

$$\text{(II)} \qquad \begin{cases} \dfrac{\partial}{\partial x} U(t, x) = U(t, x)\{C(x) + D(x)\, R(x)\}, \quad t \leqslant x, \\ U(t, t) = H(t)\, R(t); \end{cases}$$

$$\text{(III)} \qquad \begin{cases} \dfrac{\partial}{\partial x} V(s, x) = \{B(x) + R(x)\, D(x)\}\, V(s, x), \quad s \leqslant x, \\ V(s, s) = R(s)\, G(s). \end{cases}$$

Then the Fredholm resolvent $K(t, s, x)$ *of* T_x *exists for each* $x \in m_{\mathrm{I}}$ *and is given by*

$$\text{(IV)} \qquad \begin{cases} \dfrac{\partial}{\partial x} K(t, s, x) = U(t, x)\, D(x)\, V(s, x), & a \leqslant t, s \leqslant x, \\ K(t, x, x) = U(t, x)\, G(x), \\ K(x, s, x) = H(x)\, V(s, x). \end{cases}$$

(b) *For* $x \in m_{\mathrm{II}}$, *let* $K(t, s, x)$ *be the Fredholm resolvent of* T_x. *Set*

$$U(t, x) = H(t)\, k(t, x) + \int_a^x K(t, s, x)\, H(s)\, k(s, x)\, ds, \qquad a \leqslant t \leqslant x;$$

equivalently, $U(t, x)$ *is the* (*unique*) *solution to*

$$U(t, x) = H(t)\, k(t, x) + \int_a^x T(t, s)\, U(s, x)\, ds, \qquad a \leqslant t \leqslant x. \tag{2.3}$$

Then the function

$$R(x) = k(x, x) + \int_a^x k(x, s)\, G(s)\, U(s, x)\, ds$$

is a continuously differentiable solution to (I) *on* m_{II}.

COROLLARY 1. *The operator* $(I - T_x)$ *has a bounded inverse for all* $x \in [a, b]$ *if and only if the Riccati equation* (I) *has a solution on* $[a, b]$. *Hence* $m_{\mathrm{I}} = m_{\mathrm{II}}$.

Any condition which keeps the number 1 out of the spectrum of T_x, for all $x \geqslant a$, will force global existence for the solution to (I). This will be the case, in particular, if for all $x \geqslant a$, either $\| T_x \| < 1$ or T_x is equivalent to a negative-semidefinite operator. An example of the latter is given by

COROLLARY 2. *Let* A, B, C, D *and* F *be* $n \times n$ *matrices such that*

(i) $C(t)$ = adjoint of $B(t)$,

(ii) $A(t)$ *and* $(-D(t))$ *are nonnegative-definite for all* $t \geqslant a$,

(iii) F *is nonnegative-definite.*

Then the Riccati equation (I) *has a solution for all* $t \geqslant a$.

The Riccati equations occuring in the theory of optimal filtering and regulation (cf. [*1*]) satisfy the hypotheses of Corollary 2.

Theorem 1 and its corollaries are proved in Section 5.

Remarks. (i) Equations (I)–(IV) are related to the Riccati systems considered by Redheffer and Reid [*8*].

(ii) The algorithm contained in part (a) is as follows: To compute the value of $K(t_0, s_0, b)$, where $b \in m_{\mathrm{I}}$ and $t_0 < s_0$, say, integrate (I) from $x = a$ to $x = t_0$, at which point the initial condition for (II) is known. Adjoin (II) and integrate both (I) and (II) from $x = t_0$ to $x = s_0$, at which point the initial conditions for (III) and (IV) are known. Adjoin (III) and (IV) and integrate the entire system from $x = s_0$ to $x = b$.

(iii) In terms of the resolvent kernel $K(t, s, x)$, the (unique) solution to the equation

$$f(t, x) = g(t) + \int_a^x T(t, s) f(s, x)\, ds, \qquad x \in m_{\mathrm{II}}, \tag{2.4}$$

may be expressed by

$$f(t, x) = g(t) + \int_a^x K(t, s, x)\, g(s)\, ds.$$

Alternatively, following the same procedure as in Theorem 1, one may compute the solution of (2.4) from the initial-value problem

$$\begin{cases} \dfrac{\partial}{\partial x} f(t, x) = U(t, x)\, G(x) f(x, x), \qquad t \leqslant x, \quad x \in m_{\mathrm{II}}, \\ f(t, t) = g(t) + H(t)\, W(t), \\ \dfrac{d}{dt} W = \{B(t) + R(t)\, D(t)\}\, W + R(t)\, G(t) g(t), \\ W(a) = 0. \end{cases}$$

(iv) The kernel $T(t, s)$ in Theorem 1 is not as special as it may initially appear. In fact, any continuous kernel may be approximated uniformly on compacta by such a function. This is seen, for example, by approximating each component of the kernel by an exponential series. The matrix $k(t, s)$ can then be chosen as a diagonal matrix of elements of the form $\exp(zt - ws)$. The matrices H and G are then determined by the weights in the relevant series. Using the method of "differential approximation," it is also possible to base the approximation on the coefficient matrices and initial conditions of (2.1) and (2.2), rather than on special solutions (cf. [*9*]).

Further, Theorem 1 itself may be extended to the class of continuous kernels by passing to infinite-dimensional differential equations. The rigorous treatment is basically an exercise in the style of Dieudonné ([*10*], Chapter 10).

3. Examples

In this section we give some examples for scalar kernels.

First consider a kernel of the form

$$T(t, s) = \sum_{n=1}^{p} h_n(t) g_n(s) \exp(-\gamma_n \mid t - s \mid), \tag{3.1}$$

where $\{h_n\}$ and $\{g_n\}$ are continuous functions and $\{\gamma_n\}$ are constants. In matrix notation,

$$T(t, s) = H(t)\, k(t, s)\, G(s),$$

where $H(t) = (h_1(t),\dots, h_p(t))$, $G(s)$ is the transpose of

$(g_1(s),\dots, g_p(s))$ and $k(t, s) = \operatorname{Diag}[\exp(-\gamma_1 \mid t - s \mid),\dots, \exp(-\gamma_p \mid t - s \mid)]$.

The function $k(t, s)$ is a fundamental kernel for the matrices A, B, C, F where

$$-\tfrac{1}{2}A = B = C = \operatorname{Diag}(-\gamma_1 ,\dots, -\gamma_p)$$

and $F = I$. The relevant Riccati equation here is

$$\begin{cases} \dfrac{d}{dx} R = -2B + BR + RB + RG(x)\, H(x)\, R, \\ R(a) = I. \end{cases}$$

With suitable convergence of the series (3.1), the case $p = \infty$ can be handled in the same manner as Theorem 1. Similar kernels are treated in [11] and [12].

More generally, one can treat kernels of the form

$$T(t, s) = \iint h_{\pm}(t, \alpha)\, g_{\pm}(s, \beta)\, e^{-\alpha t - \beta s}\, dW_{\pm}(\alpha, \beta),$$

where the $(+)$ representation holds for $t \geqslant s$, the $(-)$ representation holds for $t < s$, and where $W_{\pm}(\alpha, \beta)$ are functions of two complex variables (cf. [6]).

The efficacy of this method lies in the choice of the representation so as to make the underlying differential equations numerically stable.

4. Special Factorizations

Following Gohberg and Krein [13], a continuous matrix kernel $T(t, s)$ defined on $[a, b] \times [a, b]$ is said to admit a *special factorization* if there exist continuous Volterra kernels $V_{\pm}(t, s)$, vanishing on $t > s$ and $t < s$, respectively, which have the property that the Fredholm resolvent $K(t, s)$ of $T(t, s)$ is representable in the form

$$K(t, s) = V_{+}(t, s) + V_{-}(t, s) + \int_a^b V_{+}(t, y)\, V_{-}(y, s)\, dy, \quad a \leqslant t, s \leqslant b. \tag{4.1}$$

In operator notation this is equivalent to the equation

$$(I - T)^{-1} = (I + V_{+})(I + V_{-}).$$

In [13] it is proved that $T(t, s)$ has a special factorization if and only if the operator T_x on $C_p[a, x]$ defined by

$$(T_x f)(t) = \int_a^x T(t, s) f(s)\, ds, \qquad a \leqslant t \leqslant x$$

has a bounded inverse for each x in $[a, b]$. For the kernels considered in Section 2, this result, in combination with Theorem 1, leads to another characterization of special factorizations.

THEOREM 2. *Let $k(t, s)$ be a fundamental kernel for the matrices $A(t)$, $B(t)$, $C(t)$, and F and let $T(t, s) = H(t)\, k(t, s)\, G(s)$, $a \leqslant t, s \leqslant b$, where $D(t) = G(t)\, H(t)$. Then $T(t, s)$ admits a special factorization if and only if the Riccati equation*

$$\begin{cases} \dfrac{d}{dx} R = A(x) + B(x)\, R + RC(x) + RD(x)\, R, \\ R(a) = F, \end{cases}$$

has a solution on $[a, b]$. In this case the Volterra kernels $V_{\pm}$ are given by

$$\begin{aligned} V_{+}(t, s) &= U(t, s)\, G(s), & t &\leqslant s, \\ &= 0, & t &> s, \\ V_{-}(t, s) &= H(t)\, V(t, s), & t &> s, \\ &= 0, & t &\leqslant s, \end{aligned}$$

where U and V are defined by Eq. (II) *and* (III) of Theorem 1.

5. PROOF OF THEOREM 1

Throughout this proof, we make constant use of two well-known facts about Fredholm integral equations.

(i) Uniqueness: For each $x \in m_{\text{II}}$ and $g \in C_p[a, x]$, the equation

$$f(t) = g(t) + \int_a^x T(t, s) f(s)\, ds, \qquad a \leqslant t \leqslant x, \tag{5.1}$$

has a unique solution in $C_p[a, x]$.

(ii) Superposition: If f_i is the solution to (5.1) with $g = g_i$, $i = 1, 2$ and A_1, A_2 are constant matrices, then $A_1 f_1 + A_2 f_2$ is the solution to (5.1) with $g = A_1 g_1 + A_2 g_2$.

We first prove Part (b). For $x \in m_{\text{II}}$, let $K(t, s, x)$ be the resolvent kernel of $T(t, s)$. The resolvent equation gives

$$K(t, s, x) = T(t, s) + \int_a^x T(t, r)\, K(r, s, x)\, dr, \qquad a \leqslant t, \quad s \leqslant x. \tag{5.2}$$

For $s > x$, $s \in m_{\text{II}}$, we also let $K(t, s, x)$ denote the unique solution to (5.2). Similarly, for $y \in m_{\text{II}}$,

$$K(t, s, y) = T(t, s) + \int_a^y T(t, r)\, K(r, s, y)\, dr, \qquad a \leqslant t \leqslant y, \quad s \in m_{\text{II}}. \tag{5.3}$$

Subtracting (5.2) from (5.3) gives, for $a \leqslant t \leqslant x \leqslant y$,

$$\begin{aligned}\{K(t, s, y) - K(t, s, x)\} &= \int_x^y T(t, r)\, K(r, s, y)\, dr \\ &\quad + \int_a^x T(t, r)\{K(r, s, y) - K(r, s, x)\}\, dr.\end{aligned} \tag{5.4}$$

It is easily checked that $K(t, s, x)$ is continuous in the variables t, s, x simultaneously. By superposition and uniqueness, it follows that (5.4), considered as an integral equation in t with s, x, y fixed, has the solution

$$\{K(t, s, y) - K(t, s, x)\} = \int_x^y K(t, r, x)\, K(r, s, y)\, dr. \tag{5.5}$$

This, in turn, implies that $K(t, s, x)$ is continuously differentiable in x and

$$\frac{\partial}{\partial x} K(t, s, x) = K(t, x, x)\, K(x, s, x). \tag{5.6}$$

(This is the variational formula for Fredholm resolvents due independently to Bellman [4] and Krein [7].)

Now let $U(t, x)$ be defined by

$$U(t, x) = H(t)\, k(t, x) + \int_a^x K(t, s, x)\, H(s)\, k(s, x), \qquad a \leqslant t \leqslant x. \tag{5.7}$$

Therefore $U(t, x)$ is continuous in t and x and since $k(t, x)$ is continuously differentiable in x for $t \leqslant x$, the same is true for $U(t, x)$. Differentiating (5.7) and using (2.1) and (5.6) gives

$$\frac{\partial}{\partial x} U(t, x) = H(t)\, k(t, x)\, C(x) + K(t, x, x)\, H(x)\, k(x, x) + K(t, x, x) \int_a^x K(x, s, x)\, H(s)\, k(s, x)\, ds + \int_a^x K(t, s, x)\, H(s)\, k(s, x)\, ds\, C(x). \tag{5.8}$$

Since $T(t, s) = H(t)\, k(t, s)\, G(s)$, we see, on comparing (5.7) and (5.2), that $K(t, x, x) = U(t, x)\, G(x)$. Using this in (5.8) then gives

$$\frac{\partial}{\partial x} U(t, x) = U(t, x)\{C(x) + G(x)\, U(x, x)\}, \qquad t \leqslant x. \tag{5.9}$$

Now set

$$R(x) = k(x, x) + \int_a^x k(x, s)\, G(s)\, U(s, x)\, ds, \tag{5.10}$$

so that $U(x, x) = H(x)\, R(x)$. Equation (5.10) shows that R is continuously differentiable on m_{II}; further, on using (2.1) and (5.9), we have

$$\begin{aligned} \frac{d}{dx} R &= A(x) + B(x)\, k(x, x) + k(x, x)\, C(x) + k(x, x)\, G(x)\, U(x, x) \\ &\quad + B(x) \int_a^x k(x, s)\, G(s)\, U(s, x)\, ds \\ &\quad + \int_a^x k(x, s)\, G(x)\, U(s, x)\, ds\, \{C(x) + G(x)\, U(x, x)\} \\ &= A(x) + B(x)\, R + RC(x) + RG(x)\, H(x)\, R. \end{aligned}$$

Equation (I) follows on noting $D(x) = G(x)\, H(x)$ and $R(a) = k(a, a) = F$.

Now we turn to part (a). Let $R(x)$ be the solution to (I) on m_{I} and let $U(t, x)$ be defined by (II). It is important to note that $U(t, x)$ is uniquely defined by this initial-value problem and is continuously differentiable in $x \geqslant t$, $x \in m_{\mathrm{I}}$.

Now put

$$S(x) = k(x, x) + \int_a^x k(x, s)\, G(s)\, U(s, x)\, ds, \qquad x \in m_{\mathrm{I}}.$$

Thus the function $Y(x) = S(x) - R(x)$ satisfies the linear equation

$$\frac{d}{dx} Y = BY + Y(C + DR), \qquad x \in m_{\mathrm{I}}.$$

The initial condition $Y(a) = S(a) - R(a) = 0$, then forces $Y \equiv 0$, i.e.,

$$S(x) \equiv R(x), \qquad x \in m_{\mathrm{I}}. \tag{5.11}$$

Now define $W(t, x)$ by

$$W(t, x) = H(t)\, k(t, x) + \int_a^x H(t)\, k(t, s)\, G(s)\, U(s, x)\, ds, \qquad a \leqslant t \leqslant x. \tag{5.12}$$

On differentiating (5.12), as in part (b) above, we have

$$\frac{\partial}{\partial x} W(t, x) = W(t, x)\{C(x) + D(x)\, R(x)\}, \qquad a \leqslant t \leqslant x.$$

Further, $W(t, t) = H(t)\, S(t) = H(t)\, R(t)$ by (5.11). Thus $W(t, x)$ is a continuously differentiable solution to (II) and by the uniqueness already noted it follows that $W(t, x) \equiv U(t, x)$, $a \leqslant t \leqslant x$. In other words $U(t, x)$ is the unique solution to (2.3) on m_{I}.

A similar argument shows that $V(s, x)$ defined by (III) is the unique solution to

$$V(s, x) = k(x, s)\, G(s) + \int_a^x V(y, x)\, T(y, s)\, dy, \qquad a \leqslant s \leqslant x, \quad x \in m_{\mathrm{I}}. \tag{5.13}$$

From (2.3) and (5.13) it is readily checked that the function $K(t, s, x)$, defined by (IV), satisfies the resolvent equations:

$$K(t, s, x) = \begin{cases} T(t, s) + \int_a^x T(t, y)\, K(y, s, x)\, dy, \\ T(t, s) + \int_a^x K(t, y, x)\, T(y, s)\, dy, \qquad a \leqslant t, s \leqslant x. \end{cases}$$

Thus, $K(t, s, x)$ is the Fredholm resolvent of T_x for $x \in m_{\mathrm{I}}$. Q.E.D.

COROLLARY 1. Part (*a*) implies that $m_{\mathrm{I}} \subset m_{\mathrm{II}}$, while Part (*b*) gives the opposite inclusion.

COROLLARY 2. Let $\Phi(t, \xi)$ be the principle matrix for $B(t)$; that is,

$$\frac{\partial}{\partial t} \Phi(t, \xi) = B(t)\, \Phi(t, \xi), \qquad t \geqslant \xi,$$

$$\Phi(\xi, \xi) = I_{n \times n},$$

and let * denote the operation of conjugate-transpose. Then, in the terminology of Section 2, the kernel

$$k(t, s) = \Phi(t, a) F \Phi^*(s, a) + \int_a^{\min(t,s)} \Phi(t, \xi)\, A(\xi)\, \Phi^*(s, \xi)\, d\xi, \tag{5.14}$$

is fundamental for the matrices $A(t)$, $B(t)$, $B^*(t)$ and F.

Now set $T(t, s) = k(t, s)\, D(s)$. It follows from Theorem 1 that the Riccati equation

$$\begin{cases} \dfrac{d}{dx} R = A(x) + B(x)\, R + RB^*(x) + RD(x)\, R, \\ R(a) = F \end{cases}$$

has a solution on m_{II}. We will prove, under the hypotheses of this corollary, that the spectrum of T_x is contained in the negative real axis, for all $x \geqslant a$. The Fredholm alternative then implies $m_{\text{II}} = [a, \infty)$.

By hypothesis (ii) on D we may write

$$(-D(s)) = J(s)\, J(s),$$

where J is nonnegative-definite matrix for all $s \geqslant a$. Since the spectrum of the integral operator with kernel $k(t, s)(-D(s))$ is identical to that of operator with kernel $J(t)\, k(t, s)\, J(s)$, it suffices to prove that $k(t, s)$ is the kernel of a nonnegative-definite operator; precisely, for every $f \in C_p[a, x]$,

$$\int_a^x f^*(t)\, dt \int_a^x k(t, s) f(s)\, ds \geqslant 0. \tag{5.15}$$

Using (5.14), we can write the left-hand side of (5.15) as $N_1 + N_2$, where

$$N_1 = \left(\int_a^x f^*(t)\, \Phi(t, a)\, dt\right) F \left(\int_a^x \Phi^*(s, a) f(s)\, ds\right),$$

and

$$N_2 = \int_a^x f^*(t)\, dt \int_a^x \left(\int_a^{\min(t,s)} \Phi(t, \xi)\, A(\xi)\, \Phi^*(s, \xi)\, d\xi\right) f(s)\, ds. \tag{5.16}$$

The number N_1 is nonnegative by hypothesis (iii). On interchanging the orders of integration in (5.16), we have

$$N_2 = \int_a^x d\xi \left(\int_\xi^x f^*(t)\, \Phi(t, \xi)\, dt\right) A(\xi) \left(\int_\xi^x \Phi^*(s, a) f(s)\, ds\right).$$

Hypothesis (ii) on A shows that N_2 is nonnegative, which then completes the proof.

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